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An extended stochastic Allee model with harvesting and the risk of extinction of the herring population

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Abstract

Overexploitation of commercially beneficial fish is a serious ecological problem around the world. The growth profiles of most of the species are likely to follow density regulated theta-logistic model irrespective of any taxonomy group [Sibly et al., Science, 2005]. Rapid depletion of population size may cause reduced fitness, and the species is exposed to Allee phenomena. Here sustainability is addressed by modelling the herring population as a stochastic process and computing the probability of extinction and expected time to extinction. The models incorporate an Allee effect, crowding effects which reduce birth and death rates at large populations, and two possible choices of harvesting models viz. linear harvesting and nonlinear harvesting. A seminal attempt is made by Saha [Saha et al., Ecol. Model, 2013] for this economically beneficial fish, but ignored the vital phenomena of harvesting. Moreover, in this model, the demographic stochasticity is introduced through the white-noise term, which has certain limitations when harvesting is introduced into the system. White noise is appropriate for such a system where immigration and emigration are allowed, but a harvesting model is rational for a closed system. The demographic stochasticity is introduced by replacing an ordinary differential equation model with a stochastic differential equation model, where the instantaneous variance in the SDE is derived directly from the birth and death rates of a birth-death process. The modelling parameters are fit to data of the herring populations collected from Global Population Dynamics Database (GPDD), and the risk of extinction of each population is computed under different harvesting protocols. A threshold for handling times is computed beneath which the risk of extinction is high. This is proposed as a recommendation to management for sustainable harvesting.

Keyword:

Allee effect; Birth-death process; Handling time; Theta-Logistic model.

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1 Introduction

One of the major problems worldwide is the overexploitation of biological resources. The proper utilization and exploitation of renewable resources are intricate research areas to be explored. The exploitation of resources should be carried out in such a way that it does not hamper the sustainability of the resources. Herring is one of the beneficial fish populations, which is in threatened condition for overexploitation in various geographical zones [47].

The overall status of the herring population can be well understood from the data of the Global Population Dynamics Database (GPDD) maintained by the Imperial College of London, European Ecological Society. Although the data set may not be available for the same time window, it could give us an exhaustive picture of the herring population over the entire globe.

The herring fish is showing threatened status in some places such as the Palaearctic Biogeographic Zone of Iceland (GPDD ID-1765), Nearctic Biogeographic Zone of Canada, North America (GPDD ID-1759) [47]. Both sides of the Atlantic ocean (GPDD ID-1758), Coast of southern Africa (GPDD ID-1913), Prince Robert district of British Columbia (GPDD ID-1772) they are the least concern. In a few places such as Venezuela (GPDD ID-1978), rivers of Benin, Nigeria, Western Cameroon (GPDD ID-1959) and some regions of the Baltic sea (GPDD ID-1741) they are vulnerable (www.iucnredlist.org). Note that, GPDD contains records of these population sizes over different years (<http://www.sw.ic.ac.uk/cpb/cpb/gpdd.html>). Mathematical modelling and exploratory data analysis are the basis of understanding the population status and extinction threats of this species.

The available literature on developing the mathematical models for capturing the dynamics of this economically beneficial fish is limited except for the seminal contribution by Saha et al. [47]. In a study on individual population dynamics by Sibly et al. [50] concludes that most of the density and per-capita growth rate (pgr) relationship of different species are concave in nature. This relationship can be well explained by the theta-logistic model. We expect that, when the species is of least concern and not exposed to the Allee effect, it is likely to follow the density regulated theta-logistic model. However, when the species is exposed to the Allee effect, the density regulated Allee model is suitable to explain the growth status of the population [47]. Saha et al. [47] explored the evidence of an Allee effect in herring fish through the density regulated Allee equation but unfortunately, this study is far apart from reality, because of neglecting a vital event of harvesting in the model formation.

The characteristics of group defense and schooling are the natural properties of the herring fish. These characteristics are also common to many other commercially beneficial fishes [43, 15]. The extinction threat to the species will be increased if they possess these characteristics and are exposed to Allee effect and severe exploitation. The schooling fish are more likely to be susceptible to the strong Allee effect because of over-harvesting [15]. Schooling is generally considered as a defense mechanism against the attack of predators [45]. However, when a fisherman harvest on a school, a large number of fishes are captured at a time. This may not be the case when the fishes do not form school. So the fishes in school may become more vulnerable to extinction because of uncontrolled harvesting [15, 26].

In this article, we propose a density regulated stochastic θ -logistic model and extended stochastic Allee model incorporating both the linear and nonlinear harvesting terms in the herring fish popula-

tion. We introduce demographic variation in our proposed equation for incorporating stochasticity in the model. Demographic stochasticity is an inherent property of the species as the individuals go through life cycle transitions randomly, regardless of the environment from where it comes from. Under demographic uncertainty, populations can always become extinct and the probability of extinction depends on population size and structure [34, 36]. For this, demographic stochasticity is described as one of the four extinction vortices [16]. Moreover, the species are more likely to stay near low population size in the presence of the Allee effect along with the additional impact of harvesting phenomena. The environmental stochasticity results from the temporal fluctuation in mortality and reproductive rates of all individuals in a population in the same or similar fashion. So environmental stochasticity is prominent in population for all sizes [15]. Moreover, the genetic heterogeneity and the variation in the sex ratio are two essential features of the fish population like herring [19, 54]. Thus it is important to understand the factors which influence the demographic stochasticity to identify the pattern of extinction risk when the population size is low. For this reason, we assume a constant environment to understand the process which influences demographic stochasticity better and ignore the environmental stochasticity and the random catastrophic event in the model [35].

We are representing the entire system in the framework of the birth-death process. We separate the growth rate of the population into the birth and death rates based on an ecological perspective. For this separation, we consider the crowding effect in the birth and death rate for the θ -logistic model as the population size is sufficiently large, the crowding effect effective in both the birth and death rates. In the Allee model, we consider the crowding effect in the birth term only as at the low population size the crowding effect hampers the birth rate only. Here the harvesting term is used as the proxy of the additional death term. In this study, we use four data sets (GPDD ID-1759, ID-1765, ID-1741 and ID-1772) of four distinct geographic locations. The first two IDs are same with the study as [47]. The remaining two IDs can provide additional information on the species status of the other two geographical regions. Moreover, the density regulation and the Allee effect are two natural phenomena, and we cannot control them. However, in comparison, the fisherman and the management personnel have control over the handling time which is a function of the nonlinear harvesting term. Treating handling time as a control parameter, we will discuss its association with the extinction threats so that the management policy can be determined and the species conservation status is preserved.

The different sections of the article are as follows. In section 2, we give some ideas about both the theta-logistic and Allee model considering the linear and nonlinear harvesting. Then we propose our models in both the deterministic and stochastic frameworks and give some empirical estimate of the probability of extinction and expected time to extinction, which we have used for numerical results. In section 3, we find stability conditions for the equilibrium points. In section 4, we interpret our results through real data of the herring fish population. In section 5, we simulate our results through numerical technique, and in the last section 6, we conclude based on our results.

2 Model formulation

2.1 The deterministic model

We consider the general θ -logistic model, which is widely used to clarify the density dependence in the real population as well as time-series data. The mathematical representation of the θ -logistic model can be written as [9]

$$\frac{dx}{dt} = rx \left[1 - \left(\frac{x}{K} \right)^\theta \right], \quad (2.1)$$

where $x(t)$ denotes the density of resource population at time t , r is the intrinsic growth rate, K is the carrying capacity and θ is called the density regulation around carrying capacity. It should be noted that when $\theta = 1$, the equation (2.1) becomes the classical logistic model. In the θ -logistic model, the per-capita growth rate (pgr) is a monotonically decreasing function of population size. However, when the population size is low, an Allee effect may be ubiquitous and a critical population size may exist such that if the population size lies below that threshold value, pgr becomes negative. This critical depensation level can be the reason for the population to be more vulnerable to extinction. It is observed that schooling fishes are nearer to the critical depensation level due to overfishing that leads them into danger [15]. Indiscriminate harvesting in dense schools causes the population to decline severely. In such cases, the strength of density dependence may be significant and can have potential implications for the preservation of animal populations [47]. In addition to the Allee threshold, the strength of density regulation also plays a consequential role. We consider the following model to include the Allee effect due to Saha et al. [47]

$$\frac{dx}{dt} = rx \left(\frac{x}{K} - \frac{A}{K} \right) \left(1 - \left(\frac{x}{K} \right)^\theta \right). \quad (2.2)$$

Henceforth, we call the model (2.2) as Allee-Saha model (ASM). We consider two types of harvesting strategies viz. proportional or linear harvest rate and nonlinear harvest rate. In general, the linear harvesting or proportional harvesting strategy (H) is determined by catch-per-unit effort hypothesis [13] and presented as $H(x, E) = qEx$, where q is the catchability coefficient and E is the harvesting effort. This form of harvesting contains the following defects [22]

1. It considers the random search for every resource population.
2. It assumes that every individual to the resource population is equally likely to be captured.
3. H is unbounded function of E for a fixed x .
4. H is unbounded function of x for a fixed E .

The above four unrealistic features can be removed by introducing handling time. In such situation, the functional form of the harvesting policy is given by $H(x, E) = \frac{qEx}{aE + Lx}$ [22], where a is the degree of competition among the boats, fishermen and other technology used in fishing [4] and L is the product of capture rate and handling time [3]. If the capture rate is assumed to be constant, L is solely dependent on handling time and hence L can be treated as a representative of handling time. This modified form of the harvesting exhibit saturation concerning both resource size and effort. It is observed that, $h \rightarrow \left(\frac{q}{a} \right) x$ as $E \rightarrow \infty$ when x is fixed and $h \rightarrow \left(\frac{q}{L} \right) E$ as $x \rightarrow \infty$ when L is fixed [22].

Hereafter we call this modified harvest rate as non-proportional or nonlinear harvest rate. We consider harvesting as a death process, i.e., additional mortality, along with natural mortality. Introducing the harvesting phenomena in the models (2.1) and (2.2), we have

$$\frac{dx}{dt} = rx \left[1 - \left(\frac{x}{K} \right)^\theta \right] - H(x), \quad (2.3)$$

$$\frac{dx}{dt} = rx \left(\frac{x}{K} - \frac{A}{K} \right) \left(1 - \left(\frac{x}{K} \right)^\theta \right) - H(x), \quad (2.4)$$

where $H(x)$ is the harvesting component. In case of our model, we consider both linear and nonlinear harvesting.

2.2 Stochastic model

We can introduce the stochasticity in the model either by incorporating the demographic and environmental stochastic terms. The incorporation of either demographic or environmental stochasticity or both elucidate some debates and cross debates in several studies in the existing literature. The notable feature of the demographic stochasticity is that it would accelerate the risk of population decline or population extinction, even in the population exposed to a positive growth rate. Populations naturally consist of discrete entities, and demographic stochasticity is the consequence of small population size and discreteness [24]. Substantial studies [44, 32] have been done on demographic stochasticity concerning species conservation, as it is a problem of small populations. When the population growth is positive, the mean time to extinction in a model with demographic stochasticity scales exponentially with carrying capacity [23, 32, 29]. For a better understanding of the reader, we describe the relative comparison between demographic and environmental stochasticity in brief concerning the herring population in the subsequent section.

Allee effect in association with demographic stochasticity

Determination of the relative risks of extinction from demographic stochasticity, environmental stochasticity, and random catastrophes of a single population is of prime importance in statistical ecology. A seminal study by Lande et al. [33] assessed the relative risks by comparing asymptotic scaling relationships describing how the average time to extinction, increases with the carrying capacity of a population, under each stochastic factor alone. The relative importance of demographic and environmental stochasticity and random catastrophes discussed by Shaffer [49] is intuitively appealing and eventually may be supported by empirical evidence. In a study with a stochastic population dynamics model, Allen and Allen [5] take into account the random nature of the individual birth and death processes with demographic variability instead of the random fluctuations in the environment affect the entire population in a uniform pattern. Using the method of maximum likelihood estimation, Brännström and Sumpter [11] claim that their models with demographic noise have better proximity with both the simulated and real data in comparison with the environmental stochasticity. The authors also admit that the traditionally chosen stochastic analogues to deterministic models additive normally distributed noise and multiplicative lognormally distributed noise (environmental noise) generally fit all data sets well, but may not be important in practice always.

Chance independent events of individual mortality and reproduction cause random fluctuations in population growth rate, which is the foundation of demographic stochasticity, primarily in small and sparse populations. Note that, Allee effects are expected to operate only in a small population. On the other hand, environmental stochasticity is prominent in populations of all sizes. The relationship between Allee effects and demographic stochasticity has been critically discussed by ecologists over the last few years [15]. In this connection, lots of debates and cross debates came upon the inclusion of demographic stochasticity in the list of Allee effect mechanisms [32, 18, 8, 51]. To get an idea of how demographic stochasticity and Allee effect combine to affect the dynamic properties of a population, we will study the stochastic counterpart of the pgr function. Substantial references suggest that for the herring fish, sex ratio changes over the years. In an earlier study, the biased sex ratio of the herring spawning population in the Georges Bank area was observed. Extreme male-skewed sex ratios on spawning grounds for Atlantic cod *Gadus morhua* with typical coastal cod signatures are common [19]. Long-term changes in sex ratios of Pacific herring in Prince William Sound, Alaska has been recently assessed by Ward et al.[54]. Since in the proposed model, the rates are derived by partitioning a mean-field model this further implies that any effect of varying sex ratio is incorporated via a depression in per capita population growth at low populations due to reduced reproductive potential. Then the variance in sex ratios can be considered as a component of the Allee effect. Therefore the importance of varying sex ratios in herring is good motivation for incorporating an Allee effect in the model. Generally, the researcher should follow the study of Allen et al. [6] for technical details on the discrete-time branching process. However, in stochastic population models, the two most commonly used statistics in evaluating the population viability are (i) the extinction probability (i.e., the probability that the population goes extinct within a specified time interval), (ii) the mean time to extinction. As regards Allee effects, we are interested in how these characteristics respond to initial population size [15]. It is worthy to mention that environmental stochasticity is acting on the species constantly irrespective of the population size, so it does not have any special connection with the Allee effect. So substantial studies [5] ignored the environmental stochasticity in modelling population viability vortex. However, in comparison, demographic stochasticity is strongly attached to the Allee effect, as both are prominent in small populations.

The objective of the present study is to understand how demographic stochasticity and Allee effects contribute to extinction risk; then, it is reasonable to exclude environmental stochasticity as a simplifying modelling choice. This argument would be well motivated by the connection between demographic stochasticity and Allee effects that we already highlighted with the above discussions through supportive literature.

Birth-death process framework

In many research articles on harvesting, authors introduced stochasticity through white noise [14, 27]. Introducing demographic stochasticity through white noise has certain limitations. The white noise is a random fluctuation independent of the population density and may be caused due to the density independent immigration and emigration in the biological point of view. Therefore, in such a situation, the extinction state is not absorbing. Under this circumstance, there is a chance to recover the population

from local extinction due to the density independent immigration and emigration. So in the stochastic model, this local extinction is not permanent [25, 2]. In other words, the positive perturbation can save the population from extinction. Harvesting is rational for the closed system where the immigration and emigration do not occur directly in the model. So the incorporation of demographic stochasticity through the white noise is not appropriate for the harvesting model. This complication can be averted if the demographic stochasticity is incorporated through the birth-death process instead of any other commonly used noises [5, 52, 53].

Thus, in this work, we frame a suitable birth-death process using the deterministic model (3.1) and (3.2). We assume both the time and state space are continuous variables and the total population at time t is $X(t) \in [0, N]$. Let $p(x, t)$ denotes the probability density function for the continuous random variable $X(t)$. Then $\text{Prob}\{X(t) \in [a, b]\} = \int_a^b p(x, t) dx$.

This probability density function for the general birth-death process satisfies the well-known Kolmogorov forward differential equation [5]

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial[(b(x) - d(x))p(x, t)]}{\partial x} + \frac{1}{2} \frac{\partial^2[(b(x) + d(x))p(x, t)]}{\partial x^2}, \quad (2.5)$$

where $X(t) \in (0, N)$, $t \in (0, \infty)$, $p(x, 0) = \delta(x - x_0)$, $b(x)$ and $d(x)$ are the birth and death rates. Here the term $\mu(x) = b(x) - d(x)$ is called the infinitesimal mean and $v^2(x) = b(x) + d(x)$ is called infinitesimal variance.

For the Kolmogorov forward equation it can be easily found that the solution or the sample paths, $X(t)$ of a stochastic process satisfy the Ito stochastic integral equation [5, 21, 56]

$$X(t) = x_0 + \int_0^t [b(X(u)) - d(X(u))] du + \int_0^t \sqrt{b(X(u)) + d(X(u))} dW(u), \quad X(0) = x_0 > 0. \quad (2.6)$$

The integrals that appeared in the equation (2.6) consist of two parts among which the first part is deterministic and Riemann integrable, and the second part is a stochastic Ito integral. In the present work, the population size variable $X(t)$, is a random variable and satisfies the stochastic differential equation (2.6). Here, $X(t)$ for $t > t_0$ is characterised by the particular sample path of the Wiener process $W(t)$ for $t > t_0$ and the initial value x_0 . The process $W(t)$ for $t > t_0$ is independent of $X(t)$ for $t < t_0$ as the process $X(t)$ is a nonanticipating function of t . This shows that, for a given initial condition x_0 , the future propagation of $X(t)$ with time for $t > t_0$ is independent of $X(t)$ for $t < t_0$. Thus the process $X(t)$ is a Markov process [21]. Here, the Wiener process W is defined as $\Delta W(t) = W(t + \Delta t) - W(t)$, and it satisfies the normal distribution, $N(0, \Delta t)$. For simplicity, this equation can be expressed as the SDE [5]

$$\frac{dX(t)}{dt} = b(X(t)) - d(X(t)) + \sqrt{b(X(t)) + d(X(t))} \frac{dW}{dt}. \quad (2.7)$$

In a population model, the birth function $b(x)$ and the death function $d(x)$ should satisfy the following conditions. For the θ -logistic model, we assume that there exist two real numbers M and N with $M < N$ such that [5]

1. $b(0) = d(0) = 0$ and $b(x) \leq 0$ for $x \geq N$,
2. $b(x) > 0$ for $x \in (0, N]$ and $d(x) > 0$ for $x \in (0, N]$,
3. $b(x) > d(x)$ for $x \in (0, M]$,

242 4. $b(x) < d(x)$ for $x \in (M, N)$.

243 In addition to the above four properties, $b(x)$ and $d(x)$ are $C^2([0, N])$. For the ASM, we assume that
244 there exist real numbers M , N and A^* such that,

245 1. $b(0) = d(0) = 0$ and $b(x) \leq 0$ for $x \geq N$,

246 2. $b(x) > 0$ for $x \in (0, N]$ and $d(x) > 0$ for $x \in (0, N]$,

247 3. $b(x) < d(x)$ for $x \in [0, A^*)$ and (M, N) ,

248 4. we obtain a threshold value A^* such that $b(x) > d(x)$ for $x \in (A^*, M]$.

249 In addition to the above four properties, $b(x)$ and $d(x)$ are $C^2([0, N])$. The feasibility of these regularity
250 conditions is well discussed in Appendix A.

251 **Remark 1.** *The regularity conditions for birth and death rates of two models (3.1) and (3.2) are not*
252 *exactly the same. They differ in the conditions (3) and (4). This is because, for the Allee model, the*
253 *chance of species extinction is more below the critical threshold A^* and in such case, $b(x)$ is less than*
254 *$d(x)$. Beyond the critical threshold, the $b(x)$ is greater than $d(x)$.*

255 There are infinitely many forms of the birth and death function for the θ -logistic and the ASM.
256 However, we will consider only that form which is ecologically meaningful for the dynamical system
257 considered here. The following ecological facts have been taken under consideration to chose suitable
258 birth function and death function. The Allee effect is typically defined as the positive correlation between
259 population density and population fitness (survivability, natality, etc.) at low population size, whereas
260 crowding effect is the intraspecific competition induced a negative impact on the population growth and
261 fitness [1]. Individuals from a population with low density are unlikely to compete for resources and
262 will show an Allee effect [15, 31]. However, in small population crowding may act via hormonal stress
263 and other physiological altered activities instead of via competition for food. In particular, for a fish
264 population with a biased sex ratio towards males, crowding acts on the individual level in the presence of
265 any member of the same sex [39, 42]. Crowding even at a small population size may affect by triggering
266 the hormonal and physiological stress in female fishes [39]. The presence of another female fish reduces
267 the ovum production rate of a female fish in both large and small populations [38]. The birth rate of such
268 species like fish is dependent on and represented by the fecundity. This shows that the crowding affects
269 the birth rate equally irrespective of the size and growth pattern of the population. So, we must consider
270 the crowding effect in a population both in the presence and absence of the Allee effect. In the θ -logistic
271 model, the per capita growth rate (pgr) is high at low population density; in fact, it is maximum at zero
272 population size. One can expect that in the presence of the θ -logistic growth law the population size
273 is more likely to stay at high density. The large population size in fish school induces the competition
274 for food and defense in both the sexes of fishes in addition to the physiological stress in females. The
275 competition for inadequate food and sacrificed defense lead to an increment in death rate too. Only
276 cooperation at a low population size can negotiate this impact on death due to crowding. Allee-prone
277 herring population with a small school show the cooperation to overcome the impact of the crowding
278 on the death rate. So, in the local herring population with an Allee effect, the impact of crowding on
279 the death rate can be ignored. However, the population in the absence of the Allee effect will face the

competition for resources which raises its death rate.

These ecological aspects and the four properties mentioned earlier in this section lead us to separate the growth rate into the birth and death function as follows. The birth and the death functions for the θ -logistic model (3.1) are respectively

$$b(x) = (r+1)x - \frac{rx^{\theta+1}}{2K^\theta}, \quad d(x) = x + \frac{rx^{\theta+1}}{2K^\theta} + H(x) \quad (2.8)$$

and for the model (3.2) are respectively [57]

$$b(x) = \frac{rx^2}{K^{\theta+1}} (K^\theta + Ax^{\theta-1}) \left[1 - \frac{x^\theta}{K^\theta + Ax^{\theta-1}} \right], \quad d(x) = \frac{rAx}{K} + H(x). \quad (2.9)$$

Using two sets of birth and death rates, the θ -logistic and the ASM model with harvesting term are governed by the following stochastic differential equations

$$dX = \left[rX \left(1 - \left(\frac{X}{K} \right)^\theta \right) - H(X) \right] dt + \sqrt{\left[(r+1)X - \frac{X^{\theta+1}}{2K^\theta} \right] + \left[X + \frac{X^{\theta+1}}{2K^\theta} + H(X) \right]} dW(t) \quad (2.10)$$

and for the ASM

$$dX = \left[rX \left(\frac{X}{K} - \frac{A}{K} \right) \left(1 - \left(\frac{X}{K} \right)^\theta \right) - H(X) \right] dt + \sqrt{\left[\frac{rX^2}{K^{\theta+1}} (K^\theta + AX^{\theta-1}) \left(1 - \frac{X^\theta}{K^\theta + AX^{\theta-1}} \right) + \frac{rAX}{K} + H(X) \right]} dW(t). \quad (2.11)$$

Here the parameters of this model are described as earlier.

Probability of extinction

We often use the term first passage probability (the probability of reaching a large population size before reaching a small one) and the expected time to extinction to estimate the population viability [17] in the stochastic population model. Here, we are interested to evaluate the probability of extinction for the different initial sizes of the population. Consider $\xi(n; a, b)$ as the probability when the population arrives a before arriving at an upper population size b , where the initial population is n , and $0 < a \leq n \leq b$. A well-known formula gives,

$$\xi(n; a, b) = \frac{\int_n^b \exp[-\phi(x)] dx}{\int_a^b \exp[-\phi(x)] dx}$$

$$\phi(x) = 2 \int^x \frac{\mu(u)}{v^2(u)} du$$

$\xi(n; a, b)$ is equal to 1 when $n = a$ and is equal to 0 when $n = b$, is a strictly monotonic decreasing function of n in the (a, b) . We obtain the probability of extinction for our models by assuming $a \rightarrow 0$ and $b \rightarrow \infty$. We do not obtain an explicit analytic form for the probability of extinction due to the complexity of our models.

Expected time to extinction:

The expected time to extinction is one of the primary measurements of the vulnerability of a species to extinction in a stochastic environment. It gives a convenient assessment of the random time remaining

until extinction. It is often used as a proxy for more complicated measures of the risk of extinction [12]. The diffusion approximation is a well-known mathematical tool to describe the population fluctuations in the discrete-time birth-death process. Our proposed stochastic models can be well approximated by a diffusion process (2.7), where the mathematical expression of the corresponding infinitesimal mean ($\mu(x)$) and infinitesimal variance ($v^2(x)$) are mentioned earlier.

In this article, $\mu(x)$ are respectively $\left[rx \left(1 - \left(\frac{x}{K}\right)^\theta\right) - H(x)\right]$ and $\left[rx \left(\frac{x}{K} - \frac{A}{K}\right) \left(1 - \left(\frac{x}{K}\right)^\theta\right) - H(x)\right]$, where as $v^2(x)$ are respectively $\left[(r+1)x - \frac{x^{\theta+1}}{2K^\theta}\right] + \left[x + \frac{x^{\theta+1}}{2K^\theta} + H(x)\right]$ and $\left[\frac{rx^2}{K^{\theta+1}} (K^\theta + Ax^{\theta-1}) \left(1 - \frac{x^\theta}{K^\theta + Ax^{\theta-1}}\right) + \frac{rAx}{K} + H(x)\right]$ for the θ -logistic and the Allee models. The expected time to extinction of the population can be obtained using the distribution of the sojourn time (waiting time), using the following Green's function [30].

$$\begin{aligned} G(x; x_0) &= 2m(x)S(x), \quad x < x_0 \\ &= 2m(x)S(x_0), \quad x \geq x_0 \end{aligned}$$

where,

$S(x) = \int_1^x s(u)du$, $s(u) = \exp\left[-2 \int_1^u \frac{\mu(z)}{v^2(z)} dz\right] = \exp\left[-2 \int_1^u \frac{b(z)-d(z)}{b(z)+d(z)} dz\right]$ and $m(x) = \frac{1}{v^2(x)s(x)}$. We assume the population will be extinct if $x = 1$ [46]. Hence the expression of expected time to extinction will be $\int_1^\infty G(x; x_0)dx = \int_1^{x_0} G(x; x_0)dx + \int_{x_0}^\infty G(x; x_0)dx$

3 The stability analysis for the deterministic model:

Introducing the linear and nonlinear harvesting terms in the model (3.1), the equations become,

$$\begin{aligned} \frac{dx}{dt} &= rx \left[1 - \left(\frac{x}{K}\right)^\theta\right] - qEx \\ \frac{dx}{dt} &= rx \left[1 - \left(\frac{x}{K}\right)^\theta\right] - \frac{qEx}{aE + Lx} \end{aligned} \quad (3.1)$$

and introducing the same in equations (3.2) it become,

$$\begin{aligned} \frac{dx}{dt} &= rx \left(\frac{x}{K} - \frac{A}{K}\right) \left(1 - \left(\frac{x}{K}\right)^\theta\right) - qEx \\ \frac{dx}{dt} &= rx \left(\frac{x}{K} - \frac{A}{K}\right) \left(1 - \left(\frac{x}{K}\right)^\theta\right) - \frac{qEx}{aE + Lx} \end{aligned} \quad (3.2)$$

Let $f(x) = \frac{dx}{dt}$, we have obtained the equilibria of (3.1) and (3.2) by solving for the roots of $f(x)$, and the stability is determined by analyzing the sign of $f'(x)$. In both the cases, we have found the stability including the linear and nonlinear harvesting.

3.1 The θ -logistic harvesting model

Lemma 1. *For the linear harvest rate in equation (3.1), the trivial equilibrium $x = 0$ is stable (or unstable) according as*

$$r - qE < 0 \text{ (or } > 0),$$

and the nontrivial equilibria are stable (or unstable) according as

$$\theta(qE - r) < 0 \text{ (or } > 0).$$

Proof. See the appendix B(i). □

Lemma 2. *For the nonlinear harvest rate in equation (3.1), the trivial equilibrium point $x = 0$ is stable (or unstable) according as*

$$r - \frac{q}{a} < 0 \text{ (or } > 0),$$

and the nontrivial equilibria x^ are stable (or unstable) according as*

$$r - \frac{r}{K^\theta} x^{*\theta} (1 + \theta) - \frac{aqE^2}{(aE + Lx^*)^2} < 0 \text{ (or } > 0).$$

Proof. See the appendix B (i). □

3.2 The ASM with harvesting

Now we consider the ASM under harvesting. Note that, the ASM (2.2) has three equilibrium points, viz. 0, A , K . Among these equilibrium points 0 and K are the stable equilibrium points and A is the unstable equilibrium point. Here again $x = 0$ is the trivial equilibrium point for the model (2.2) with harvesting. The other equilibrium points are not analytically solvable but it can be evaluated numerically. We will comment on these equilibrium points in the numerical analysis section.

Lemma 3. *In the system (3.2), 0 is always a stable equilibrium point and the other nonzero equilibria x^* are stable (or unstable) according as*

$$\frac{r}{K} \left[-\frac{r}{K^{\theta+1}} x^{*2\theta} + \frac{rA}{K^{\theta+1}} x^{*2\theta-1} - \frac{K+r}{K^{\theta+1}} x^{*\theta+1} + \frac{A}{K^\theta} x^{*\theta} + \frac{r+K}{K} x^* - A \right] - qE < 0 \text{ (or } > 0)$$

for linear harvest rate.

Proof. See the appendix B(ii). □

Lemma 4. *In the equation (3.2), 0 is always a stable equilibrium point for nonlinear harvest rate and the nontrivial equilibria x^* are stable (or unstable) according as*

$$\frac{r}{K} \left[-\frac{r}{K^{\theta+1}} x^{*2\theta} + \frac{rA}{K^{\theta+1}} x^{*2\theta-1} - \frac{K+r}{K^{\theta+1}} x^{*\theta+1} + \frac{A}{K^\theta} x^{*\theta} + \frac{r+K}{K} x^* - A \right] - \frac{aqE^2}{(aE + Lx^*)^2} < 0 \text{ (or } > 0).$$

Proof. See the appendix B(ii). □

Remark 2. *For the theta-logistic model, the MSY always exists when we apply the nonlinear harvesting. However, when the species follow the Allee mechanism, the MSY may or may not exist for different choice of parameter values. As it is not relevant to our paper, we can ignore the detailing of the statement.*

4 Data analysis

In this paper, we validate our study through the population time-series data of the herring fish population in some specific regions available in the GPDD. We consider the population time-series data of the herring fish with the GPDD ID-1741, ID-1759, ID-1765, and ID-1772. For the data set GPDD ID-1741, the experiment was performed in the Baltic sea areas covering 28 and 29 S area and for the data set GPDD ID-1772, it was done in the Prince Robert district, British Columbia. For the GPDD ID-1765 and GPDD

ID-1759, the herring populations are obtained in the region of Canada and Iceland respectively. We have fitted these population data to the theta-logistic and the ASM considering both the linear and nonlinear harvest rates. We rescale the population size data by dividing each population size value by the common factor 10^3 .

For fitting, we can use either size or RGR equations. Note that, the rate modelling is often superior to the size modelling. Sometimes it is difficult and may be misleading for the experimenter to choose the underlying model – among the number of different choices by observing the shape of the size profile curves. On the contrary, the RGR plotted against the size or the time, which often gives a better recognition of the growth curve models suitable for the given data. Additionally, in most of the cases, the errors in the RGR have a significantly smaller variation in comparison with the errors associated with size data and the errors in the absolute growth rates [55, 48]. These authors have observed that, errors in the usual size models required a complicated autocorrelation structure whereas in the relative growth rate analysis, independent errors or a simple autocorrelation structure in the errors, usually fit the data. Hence, estimation based on the RGR data is more popular than the usual population size based estimation.

Raw size data for different species are available in the GPDD database but not the RGR data. For empirical estimates of RGR, we use Fisher's approximation [20]. The scatter plots of size vs empirical RGR are exhibiting more or less a quadratic trend for the IDs 1741, 1759, 1765 which is synergistic with the Allee effect phenomena (see figure: 1, 2, 3). On the other hand, the scatter plot for the ID-1765 exhibits a concave upward and decreasing trend, which has proximity with the θ -logistic model (figure: 4). So these plots give us a clue that, the ASM might be applicable for the GPDD ID-1741, ID-1759 and ID-1765 but the θ -logistic model is applicable for the ID-1772. As the ASM is not analytically solvable, so we use the RGR equation for the IDs 1741, 1759, 1765 and the size equation for the ID-1772. Here we obtain, if we consider the nonlinear harvesting for all the GPDD ID sets, the mean square error is minimum. Henceforth among all models, the best fitted curve is the ASM with the IDs 1741, 1759, 1765 and the θ -logistic model for the 1772 considering nonlinear harvesting.

Three model fitting curves for the ASM (see subplots d, e, f for figure: 1) can be clearly distinguishable in the bared eye. It is hard to distinguish the two subplots A1 and C1 for figure: 2, which are looking pretty similar. We think that it is reasonable to represent all the three subplots (A1, B1, C1) of the figure: 2 in a magnified version, which is depicted in the subplots A2, B2, C2 respectively. B2 is clearly distinguishable from A2 and C2 where, the entire RGR fitted curve lies in a negative Y axis which means fitness of the species is negative and so prone to extinction. The subplots A1 and C1 are not distinguishable in a bared eye even after magnification. However, from the model fitting criteria, there is a 10 percent (approx.) reduction in the residual sum of square. This implies the ASM with nonlinear harvesting provides the best-fitting ID for the data sets 1759.

It is indistinguishable for the two subplots A1 and C1 for the figure: 3, which are looking pretty similar. Here also we have magnified these two indistinguishable figures in the amplified subplots A2 and C2 respectively. The magnified figures exhibit that for all the three models, RGR-fitted curves lie in the negative Y axis. Note that, most of the raw data of RGR are negative in the figure: 3. Hence, the

species are under threat according to the model fitting profiles.

For the ID-1772, the θ -logistic model with nonlinear harvesting provides the best fitted compare with the ASM. As we already mentioned that RGR trend is approximately decreasing, so it is not rational to include the other figures for fitting with the ASM.

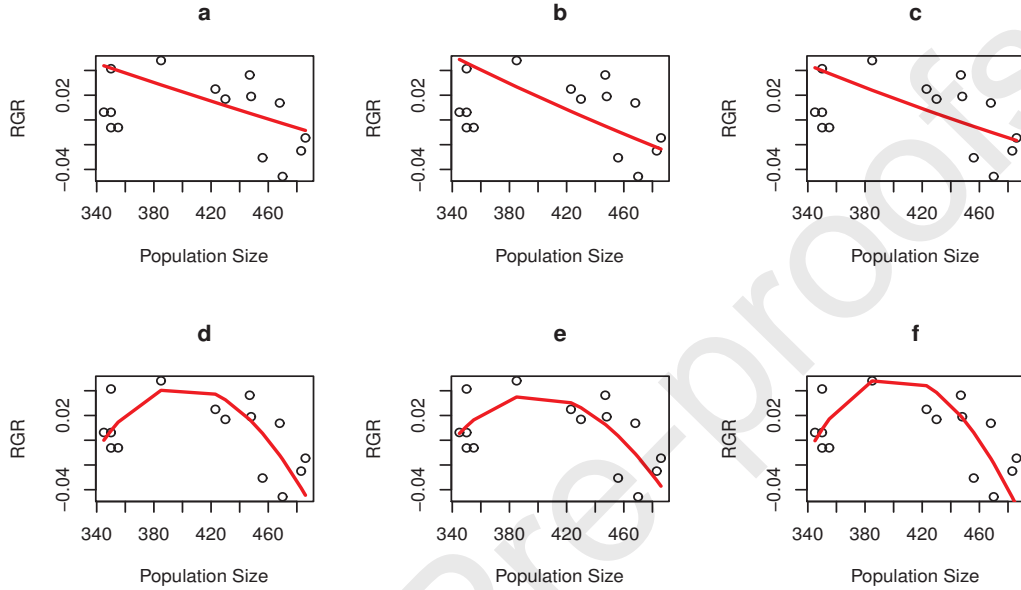


Figure 1: Fitting of the data for the herring fish population with the GPDD ID-1741. (a) Fitting with the θ -logistic model without harvesting. The estimated parameter values are: $r = 0.25$, $K = 462$, $\theta = 0.66$ and the residual sum of square error is 0.0291. (b) It represents the fitting with the θ -logistic model with linear harvesting. The corresponding estimated parameters are $r = 0.6$, $K = 500$, $\theta = 0.38$, $h = 0.03$ and the residual sum of square error is 0.0074. (c) Fitting with the θ -logistic model taking nonlinear harvesting. The estimated values are: $r = 0.65$, $K = 460$, $\theta = 0.23$, $q = 1.2$, $E = 5.30$, $a = 1.5$, $L = 5$ and the residual sum of square error is 0.0081. (d) Fitting of the ASM without harvesting. The estimated parameter values are: $r = 0.052$, $K = 460$, $\theta = 0.13$, $A = 345$ and the residual sum of square error is 1.687×10^{-5} . (e) Fitting with the ASM with the linear harvesting. The estimated values are: $r = 0.15$, $K = 460$, $\theta = 0.03$, $h = 0.001$, $A = 340$, and the residual sum of square error is 1.95621×10^{-6} . (f) Fitting of the ASM using the nonlinear harvesting. The estimated values are: $r = 0.40$, $K = 460$, $E = 0.446$, $\theta = 0.017$, $A = 345$, $q = 2.106$, $a = 1.01$, $L = 5.58$ and the residual sum of square error is 5.431193×10^{-7} .

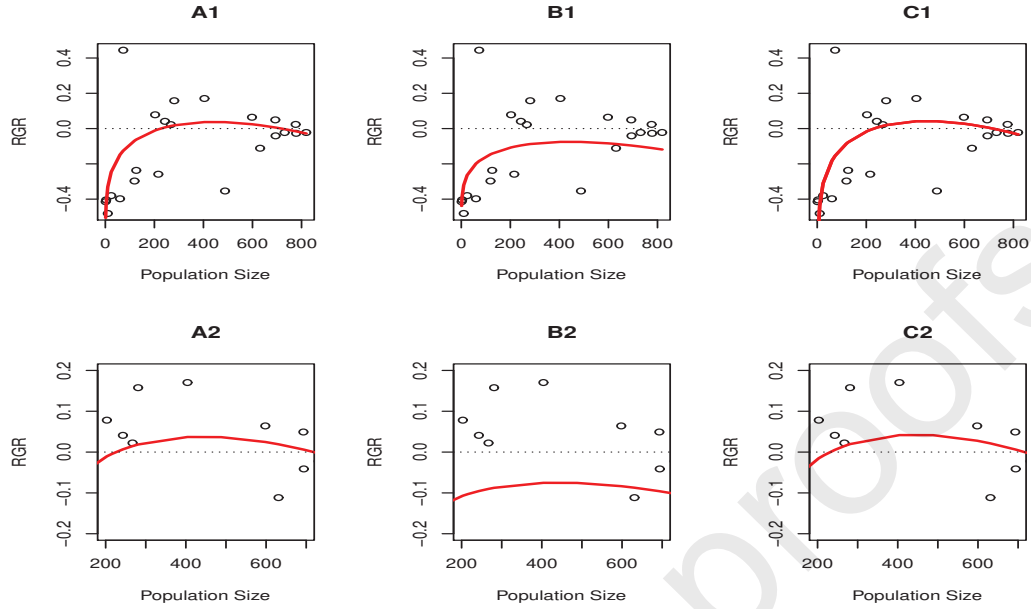


Figure 2: Fitting of the model considering different harvesting strategies for the herring fish population with the GPDD ID-1765. We obtain from the previous literature that GPDD ID-1765 has fitted the Allee model. Hence we consider the harvesting on the Allee model. $A1, B1, C1$ are the fitting of the Allee model without harvesting, linear harvesting and nonlinear harvesting respectively. $A2, B2, C2$ are the magnified figures for the subplots $A1, B1, C1$ respectively for a small region. The estimated parameter values for $A1$ are: $r = 0.12$, $K = 719$, $\theta = 0.003$, $A = 226$ and the residual sum of square is 0.334. The estimated values for $B1$ are: $r = 0.12$, $K = 719$, $\theta = 0.002$, $h = 0.005$, $A = 226$ and the residual sum of square is 1.24. For $C1$, the estimated values are: $r = 0.12$, $K = 719$, $E = .84$, $\theta = 0.0005$, $A = 226$, $q = .5$, $a = 9.03$, $L = 17.39$ and the residual sum of square is 0.032.

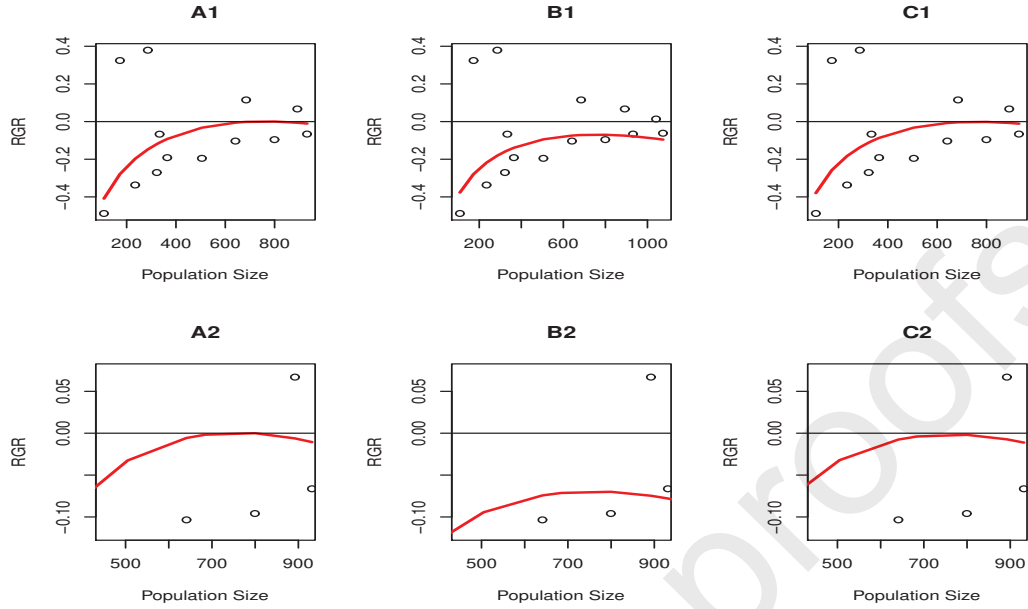


Figure 3: This figure represents the fitting of the model on data for the herring fish population with the GPDD ID-1759. We obtain from the previous literature that GPDD ID-1759 follows the Allee model. Hence we fit this data set with the Allee model. $A1, B1, C1$ are the fitted curve with the Allee model without harvesting, linear harvesting and nonlinear harvesting respectively. $A2, B2, C2$ are the magnified figures for the subplots $A1, B1, C1$ respectively for a small region. $A1$ is the fitting of the model on data for the herring fish population with GPDD ID-1759. Here the estimated parameter values are $r = 0.52$, $K = 1210$, $\theta = 0.0006$, $A = 800$ and the residual sum of square is 1.174. The estimated values in $B1$ are $r = 0.52$, $K = 1210$, $\theta = 0.0004$, $h = 0.07$, $A = 800$ and the residual sum of square is 2.61. In $C1$, the estimated values are $r = 0.78$, $K = 1210$, $E = 2.5$, $\theta = 0.0005$, $A = 710$, $q = 6.03$, $a = 4.1$, $L = 9.5$ and the residual sum of square is 0.9885.

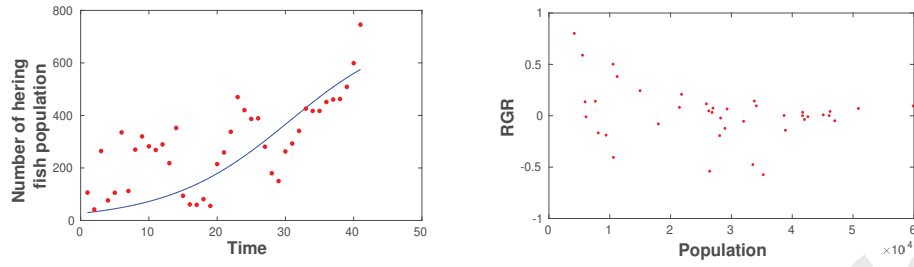


Figure 4: This figure represents the estimation of the size profile and RGR profile for the data set GPDD ID-1772. The parameter values are obtained from the best-fitted curve (the θ -logistic model with nonlinear harvest rate). The estimated values are: $r = 1.10$, $K = 672.2$, $E = 0.33$, $\theta = 0.104$, $q = 0.9$, $a = 0.06$, $L = 14.26$. Here we observe that all the sample paths are eventually become stable at carrying capacity. The second figure represents the RGR against the population size.

5 Results and discussion

In the present paper, we focus our attention on the extinction status of the herring fish population from four different geographic locations of Europe and America as mentioned earlier in the section 4. In the earlier work on the herring population, the harvesting was ignored and hence the chance of underestimation of the probability of extinction cannot be ruled out. We observe that, the growth profile of the herring fish from studied geographic locations can be better represented by the extended ASM incorporating linear/ nonlinear harvesting. The estimated probabilities of extinction of the herring fish from the four locations are provided in the table 1. Note that, the probability of extinction of the Canadian herring when harvesting is performed is found to be 0.99, which is pretty high and larger than its previous estimated value (0.93) as mentioned in Saha et al. [47]. The expected time to extinction of this species is also found to be significantly small when harvesting is considered. This time to extinction is also smaller than its previous estimate (133 years). For the Icelandic herring, the probability of extinction and the expected time to extinction are found to be 0.87 and 105 years respectively instead of the previous estimates 0.80 and 206 years. In these two cases, we observe that there is a possibility of underestimation of the extinction threats of herring population. A species under potential threat may be of least concern due to the underestimation of extinction probability. This study can give a caution to the management expert to look upon the matter seriously.

The handling time in nonlinear harvesting plays a key role to frame a suitable harvesting strategy. For any particular fish species there exists a threshold value (LP) of the handling time such that if the handling time goes below that threshold value, the only stable equilibrium is the extinction state. For higher values of L two stable equilibrium, including one extinction equilibrium exists. This can be well understood from the saddle-node bifurcation diagrams, which are depicted in the figure: 5, 6, 7. The two stable equilibria are separated by one unstable equilibrium (blue line in the bifurcation figure).

Thus, very low handling time may be an indirect indication of the extinction of the herring population. For example, in the case of the herring fish with GPDD ID-1759 and ID-1765, the ASM with nonlinear harvest rate is found to be the best fitted model. The estimated L values for these two fish populations are 20.5 and 28 respectively, which are much smaller than the respective threshold values 67 and 57. Thus these two fish species are expected to be under strong threat.

Fitting through RGR model using the grid search technique for the species GPDD ID-1741 is depicted in the figure: 1. Here also the ASM with nonlinear harvest rate is found to be the best fitted model in this case. The parameter estimation and the residual sum of square of the fitted model are thoroughly discussed in the figure caption. The estimated value of handling time(L) is 5.58. From the bifurcation diagram (figure: 5), we observe that the threshold value of L (limit point) is 1.616 from where the bistability starts and the population stay in the same state for any higher value of L . The only stable equilibrium point is at 0, which is occurred when L is less than 1.616. Note that, the estimated L value for this species is sufficiently larger than 1.616 and hence the possibility of the chance of extinction is relatively small compared to the herring fish with GPDD ID-1759 and 1765 (see table 1). When the size of the population stays at the upper stable region, the possibility of extinction cannot be ruled out since the sufficient amount of perturbation may lead the population size to shift to the other stable state that is an extinction state.

We know a species with the θ -logistic growth profile has low chance of extinction if the harvesting is absent [10]. Actually, the concept of critical Allee threshold below which the fitness is negative is non-existent in this model. We transform the model equation (θ -logistic harvesting) into the logistic map setup. If the transition graph shows the existence of the period three orbits, then it implies the existence of the points of every period, and the system goes to the chaos [7]. In this paper for the θ -logistic model, we claim that the model has no period doubling or period three orbits for the handling time L [37]. We have to choose the density regulation parameter θ and the handling time L so that the equilibrium points are in the stable region. To understand the combined effect of θ and L on the stability, we draw two parameters bifurcation diagram, which is depicted in the figure: 8. This diagram for the general θ -logistic harvesting model shows that the moderate value of the handling time L and θ is appropriate conditions for the species sustainability.

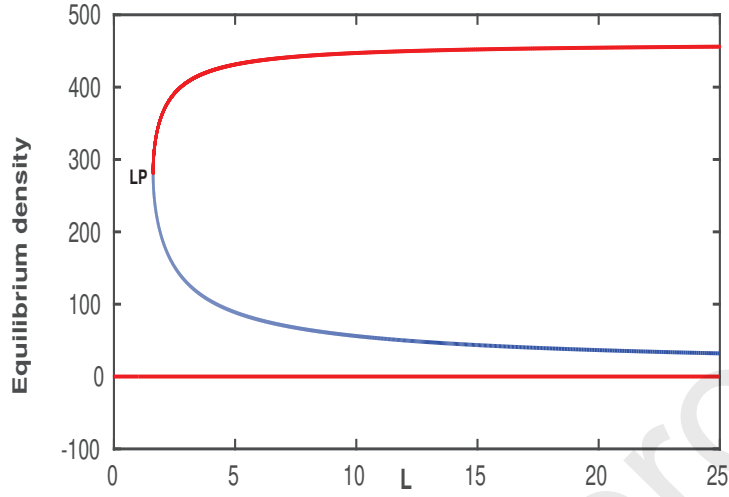


Figure 5: This figure represents the bifurcation diagram (saddle-node) of equilibrium density vs. L (for GPDD ID-1741). The red line represents the stable equilibrium points, and the blue line represents the unstable equilibrium points. The other parameters are $A = 345$, $r = 0.40$, $K = 460$, $a = 1.01$, $q = 2.106$, $E = .446$, $\theta = 0.017$. Here LP is the limit point of the saddle-node bifurcation.

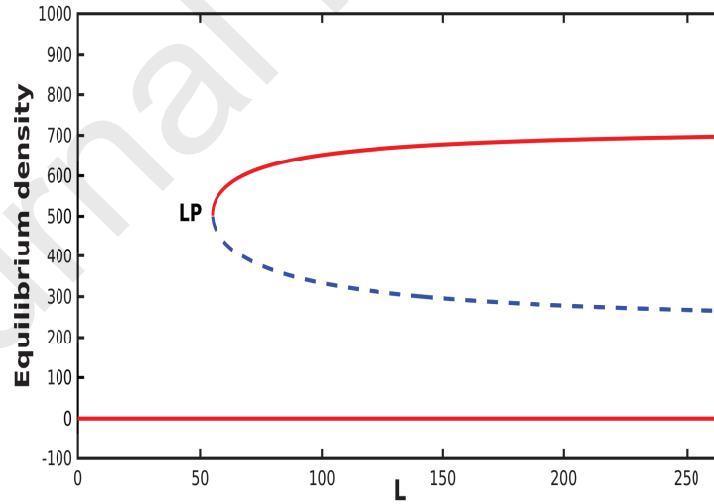


Figure 6: This figure represents the bifurcation diagram of equilibrium density vs. L (for GPDD ID-1765). The red line represents the stable equilibrium points, and the blue dots represent the unstable equilibrium points. The parameters are $r = 0.12$, $K = 719$, $E = .84$, $\theta = 0.0005$, $A = 226$, $q = .5$, $a = 9.03$, $L = 17.39$.

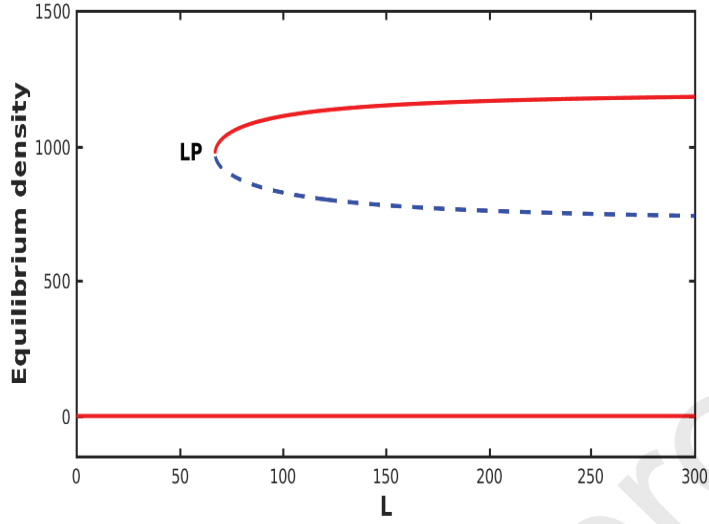


Figure 7: This is the bifurcation diagram of equilibrium density vs. L for the data set GPDD ID-1759. The red line represents the stable equilibrium points, and the blue dots represent the unstable equilibrium points. The other parameters are $r = 0.78$, $K = 1210$, $E = 2.5$, $\theta = 0.00048$, $A = 710$, $q = 6.03$, $a = 4.1$, $L = 9.5$.

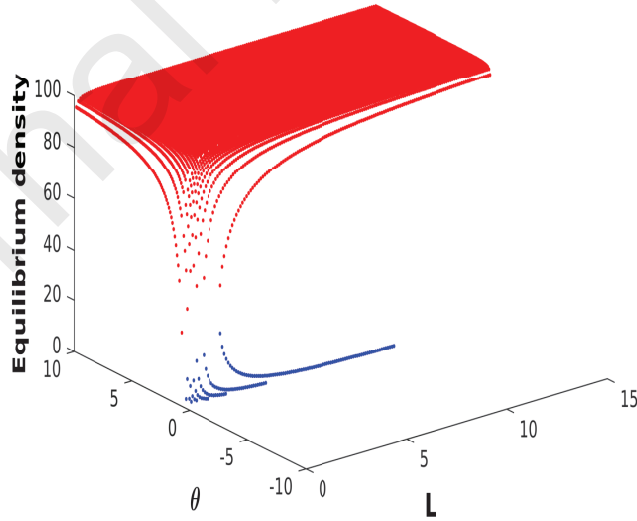


Figure 8: The plot exhibits the equilibrium density against the two important parameters L , θ . The red dots indicate the stable equilibrium states whereas the blue dots represent the unstable equilibrium.

Table 1: Probability and expected time to extinction

GPDD ID	Expected time to extinction	Probability of extinction
GPDD ID-1741	Without harvesting : 540 years	*
	With linear harvesting : 175 years	*
	With nonlinear harvesting : 263 years	0.66
GPDD-ID-1765	Without harvesting : 320 years	*
	With linear harvesting : 55 years	*
	With nonlinear harvesting : 105 years	0.87
GPDD ID-1759	Negligible	0.99

Remark 3. * : The interpretation of the symbol '*' is described under this remark. The probability of extinction is calculated only for the entities related to the nonlinear harvesting as we observed that all the data sets have a good proximity with the models having nonlinear harvesting terms. We are not interested in the probability of extinction for the entities associated with the linear and without harvesting for the three GPDD IDs. So these entities of table 1 with the subtitle probability of extinction are filled up with the symbol '*'. The row with GPDD ID-1772, has not been included in this table as the data having good proximity with the θ -logistic model instead of the extinction-prone Allee model. Note that, the estimated values of the expected time to extinction and the probability of extinction has a good agreement in the output.

6 Conclusion

The indiscriminate usage of biological resources is increasing. Thus, the sustainability of such resources is at stake [47]. In this context, applying an appropriate harvesting strategy is essential. Saha et al.[47] analyzed the extinction status of the herring fish ignoring the harvesting issue. Herring fish is a commercially beneficial fish and harvested substantially throughout the year. Hence proper understanding of the herring fish dynamics through a theoretical model has immense practical value. We propose a theoretical extended density regulated stochastic Allee model incorporating harvesting issue. We introduce demographic stochasticity instead of environmental variation. A recent study [40] on flour beetle (*Tribolium*) concludes that many species currently viewed as at risk of extinction from environmental stochasticity could instead be at much higher risk from undetected demographic variance. This demographic variance is determined by sex ratio variation and demographic heterogeneity that has been mistakenly endorsed to environmental stochasticity. Note that the evidence of the sex ratio variation and the genetic heterogeneity motivate us to incorporate demographic stochasticity in our proposed model [41, 28].

The density regulation (θ) is the inherent property of the species, and it is fixed for a specific location. However, the experimenter has some control over the handling time. From the present study,

the biologists could get some clue for choosing optimal handling time so that the species can sustain with regular harvesting. This claim is well established through the population data of the beneficial herring fish at the Prince Robert district, British Columbia, Canada and Iceland region. In this article, we identify some local species of herring in the Baltic sea, Canadian and Iceland region, where the Allee model (ASM) with nonlinear harvesting is evident. In this case, the model exhibits bistable equilibrium and for this, the system may undergo a regime shift from the stable non-zero state to the extinction state. We can prevent the system from entering into the extinction state from a non-zero stable state by increasing handling time provided the initial size of the population is at a moderate level. We conclude that the conservation biologists should prefer the nonlinear harvesting strategy in comparison with the proportional harvesting as an adequate handling time is safe for the resource population.

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Appendix

Appendix A (i): The condition for birth rate and death rate for the θ -logistic model

For the model 3.1, we consider the birth rate as

$$b(x) = (r+1)x - \frac{x^{\theta+1}}{2K^\theta},$$

and the death rates corresponding to linear and nonlinear harvest rate are respectively

$$\begin{aligned} d(x) &= x + \frac{x^{\theta+1}}{2K^\theta} + qex \\ d(x) &= x + \frac{x^{\theta+1}}{2K^\theta} + \frac{qex}{aE + Lx}. \end{aligned}$$

Theorem 1. If $(r - \frac{q}{a}) > 0$ then $\exists M, N$ such that $b(x) > d(x), \forall x \in (0, M)$ and $b(x) < d(x), \forall x \in (M, N)$.

Proof. $b(x) - d(x) = rx \left(1 - \left(\frac{x}{K}\right)^\theta\right) - \frac{qEx}{aE+Lx} = f(x)$ (say). Now $f(0) = 0$ and $f'(0) = (r - \frac{q}{a}) > 0$ (from our assumption). Again $f(x) < 0 \forall x \geq K$. Hence $\exists M \in (0, K)$ such that $f(M) = 0$ and $f(x) > 0 \forall x \in (0, M)$ and $f(x) < 0 \forall x > M$.

We have,

$$b(x) = x \left[(r+1) - \frac{1}{2} \left(\frac{x}{K}\right)^\theta \right] > 0 \Rightarrow x < K [2(r+1)]^{\frac{1}{\theta}}.$$

We assign $N = K [2(r+1)]^{\frac{1}{\theta}}$. Clearly $N > 0$ as $r > 0$ and $b(x) > 0$ if $x \in (0, N)$. \square

Remark: Using similar process as above we can find M and N for proportional harvesting also.

Appendix (ii): The condition for birth rate and death rate for the ASM

We consider the ASM as

$$\frac{dx}{dt} = rx \left(\frac{x}{K} - \frac{A}{K}\right) \left(1 - \left(\frac{x}{K}\right)^\theta\right) = r \frac{A+K}{K^{\theta+1}} x^{\theta+1} \left(1 - \frac{x}{A+x}\right) - \frac{rA}{K^\theta} x^\theta = \frac{rx^2}{K^{\theta+1}} (K^\theta + Ax^{\theta-1}) \left[1 - \frac{x^\theta}{K^\theta + Ax^{\theta-1}}\right] + \frac{rAx}{K}.$$

When we introduce the nonlinear harvesting phenomena then the model reduce to

$$\frac{dx}{dt} = rx \left(\frac{x}{K} - \frac{A}{K}\right) \left(1 - \left(\frac{x}{K}\right)^\theta\right) + \frac{qEx}{aE+Lx}.$$

In this case we can write the RHS of the above model of the form $b(x) - d(x)$,

$$\text{where, } b(x) = \frac{rx^2}{K^{\theta+1}} (K^\theta + Ax^{\theta-1}) \left[1 - \frac{x^\theta}{K^\theta + Ax^{\theta-1}}\right] \text{ and } d(x) = \frac{rAx}{K} + \frac{qEx}{aE+Lx}.$$

Here we observe that $b(x)$ and $d(x)$ are always positive in the interval $(0, N]$. To justify $b(x) < d(x)$, we

$$\text{have, } rx \left(\frac{x}{K} - \frac{A}{K}\right) \left(1 - \left(\frac{x}{K}\right)^\theta\right) - \frac{qEx}{aE+Lx} < 0.$$

The sufficient condition for the above inequality is

$$\frac{r}{K}(x - A) - \frac{qE}{aE + Lx} < 0 \quad [\text{Putting } \left(\frac{x}{K}\right)^\theta = 0].$$

Which implies,

$$\begin{aligned}
&\Rightarrow (x - A)(aE + Lx) - \frac{qEK}{r} < 0, \\
&\Rightarrow Lx^2 - (LA - aE) - \left(AaE + \frac{qEK}{r}\right) < 0, \\
&\Rightarrow x = \frac{(LA - aE) \pm \sqrt{(LA - aE)^2 + 4L(AaE + \frac{qEK}{r})}}{2L}, \\
&\Rightarrow (x - \alpha)(x - \beta) < 0,
\end{aligned}$$

where,

$$\alpha = \frac{(LA - aE) + \sqrt{(LA - aE)^2 + 4L(AaE + \frac{qEK}{r})}}{2L}, \quad \beta = \frac{(LA - aE) - \sqrt{(LA - aE)^2 + 4L(AaE + \frac{qEK}{r})}}{2L}.$$

Consider $\beta = -m$ (say) and $m > 0$. Therefore $(x - \beta) > 0$, implies $(x - \alpha) < 0$, $\Rightarrow x < \alpha$. We can assume $A^* = \alpha$, which implies,

$$r \left(\frac{A^*}{K} - \frac{A}{K} \right) \left(1 - \left(\frac{M}{K} \right)^\theta \right) > \frac{qE}{aE + LA^*} \quad (6.1)$$

Remark: Using similar process as above we can find M and N for proportional harvesting also.

Appendix B: The stability analysis for the deterministic model

(i) The θ -logistic harvesting model

Case 1: Linear harvest rate

From the equation (3.1),

$$\frac{dx}{dt} = rx \left[1 - \left(\frac{x}{K} \right)^\theta \right] - qEx = f(x) \text{ (say)}. \quad (6.2)$$

The θ -logistic model has two equilibrium points, viz. 0, and K . For the model (6.2), $x = 0$ is also a trivial equilibrium point. The nonzero equilibrium points are obtained as follows:

$$\begin{aligned}
&r \left(1 - \left(\frac{x}{K} \right)^\theta \right) - qE = 0 \\
&\Rightarrow x^\theta = K^\theta \left(1 - \frac{qE}{r} \right) \\
&\Rightarrow x = K \left(1 - \frac{qE}{r} \right)^{\frac{1}{\theta}}
\end{aligned}$$

So, the nonzero equilibrium point is given by $x^* = K \left(1 - \frac{qE}{r} \right)^{\frac{1}{\theta}}$.

$$\text{Now } f'(x) = r \left(1 - \left(\frac{x}{K} \right)^\theta \right) - \frac{r\theta}{K^\theta} x^{\theta-1} - qE$$

So $f'(0) = r - qE$. Hence $x = 0$ is stable (or unstable) equilibrium point according to

$$r - qE < 0 \text{ (or } > 0 \text{)}.$$

For the other nonzero equilibrium points,

$$f' \left(K \left(1 - \frac{qE}{r} \right)^{\frac{1}{\theta}} \right) = \theta(qE - r).$$

Hence x^* is stable or unstable according to

$$\theta(qE - r) < 0 \text{ (or } > 0 \text{)}.$$

Case 2: Nonlinear harvest rate

From the equation (3.1),

$$\frac{dx}{dt} = rx \left[1 - \left(\frac{x}{K} \right)^\theta \right] - \frac{qEx}{aE + Lx} = f(x) \text{ (say)}, \quad (6.3)$$

where, $x = 0$ is a trivial equilibrium point of the above equation. We obtain the non trivial equilibrium points corresponding to the model (6.3) setting

$$f(x) = 0. \quad (6.4)$$

$$\begin{aligned} &\Rightarrow r \left(1 - \left(\frac{x}{K} \right)^\theta \right) (aE + Lx) - qE = 0 \\ &\Rightarrow \frac{rL}{K^\theta} x^{\theta+1} + \frac{aEr}{K^\theta} x^\theta - rLx + (qE - raE) = 0 \end{aligned}$$

The roots of the equation (6.4) are the equilibrium points which can be obtained numerically. For the trivial solution $x = 0$, $f'(0) = (r - \frac{q}{a})$. Hence $x = 0$ is stable or unstable equilibrium point according to

$$r - \frac{q}{a} < 0 \text{ (or } > 0 \text{)}.$$

We have,

$$f'(x) = r \left(1 - \left(\frac{x}{K} \right)^\theta \right) - \frac{r\theta}{K^\theta} x^\theta - \frac{aqE^2}{(aE + Lx)^2} = r - \frac{r}{K^\theta} x^\theta (1 + \theta) - \frac{aqE^2}{(aE + Lx)^2}.$$

So, if x^* is an equilibrium point, then x^* is stable (or unstable) according to

$$r - \frac{r}{K^\theta} x^{*\theta} (1 + \theta) - \frac{aqE^2}{(aE + Lx^*)^2} < 0 \text{ (or } > 0 \text{)}.$$

(ii) The ASM with harvesting

Case 1: Linear harvest rate

Considering the model (3.2), with linear harvest rate,

$$\frac{dx}{dt} = rx \left(\frac{x}{K} - \frac{A}{K} \right) \left(1 - \left(\frac{x}{K} \right)^\theta \right) - qEx = f(x) \text{ (say)}. \quad (6.5)$$

The equilibrium points are given by,

$$\begin{aligned} &rx \left(\frac{x}{K} - \frac{A}{K} \right) \left(1 - \left(\frac{x}{K} \right)^\theta \right) - qEx = 0, \\ &\Rightarrow \frac{r}{K^{\theta+1}} x^{\theta+1} - \frac{rA}{K^{\theta+1}} x^\theta - \frac{r}{K} x + \left(\frac{rA}{K} + qE \right) = 0. \end{aligned}$$

Hence $x = 0$ is the trivial solution and we get the other nontrivial solution by solving the value of x using numerical technique.

$$\text{Now } f'(x) = \frac{r}{K} \left[-\frac{r}{K^{\theta+1}} x^{2\theta} + \frac{rA}{K^{\theta+1}} x^{2\theta-1} - \frac{K+r}{K^{\theta+1}} x^{\theta+1} + \frac{A}{K^\theta} x^\theta + \frac{r+K}{K} x - A \right] - qE,$$

and $f'(0) = -\left(\frac{rA}{K} + qE\right)$. Therefore 0 is always a stable equilibrium point. In case of other nontrivial solution if x^* is an equilibrium point, then from the stability criteria, x^* will be stable (or unstable) according to

$$f'(x^*) = \frac{r}{K} \left[-\frac{r}{K^{\theta+1}} x^{*2\theta} + \frac{rA}{K^{\theta+1}} x^{*2\theta-1} - \frac{K+r}{K^{\theta+1}} x^{*\theta+1} + \frac{A}{K^\theta} x^{*\theta} + \frac{r+K}{K} x^* - A \right] - qE < 0 \text{ (or } > 0 \text{)}.$$

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Case 2: Nonlinear harvest rate

If we take the nonlinear harvest rate for the ASM, the corresponding equilibria are obtained from the roots of the equation

$$rx \left(\frac{x}{K} - \frac{A}{K} \right) \left[1 - \left(\frac{x}{K} \right)^\theta \right] - \frac{qEx}{aE + Lx} = 0.$$

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This implies,

$$\begin{aligned} & \frac{r}{K} \left[\left(1 - \left(\frac{x}{K} \right)^\theta \right) (aEx + Lx^2 - aEa - ALx) \right] - qE = 0 \\ \Rightarrow & \frac{rL}{K^{\theta+1}} x^{\theta+2} + \left(\frac{r(aE - AL)}{K^{\theta+1}} \right) x^{\theta+1} - \frac{aEAr}{K^{\theta+1}} x^\theta - \frac{Lr}{K} x^2 + \frac{r(AL - aE)}{K} x + \left(\frac{rqEA}{K} + qE \right) = 0. \end{aligned}$$

Hence $x = 0$ is also a trivial solution and other non-zero solutions can be obtained by numerically. Now,

$$f'(x) = \frac{r}{K} \left[-\frac{r}{K^{\theta+1}} x^{2\theta} + \frac{rA}{K^{\theta+1}} x^{2\theta-1} - \frac{K+r}{K^{\theta+1}} x^{\theta+1} + \frac{A}{K^\theta} x^\theta + \frac{r+K}{K} x - A \right] - \frac{aqE^2}{(aE + Lx)^2},$$

and $f'(0) = -\left(\frac{rA}{K} + \frac{q}{a}\right)$. Hence 0 is also a stable equilibrium point. For the case of other nontrivial solution, if x^* is an equilibrium point then x^* will be stable (or unstable) according to

$$f'(x^*) = \frac{r}{K} \left[-\frac{r}{K^{\theta+1}} x^{*2\theta} + \frac{rA}{K^{\theta+1}} x^{*2\theta-1} - \frac{K+r}{K^{\theta+1}} x^{*\theta+1} + \frac{A}{K^\theta} x^{*\theta} + \frac{r+K}{K} x^* - A \right] - \frac{aqE^2}{(aE + Lx^*)^2} < 0 \text{ (or } > 0).$$