Chapter 2 - Operations

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From "Book of Abstract Algebra" by Charles C. Pinter

A. Examples of Operations

Which of the following rules are operations on the indicated set? ($\mathbb{Z}$ designates the set of integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers.) For each rule which is not an operation, explain why it is not.

1. $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$.
2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.
3. $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set $\mathbb{R}$.
4. Subtraction, on the set $\mathbb{Z}$.
5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution

1. This is not an operation on $\mathbb{Q}$ because $a * b$ is not uniquely defined and $\mathbb{Q}$ is not closed under $*$. If a and b are rational numbers they can be written as $a = \frac{c}{d}$ and $b = \frac{e}{f}$ where $c, d, e, \text{and} f$ are integers, $d \neq 0$, and $f \neq 0$. If we let $c = 2$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{2}{1} \cdot \frac{2}{1}} = \sqrt{2 \cdot 2} = \sqrt{4}$$

and since $\sqrt{4} = \pm 2$ we see that $a * b$ is not uniquely defined. Now let $c = 3$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{3}{1} \cdot \frac{2}{1}} = \sqrt{6}$$

and we see that there is no rational number f such that $f \cdot f = 6$, therefore $\sqrt{6}$ is not a rational number. Thus, $\mathbb{Q}$ is not closed under $*$.

2. This is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$ because the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$. For example, if we let $a = 2$ and $b = 1$ then

$$a \ln b = 2 \ln 1 = 0$$

and $0 \notin \{x \in \mathbb{R} : x > 0\}$. Therefore the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$ and $*$ is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$.

3. This is not an operation on $\mathbb{R}$ because $a * b$ is not uniquely defined. If we solve for x we see

$$\begin{aligned} x^2 - a^2b^2 &= 0 \\ x^2 &= a^2b^2 \end{aligned}$$

since a^2 and b^2 will always be positive numbers, then a^2b^2 will be positive as well. Then solving for x we have

$$x = (ab, -ab)$$

and we see that $a * b$ is not uniquely defined and therefore $*$ is not an operation on $\mathbb{R}$.

4. This is an operation on $\mathbb{Z}$.

5. This is not an operation on $\mathbb{Z}$ because the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$. For example, if we let $a = 10$ and $b = 5$ then $5 - 10 = -5$ and $-5 \notin \{n \in \mathbb{Z} : n \geq 0\}$. Therefore the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$.

6. This is an operation on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

B. Properties of Operations

Each of the following is an operation $*$ on $\mathbb{R}$. Indicate whether or not

(i). it is commutative,

(ii). it is associative,

(iii). $\mathbb{R}$ has and identity element with respect to $*$,

(iv). every $x \in \mathbb{R}$ has an inverse with respect to $*$.

Instructions For (i), compute $x * y$ and $y * x$, and verify whether or not they are equal. For (ii), compute $x * (y * z)$ and $(x * y) * z$, and verify whether or not they are equal. For (iii), first solve the equation $x * e = x$ for e; if the equation cannot be solved, there is identity element. If it *can* be solved, it is still necessary to check that $e * x = x * e = x$ for any $x \in \mathbb{R}$. If it checks, then e is an identity element. For (iv), first not that it there is no identity element, there can be no inverses. If there is an identity element e, first solve the equation $x * x' = e$ for x'; if the equation cannot be solved, x does not have an inverse. If it can be solved, check to make sure that $x * x' = x' * x = e$. If this checks, x' is the inverse of x.

1. $x * y = \sqrt{x^2 + y^2}$
2. $x * y = |x + y|$
3. $x * y = |xy|$
4. $x * y = x - y$
5. $x * y = xy + 1$
6. $x * y = \max\{x, y\}$
7. $x * y = \frac{xy}{x+y+1}$

Solution

	Commutative	*Associative*	*Identity*	*Inverses*
1.	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐

(i). $x * y = \sqrt{x^2 + y^2}$
$y * x = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2}$ because $+$ is a commutative operation on $\mathbb{R}$
(*Thus* $*$ *is commutative.*)

(ii). $x * (y * z) = x * \left(\sqrt{y^2 + z^2}\right) = \sqrt{x^2 + \left(\sqrt{y^2 + z^2}\right)^2} = \sqrt{x^2 + y^2 + z^2}$
$(x * y) * z = \sqrt{(x^2 + y^2)} * z = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$
(*Thus* $*$ *is associative.*)

(iii). Solve $x * e = x$ for x: $\sqrt{x^2 + e^2} = x$ therefore $e = 0$.
Check: $x * 0 = \sqrt{x^2 + 0} = \sqrt{x^2} = x$; $0 * x = \sqrt{0 + x^2} = \sqrt{x^2} = x$
Therefore, 0 is the identity element.
($*$ *has an identity element*)

(iv). Solve $x * x^{'} = e$ for $x^{'}$.

$$
\begin{aligned}
x * x^{'} &= e \\
\sqrt{x^2 + x^{'2}} &= 0 \\
x^{'} &= \sqrt{-x^2} \\
x^{'} &= \pm ix
\end{aligned}
$$

where i is the imaginary unit.

Now check that $x * x^{'} = x^{'} * x = e$.

If $x^{'} = ix$ then

$$
\begin{aligned}
x * x' &= \sqrt{x^2 + (xi)^2} \\
&= \sqrt{x^2 + (x^2 \cdot i^2)} \\
&= \sqrt{x^2(1 + i^2)} \\
&= \sqrt{x^2(1 + (-1))} \\
&= \sqrt{x^2 \cdot 0} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
x' * x &= \sqrt{(xi)^2 + x^2} \\
&= \sqrt{(x^2 \cdot i^2) + x^2} \\
&= \sqrt{(1 + i^2)x^2} \\
&= \sqrt{(1 + (-1))x^2} \\
&= \sqrt{0 \cdot x^2} = 0
\end{aligned}
$$

If $x' = -ix$ then

$$
\begin{aligned}
x * x' &= \sqrt{x^2 + (-xi)^2} \\
&= \sqrt{x^2 + (xi)^2} && \text{exponent rule: } (-a)^n = a^n, \text{if } n \text{ is even} \\
&= \sqrt{x^2 + (x^2 \cdot i^2)} \\
&= \sqrt{x^2(1 + i^2)} \\
&= \sqrt{x^2(1 + (-1))} \\
&= \sqrt{x^2 \cdot 0} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
x' * x &= \sqrt{(-xi)^2 + x^2} \\
&= \sqrt{(xi)^2 + x^2} && \text{exponent rule: } (-a)^n = a^n, \text{if } n \text{ is even} \\
&= \sqrt{(x^2 \cdot i^2) + x^2} \\
&= \sqrt{(1 + i^2)x^2} \\
&= \sqrt{(1 + (-1))x^2} \\
&= \sqrt{0 \cdot x^2} = 0
\end{aligned}
$$

Therefore, $\pm ix$ is the inverse of x.
(*Every element has an inverse*)

2. | *Commutative* | *Associative* | *Identity* | *Inverses* |
|---|---|---|---|
| Yes ☒ No ☐ | Yes ☐ No ☒ | Yes ☒ No ☐ | Yes ☒ No ☐ |

(i). $x * y = |x + y|$
$y * x = |y + x| = |x + y|$ because $+$ is a commutative operation on $\mathbb{R}$
(*Thus* $*$ *is commutative.*)

(ii). $x * (y * z) = x * |y + z| = |x + |y + z||$
Let $x = -5$, $y = 10$, and $z = -20$, then
$|-5 + |10 - 20|| = |-5 + |-10|| = |-5 + 10| = |5| = 5$
$(x * y) * z = |x + y| * z = ||x + y| + z|$
Again, let $x = -5$, $y = 10$, and $z = -20$, then
$||-5 + 10| - 20| = ||5| - 20| = |5 - 20| = |-15| = 15$
Since $5 \neq 15$ then $*$ is not associative on $\mathbb{R}$

(iii). Solve $x * e = x$ for e: $\quad |x + e| = x \Rightarrow e = 0$
Check: $x * 0 = |x + 0| = x$; $0 * x = |0 + x| = |x| = x$.
Therefore 0 is the identity element.

(iv). Solve $x * x^{'} = 0$ for $x^{'}$: $\quad |x + x^{'}| = 0 \Rightarrow x^{'} = -x$
Check: $x * x^{'} = |x - x| = 0$
$x^{'} * x = |-x + x| = 0$. Therefore, $-x$ is the inverse of x.

3. | *Commutative* | *Associative* | *Identity* | *Inverses* |
|---|---|---|---|
| Yes ☒ No ☐ | Yes ☐ No ☒ | Yes ☒ No ☐ | Yes ☒ No ☐ |