Chapter 2 - Operations

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From "Book of Abstract Algebra" by Charles C. Pinter

A. Examples of Operations

Which of the following rules are operations on the indicated set? ($\mathbb{Z}$ designates the set of integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers.) For each rule which is not an operation, explain why it is not.

1 $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$.

2 $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.

3 $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set $\mathbb{R}$.

4 Subtraction, on the set $\mathbb{Z}$.

5 Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

6 $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution

1 This is not an operation on $\mathbb{Q}$ because $a * b$ is not uniquely defined and $\mathbb{Q}$ is not closed under $*$. If a and b are rational numbers they can be written as $a = \frac{c}{d}$ and $b = \frac{e}{f}$ where $c, d, e, \text{and} f$ are integers, $d \neq 0$, and $f \neq 0$. If we let $c = 2$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{2}{1} \cdot \frac{2}{1}} = \sqrt{2 \cdot 2} = \sqrt{4}$$

and since $\sqrt{4} = \pm 2$ we see that $a * b$ is not uniquely defined. Now let $c = 3$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{3}{1} \cdot \frac{2}{1}} = \sqrt{6}$$

and we see that there is no rational number f such that $f \cdot f = 6$, therefore $\sqrt{6}$ is not a rational number. Thus, $\mathbb{Q}$ is not closed under $*$.

2 This is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$ because the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$. For example, if we let $a = 2$ and $b = 1$ then

$$a \ln b = 2 \ln 1 = 0$$

and $0 \notin \{x \in \mathbb{R} : x > 0\}$. Therefore the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$ and $*$ is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$.

3 This is not an operation on $\mathbb{R}$ because $a * b$ is not uniquely defined. If we solve for x we see

$$\begin{aligned} x^2 - a^2b^2 &= 0 \\ x^2 &= a^2b^2 \end{aligned}$$

since a^2 and b^2 will always be positive numbers, then a^2b^2 will be positive as well. Then solving for x we have

$$x = (ab, -ab)$$

and we see that $a * b$ is not uniquely defined and therefore $*$ is not an operation on $\mathbb{R}$.

4 This is an operation on $\mathbb{Z}$.

5 This is not an operation on $\mathbb{Z}$ because the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$. For example, if we let $a = 10$ and $b = 5$ then $5 - 10 = -5$ and $-5 \notin \{n \in \mathbb{Z} : n \geq 0\}$. Therefore the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$.

6 This is an operation on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

B. Properties of Operations

Each of the following is an operation $*$ on $\mathbb{R}$. Indicate whether or not

(i) it is commutative,

(ii) it is associative,

(iii) $\mathbb{R}$ has and identity element with respect to $*$,

(iv) every $x \in \mathbb{R}$ has an inverse with respect to $*$.

Instructions For (i), compute $x * y$ and $y * x$, and verify whether or not they are equal. For (ii), compute $x * (y * z)$ and $(x * y) * z$, and verify whether or not they are equal. For (iii), first solve the equation $x * e = x$ for e; if the equation cannot be solved, there is identity element. If it *can* be solved, it is still necessary to check that $e * x = x * e = x$ for any $x \in \mathbb{R}$. If it checks, then e is an identity element. For (iv), first not that it there is no identity element, there can be no inverses. If there is an identity element e, first solve the equation $x * x' = e$ for x'; if the equation cannot be solved, x does not have an inverse. If it can be solved, check to make sure that $x * x' = x' * x = e$. If this checks, x' is the inverse of x.

1) $x * y = \sqrt{x^2 + y^2}$

Solution Although the instructions for this section state that $x * y = \sqrt{x^2 + y^2}$ is an operation on $\mathbb{R}$, I don't think it is an operation on $\mathbb{R}$ because $\sqrt{x^2 + y^2}$ is not uniquely defined on $\mathbb{R}$. For example, if we let $x = 1$ and $y = 0$, then $x * y = \sqrt{1^2 + 0^2} = \sqrt{1} = (-1, 1)$ and we see that $\sqrt{x^2 + y^2}$ is not uniquely defined on $\mathbb{R}$. Therefore, $x * y = \sqrt{x^2 + y^2}$ is not an operation on $\mathbb{R}$.

2) $x * y = |x + y|$

Solution	*Commutative*	*Associative*	*Identity*	*Inverses*
	Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒	Yes ☐ No ☒

(a) $x * y = |x + y|$
$y * x = |y + x| = |x + y|$ because $+$ is a commutative operation on $\mathbb{R}$
Therefore $*$ is commutative on $\mathbb{R}$.

(b) $x * (y * z) = x * |y + z| = |x + |y + z||$
Let $x = -5$, $y = 10$, and $z = -20$, then
$|-5 + |10 - 20|| = |-5 + |-10|| = |-5 + 10| = |5| = 5$
$(x * y) * z = |x + y| * z = ||x + y| + z|$
Again, let $x = -5$, $y = 10$, and $z = -20$, then
$||-5 + 10| - 20| = ||5| - 20| = |5 - 20| = |-15| = 15$
Since $5 \neq 15$ then $*$ is not associative on $\mathbb{R}$

(c) There is no identity element with respect to $*$ on $\mathbb{R}$ because there is no $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in $\mathbb{R}$. For example, if $a = -1$ there is no $e \in \mathbb{R}$ such that $|e + (-1)| = -1$ because the operation $*$ defined by $x * y = |x + y|$ always returns a positive number.

(d) Since there is no identity element with respect to $*$ on $\mathbb{R}$ then there is no inverse with respect to $*$ on $\mathbb{R}$.

3) $x * y = |xy|$

Solution	*Commutative*	*Associative*	*Identity*	*Inverses*
	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒

(a) $x * y = |xy| = \begin{cases} xy & \text{if } xy \geq 0 \\ -(xy) & \text{if } xy < 0 \end{cases}$

$y * x = |xy| = \begin{cases} yx & \text{if } yx \geq 0 \\ -(yx) & \text{if } yx < 0 \end{cases}$

Assuming multiplication is commutative on $\mathbb{R}$ (not proven here) then $xy = yx$ and $-(xy) = -(yx)$. Therefore, $*$ is commutative on $\mathbb{R}$.

(b) $x * (y * z) = x * |yz| = |x|yz|| = xyz$
$(x * y) * z = |xy| * z = ||xy|z| = xyz$
Therefore $*$ is associative on $\mathbb{R}$.

(c) There is no identity element with respect to $*$ on $\mathbb{R}$ because there is no $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in $\mathbb{R}$. For example, if $a = -5$ there is no $e \in \mathbb{R}$ such that $|-5e| = -5$. Therefore there is no identity element with respect to $*$ on $\mathbb{R}$.

(d) Since there is no identity element with respect to $*$ on $\mathbb{R}$ then there is no inverse with respect to $*$ on $\mathbb{R}$.

4) $x * y = x - y$

Solution	*Commutative*	*Associative*	*Identity*	*Inverses*
	Yes ☐ No ☒	Yes ☐ No ☒	Yes ☐ No ☒	Yes ☐ No ☒

(a) The operation $*$ defined by $x*y = x-y$ is not commutative because $x - y \neq y - x$ for all x and y in $\mathbb{R}$. For example, if $x = 1$ and $y = 2$, then $x - y = 1 - 2 = -1$ and $y - x = 2 - 1 = 1$. Since $-1 \neq 1$ then the operation $*$ defined by $x * y = x - y$ is not commutative on $\mathbb{R}$.

(b) The operation $*$ defined by $x*y = x-y$ is not associative on $\mathbb{R}$ because $x*(y*z) \neq (x * y) * z$ for all x, y, and z in $\mathbb{R}$. For example, if $x = 5$, $y = -1$, and $z = 2$ then $x * (y * z) = 5 - (-1 - 2) = 8$ and $(x * y) * z = (5 - (-1)) - 2) = 4$. Since $8 \neq 4$ then the operation $*$ defined by $x * y = x - y$ is not associative on $\mathbb{R}$.

(c) Solving $x * e = x$ for e we have that $x - e = x \Rightarrow e = 0$. However checking that $x*e = e*x = x$ we see that $x - 0 = x$ and $0 - x = -x$. Since $x \neq -x$ then 0 is not the identity element with respect to the operation $*$ defined by $x * y = x - y$ on $\mathbb{R}$. Therefore there is no identity element with respect to the operation $*$ defined by $x * y = x - y$ on $\mathbb{R}$.

(d) Since there is no identity element with respect to $*$ on $\mathbb{R}$ then there is no inverse with respect to $*$ on $\mathbb{R}$.

5) $x * y = xy + 1$

Solution	*Commutative*	*Associative*	*Identity*	*Inverses*
	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒

(a) $x * y = xy + 1$
$y * x = yx + 1 = xy + 1$ because $\times$ is commutative on $\mathbb{R}$.
Therefore, the operation $*$ defined by $xy + 1$ is commutative on $\mathbb{R}$.

(b) The operation $*$ defined by $x*y = x-y$ is not associative on $\mathbb{R}$ because $x*(y*z) \neq (x * y) * z$.
$x * (y * z) = x * (yz + 1) = x(yz + 1) + 1 = xyz + x + 1$
$(x * y) * z = (xy + 1) * z = (xy + 1)z + 1 = xyz + z + 1$
Since $xyz + x + 1 \neq xyz + z + 1$ then the operation $*$ defined by $x * y = x - y$ is not associative on $\mathbb{R}$.

(c) Since there is no identity element with respect to $*$ on $\mathbb{R}$ then there is no inverse with respect to $*$ on $\mathbb{R}$.

6) $x*y = \max\{x, y\}$ *Solution*

Commutative	*Associative*	*Identity*	*Inverses*
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒

(a) If $x \neq y$ then $x * y = \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{if } y > x \end{cases}$

$y * x = \max(y, x) = \begin{cases} y & \text{if } y > x \\ x & \text{if } x > y \end{cases}$

If $x = y$ then $\max(x, y) = z$ where $z = x = y$.
Therefore $x * y = y * x$ and the operation $*$ defined by $\max\{x, y\}$ is commutative on $\mathbb{R}$.

(b) The operation $*$ defined by $\max\{x, y\}$ is associative on $\mathbb{R}$
$x * (y * z) = x * \max\{y, z\} = \max\{x, \max\{y, z\}\} = \max\{x, y, z\}$
$(x * y) * z = \max\{x, y\} * z = \max\{\max\{x, y\}, z\} = \max\{x, y, z\}$
Since $\max\{x, y, z\} = \max\{x, y, z\}$ the operation $*$ defined by $\max\{x, y\}$ is associative on $\mathbb{R}$.

(c) Solving $x * e = x$ for e we see that $max\{x, e\} = x \Rightarrow e = -\infty$ However, $-\infty \notin \mathbb{R}$ as is required for an identity element. There is no other $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in $\mathbb{R}$. Therefore there is not an identity element with respect to the operation $*$ defined by $\max\{x, y\}$ on $\mathbb{R}$.

(d) Since there is no identity element with respect to $*$ on $\mathbb{R}$ then there is no inverse with respect to $*$ on $\mathbb{R}$.

7) $x * y = \frac{xy}{x+y+1}$

Commutative	*Associative*	*Identity*	*Inverses*
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒

(a) $x * y = \frac{xy}{x+y+1}$; $y * x = \frac{yx}{y+x+1}$
Since $\times$ and $+$ are commutative on $\mathbb{R}$ then $\frac{xy}{x+y+1} = \frac{yx}{y+x+1}$. Therefore, the operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ is commutative on $\mathbb{R}$.

(b) The operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ is associative on $\mathbb{R}$.

$$x * (y * z) = x * \frac{yz}{y + z + 1} = \frac{x\left(\frac{yz}{y+z+1}\right)}{x + \frac{yz}{y+z+1} + 1} = \frac{xyz}{1 + x + y + z + xy + xz + yz}$$

$$(x * y) * z = \frac{xy}{x + y + 1} * z = \frac{z\left(\frac{xy}{x+y+1}\right)}{\frac{xy}{x+y+1} + z + 1} = \frac{xyz}{1 + x + y + z + xy + xz + yz}$$

$x * (y * z) = (x * y) * z$ and therefore the operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ is associative on $\mathbb{R}$.

(c) Solving $x * e = x$ for e we see that $\frac{xe}{x+e+1} = x \Rightarrow x = -1$ and therefore there is no $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in $\mathbb{R}$. Therefore there is not an identity element with respect to the operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ on $\mathbb{R}$.

(d) Since there is no identity element with respect to $*$ on $\mathbb{R}$ then there is no inverse with respect to $*$ on $\mathbb{R}$.

C. Operations on a Two-Element Set

Let A be the two-element set $A = \{a, b\}$.

1) Write the tables of all 16 operations on A

Solution

O_1

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	a

O_2

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	b

O_3

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	b

O_4

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	b

O_5

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	b

O_6

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	a

O_7

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	a

O_8

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	a

O_9

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	b

O_{10}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	a

O_{11}

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	a

O_{12}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	b

O_{13}

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	a

O_{14}

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	a

O_{15}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	b

O_{16}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	b

2) Identify which of the operations O_1 to O_{16} are commutative.
The operations O_1, O_2, O_3, O_5, O_6, O_8, O_{11}, and O_{12} are commutative.

3) Identify which operations, among O_1 to O_{16}, are associative.
The operations O_1, O_2, O_3, O_4, O_5, O_9, O_{11}, and O_{12} are associative

4) For which of the operations O_1 to O_{16} is there an identity element?
There is and identity element for O_3, O_5, O_{11}, and O_{12} with respect to the operation $*$ on A

5) For which of the operations O_1 to O_{16} does every element have and inverse?
We only need to consider the operations that have an identity element. For the operations O_{11} and O_{12} every element has an inverse. For operation O_3, there is no inverse for b and for operation O_5 there is no inverse for a.

D. Automata: The Algebra of Input/Output Sequences (2^{nd} edition)

The set of all sequences of symbols in the alphabet A is denoted by A^*.

There is an operation on A^* called *concatenation*: if **a** and **b** are in A^*, say $\mathbf{a} = a_1a_2...a_n$ and $\mathbf{b} = b_1b_2...b_m$, then

$$\mathbf{a} * \mathbf{b} = a_1a_2...a_nb_1b_2...b_m.$$

The symbol λ denotes the empty sequence.

(a)

Proposition. *The operation $*$ defined by $\boldsymbol{ab} = a_1a_2...a_nb_1b_2...b_m$ is associative.*

Proof. To prove that $*$ is associative, let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in A$. Then

$$\mathbf{x} * (\mathbf{y} * \mathbf{z}) = \mathbf{x} * y_1y_2...y_nz_1z_2...z_m = x_1x_2...x_ky_1y_2...y_nz_1z_2...z_m$$

and

$$(\mathbf{x} * \mathbf{y}) * \mathbf{z} = x_1x_2...x_ky_1y_2...y_n * \mathbf{z} = x_1x_2...x_ky_1y_2...y_nz_1z_2...z_m.$$

Since the two results are the same, the operation $*$ is associative. □

(b) The operation $*$ defined by $\mathbf{a} * \mathbf{b} = a_1a_2...a_nb_1b_2...b_m$ is not commutative because switching the order of **a** and **b** in the operation will lead to different results.

(c)

Proposition. *There is an identity element with respect to the operation $*$ defined by $\boldsymbol{ab} = a_1a_2...a_nb_1b_2...b_m$ on A^*.*

Proof. We must show that there is an element $e \in A^*$ such that $e * a = a$ and $a * e = a$ for every $a \in A^*$. Let **a** be an arbitrary element of A^*, $e = \lambda$, and $e \in A^*$. Since λ is the empty sequence, then

$$\mathbf{a} * e = \mathbf{a} * \lambda = a_1a_2...a_m\lambda = a_1a_2...a_m = \mathbf{a}$$

and

$$e * \mathbf{a} = \lambda * \mathbf{a} = \lambda a_1a_2...a_m = a_1a_2...a_m = \mathbf{a}.$$

Therefore we conclude that λ is the identity element with respect to the operation $*$ defined by $\mathbf{ab} = a_1a_2...a_nb_1b_2...b_m$ on A^*.

□