Math 343 - Introduction to Modern Algebra **Notes**

Binary Operations and Binary Structures

ALL SETS ARE ASSUMED TO BE NONEMPTY!

Let X be a set. $*$ is a binary operation on X if $*$ is a function from $X \times X$ to X. That is, $\forall x, y \in X$, $x * y \in X$.

A binary operation on a set X is commutative if $\forall x, y \in X$, $x * y = y * x$.

Examples:

(1) Let $X = \mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and $*$ be the addition operator $+$ or the multiplication operator $\cdot$. Both are commutative binary operations.

(2) Let X be the set of $m \times n$ matrices, denoted M_{mn}, and $*$ be matrix addition. This is a commutative operation. Note: We can distinguish between $m \times n$ matrices with real entries versus complex entries with the notations $M_{mn}(\mathbb{R})$ and $M_{mn}(\mathbb{C})$.

(3) Let X be the set of $n \times n$ matrices, denoted M_{nn} or M_n, and $*$ be matrix multiplication. This is not a commutative operation (for $n > 1$):

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(4) Let X be the set of functions $f : \mathbb{R} \to \mathbb{R}$ and $*$ be composition. This is not a commutative operation:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x - 1$. Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is given by $(f \circ g)(x) = (x-1)^2 = x^2 - 2x + 1$ and $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $(g \circ f)(x) = x^2 - 1$.

(5) Let X be the set of alphabet strings (finite sequences of letters) and $*$ be concatenation (putting two strings together in order). For example, if $\alpha =$ race and $\beta =$ car then $\alpha * \beta =$ racecar. Clearly this binary operation is not commutative.

Let $*$ be a binary operation on a set X. Let Y be a subset of X. Y is closed under $*$ if $\forall x, y \in Y$, $x * y \in Y$.

Examples:

-Consider the interval $[0, 1] \subseteq \mathbb{R}$. Clearly addition is not closed on this subset of $\mathbb{R}$.

-Consider the set Y of all $n \times n$ matrices with non-negative entries as a subset of M_n. Clearly matrix multiplication is closed on Y.

Let X be set with a binary operation $*$ on X. We call $(X, *)$ a binary structure and often out of laziness just refer to it as X.

Let $(X, *)$ and $(X', *')$ be binary structures. A bijection $(1-1$ and onto$)$ $f : X \to X'$ satisfying $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$ is called a isomorphism of X and X' (as binary structures with respect to the binary operators) and X and X' are called isomorphic.

Let's show that $(\mathbb{R}, +)$ and $(X, \cdot)$ where $X = \{x \in \mathbb{R} | x > 0\}$ are isomorphic:

Define $f : \mathbb{R} \to X$ by $f(x) = e^x$. f is well-defined since $e^x \in X$ for all $x \in \mathbb{R}$ as $e^x > 0$. Let $x, y \in \mathbb{R}$ and suppose $f(x) = f(y)$. Then $e^x = e^y$. Hence $x = y$, so f is $1-1$. Let $r \in X$. Then $\ln(r) \in \mathbb{R}$ and $e^{\ln(r)} = r$. So f is onto. Finally, let $x, y \in \mathbb{R}$. $f(x+y) = e^{x+y} = e^x e^y = f(x)f(y)$.

Let $(X, *)$ be a binary structure. $e \in X$ is called an identity element (for $*$) if $\forall x \in X$, $e * x = x * e = x$.

For example, the identity matrix I_2 acts as the identity element in $(M_2, \cdot)$, but not in $(M_2, +)$. What acts as the identity element in $(M_2, +)$?

Not all binary structures have identity elements. Consider the interval $(1, \infty) \subseteq \mathbb{R}$ with the binary operation $\cdot$. There is no identity element in this binary structure since for all $x, y \in (1, \infty)$, $x \cdot y > Max(x, y)$, so $x \cdot y \neq x$ and $x \cdot y \neq y$.

Let $(X, *)$ be a binary structure with an identity element e. Suppose e' is also an identity element of $(X, *)$. Then $e * e' = e$ since e' is an identity element. But since e is an identity element, $e * e' = e'$. Putting these together, $e' = e$. So an identity element is unique!

Proposition 0.1: Let $(X, *)$ be a binary structure with an identity element e. Suppose $(X', *')$ is an isomorphic binary structures given by the isomorphism $\phi : X \to X'$. Then $\phi(e)$ is an identity element of $(X', *')$.

Proof: Let $y \in X'$. It suffices to show that $\phi(e) *' y = y *' \phi(e) = y$. Since ϕ is onto there exists $x \in X$ such that $\phi(x) = y$. Then $\phi(e) *' y = \phi(e) *' \phi(x) = \phi(e * x) = \phi(x) = y$ and $y *' \phi(e) = \phi(x) *' \phi(e) = \phi(x * e) = \phi(x) = y$ $\square$

Groups

A group, G, is a set together with a binary operation $*$ on G (so a binary structure) such that the following three axioms are satisfied:

(A) For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$. We say $*$ is associative.

(B) There exists an identity element $e \in G$.

(C) For all $x \in G$, there exists an element $x' \in G$ such that $x * x' = x' * x = e$. Such an element x' is called an inverse of x.

If finite, $|G|$ is the order of the group.

Note: Formally, the group is $(G, *)$, but often when $*$ is understood by context we just write G for the group. Also, the identity element is an inverse of itself.

Examples:

(1) Let $(G, *)$ consists of $G = \{e\}$ where $e * e = e$. This is the trivial group of order 1. Think $(\{1\}, \cdot)$ or $(\{0\}, +)$ or $(\{f\}, \circ)$ where $f : \mathbb{R} \to \mathbb{R}$ is the function given by $f(x) = x$.

(2) $(\mathbb{Z}, +)$ (or replace $\mathbb{Z}$ with $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$) forms a group as addition is an associative operator, 0 is the identity element, and an inverse of $x \in \mathbb{Z}$ is $-x \in \mathbb{Z}$.

(3) Let $X \subseteq \mathbb{C}$ containing the number 0. Let $X^* = X \setminus \{0\}$. $(\mathbb{Q}^*, \cdot)$ (or replace $\mathbb{Q}^*$ with $\mathbb{R}^*$ or $\mathbb{C}^*$) forms an infinite group as multiplication is an associative operator, 1 is the identity element, and an inverse of $x \in \mathbb{Q}^*$ is $1/x \in \mathbb{Q}^*$.

(4) Let X be either $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Let $X^+ = \{x \in X | x > 0\}$. $(\mathbb{Q}^+, \cdot)$ (or replace $\mathbb{Q}^+$ with $\mathbb{R}^+$) forms an infinite group as multiplication is an associative operator, 1 is the identity element, and an inverse of $x \in \mathbb{Q}^+$ is $1/x \in \mathbb{Q}^+$.

(5) Let $n \in \mathbb{N}$. Let $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ where the closed binary operation on $\mathbb{Z}_n$ is addition modulo n. $\mathbb{Z}_n$ is a finite group or order n. Associativity is inherited from the addition operator, the identity element is 0, and an inverse of $a \in \mathbb{Z}_n$ is the element $(n-a)(\text{mod } n) \in \mathbb{Z}_n$.

(6) Let $m, n \in \mathbb{N}$. $(M_{mn}(\mathbb{R}), +)$ forms a group as matrix addition is associative, 0_{mn} ($m \times n$ matrix consisting of all zeros) is the identity, and an inverse of $A \in M_{mn}(\mathbb{R})$ is $-A \in M_{mn}(\mathbb{R})$. Note: This works with $\mathbb{R}$ replaced by $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$ (or any other so-called *ring* – to be defined later).

(7) Let $n \in \mathbb{N}$. Let $GL_n(\mathbb{R}) = \{A \in M_n | \det(A) \neq 0\}$. With matrix multiplication as the binary operator, $GL_n(\mathbb{R})$ is called the general linear group of $n \times n$ matrices. Matrix multiplication is associative, the identity element is I_n ($\det(I_n) = 1$), and an inverse of $A \in GL_n(\mathbb{R})$ is $A^{-1} \in GL_n(\mathbb{R})$ which exists since $\det(A) \neq 0$ and is in $GL_n(\mathbb{R})$ since $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)} \neq 0$. Note: This works with $\mathbb{R}$ replaced by $\mathbb{Q}$ or $\mathbb{C}$ (or any other so-called *field* – to be defined later).

(8) $U = \{z \in \mathbb{C} : |z| = 1\}$ forms a group under multiplication. Complex number multiplication is associative, the identity element is 1, and an inverse of $z = a+bi \in U$ is $\bar{z} = a - bi \in U$ (since $1 = a^2 + b^2$). Exercise left for you (ELFY): Show that $z\bar{z} = |z| = 1$ for $z \in U$. This group is sometimes referred to as the *circle group* as geometrically it's the unit circle in the complex plane.

(9) Let $n \in \mathbb{N}$. The so-called *nth Roots of Unity*, $U_n = \{z \in \mathbb{C} | z^n = 1\}$, form a finite group under multiplication. Associativity is inherited from complex number multiplication, $1 \in U_n$ is the identity element, and an inverse of $z \in U_n$ is $\dfrac{1}{z}$ (defined since $z \neq 0$) which is in U_n since $\left(\dfrac{1}{z}\right)^n = \dfrac{1}{z^n} = 1$. ELFY: Show that for $z \in U_n$, $|z| = 1$ (so $z \in U$) and $\dfrac{1}{z} = \bar{z}$.

Let's have some more fun with U_n: Let $z \in U_n$. Since $z \in \mathbb{C}$, $z = re^{i\theta}$ where $r, \theta \in \mathbb{R}$ ($r \geq 0$). Then

$$z^n = r^n e^{in\theta} = 1 \text{ so } r = 1 \text{ and } n\theta = 2k\pi \text{ where } k \in \{0, 1, 2, ..., n-1\}.$$

Note: If $n\theta = 2k\pi$ for an integer k then $\theta = \frac{2k\pi}{n}$, so by periodicity, there is no need to consider θ outside of $[0, 2\pi)$ and hence there is no need to consider k outside of $\{0, 1, 2, ..., n-1\}$.

So $U_n = \{e^{i(2k\pi/n)} | k = 0, 1, 2, ..., n-1\}$ is a finite group of order n. The root of unity $e^{i(2k\pi/n)}$ is primitive if the $\gcd(k, n) = 1$.

(10) Let X be a set. Then the set $B = \{f : X \to X | f \text{ is a bijection}\}$ forms a group under function composition $\circ$. Function composition is associative (ELFY), the identity element is $Id : X \to X$ given by $Id(x) = x$ (ELFY), and an inverse of $f \in B$ is $f^{-1} \in B$ (ELFY).

(11) Let X be the set $\{1, 2, ..., n\}$. Then $S_n = \{\sigma : X \to X | \sigma \text{ is a bijection}\}$ under function composition is called the symmetric group of permutations of n (a finite group of order $n!$). We will spend a lot of time with these groups.

Notes for the symmetric group:

-Let $\sigma \in S_n$. We adopt the convention that σ *acts on the right*:

So if σ sends $i \in \{1, 2, ..., n\}$ to $j \in \{1, 2, ..., n\}$ we write $(i)\sigma = j$.

-The best notation for elements of S_n is *cycle notation*:

$c \in S_n$ is a cycle if $c = (i_1\ i_2\ \cdots\ i_\ell)$ where $\ell \in \{1, 2, ..., n\}$ is the length of the cycle and $(i_j)c = i_{(j+1)}$ for $j = 1, 2, ..., \ell - 1$ and $(i_\ell)c = i_1$, while c fixes elements (sends the element to itself) in $\{1, 2, ..., n\} \setminus \{i_1\ i_2\ \cdots\ i_\ell\}$.

For example, the cycle $c = (1\ 2\ 4)$ of length 3 in S_4 sends 1 to 2, 2 to 4, and 4 to 1, while fixing 3. Of course this cycle can be written in three equivalent ways: (1 2 4) or (2 4 1) or (4 1 2).

Now let $\sigma \in S_n$. Determine $(1)\sigma, ((1)\sigma)\sigma = (1)\sigma^2, ...$ until the first $j_1 \in \{1, 2, ..., n\}$ such that $(1)\sigma^{j_1} = 1$. This gives the cycle $c_1 = (1\ (1)\sigma\ (1)\sigma^2\ \cdots\ (1)\sigma^{j_1-1})$. Then pick the smallest $i \in \{1, 2, ..., n\} \setminus \{1, (1)\sigma, (1)\sigma^2, ..., (1)\sigma^{j_1-1}\}$ (if possible; if not, then $\sigma = c_1$) and do the same thing to get the cycle $c_2 = (i\ (i)\sigma\ (i)\sigma^2\ \cdots\ (i)\sigma^{j_2-1})$ where $j_2 \in \{1, 2, ..., n\}$ is the smallest integer such that $(i)\sigma^{j_2} = i$. Note: c_1 and c_2 are disjoint cycles as they do not share any entries. Keep doing this to produce pairwise disjoint cycles $c_1, c_2, ..., c_k$ that account for all the elements of $\{1, 2, ..., n\}$. Then $\sigma = c_1 c_2 \cdots c_k$ (the product can be done in any order since disjoint cycles commute: $(j)\sigma = (j)c_\ell$ where j is involved in c_ℓ). Note: Convention is to drop all cycles of length 1 and to write Id for the identity permutation $(1)(2)\cdots(n)$.

For example, let $\sigma \in S_6$ be the permutation that sends 1 to 2, 2 to 4, 3 to 5, 4 to 1, 5 to 3, and 6 to 6. Then in cycle notation $\sigma = (1\ 2\ 4)(3\ 5)$.

Here are all 6 elements of S_3 in cycle notation:

Id, (1 2), (2 3), (1 3), (1 2 3), and (1 3 2).

Here are all 24 elements of S_4 in cycle notation:

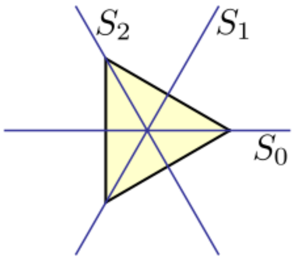
Id, (1 2), (1 3), (1 4), (2 3), (2 4), (3 4), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), (1 2 3), (1 3 2), (1 2 4), (1 4 2), (1 3 4), (1 4 3), (2 3 4), (2 4 3), (1 2 3 4), (1 2 4 3), (1 3 2 4), (1 3 4 2), (1 4 2 3), (1 4 3 2).

Note: Cycles of length two are called transpositions.

(12) Dihedral groups: Consider a regular polygon P with $n \geq 3$ sides (e.g. an equilateral triangle, a square, etc...). A symmetry of P is either a rotation or a reflection of P such that the result looks the same (ignoring labeling of the vertices). Clearly the composition of two symmetries produces a symmetry, and each symmetry has an inverse symmetry bringing the polygon back to it's original state (where vertices are kept track of by labels).

There are n (ccw) rotations (including the *trivial rotation* by 0 radians) and n reflections for a total of $2n$ elements in the dihedral group of order $2n$, denoted D_{2n}. The identity element is the trivial rotation, the inverse of a reflection is itself, and the inverse of a non-identity rotation by θ radians is the non-identity rotation by $2\pi - \theta$ radians.

For example, D_6 consists of the symmetries of an equilateral triangle: Let R_0, R_1, R_2 be the (ccw) rotations by 0, $2\pi/3$, $4\pi/3$ radians respectively and S_0, S_1, S_2 be the 3 reflections (as seen in the figure). Consider the product R_1S_0. It's important to realize that this is the composition of R_1 after S_0. A quick check on the diagram below yields $R_1S_0 = S_1$.



Here is a multiplication table for the D_6 (product order is row by column):

$\cdot$	R_0	R_1	R_2	S_0	S_1	S_2
R_0	R_0	R_1	R_2	S_0	S_1	S_2
R_1	R_1	R_2	R_0	S_1	S_2	S_0
R_2	R_2	R_0	R_1	S_2	S_0	S_1
S_0	S_0	S_2	S_1	R_0	R_2	R_1
S_1	S_1	S_0	S_2	R_1	R_0	R_2
S_2	S_2	S_1	S_0	R_2	R_1	R_0

(ELFY) Verify the products in the table!

The following are binary structures that are NOT groups:

-M_n where the binary operation is matrix multiplication. While matrix multiplication is associative and I_n is the identity element, not every element has an inverse: In fact, for any matrix A such that $\det(A) = 0$ there is no inverse.

-The set of positive integers, $\mathbb{Z}^+$ where the operation is multiplication. While the operation is associative and 1 is the identity, for integers $n \geq 2$ there is no inverse in $\mathbb{Z}^+$.

-The set of all functions $f : \mathbb{R} \to \mathbb{R}$ where the binary operation is function composition. While the operation is associative and $f(x) = x$ is the identity, many functions have no inverse, like $f(x) = x^2$.

A group G is called <u>abelian</u> if the binary operation is commutative.

(ELFY) Identify which groups among the examples are abelian, and which are not abelian. Note: This should not be hard.

ELEMENTARY PROPERTIES OF GROUPS:

Proposition 1.1 **(Cancelation Laws)**: Let $(G, *)$ be a group. For all $a, b, c \in G$, $a*b = a*c$ implies $b = c$. For all $a, b, c \in G$, $b * a = c * a$ implies $b = c$.

Proof: Let $a, b, c \in G$ and e be the identity. Assume $a * b = a * c$. Let a' be an inverse of a. Then $a' * a = e$. By multiplying both sides of $a * b = a * c$ on the left by a' we get $a' * (a * b) = a' * (a * c)$. By using the associative property, we get $(a' * a) * b = (a' * a) * c$. This becomes $e * b = e * c$, and thus $b = c$. Similarly, $b * a = c * a$ implies $b = c$ $\square$

Proposition 1.2: Let $(G, *)$ be a group and $a, b \in G$. Then the equations $a * x = b$ and $y * a = b$ have unique solutions x and y in G.

Proof: Let e be the identity. Assume $a * x = b$. Let a' be an inverse of a. Then $a * (a' * b) = (a * a') * b = e * b = b$. So $x = a' * b$ is a solution to $a * x = b$. Suppose x_1 and x_2 are both solutions to $a * x = b$. Then $a * x_1 = a * x_2$. Then by the previous proposition, $x_1 = x_2$, showing that $x = a' * b$ is the unique solution to $a * x = b$. A similar argument shows that $y = b * a'$ is the unique solution to $y * a = b$ $\square$

The last proposition justifies that *inverses are unique*:

Let $(G, *)$ be a group and e the identity in G. If a' is an inverse of a then it is a solution to $a * x = e$ and $y * a = e$ where e is the identity of G. Hence a' is THE unique inverse of a, so we can finally start writing "the inverse" rather than "an inverse." At this point we will also adopt the notations a^{-1} or $-a$ as the standard for the inverse of a group element a.

Proposition 1.3: Let $(G, *)$ be a group and $a, b \in G$. Then the inverse of $a * b$ is $(a * b)^{-1} = b^{-1} * a^{-1}$.

Proof: Let e be the identity. Then $(a*b)*(b^{-1}*a^{-1}) = a*(b*(b^{-1}*a^{-1})) = a*((b*b^{-1})*a^{-1}) = = a * (e * a^{-1}) = a * a^{-1} = e$. Hence $x = b^{-1} * a^{-1}$ is the unique solution to $(a * b) * x = e$. A similar calculation shows that $y = b^{-1} * a^{-1}$ is the unique solution to $y * (a * b) = e$ $\square$

Subgroups

Let's begin with some conventions. If G is a group and we are regarding the binary operation as a "multiplicative" operation then we write ab for the product of $a \in G$ and $b \in G$ (in that order) and a^{-1} for the inverse of $a \in G$. Furthermore, we write a^2 for aa, a^3 for $aaa = (aa)a = a(aa)$, and so it goes. a^0 denotes e. So for example, $a^2b^3a^{-1}$ is shorthand for $aabbba^{-1}$.

If G is a group and we are regarding the binary operation as a "additive" operation then for $a \in G$ we write $2a$ for $a + a$, $3a$ for $a + a + a = a + (a + a) = (a + a) + a$, and so it goes. 0 is the identity and $-a$ is the inverse of $a \in G$. We also write $-na$ for $n(-a)$ where n is a positive integer.

Note: The use of the additive notation implies the group is abelian! So in general, the default notation for a group's binary operation is multiplicative without assuming the group is (or isn't) abelian.

Let G be a group. If a subset $H \subseteq G$ that is closed under the binary operation of G is itself a group under the same binary operation then H is a <u>subgroup</u> of G. If $H \neq G$ is a subgroup of G it is called a <u>proper subgroup</u>.

Examples:

(1) Let G be a group. G is a subgroup of itself. Another subgroup is the <u>trivial subgroup</u> $\{e\}$ where e is identity element in G.

(2) Let $n \in \mathbb{N}$. U_n (nth roots of unity) form a subgroup of the circle group U and of $\mathbb{C}^*$ (under multiplication). The circle group U is a subgroup of $\mathbb{C}^*$.

(3) Let $G = \mathbb{Z}_4$. The subgroups of G are the trivial subgroup, $\{0\}$, the subgroup $\{0, 2\}$, and the group G itself.

(4) Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (aka the <u>Klein 4-group</u> up to isomorphism - more on this soon!) where the operation is componentwise addition modulo 2. The subgroups of G are the trivial subgroup, $\{(0, 0)\}$, the subgroup $\{(0, 0), (1, 0)\}$, the subgroup $\{(0, 0), (0, 1)\}$, the subgroup $\{(0, 0), (1, 1)\}$, and the group G itself.

Notation for a subgroup:

$H \leq G$ denotes that H is a subgroup of G (possibly equal to G). $H < G$ denotes that H is a proper subgroup of G.

Note:

Let $H \leq G$ and $a \in H$. The equation $ax = a$ must have a unique solution in H, namely the identity e of H. But this equation would also have the same unique solution in G since H is a subset of G. This means that the identity for the group and the subgroup coincide. A similar argument shows that if $a^{-1} \in H$ is the inverse of $a \in H$ then the same element a^{-1} is also the inverse of $a \in G$.

Proposition 2.1: Let G be a group and $H \subseteq G$. $H \leq G$ iff the following three conditions hold:

1. H is closed under the binary operation of G.
2. The identity element $e \in G$ is in H.
3. For all $a \in H$, $a^{-1} \in H$.

Proof: The fact that if $H \leq G$ then these three conditions hold follows directly from the definition of a subgroup and the remarks above.

Conversely suppose that we know that conditions 1, 2 and 3 above hold for $H \subseteq G$. Condition 1 tells us that the closed binary operation on G forms a closed binary operation on H. Conditions 2 and 3 tell us that H satisfies having an identity element and every element in H has an inverse in H. The only thing left to check is associativity; that's inherited from G since for all $a, b, c \in H$, it follows that $a, b, c \in G$ and thusly $a(bc) = (ab)c$ $\square$

Example showing that a subset is a subgroup:

Consider $SL_n(\mathbb{R}) = \{A \in M_{nn} | \det(A) = 1\}$. Clearly $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$. For all matrices $A, B \in SL_n(\mathbb{R})$, we have that $\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$. So matrix multiplication is a closed binary operation on $SL_n(\mathbb{R})$. Clearly the identity $I_n \in GL_n(\mathbb{R})$ is in $SL_n(\mathbb{R})$ as $\det(I_n) = 1$. Let $A \in SL_n(\mathbb{R})$. Let A^{-1} be the inverse of A in $GL_n(\mathbb{R})$. Since $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1})$ we have that $A^{-1} \in SL_n(\mathbb{R})$. Hence by proposition 2.1, $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$. This also works over other *fields* (to be defined later) such as $\mathbb{Q}$ and $\mathbb{C}$. $SL_n(\mathbb{R})$ is called the special linear group (subgroup of the general linear group).

Proposition 2.2 **(The** 1**-step subgroup test)**: Let G be a group and let H be a nonempty subset of G. If for all $a, b \in H$, $ab^{-1} \in H$ then H is a subgroup of G.

Proof: Assume for all $a, b \in H$, $ab^{-1} \in H$. Let $h \in H$. By setting $a = b = h$ we get that $ab^{-1} = hh^{-1} = e \in H$ (where e is the identity in G). Then by setting $a = e$ and $b = h$ we get that $h^{-1} \in H$. Finally, we show that H is closed under the binary operation. Let $h_1, h_2 \in H$. By setting $a = h_1$ and $b = h_2^{-1} \in H$ we get $h_1(h_2^{-1})^{-1} = h_1h_2 \in H$ $\square$

Let's see this test in action! Let's consider the group $GL_2(\mathbb{R})$.

$$\text{Show that } UT_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\} < GL_2(\mathbb{R}) :$$

First we note that $I_2 \in UT_2(\mathbb{R})$ so $UT_2(\mathbb{R}) \neq \emptyset$. Next we note that $UT_2(\mathbb{R}) \subset GL_2(\mathbb{R})$ as for any $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in UT_2(\mathbb{R})$, $\det(A) = ad \neq 0$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{R}) \setminus UT_2(\mathbb{R})$.

Let $A, B \in UT_2(\mathbb{R})$. So $A = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$ where $a_1, a_2, b_1, b_2, d_1, d_2 \in \mathbb{R}$, $a_1d_1 \neq 0$ and $a_2d_2 \neq 0$. Hence $a_1 \neq 0$, $a_2 \neq 0$, $d_1 \neq 0$ and $d_2 \neq 0$.

Finally, $B^{-1} = \begin{pmatrix} \frac{1}{a_2} & -\frac{b_2}{a_2d_2} \\ 0 & \frac{1}{d_2} \end{pmatrix}$ and thus

$$AB^{-1} = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} \frac{1}{a_2} & -\frac{b_2}{a_2d_2} \\ 0 & \frac{1}{d_2} \end{pmatrix} = \begin{pmatrix} \frac{a_1}{a_2} & -\frac{a_1b_2}{a_2d_2} + \frac{b_1}{d_2} \\ 0 & \frac{d_1}{d_2} \end{pmatrix} \in UT_2(\mathbb{R}) \text{ since } \frac{a_1d_1}{a_2d_2} \neq 0$$

So the set $UT_2(\mathbb{R})$ of invertible, upper-triangular matrices is a proper subgroup of the general linear group $GL_2(\mathbb{R})$.

Note:

Let G be a group and $a \in G$. Clearly a subgroup $H \leq G$ that contains a must also contain all integer powers of a (these may not all be distinct!) such as the identity a^0, the inverse a^{-1}, a^2, $a^{-2} = (a^{-1})^2$, and so on. So if $a \in H$ and $H \leq G$ then $\{a^n | n \in \mathbb{Z}\} \subseteq H$.

Proposition 2.3: Let G be a group and $a \in G$. Then

$$< a >= \{a^n | n \in \mathbb{Z}\}$$

is the smallest subgroup of G that contains a (called the cyclic subgroup of G generated by $a \in G$).

Proof: First we check that $< a >$ is closed under the binary operation: Let $x, y \in < a >$. Then there exists $r, s \in \mathbb{Z}$ such that $x = a^r$ and $y = a^s$. Then $xy = a^r a^s = a^{r+s} \in < a >$ since $r+s \in \mathbb{Z}$, so $< a >$ is closed under the binary operation. The identity element is $a^0 = e$ so the identity element of G is in $< a >$. For $a^r \in < a >$, $a^{-r} \in < a >$ and $a^r a^{-r} = a^0 = e$. Finally, by the note above the proposition, if $H \leq G$ contains a then H contains $< a >$. So $< a >$ is the smallest subgroup of G that contains a $\square$

Let G be a group (maybe finite, maybe not). Let $a \in G$. If the cyclic subgroup $< a >$ is finite then the order of the element a is $| < a > |$.

Let G be a group. An element $a \in G$ generates G (aka is a generator of G) if $G =< a >$. Furthermore a group G is a cyclic group when there is an element $a \in G$ that generates G.

Note: In additive notation, $< a >= \{na | n \in \mathbb{Z}\}$.

Consider the group $\mathbb{Z}_4$ of order 4 (integers $\{0, 1, 2, 3\}$ under addition modulo 4). $< 1 >$ contains 1, $1+1 = 2$, $1+1+1 = 3$, $1+1+1+1 = 0$. Similarly, $< 3 >$ contains 3, $3+3 = 2$, $3+3+3 = 1$ and $3+3+3+3 = 0$. So $< 1 >=< 3 >= \mathbb{Z}_4$. $< 2 >$ contains exactly 2 and $2+2 = 0$ $(-2 = 2)$. Finally, $< 0 >$ is the trivial subgroup. Hence $\mathbb{Z}_4$ is a cyclic subgroup of order 4 generated by 1 or 3, but not by 2 or 0.

Consider the group $V = \mathbb{Z}_2 \times \mathbb{Z}_2$. Clearly $< (0, 0) >$ is the trivial subgroup. Let $a \in V$ be nontrivial (not $(0, 0)$). Then $a + a = 0$ so $-a = a$. Hence $< a >= \{a, 0\}$. So V is not cyclic as there is no single generator of V (all nontrivial elements are order 2).

Proposition 2.4: Every cyclic group is abelian.

Proof: Let G a cyclic group generated by $a \in G$. Let $x, y \in G$. Then there exists $r, s \in \mathbb{Z}$ such that $x = a^r$ and $y = a^s$. Then $xy = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = yx$ $\square$

Lemma 2.5 **(Division Algorithm for $\mathbb{Z}$)**: If $d \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$ then there exists unique integers q, r such that $n = qd + r$ where $0 \leq r \leq d - 1$.

Proof: Let $X = \{n - dk | n - dk \geq 0, k \in \mathbb{Z}\}$. By selecting $k \in \mathbb{Z}$ so that $k < \frac{n}{d}$ (integers are unbounded) we have $dk < n$ and therefore, $n - dk \in X$. Thus X is nonempty. Let $r \in X$ be the smallest element. In particular $r \geq 0$. Label by q the integer that satisfies $n - dq = r$. Then $n = dq + r$.

Now suppose $r \geq d$. Then $r - d \geq 0$ and hence, $n - d(q+1) = r - d \in X$ and $r - d < r$. But this contradicts that r is the smallest element of X.

Thus $0 \leq r < d$.

Now for uniqueness. Suppose $r_1, r_2, q_1, q_2 \in \mathbb{Z}$ and

$n = dq_1 + r_1$ and $0 \leq r_1 < d$.

$n = dq_2 + r_2$ and $0 \leq r_2 < d$.

Then $0 = n - n = d(q_1 - q_2) + (r_1 - r_2)$ implies $d(q_1 - q_2) = (r_2 - r_1)$. But then d divides $r_2 - r_1$ and since $0 \leq r_1, r_2 < d$,

$$-d < r_2 - r_1 < d \text{ and so } r_1 = r_2.$$

Then $d(q_1 - q_2) = 0$ implying that $q_1 = q_2$ $\square$

Proposition 2.6: Every subgroup of a cyclic group is a cyclic group.

Proof: Let G a cyclic group generated by $a \in G$. Let $H \leq G$. Assume $H \neq < e >$ where e is the identity as otherwise then we are done. Let $d \in \mathbb{Z}^+$ be the smallest such that $a^d \in H$. We want to show that H is generated by a^d.

Let $b \in H$. Then $b = a^n$ for some $n \in \mathbb{Z}$. Find integers q, r such that $n = qd + r$ and $0 \leq r < d$. Then $b = a^{qd+r} = \left(a^d\right)^q a^r$, so $a^r = b\left(a^d\right)^{-q} \in H$ as $b \in H$ and $\left(a^d\right)^{-q} \in H$ as H contains $< a^d >$. Hence $r = 0$ as otherwise it contradicts how d was chosen. Consequently, $a^n = \left(a^d\right)^q$ showing that $< a^d >$ contains H, and hence is equal to H. Hence H is a cyclic group generated by a^d $\square$

Corollary 2.7: The subgroups of $\mathbb{Z}$ (as a group under addition) are $n\mathbb{Z} =< n >= \{kn | k \in \mathbb{Z}\}$ where $n \in \mathbb{Z}$.

Proof: Since $\mathbb{Z} =< 1 >$ is cyclic, every subgroup must be cyclic by the previous proposition. That completes the proof $\square$

Let m, n be positive integers. Consider $H = \{am + bn | a, b \in \mathbb{Z}\}$. ELFY: Show that $H \leq \mathbb{Z}$ (under addition).

Let $d = \gcd(m, n) \in \mathbb{Z}^+$ (greatest common divisor). We know that $H =< \ell >= \ell\mathbb{Z}$ for some $\ell \in \mathbb{Z}$. Assume without loss of generality (WLOG) that $\ell > 0$. In particular, $m \in H$ (set $a = 1$, $b = 0$) and $n \in H$ (set $a = 0$, $b = 1$), so $m = j\ell$ and $n = k\ell$ for some $j, k \in \mathbb{Z}$. So ℓ must be a common divisor of m and n implying $\ell \leq d$. Since $\ell \in H$, there exists $a, b \in \mathbb{Z}$ such that $\ell = am + bn$. Since d divides the RHS of this equation, it must divide the LHS of this equation. So d divides ℓ and thus ℓ is a positive-integer multiple of d. From earlier we have that $\ell \leq d$. Hence it must be that $\ell = d$. So the generator of H is the greatest common divisor of m, n.

As a consequence of this, there exists $r, s \in \mathbb{Z}$ such that $\gcd(m, n) = rm + sn$.

[The greatest common divisor of two positive integers is a linear combination of the integers!]

Let d be the least positive integer such that there exists integers k, j such that $d = km + jn$. Let integers a, b satisfy $d = am + bn$

Claim: d is the $\gcd(m, n)$.

Let's prove the claim. Let's show first that d divides m. By the division algorithm, there exists unique integers q, r such that $m = qd + r$ and $0 \leq r < d$. Then it follows that $r = m - qd = m - q(am + bn) = (1 - qa)m + (-qb)n$. Since d was picked to be the least positive integer such that there exists integers k, j such that $d = km + jn$ we get that $r = 0$. So d divides m. A symmetric argument shows d divides n. So d is a common divisor of m and n.

Suppose $d' = \gcd(m, n)$. Then $d' \geq d$. Well, d' divides m and n, so it follows that d' divides $d = am + bn$. Therefore $d' \leq d$. So if follows that $d = d'$, finishing the proof of the claim.

Suppose the $\gcd(m, n) = 1$ (that is, m, n are relatively prime) and suppose m divides nk where $k \in \mathbb{Z}$. Let's show that it must be the case that m divides k:

First we can note that there exists integers r, s such that $1 = rm + sn$.

Multiplying both sides by the integer k we get $k = krm + ksn$ Since m divides the RHS it divides the LHS.

We will use these results (on this page) in the next section.

Introduction to Group Isomorphisms

Let G and G' be groups. A bijection ($1-1$ and onto) $\phi : G \to G'$ satisfying $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$ is called a group isomorphism between G and G' (as groups with respect to their operations) and G and G' are called isomorphic groups and we use the notation $G \cong G'$.

Note: All we really require is that they are isomorphic as binary structures!

Examples:

(1) Let $G = S_3 = \{Id, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ (a symmetric group) and $G' = D_6 = \{R_0, R_1, R_2, S_0, S_1, S_2\}$ with the multiplication table on page 6 (a dihedral group).

By having $\phi(Id) = R_0$, $\phi((1\ 2)) = S_0$, $\phi((2\ 3)) = S_2$, $\phi((1\ 3)) = S_1$, $\phi((1\ 2\ 3)) = R_2$, and $\phi((1\ 3\ 2)) = R_1$ we get an isomorphism.

ELFY: Show that ϕ is indeed a group isomorphism. So $S_3 \cong D_6$.

(2) Let $n \in \mathbb{N}$. The nth roots of unity $U_n = \{e^{i(2k\pi/n)} | k = 0, 1, 2, ..., n-1\}$ (with complex number multiplication) is isomorphic to the cyclic group $\mathbb{Z}_n$ (modulo n addition):

Let $\phi : U_n \to \mathbb{Z}_n$ be given by $\phi\left(e^{i(2k\pi/n)}\right) = k$ for $k = 0, 1, ..., n-1$. By explicit construction this is a bijection from U_n to $\mathbb{Z}_n$. We just require that the "morphism" property is satisfied:

Let $z_1, z_2 \in U_n$. Then there exists $a, b \in \{0, 1, ..., n-1\}$ such that $z_1 = e^{i(2a\pi/n)}$ and $z_2 = e^{i(2b\pi/n)}$. Then

$$\phi(z_1 z_2) = \phi\left(e^{i(2a\pi/n)} e^{i(2b\pi/n)}\right) = \phi\left(e^{i(2(a+b)\pi/n)}\right) = (a+b) \text{ mod n} = (\phi(z_1) + \phi(z_2)) \bmod n.$$

So $U_n \cong \mathbb{Z}_n$.

Proposition 3.1: Let G and G' be isomorphic groups and $\phi : G \to G'$ be an isomorphism. Let e and e' be the identities in G and G' respectively. The following hold true:

1. $\phi(e) = e'$
2. For all $g \in G$, $\phi(g^{-1}) = \phi(g)^{-1}$.

Proof: Let $g \in G$. $\phi(g) = \phi(ge) = \phi(g)\phi(e)$. By left-cancelation by $\phi(g)$ we get that $e' = \phi(e)$ proving 1. Now consider that we have $e' = \phi(e) = \phi(g^{-1}g) = \phi(g^{-1})\phi(g)$ and $e' = \phi(e) = \phi(gg^{-1}) = \phi(g)\phi(g^{-1})$. So using either one $\phi(g)^{-1} = \phi(g^{-1})$ $\square$

Proposition 3.2: Let G be a cyclic group with generator $a \in G$. If G is infinite then $G \cong \mathbb{Z}$ (under addition). If G is finite with $|G| = n$ then $G \cong \mathbb{Z}_n$ (under addition modulo n).

Proof: Let $e = a^0$ be the identity. Two cases: Either a is of order $m \in \mathbb{N}$ or not (meaning $a^m \neq e$ for all $m \in \mathbb{N}$).

Let's start with the latter. Define $\phi : G \to \mathbb{Z}$ by $\phi(a^n) = n$ for $n \in \mathbb{Z}$. First we show this is a well-defined function. $a^n \neq a^m$ for all distinct integers m, n because otherwise, a would be of order $|m - n| \in \mathbb{N}$. So every input is unique and hence we don't have one input going to multiple outputs. This makes ϕ a well-defined function. By construction ϕ is $1-1$ and onto. So ϕ bijection. Finally, for $a^m, a^n \in G$, $\phi(a^m a^n) = \phi(a^{m+n}) = m + n = \phi(a^m) + \phi(a^n)$. So ϕ is an isomorphism showing $G \cong \mathbb{Z}$.

Now let's assume a is of order m. Let n be the smallest positive integer such that $a^n = e$. Let $s \in \mathbb{Z}$. Then $s = qn + r$ for unique integers q, r such that $0 \leq r < n$. Then $a^s = a^{qn+r} = (a^n)^q a^r = e^q a^r = a^r$. So for all $a^s \in G$ there exists $r \in \{0, 1, 2, ..., n-1\}$ such that $a^s = a^r$. This means that $G = \{a^i | i = 0, 1, ..., n-1\}$. Suppose $0 \leq i < j < n$ and $a^i = a^j$. Then $a^{j-i} = a^0$ for $0 < j - i < n$ contradicting that the order of a is n. Thus the elements $a^0, a^1, ..., a^{n-1}$ are all distinct (so $G =< a >$ is of order $m = n$). Thus the map given by $\phi(a^i) = i$ for $i \in \{0, 1, ..., n-1\}$ is well-defined, $1-1$ and onto. Finally, for $i, j \in \mathbb{Z}$, $\phi(a^{i+j}) = \phi(a^{(i+j) \bmod n}) = (i + j) \bmod n = (\phi(a^i) + \phi(a^j)) \bmod n$. So ϕ is an isomorphism proving that $G \cong \mathbb{Z}_n$ $\square$

Proposition 3.3: Let G be a cyclic group of order n generated by a. Let $b = a^s \in G$ and $H =< b >$. Then H is of order n/d where $d = \gcd(n, s)$.

Proof: Let e be the identity of G. As we have seen in the proof of the previous proposition, the order of H is the smallest positive integer m where $b^m = e$. Then $e = (a^s)^m = a^{sm}$. $a^{sm} = e$ iff n divides sm. So n divides sm and m is the smallest positive integer such that n divides sm. That is, there exists an integer k such that $nk = sm$. Let $d = \gcd(s, n)$. Then there exists integers u, v such that $d = un + vs$. Then $1 = u(n/d) + v(s/d)$ where $n/d, s/d \in \mathbb{Z}$. So n/d and s/d are relatively prime as their greatest common divisor is 1.

So we want to find the smallest positive integer m such that $\dfrac{sm}{n}$ is an integer.

This is equivalent to finding the smallest positive integer m such that $\dfrac{(s/d)m}{(n/d)}$ is an integer.

That is, find the smallest positive integer m such that n/d divides $(s/d)m$.

Since n/d and s/d are relatively prime, m would need to be the smallest positive integer such that n/d divides m. Thus $m = n/d$ $\square$

Corollary 3.4: Let G be a cyclic group of order n generated by a. Let $b = a^s \in G$ where $gcd(s, n) = 1$. Then b generates G.

Proof: Simply apply the last theorem to see that the order of $H =< b >$ is n, so it must contain all n elements of G, making $H = G$. Hence b generates G $\square$