Chapter 3 Solutions
From "Book of Abstract Algebra" by Charles C. Pinter
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A. Examples of Abelian Groups

1. Let $G = \mathbb{R}$, and let $*$ be the binary operation on G defined by

$$x * y \ = \ x + y + k$$

for all $x, y \in G$ and where k is a fixed constant.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + k$$

and

$$y * x = y + x + k = x + y + k.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + k) \\ &= x + (y + z + k) + k \\ &= x + y + z + 2k \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + k) * z \\ &= (x + y + k) + z + k \\ &= x + y + z + 2k. \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $-k$ because

$$x * -k = x + (-k) + k = x$$

and

$$-k * x = -k + x + k = x.$$

Finally, we see that the inverse with respect to $*$ is $-x - 2k$ because

$$x * x^{-1} = x + (-x - 2k) + k = x - x - 2k = -k$$

and

$$x^{-1} * x = (-x - 2k) + x + k = -x - 2k + x + k = -k$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. $\square$

2. Let $G = \mathbb{R}$, and let $*$ be the binary operation on G defined by