Chapter 3 Solutions
From "Book of Abstract Algebra" by Charles C. Pinter
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A. Examples of Abelian Groups

1. Let $G = \mathbb{R}$, and let $*$ be the binary operation on G defined by

$$x * y \;=\; x + y + k$$

for all $x, y \in G$ and where k is a fixed constant.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + k$$

and

$$y * x = y + x + k = x + y + k.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + k) \\ &= x + (y + z + k) + k \\ &= x + y + z + 2k \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + k) * z \\ &= (x + y + k) + z + k \\ &= x + y + z + 2k. \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $-k$ because

$$x * -k = x + (-k) + k = x$$

and

$$-k * x = -k + x + k = x.$$

Finally, we see that the inverse with respect to $*$ is $-x - 2k$ because

$$x * x^{-1} = x + (-x - 2k) + k = x - x - 2k = -k$$

and

$$x^{-1} * x = (-x - 2k) + x + k = -x - 2k + x + k = -k$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. $\square$

2. Let $G = \{x \in \mathbb{R} : x \neq 0\}$, and let $*$ be the binary operation on G defined by

$$x * y = \frac{xy}{2} \tag{1}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = \frac{xy}{2}$$

and

$$y * x = \frac{xy}{2} = \frac{yx}{2}.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$x * (y * z) = x * \frac{yz}{2} = \frac{\frac{xyz}{2}}{2} = \frac{xyz}{4}$$

and

$$(x * y) * z = \frac{xy}{2} * z = \frac{\frac{xyz}{2}}{2} = \frac{xyz}{4}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 2 because

$$x * 2 = \frac{x2}{2} = x$$

and

$$2 * x = \frac{2x}{2} = x.$$

Finally we see that $\frac{4}{x}$ is the inverse of the operation $*$ because

$$x * \frac{4}{x} = \frac{\frac{4x}{x}}{2} = \frac{4}{2} = 2$$

and

$$\frac{4}{x} * x = \frac{\frac{4x}{x}}{2} = \frac{4}{2}.$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. □

3. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $*$ be the binary operation on G defined by

$$x * y = x + y + xy \tag{2}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + xy$$

and

$$y * x = y + x + yx = x + y + xy$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + yz) \\ &= x + (y + z + yz) = x + (y + z + yz) + x(y + z + yz) \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + xy) * z \\ &= (x + y + xy) + z + (x + y + xy)z \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 0 because

$$x * 0 = x + 0 + x \cdot 0 = x$$

and

$$0 * x = 0 + x + 0 \cdot x = x.$$

Finally we see that $-\frac{x}{1+x}$ is the inverse of the operation $*$ because

$$\begin{aligned} x * \left(-\frac{x}{1+x}\right) &= x + \left(-\frac{x}{1+x}\right) + x \cdot \left(-\frac{x}{1+x}\right) \\ &= \frac{x + x^2 - x + x^2}{1+x} = 0 \end{aligned}$$

and

$$\begin{aligned} \left(-\frac{x}{1+x}\right) * x &= \left(-\frac{x}{1+x}\right) + x + \left(-\frac{x^2}{1+x}\right) \\ &= \frac{x + x^2 - x - x^2}{1+x} = 0 \end{aligned}$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group.

$\square$

4. Let $G = \{x \in \mathbb{R} : -1 < x < 1\}$, and let $*$ be the binary operation on G defined by

$$x * y = \frac{x+y}{xy+1} \tag{3}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = \frac{x+y}{xy+1}$$

and

$$y * x = \frac{y+x}{yx+1} = \frac{x+y}{xy+1}$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * \frac{y+z}{yz+1} \\ &= \frac{x + \frac{y+z}{yz+1}}{\frac{y+z}{yz+1} + 1} \\ &= x + y + z + xy + xz + yz + xyz + 1 \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= \frac{x+y}{xy+1} * z \\ &= \frac{\frac{x+y}{xy+1} + z}{\frac{x+y}{xy+1} z + 1} \\ &= x + y + z + xy + xz + yz + xyz + 1 \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 0 because

$$x * 0 = \frac{x+0}{x \cdot 0 + 1} = \frac{x}{1} = x$$

and

$$0 * x = \frac{0+x}{0 \cdot x + 1} = \frac{x}{1} = x$$

Finally we see that $-x$ is the inverse of the operation $*$ because

$$x * -x = \frac{x - x}{x \cdot -x + 1} = \frac{0}{-x^2 + 1} = 0$$

and

$$-x * x = \frac{-x + x}{-x \cdot x + 1} = \frac{0}{-x^2 + 1} = 0$$

$\square$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group.

B. Groups on the Set $\mathbb{R} \times \mathbb{R}$

1. Let $(a, b) * (c, d) = (ad + bc, bd)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \neq 0\}$

(a)

Proposition. *The set G, with the operation $*$, is a group.*

Proof. To prove the set G, with the operation $*$, is a group we must show that $*$ is associative and that G has an identity element and inverses.

To prove that $*$ is associative let a, b, c and d be arbitrary elements of G. Then

$$\begin{aligned}(a, b) * ((c, d) * (e, f)) &= (a, b) * (cf + de, df) = (a \cdot df + b \cdot (cf + de), b \cdot df) \\ &= (adf + bcf + bde, bdf)\end{aligned}$$

and

$$\begin{aligned}((a, b) * (c, d)) * (e, f) &= (ad + bc, bd) * (e, f) = ((ad + bc) \cdot f + bd \cdot e, bd \cdot f) \\ &= (adf + bcf + bde, bdf)\end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $(0, 1)$ because

$$(a, b) * (0, 1) = (a + 0, b) = (a, b)$$

and

$$(0, 1) * (a, b) = (0 + a, b) = (a, b)$$

Finally we see that $(\frac{-a}{b^2}, \frac{1}{b})$ is the inverse of the operation $*$ because

$$(a, b) * \left(-\frac{a}{b^2}, \frac{1}{b}\right) = \left(a \cdot \frac{1}{b} + b \cdot -\frac{a}{b^2}, b \cdot \frac{1}{b}\right) = \left(\frac{ab}{b^2} - \frac{ab}{b^2}, 1\right) = (0, 1)$$

and

$$\left(-\frac{a}{b^2}, \frac{1}{b}\right) * (a, b) = \left(-\frac{a}{b^2} \cdot b + \frac{a}{b}, \frac{1}{b} \cdot b\right) = \left(-\frac{ab}{b^2} + \frac{ab}{b^2}, 1\right) = (0, 1)$$

Therefore, we conclude that the set G, with the operation $*$, is a group. □

(b)

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses. In part (a), we have already shown that the operation $*$ is associative and that G has an identity element and inverses. Therefore we must show that the operation $*$ is commutative.

To prove that $*$ is commutative let a and b be arbitrary elements of G. Then

$$(a, b) * (c, d) = (ad + bc, bd)$$

and

$$(c, d) * (a, b) = (cb + da, db) = (ad + bc, bd)$$

Since the two results are the same, the operation $*$ is commutative.

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. $\square$

2. Let $(a, b) * (c, d) = (ac, bc + d)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq 0\}$

(a)

Proposition. *The set G, with the operation $*$, is a group.*

Proof. To prove the set G, with the operation $*$, is a group we must show that $*$ is associative and that G has an identity element and inverses.

To prove that $*$ is associative let a, b, c and d be arbitrary elements of G. Then

$$\begin{aligned}(a, b) * ((c, d) * (e, f)) &= (a, b) * (ce, de + f) = (a \cdot ce, b \cdot ce + de + f) \\ &= (ace, bce + de + f)\end{aligned}$$

and

$$\begin{aligned}((a, b) * (c, d)) * (e, f) &= (ac, bc + d) * (e, f) = (ac \cdot e, (bc + d) \cdot e + f) \\ &= (ace, bce + de + f)\end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $(1, 0)$ because

$$(a, b) * (1, 0) = (a \cdot 1, b \cdot 1 + 0) = (a, b)$$

and

$$(1, 0) * (a, b) = (1 \cdot a, 0 \cdot a + b) = (a, b)$$

Finally we see that $(\frac{1}{a}, \frac{-b}{a})$ is the inverse of the operation $*$ because

$$(a, b) * \left(\frac{1}{a}, -\frac{b}{a}\right) = \left(a \cdot \frac{1}{a}, \frac{b}{a} + \left(-\frac{b}{a}\right)\right) = (1, 0)$$

and

$$\left(\frac{1}{a}, -\frac{b}{a}\right) * (a, b) = \left(\frac{1}{a} \cdot a, -\frac{b}{a} \cdot a + b\right) = (1, -b + b) = (1, 0)$$

Therefore, we conclude that the set G, with the operation $*$, is a group. $\square$

(b)

Proposition. *The set G, with the operation $*$, is not an abelian group.*

Proof. To prove that the set G, with the operation $*$, is not an abelian group we must show that the operation $*$ is not commutative.

Let a, b, c and d be arbitrary elements of G. Then

$$(a, b) * (c, d) = (ac, bc + d) \tag{4}$$

and

$$(c, d) * (a, b) = (ca, da + b) \tag{5}$$

Since the two results are not the same, the operation $*$ is not commutative.

Therefore, we conclude that the set G, with the operation $*$, is not an abelian group. □

3. Let $(a, b) * (c, d) = (ac, bc + d)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R}\}$

(a)

Proposition. *The set G, with the operation $*$, is not a group.*

Proof. For the set G with the operation $*$ to be a group, all elements in G must have an inverse with respect to the operation $*$. We have shown in problem 2 that the inverse of the operation $*$ defined by $(a, b) * (c, d) = (ac, bc + d)$ is $(\frac{1}{a}, \frac{-b}{a})$. If we let $(x, y) = (0, y)$ then we see that the inverse is not defined and it follows that not every element in G has an inverse with respect to the operation $*$. Therefore, the set G, with the operation $*$, is not a group. □

(b)

Proposition. *The set G, with the operation $*$, is not an abelian group.*

Proof. To be an abelian group, the set G with the operation $*$ must be a group and the operation $*$ must be commutative. We have shown in part (a) that the set G with the operation $*$ is not a group. Therefore, the set G with the operation $*$ is also not an abelian group. □

4. Let $(a, b) * (c, d) = (ac - bd, ad + bc)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R}\}$ with the origin deleted.

(a)

Proposition. *The set G, with the operation $*$, is a group.*

Proof. To prove the set G, with the operation $*$, is a group we must show that $*$ is associative and that G has an identity element and inverses.

To prove that $*$ is associative let a, b, c and d be arbitrary elements of G. Then

$$\begin{aligned}(a,b)*((c,d)*(e,f)) &= (a,b)*(ce-df, cf+de)\\ &= (a\cdot(ce-df)-b\cdot(cf+de), a\cdot(cf+de)+(ce-df)\cdot b)\\ &= (ace-adf-bcf-bde, acf+ade+bce-bdf)\end{aligned}$$

and

$$\begin{aligned}((a,b)*(c,d))*(e,f) &= (ac-bd, ad+bc)*(e,f)\\ &= ((ac-bd)\cdot e-(ad+bc)\cdot f, (ac-bd)\cdot f+(ad+bc)\cdot e)\\ &= (ace-adf-bcf-bde, acf+ade+bce-bdf)\end{aligned}$$

The identity element for G is $(1,0)$ because

$$(a,b)*(1,0) = (a-0, 0+b) = (a,b)$$

and

$$(1,0)*(a,b) = (a-0, b+0) = (a,b)$$

Finally we see that $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ is the inverse of the operation $*$ because

$$\begin{aligned}(a,b)*\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right) &= \left(a\cdot\frac{a}{a^2+b^2}-b\cdot\frac{-b}{a^2+b^2}, a\cdot\frac{-b}{a^2+b^2}+b\cdot\frac{a}{a^2+b^2}\right)\\ &= (1,0)\end{aligned}$$

and

$$\begin{aligned}\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)*(a,b) &= \left(\frac{a}{a^2+b^2}\cdot a-\left(-\frac{b}{a^2+b^2}\cdot b\right), \frac{a}{a^2+b^2}\cdot b+\left(-\frac{b}{a^2+b^2}\cdot a\right)\right)\\ &= (1,0)\end{aligned}$$

Therefore, we conclude that the set G, with the operation $*$, is a group. □

(b)

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses. In part (a), we have already shown that the operation $*$ is associative and that G has an identity element and inverses. Therefore we must show that the operation $*$ is commutative.

To prove that $*$ is commutative let a, b, c, and d be arbitrary elements of G. Then

$$(a,b)*(c,d) = (ac-bd, ad+bc)$$

and

$$(c,d)*(a,b) = (ca-db, cb+da) = (ac-bd, ad+bc)$$

Since the two results are the same, the operation $*$ is commutative.

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. □

5. No, the operation $*$ in exercise 4, defined as $(a, b) * (c, d) = (ac - bd, ad + bc)$, on the set $\mathbb{R} \times \mathbb{R}$ is not a group because not every element in $\mathbb{R} \times \mathbb{R}$ has an inverse with respect to the operation $*$. From the solution to exercise 4 we know that the inverse with respect to $*$ is $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ and we see that the inverse for $(0, 0)$ is not defined. Therefore, the operation $*$ on the set $\mathbb{R} \times \mathbb{R}$ is not a group.

C. Groups of Subsets of a Set

The symmetric difference of any two sets A and B is defined as

$$A + B = (A - B) \cup (B - A)$$

The operation $+$ defined as above is commutative because $A + B = B + A$. This operation is also associative so that $A + (B + C) = (A + B) + C$.

If D is a set, then the *power set* of D is the set P_D of all the subsets of D. That is,

$$P_D = \{A : A \subseteq D\}$$

The operation $+$ is to be regarded as an operation on P_D.

1.

Proposition. *The identity element with respect to the operation $+$ on the set P_D is $\varnothing$.*

Proof. To prove that $\varnothing$ is the identity element with respect to the operation $+$, let A be an arbitrary element of P_D. Then

$$A + \varnothing = (A - \varnothing) \cup (\varnothing - A) = A \cup A = A$$

and

$$\varnothing + A = (\varnothing - A) \cup (A - \varnothing) = A \cup A = A$$

Therefore, we conclude that $\varnothing$ is the identity element with respect to the operation $+$. $\square$

2.

Proposition. *Every subset A of D has an inverse with respect to $+$, which is A.*

Proof. Let A be an arbitrary element of D. Then we see that A is the inverse of A with respect to $+$ because

$$A + A = (A - A) \cup (A - A) = \varnothing \cup \varnothing = \varnothing$$

and

$$A + A = (A - A) \cup (A - A) = \varnothing \cup \varnothing = \varnothing$$

Therefore, $\varnothing$ is the inverse of A with respect to $+$ on D. $\square$

3.

(a) If D is the three element set $\{a, b, c\}$ then $P_D = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(b) The operation table for $\langle P_D, +\rangle$ is

$+$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\varnothing$	$\varnothing$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a,b\}$	$\{a,c\}$	$\{b\}$	$\{c\}$	$\{a,b,c\}$	$\{b,c\}$
$\{b\}$	$\{b\}$	$\{ab\}$	$\varnothing$	$\{bc\}$	$\{a\}$	$\{abc\}$	$\{c\}$	$\{ac\}$
$\{c\}$	$\{c\}$	$\{ac\}$	$\{bc\}$	$\varnothing$	$\{abc\}$	$\{a\}$	$\{b\}$	$\{ab\}$
$\{ab\}$	$\{ab\}$	$\{b\}$	$\{a\}$	$\{abc\}$	$\varnothing$	$\{bc\}$	$\{ac\}$	$\{c\}$
$\{ac\}$	$\{ac\}$	$\{c\}$	$\{abc\}$	$\{a\}$	$\{bc\}$	$\varnothing$	$\{ab\}$	$\{b\}$
$\{bc\}$	$\{bc\}$	$\{abc\}$	$\{c\}$	$\{b\}$	$\{ac\}$	$\{ab\}$	$\varnothing$	$\{a\}$
$\{abc\}$	$\{abc\}$	$\{bc\}$	$\{ac\}$	$\{ab\}$	$\{c\}$	$\{b\}$	$\{a\}$	$\varnothing$

D. A Checkerboard Game

Consider the set $G = \{V, H, D, I\}$ where each element is defined as a move on a checkerboard that has only 4 squares. Each move is defined as as follows:

V: Move vertically; that is move from 1 to 3, or from 3 to 1, or from 2 to 4 or 4 to 2.

H: Move horizontally, that is move from 1 to 2 or vice versa, or from 3 to 4 or vice versa.

D: Move diagonally, that is, move from 2 to 3 or vice versa, or move from 1 to 4 or vice versa.

I: Stay put.

We may consider an operation on the set of these four moves which consists of performing moves successively. For example, if we move horizontally and then vertically, we end up with the same result as if we ad moved diagonally:

$$H * V = D$$

If we perform two horizontal moves in succession, we end up where we started: $H * H = I$. And so on.

(a) If $G = \{V, H, D, I\}$, and $*$ is the operation just described, write the table of G.

$*$	I	V	H	D
I	I	V	H	D
V	V	I	D	H
H	H	D	I	V
D	D	H	V	I

(b) Granted associativity, $\langle G, *\rangle$ is a group because as we see from the table in part (a), there is an identity element I and every element in G has an inverse.

E. A Coin Game