Chapter 3 Solutions
From "Book of Abstract Algebra" by Charles C. Pinter
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A. Examples of Abelian Groups

1. Let $G = \mathbb{R}$, and let $*$ be the binary operation on G defined by

$$x * y \ = \ x + y + k$$

for all $x, y \in G$ and where k is a fixed constant.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + k$$

and

$$y * x = y + x + k = x + y + k.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + k) \\ &= x + (y + z + k) + k \\ &= x + y + z + 2k \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + k) * z \\ &= (x + y + k) + z + k \\ &= x + y + z + 2k. \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $-k$ because

$$x * -k = x + (-k) + k = x$$

and

$$-k * x = -k + x + k = x.$$

Finally, we see that the inverse with respect to $*$ is $-x - 2k$ because

$$x * x^{-1} = x + (-x - 2k) + k = x - x - 2k = -k$$

and

$$x^{-1} * x = (-x - 2k) + x + k = -x - 2k + x + k = -k$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. □

2. Let $G = \{x \in \mathbb{R} : x \neq 0\}$, and let $*$ be the binary operation on G defined by

$$x * y = \frac{xy}{2} \tag{1}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = \frac{xy}{2}$$

and

$$y * x = \frac{xy}{2} = \frac{yx}{2}.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$x * (y * z) = x * \frac{yz}{2} = \frac{\frac{xyz}{2}}{2} = \frac{xyz}{4}$$

and

$$(x * y) * z = \frac{xy}{2} * z = \frac{\frac{xyz}{2}}{2} = \frac{xyz}{4}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 2 because

$$x * 2 = \frac{x2}{2} = x$$

and

$$2 * x = \frac{2x}{2} = x.$$

Finally we see that $\frac{4}{x}$ is the inverse of the operation $*$ because

$$x * \frac{4}{x} = \frac{\frac{4x}{x}}{2} = \frac{4}{2} = 2$$

and

$$\frac{4}{x} * x = \frac{\frac{4x}{x}}{2} = \frac{4}{2}.$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. □

3. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $*$ be the binary operation on G defined by

$$x * y = x + y + xy \tag{2}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + xy$$

and

$$y * x = y + x + yx = x + y + xy$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + yz) \\ &= x + (y + z + yz) = x + (y + z + yz) + x(y + z + yz) \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + xy) * z \\ &= (x + y + xy) + z + (x + y + xy)z \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 0 because

$$x * 0 = x + 0 + x \cdot 0 = x$$

and

$$0 * x = 0 + x + 0 \cdot x = x.$$

Finally we see that $-\frac{x}{1+x}$ is the inverse of the operation $*$ because

$$\begin{aligned} x * \left(-\frac{x}{1+x}\right) &= x + \left(-\frac{x}{1+x}\right) + x \cdot \left(-\frac{x}{1+x}\right) \\ &= \frac{x + x^2 - x + x^2}{1+x} = 0 \end{aligned}$$

and

$$\begin{aligned} \left(-\frac{x}{1+x}\right) * x &= \left(-\frac{x}{1+x}\right) + x + \left(-\frac{x^2}{1+x}\right) \\ &= \frac{x + x^2 - x - x^2}{1+x} = 0 \end{aligned}$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group.

$\square$

4. Let $G = \{x \in \mathbb{R} : -1 < x < 1\}$, and let $*$ be the binary operation on G defined by

$$x * y = \frac{x+y}{xy+1} \tag{3}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = \frac{x+y}{xy+1}$$

and

$$y * x = \frac{y+x}{yx+1} = \frac{x+y}{xy+1}$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * \frac{y+z}{yz+1} \\ &= \frac{x + \frac{y+z}{yz+1}}{\frac{y+z}{yz+1} + 1} \\ &= x + y + z + xy + xz + yz + xyz + 1 \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= \frac{x+y}{xy+1} * z \\ &= \frac{\frac{x+y}{xy+1} + z}{\frac{x+y}{xy+1} z + 1} \\ &= x + y + z + xy + xz + yz + xyz + 1 \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 0 because

$$x * 0 = \frac{x+0}{x \cdot 0 + 1} = \frac{x}{1} = x$$

and

$$0 * x = \frac{0+x}{0 \cdot x + 1} = \frac{x}{1} = x$$

Finally we see that $-x$ is the inverse of the operation $*$ because

$$x * -x = \frac{x-x}{x \cdot -x + 1} = \frac{0}{-x^2+1} = 0$$

and

$$-x * x = \frac{-x+x}{-x \cdot x + 1} = \frac{0}{-x^2+1} = 0$$

$\square$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group.

B. Groups on the Set $\mathbb{R} \times \mathbb{R}$

1. Let $(a, b) * (c, d) = (ad + bc, bd)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \neq 0\}$

(a)

Proposition. *The set G, with the operation $*$, is a group.*

Proof. To prove the set G, with the operation $*$, is a group we must show that $*$ is associative and that G has an identity element and inverses.

To prove that $*$ is associative let a, b, c and d be arbitrary elements of G. Then

$$\begin{aligned}(a, b) * ((c, d) * (e, f)) &= (a, b) * (cf + de, df) = (a \cdot df + b \cdot (cf + de), b \cdot df) \\ &= (adf + bcf + bde, bdf)\end{aligned}$$

and

$$\begin{aligned}((a, b) * (c, d)) * (e, f) &= (ad + bc, bd) * (e, f) = ((ad + bc) \cdot f + bd \cdot e, bd \cdot f) \\ &= (adf + bcf + bde, bdf)\end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $(0, 1)$ because

$$(a, b) * (0, 1) = (a + 0, b) = (a, b)$$

and

$$(0, 1) * (a, b) = (0 + a, b) = (a, b)$$

Finally we see that $(\frac{-a}{b^2}, \frac{1}{b})$ is the inverse of the operation $*$ because

$$(a, b) * \left(-\frac{a}{b^2}, \frac{1}{b}\right) = \left(a \cdot \frac{1}{b} + b \cdot -\frac{a}{b^2}, b \cdot \frac{1}{b}\right) = \left(\frac{ab}{b^2} - \frac{ab}{b^2}, 1\right) = (0, 1)$$

and

$$\left(-\frac{a}{b^2}, \frac{1}{b}\right) * (a, b) = \left(-\frac{a}{b^2} \cdot b + \frac{a}{b}, \frac{1}{b} \cdot b\right) = \left(-\frac{ab}{b^2} + \frac{ab}{b^2}, 1\right) = (0, 1)$$

Therefore, we conclude that the set G, with the operation $*$, is a group. □

(b)

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses. In part (a), we have already shown that the operation $*$ is associative and that G has an identity element and inverses. Therefore we must show that the operation $*$ is commutative.

To prove that $*$ is commutative let a and b be arbitrary elements of G. Then

$$(a,b)*(c,d) = (ad+bc, bd)$$

and

$$(c,d)*(a,b) = (cb+da, db) = (ad+bc, bd)$$

Since the two results are the same, the operation $*$ is commutative.

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. □

2. Let $(a,b)*(c,d) = (ac, bc+d)$ be a binary operation $*$ on the set $G = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x \neq 0\}$

(a)

Proposition. *The set G, with the operation $*$, is a group.*

Proof. To prove the set G, with the operation $*$, is a group we must show that $*$ is associative and that G has an identity element and inverses.

To prove that $*$ is associative let a, b, c and d be arbitrary elements of G. Then

$$\begin{aligned}(a,b)*((c,d)*(e,f)) &= (a,b)*(ce, de+f) = (a\cdot ce, b\cdot ce + de + f)\\ &= (ace, bce+de+f)\end{aligned}$$

and

$$\begin{aligned}((a,b)*(c,d))*(e,f) &= (ac, bc+d)*(e,f) = (ac\cdot e, (bc+d)\cdot e + f)\\ &= (ace, bce+de+f)\end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $(1,0)$ because

$$(a,b)*(1,0) = (a\cdot 1, b\cdot 1 + 0) = (a,b)$$

and

$$(1,0)*(a,b) = (1\cdot a, 0\cdot a + b) = (a,b)$$

Finally we see that $(\frac{1}{a}, \frac{-b}{a})$ is the inverse of the operation $*$ because

$$(a,b)*\left(\frac{1}{a}, -\frac{b}{a}\right) = \left(a\cdot\frac{1}{a}, \frac{b}{a} + \left(-\frac{b}{a}\right)\right) = (1,0)$$

and

$$\left(\frac{1}{a}, -\frac{b}{a}\right)*(a,b) = \left(\frac{1}{a}\cdot a, -\frac{b}{a}\cdot a + b\right) = (1, -b+b) = (1,0)$$

Therefore, we conclude that the set G, with the operation $*$, is a group. □

(b)

Proposition. *The set G, with the operation $*$, is not an abelian group.*

Proof. To prove that the set G, with the operation $*$, is not an abelian group we must show that the operation $*$ is not commutative.

Let a, b, c and d be arbitrary elements of G. Then

$$(a, b) * (c, d) = (ac, bc + d) \tag{4}$$

and

$$(c, d) * (a, b) = (ca, da + b) \tag{5}$$

Since the two results are not the same, the operation $*$ is not commutative.

Therefore, we conclude that the set G, with the operation $*$, is not an abelian group. □

3. Let $(a, b) * (c, d) = (ac, bc + d)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R}\}$

(a)

Proposition. *The set G, with the operation $*$, is not a group.*

Proof. For the set G with the operation $*$ to be a group, all elements in G must have an inverse with respect to the operation $*$. We have shown in problem 2 that the inverse of the operation $*$ defined by $(a, b) * (c, d) = (ac, bc + d)$ is $(\frac{1}{a}, \frac{-b}{a})$. If we let $(x, y) = (0, y)$ then we see that the inverse is not defined and it follows that not every element in G has an inverse with respect to the operation $*$. Therefore, the set G, with the operation $*$, is not a group. □

(b)

Proposition. *The set G, with the operation $*$, is not an abelian group.*

Proof. To be an abelian group, the set G with the operation $*$ must be a group and the operation $*$ must be commutative. We have shown in part (a) that the set G with the operation $*$ is not a group. Therefore, the set G with the operation $*$ is also not an abelian group. □

4. Let $(a, b) * (c, d) = (ac - bd, ad + bc)$ be a binary operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R}\}$ with the origin deleted.

(a)

Proposition. *The set G, with the operation $*$, is a group.*

Proof. To prove the set G, with the operation $*$, is a group we must show that $*$ is associative and that G has an identity element and inverses.

To prove that $*$ is associative let a, b, c and d be arbitrary elements of G. Then

$$\begin{aligned}(a,b)*((c,d)*(e,f)) &= (a,b)*(ce-df, cf+de)\\ &= (a\cdot(ce-df)-b\cdot(cf+de), a\cdot(cf+de)+(ce-df)\cdot b)\\ &= (ace-adf-bcf-bde, acf+ade+bce-bdf)\end{aligned}$$

and

$$\begin{aligned}((a,b)*(c,d))*(e,f) &= (ac-bd, ad+bc)*(e,f)\\ &= ((ac-bd)\cdot e-(ad+bc)\cdot f, (ac-bd)\cdot f+(ad+bc)\cdot e)\\ &= (ace-adf-bcf-bde, acf+ade+bce-bdf)\end{aligned}$$

The identity element for G is $(1,0)$ because

$$(a,b)*(1,0) = (a-0, 0+b) = (a,b)$$

and

$$(1,0)*(a,b) = (a-0, b+0) = (a,b)$$

Finally we see that $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ is the inverse of the operation $*$ because

$$\begin{aligned}(a,b)*\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right) &= \left(a\cdot\frac{a}{a^2+b^2}-b\cdot\frac{-b}{a^2+b^2}, a\cdot\frac{-b}{a^2+b^2}+b\cdot\frac{a}{a^2+b^2}\right)\\ &= (1,0)\end{aligned}$$

and

$$\begin{aligned}\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)*(a,b) &= \left(\frac{a}{a^2+b^2}\cdot a-\left(-\frac{b}{a^2+b^2}\cdot b\right), \frac{a}{a^2+b^2}\cdot b+\left(-\frac{b}{a^2+b^2}\cdot a\right)\right)\\ &= (1,0)\end{aligned}$$

Therefore, we conclude that the set G, with the operation $*$, is a group. □

(b)

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses. In part (a), we have already shown that the operation $*$ is associative and that G has an identity element and inverses. Therefore we must show that the operation $*$ is commutative.

To prove that $*$ is commutative let a, b, c, and d be arbitrary elements of G. Then

$$(a,b)*(c,d) = (ac-bd, ad+bc)$$

and

$$(c,d)*(a,b) = (ca-db, cb+da) = (ac-bd, ad+bc)$$

Since the two results are the same, the operation $*$ is commutative.

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. □

5. No, the operation $*$ in exercise 4, defined as $(a, b) * (c, d) = (ac - bd, ad + bc)$, on the set $\mathbb{R} \times \mathbb{R}$ is not a group because not every element in $\mathbb{R} \times \mathbb{R}$ has an inverse with respect to the operation $*$. From the solution to exercise 4 we know that the inverse with respect to $*$ is $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ and we see that the inverse for $(0, 0)$ is not defined. Therefore, the operation $*$ on the set $\mathbb{R} \times \mathbb{R}$ is not a group.