

Chapter 2 - Operations

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From "Book of Abstract Algebra" by Charles C. Pinter

A. Examples of Operations

Which of the following rules are operations on the indicated set? (\mathbb{Z} designates the set of integers, \mathbb{Q} the rational numbers, and \mathbb{R} the real numbers.) For each rule which is not an operation, explain why it is not.

- 1 $a * b = \sqrt{|ab|}$, on the set \mathbb{Q} .
- 2 $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.
- 3 $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set \mathbb{R} .
- 4 Subtraction, on the set \mathbb{Z} .
- 5 Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
- 6 $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution

- 1 This is not an operation on \mathbb{Q} because $a * b$ is not uniquely defined and \mathbb{Q} is not closed under $*$. If a and b are rational numbers they can be written as $a = \frac{c}{d}$ and $b = \frac{e}{f}$ where c, d, e , and f are integers, $d \neq 0$, and $f \neq 0$. If we let $c = 2$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{2}{1} \cdot \frac{2}{1}} = \sqrt{2 \cdot 2} = \sqrt{4}$$

and since $\sqrt{4} = \pm 2$ we see that $a * b$ is not uniquely defined. Now let $c = 3$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{3}{1} \cdot \frac{2}{1}} = \sqrt{6}$$

and we see that there is no rational number f such that $f \cdot f = 6$, therefore $\sqrt{6}$ is not a rational number. Thus, \mathbb{Q} is not closed under $*$.

- 2 This is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$ because the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$. For example, if we let $a = 2$ and $b = 1$ then

$$a \ln b = 2 \ln 1 = 0$$

and $0 \notin \{x \in \mathbb{R} : x > 0\}$. Therefore the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$ and $*$ is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$.

- 3 This is not an operation on \mathbb{R} because $a * b$ is not uniquely defined. If we solve for x we see

$$\begin{aligned} x^2 - a^2 b^2 &= 0 \\ x^2 &= a^2 b^2 \end{aligned}$$

since a^2 and b^2 will always be positive numbers, then $a^2 b^2$ will be positive as well. Then solving for x we have

$$x = (ab, -ab)$$

and we see that $a * b$ is not uniquely defined and therefore $*$ is not an operation on \mathbb{R} .

- 4 This is an operation on \mathbb{Z} .

- 5 This is not an operation on \mathbb{Z} because the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$. For example, if we let $a = 10$ and $b = 5$ then $5 - 10 = -5$ and $-5 \notin \{n \in \mathbb{Z} : n \geq 0\}$. Therefore the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$.

- 6 This is an operation on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

B. Properties of Operations

Each of the following is an operation $*$ on \mathbb{R} . Indicate whether or not

- (i) it is commutative,
- (ii) it is associative,
- (iii) \mathbb{R} has an identity element with respect to $*$,
- (iv) every $x \in \mathbb{R}$ has an inverse with respect to $*$.

Instructions For (i), compute $x * y$ and $y * x$, and verify whether or not they are equal. For (ii), compute $x * (y * z)$ and $(x * y) * z$, and verify whether or not they are equal. For (iii), first solve the equation $x * e = x$ for e ; if the equation cannot be solved, there is no identity element. If it *can* be solved, it is still necessary to check that $e * x = x * e = x$ for any $x \in \mathbb{R}$. If it checks, then e is an identity element. For (iv), first note that if there is no identity element, there can be no inverses. If there is an identity element e , first solve the equation $x * x' = e$ for x' ; if the equation cannot be solved, x does not have an inverse. If it can be solved, check to make sure that $x * x' = x' * x = e$. If this checks, x' is the inverse of x .

1) $x * y = \sqrt{x^2 + y^2}$

Solution

Although the instructions for this section state that $x * y = \sqrt{x^2 + y^2}$ is an operation on \mathbb{R} , I don't think it is an operation on \mathbb{R} because $\sqrt{x^2 + y^2}$ is not uniquely defined on \mathbb{R} . For example, if we let $x = 1$ and $y = 0$, then $x * y = \sqrt{1^2 + 0^2} = \sqrt{1} = (-1, 1)$ and we see that $\sqrt{x^2 + y^2}$ is not uniquely defined on \mathbb{R} . Therefore, $x * y = \sqrt{x^2 + y^2}$ is not an operation on \mathbb{R} .

2) $x * y = |x + y|$

Solution

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>

(a) $x * y = |x + y|$

$y * x = |y + x| = |x + y|$ because $+$ is a commutative operation on \mathbb{R}
Therefore $*$ is commutative on \mathbb{R} .

(b) $x * (y * z) = x * |y + z| = |x + |y + z||$

Let $x = -5$, $y = 10$, and $z = -20$, then

$$|-5 + |10 - 20|| = |-5 + |-10|| = |-5 + 10| = |5| = 5$$

$$(x * y) * z = |x + y| * z = ||x + y| + z|$$

Again, let $x = -5$, $y = 10$, and $z = -20$, then

$$||-5 + 10| - 20| = ||5| - 20| = |5 - 20| = |-15| = 15$$

Since $5 \neq 15$ then $*$ is not associative on \mathbb{R}

(c) There is no identity element with respect to $*$ on \mathbb{R} because there is no $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in \mathbb{R} . For example, if $a = -1$ there is no $e \in \mathbb{R}$ such that $|e + (-1)| = -1$ because the operation $*$ defined by $x * y = |x + y|$ always returns a positive number.

(d) Since there is no identity element with respect to $*$ on \mathbb{R} then there is no inverse with respect to $*$ on \mathbb{R} .

3) $x * y = |xy|$

Solution

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>

(a) $x * y = |xy| = \begin{cases} xy & \text{if } xy \geq 0 \\ -(xy) & \text{if } xy < 0 \end{cases}$

$$y * x = |xy| = \begin{cases} yx & \text{if } yx \geq 0 \\ -(yx) & \text{if } yx < 0 \end{cases}$$

Assuming multiplication is commutative on \mathbb{R} (not proven here) then $xy = yx$ and $-(xy) = -(yx)$. Therefore, $*$ is commutative on \mathbb{R} .

(b) $x * (y * z) = x * |yz| = |x|yz|| = xyz$

$$(x * y) * z = |xy| * z = ||xy|z| = xyz$$

Therefore $*$ is associative on \mathbb{R} .

- (c) There is no identity element with respect to $*$ on \mathbb{R} because there is no $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in \mathbb{R} . For example, if $a = -5$ there is no $e \in \mathbb{R}$ such that $|-5e| = -5$. Therefore there is no identity element with respect to $*$ on \mathbb{R} .
- (d) Since there is no identity element with respect to $*$ on \mathbb{R} then there is no inverse with respect to $*$ on \mathbb{R} .

4) $x * y = x - y$

Solution

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>

- (a) The operation $*$ defined by $x * y = x - y$ is not commutative because $x - y \neq y - x$ for all x and y in \mathbb{R} . For example, if $x = 1$ and $y = 2$, then $x - y = 1 - 2 = -1$ and $y - x = 2 - 1 = 1$. Since $-1 \neq 1$ then the operation $*$ defined by $x * y = x - y$ is not commutative on \mathbb{R} .
- (b) The operation $*$ defined by $x * y = x - y$ is not associative on \mathbb{R} because $x * (y * z) \neq (x * y) * z$ for all x, y , and z in \mathbb{R} . For example, if $x = 5$, $y = -1$, and $z = 2$ then $x * (y * z) = 5 - (-1 - 2) = 8$ and $(x * y) * z = (5 - (-1)) - 2 = 4$. Since $8 \neq 4$ then the operation $*$ defined by $x * y = x - y$ is not associative on \mathbb{R} .
- (c) Solving $x * e = x$ for e we have that $x - e = x \Rightarrow e = 0$. However checking that $x * e = e * x = x$ we see that $x - 0 = x$ and $0 - x = -x$. Since $x \neq -x$ then 0 is not the identity element with respect to the operation $*$ defined by $x * y = x - y$ on \mathbb{R} . Therefore there is no identity element with respect to the operation $*$ defined by $x * y = x - y$ on \mathbb{R} .
- (d) Since there is no identity element with respect to $*$ on \mathbb{R} then there is no inverse with respect to $*$ on \mathbb{R} .

5) $x * y = xy + 1$

Solution

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>

- (a) $x * y = xy + 1$
 $y * x = yx + 1 = xy + 1$ because \times is commutative on \mathbb{R} .
Therefore, the operation $*$ defined by $xy + 1$ is commutative on \mathbb{R} .
- (b) The operation $*$ defined by $x * y = x - y$ is not associative on \mathbb{R} because
 $x * (y * z) \neq (x * y) * z$.
 $x * (y * z) = x * (yz + 1) = x(yz + 1) + 1 = xyz + x + 1$
 $(x * y) * z = (xy + 1) * z = (xy + 1)z + 1 = xyz + z + 1$
Since $xyz + x + 1 \neq xyz + z + 1$ then the operation $*$ defined by $x * y = x - y$ is not associative on \mathbb{R} .
- (c) Since there is no identity element with respect to $*$ on \mathbb{R} then there is no inverse with respect to $*$ on \mathbb{R} .

6) $x * y = \max\{x, y\}$

Solution

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>

(a) If $x \neq y$ then $x * y = \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{if } y > x \end{cases}$

$$y * x = \max(y, x) = \begin{cases} y & \text{if } y > x \\ x & \text{if } x > y \end{cases}$$

If $x = y$ then $\max(x, y) = z$ where $z = x = y$.

Therefore $x * y = y * x$ and the operation $*$ defined by $\max\{x, y\}$ is commutative on \mathbb{R} .

(b) The operation $*$ defined by $\max\{x, y\}$ is associative on \mathbb{R}

$$x * (y * z) = x * \max\{y, z\} = \max\{x, \max\{y, z\}\} = \max\{x, y, z\}$$

$$(x * y) * z = \max\{x, y\} * z = \max\{\max\{x, y\}, z\} = \max\{x, y, z\}$$

Since $\max\{x, y, z\} = \max\{x, y, z\}$ the operation $*$ defined by $\max\{x, y\}$ is associative on \mathbb{R} .

(c) Solving $x * e = x$ for e we see that $\max\{x, e\} = x \Rightarrow e = -\infty$ However, $-\infty \notin \mathbb{R}$ as is required for an identity element. There is no other $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in \mathbb{R} . Therefore there is not an identity element with respect to the operation $*$ defined by $\max\{x, y\}$ on \mathbb{R} .

(d) Since there is no identity element with respect to $*$ on \mathbb{R} then there is no inverse with respect to $*$ on \mathbb{R} .

7) $x * y = \frac{xy}{x+y+1}$

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>

(a) $x * y = \frac{xy}{x+y+1}$; $y * x = \frac{yx}{y+x+1}$

Since \times and $+$ are commutative on \mathbb{R} then $\frac{xy}{x+y+1} = \frac{yx}{y+x+1}$. Therefore, the operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ is commutative on \mathbb{R} .

(b) The operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ is associative on \mathbb{R} .

$$x * (y * z) = x * \frac{yz}{y+z+1} = \frac{x \left(\frac{yz}{y+z+1} \right)}{x + \frac{yz}{y+z+1} + 1} = \frac{xyz}{1 + x + y + z + xy + xz + yz}$$

$$(x * y) * z = \frac{xy}{x+y+1} * z = \frac{z \left(\frac{xy}{x+y+1} \right)}{\frac{xy}{x+y+1} + z + 1} = \frac{xyz}{1 + x + y + z + xy + xz + yz}$$

$x * (y * z) = (x * y) * z$ and therefore the operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ is associative on \mathbb{R} .

(c) Solving $x * e = x$ for e we see that $\frac{xe}{x+e+1} = x \Rightarrow x = -1$ and therefore there is no $e \in \mathbb{R}$ such that $e * a = a$ and $a * e = a$ for every element a in \mathbb{R} . Therefore there is not an identity element with respect to the operation $*$ defined by $x * y = \frac{xy}{x+y+1}$ on \mathbb{R} .

(d) Since there is no identity element with respect to $*$ on \mathbb{R} then there is no inverse with respect to $*$ on \mathbb{R} .

C. Operations on a Two-Element Set

Let A be the two-element set $A = \{a, b\}$.