Chapter 3 Solutions
From "Book of Abstract Algebra" by Charles C. Pinter
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A. Examples of Abelian Groups

1. Let $G = \mathbb{R}$, and let $*$ be the binary operation on G defined by

$$x * y \;=\; x + y + k$$

for all $x, y \in G$ and where k is a fixed constant.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + k$$

and

$$y * x = y + x + k = x + y + k.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + k) \\ &= x + (y + z + k) + k \\ &= x + y + z + 2k \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + k) * z \\ &= (x + y + k) + z + k \\ &= x + y + z + 2k. \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is $-k$ because

$$x * -k = x + (-k) + k = x$$

and

$$-k * x = -k + x + k = x.$$

Finally, we see that the inverse with respect to $*$ is $-x - 2k$ because

$$x * x^{-1} = x + (-x - 2k) + k = x - x - 2k = -k$$

and

$$x^{-1} * x = (-x - 2k) + x + k = -x - 2k + x + k = -k$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. $\square$

2. Let $G = \{x \in \mathbb{R} : x \neq 0\}$, and let $*$ be the binary operation on G defined by

$$x * y = \frac{xy}{2} \tag{1}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = \frac{xy}{2}$$

and

$$y * x = \frac{xy}{2} = \frac{yx}{2}.$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$x * (y * z) = x * \frac{yz}{2} = \frac{\frac{xyz}{2}}{2} = \frac{xyz}{4}$$

and

$$(x * y) * z = \frac{xy}{2} * z = \frac{\frac{xyz}{2}}{2} = \frac{xyz}{4}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 2 because

$$x * 2 = \frac{x2}{2} = x$$

and

$$2 * x = \frac{2x}{2} = x.$$

Finally we see that $\frac{4}{x}$ is the inverse of the operation $*$ because

$$x * \frac{4}{x} = \frac{\frac{4x}{x}}{2} = \frac{4}{2} = 2$$

and

$$\frac{4}{x} * x = \frac{\frac{4x}{x}}{2} = \frac{4}{2}.$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group. $\square$

3. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $*$ be the binary operation on G defined by

$$x * y = x + y + xy \tag{2}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = x + y + xy$$

and

$$y * x = y + x + yx = x + y + xy$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * (y + z + yz) \\ &= x + (y + z + yz) = x + (y + z + yz) + x(y + z + yz) \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= (x + y + xy) * z \\ &= (x + y + xy) + z + (x + y + xy)z \\ &= x + y + z + xy + xz + yz + xyz \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 0 because

$$x * 0 = x + 0 + x \cdot 0 = x$$

and

$$0 * x = 0 + x + 0 \cdot x = x.$$

Finally we see that $-\frac{x}{1+x}$ is the inverse of the operation $*$ because

$$\begin{aligned} x * \left(-\frac{x}{1+x}\right) &= x + \left(-\frac{x}{1+x}\right) + x \cdot \left(-\frac{x}{1+x}\right) \\ &= \frac{x + x^2 - x + x^2}{1+x} = 0 \end{aligned}$$

and

$$\begin{aligned} \left(-\frac{x}{1+x}\right) * x &= \left(-\frac{x}{1+x}\right) + x + \left(-\frac{x^2}{1+x}\right) \\ &= \frac{x + x^2 - x - x^2}{1+x} = 0 \end{aligned}$$

Therefore, we conclude that the set G, with the operation $*$, is an abelian group.

$\square$

4. Let $G = \{x \in \mathbb{R} : -1 < x < 1\}$, and let $*$ be the binary operation on G defined by

$$x * y = \frac{x+y}{xy+1} \tag{3}$$

for all $x, y \in G$.

Proposition. *The set G, with the operation $*$, is an abelian group.*

Proof. To prove the set G, with the operation $*$, is an abelian group we must show that $*$ is commutative, associative and that G has an identity element and inverses.

To prove that $*$ is associative let x and y be arbitrary elements of G. Then

$$x * y = \frac{x+y}{xy+1}$$

and

$$y * x = \frac{y+x}{yx+1} = \frac{x+y}{xy+1}$$

Since the two results are the same, the operation $*$ is commutative.

Now let x, y, and z be arbitrary elements of G. Then

$$\begin{aligned} x * (y * z) &= x * \frac{y+z}{yz+1} \\ &= \frac{x + \frac{y+z}{yz+1}}{\frac{y+z}{yz+1} + 1} \\ &= x + y + z + xy + xz + yz + xyz + 1 \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= \frac{x+y}{xy+1} * z \\ &= \frac{\frac{x+y}{xy+1} + z}{\frac{x+y}{xy+1} z + 1} \\ &= x + y + z + xy + xz + yz + xyz + 1 \end{aligned}$$

Since the two results are the same, the operation $*$ is associative.

The identity element for G is 0 because

$$x * 0 = \frac{x+0}{x \cdot 0 + 1} = \frac{x}{1} = x$$

and

$$0 * x = \frac{0+x}{0 \cdot x + 1} = \frac{x}{1} = x$$

Finally we see that $-x$ is the inverse of the operation $*$ because

$$x * -x = \frac{x-x}{x \cdot -x + 1} = \frac{0}{-x^2+1} = 0$$

and

$$-x * x = \frac{-x+x}{-x \cdot x + 1} = \frac{0}{-x^2+1} = 0$$

□

Therefore, we conclude that the set G, with the operation $*$, is an abelian group.