Solutions to Chapter 1 exercises in "A Book of Abstract Algebra" by Pinto

November 27, 2020

A. Examples of Operations

Which of the following rules are operations on the indicated set? ($\mathbb{Z}$ designates the set of integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{R}$ the real numbers.) For each rule which is not an operation, explain why it is not.

1. $a * b = \sqrt{|ab|}$, on the set $\mathbb{Q}$.
2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.
3. $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set $\mathbb{R}$.
4. Subtraction, on the set $\mathbb{Z}$.
5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution

1. This is not an operation on $\mathbb{Q}$ because $a * b$ is not uniquely identified and $\mathbb{Q}$ is not closed under $*$. If a and b are rational numbers they can be written as $a = \frac{c}{d}$ and $b = \frac{e}{f}$ where $c, d, e, \text{and} f$ are integers, $d \neq 0$, and $f \neq 0$. If we let $c = 2$, $d = 1$, $e = 2$, and $f = 1$ then

   $$\sqrt{\frac{2}{1} \cdot \frac{2}{1}} = \sqrt{2 \cdot 2} = \sqrt{4}$$

   and since $\sqrt{4} = \pm 2$ we see that $a * b$ is not uniquely identified. Now let $c = 3$, $d = 1$, $e = 2$, and $f = 1$ then

   $$\sqrt{\frac{3}{1} \cdot \frac{2}{1}} = \sqrt{6}$$

   and we see that there is no rational number f such that $f \cdot f = 6$, therefore $\sqrt{6}$ is not a rational number. Thus, $\mathbb{Q}$ is not closed under $*$.

2. This is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$ because the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$. For example, if we let $a = 2$ and $b = 1$ then

$$a \ln b = 2 \ln 1 = 0$$

and $0 \notin \{x \in \mathbb{R} : x > 0\}$. Therefore the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$ and $*$ is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$.