

Solutions to Chapter 1 exercises in "A Book of Abstract Algebra" by Charles Pinter

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A. Examples of Operations

Which of the following rules are operations on the indicated set? (\mathbb{Z} designates the set of integers, \mathbb{Q} the rational numbers, and \mathbb{R} the real numbers.) For each rule which is not an operation, explain why it is not.

1. $a * b = \sqrt{|ab|}$, on the set \mathbb{Q} .
2. $a * b = a \ln b$, on the set $\{x \in \mathbb{R} : x > 0\}$.
3. $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set \mathbb{R} .
4. Subtraction, on the set \mathbb{Z} .
5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.
6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution

1. This is not an operation on \mathbb{Q} because $a * b$ is not uniquely defined and \mathbb{Q} is not closed under $*$. If a and b are rational numbers they can be written as $a = \frac{c}{d}$ and $b = \frac{e}{f}$ where c, d, e , and f are integers, $d \neq 0$, and $f \neq 0$. If we let $c = 2$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{2}{1} \cdot \frac{2}{1}} = \sqrt{2 \cdot 2} = \sqrt{4}$$

and since $\sqrt{4} = \pm 2$ we see that $a * b$ is not uniquely defined. Now let $c = 3$, $d = 1$, $e = 2$, and $f = 1$ then

$$\sqrt{\frac{3}{1} \cdot \frac{2}{1}} = \sqrt{6}$$

and we see that there is no rational number f such that $f \cdot f = 6$, therefore $\sqrt{6}$ is not a rational number. Thus, \mathbb{Q} is not closed under $*$.

2. This is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$ because the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$. For example, if we let $a = 2$ and $b = 1$ then

$$a \ln b = 2 \ln 1 = 0$$

and $0 \notin \{x \in \mathbb{R} : x > 0\}$. Therefore the set $\{x \in \mathbb{R} : x > 0\}$ is not closed under $*$ and $*$ is not an operation on the set $\{x \in \mathbb{R} : x > 0\}$.

3. This is not an operation on \mathbb{R} because $a * b$ is not uniquely defined. If we solve for x we see

$$\begin{aligned} x^2 - a^2 b^2 &= 0 \\ x^2 &= a^2 b^2 \end{aligned}$$

since a^2 and b^2 will always be positive numbers, then $a^2 b^2$ will be positive as well. Then solving for x we have

$$x = (ab, -ab)$$

and we see that $a * b$ is not uniquely defined and therefore $*$ is not an operation on \mathbb{R} .

4. This is an operation on \mathbb{Z} .
5. This is not an operation on \mathbb{Z} because the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$. For example, if we let $a = 10$ and $b = 5$ then $5 - 10 = -5$ and $-5 \notin \{n \in \mathbb{Z} : n \geq 0\}$. Therefore the set $\{n \in \mathbb{Z} : n \geq 0\}$ is not closed under $*$.
6. This is an operation on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

B. Properties of Operations

Each of the following is an operation $*$ on \mathbb{R} . Indicate whether or not

- (i). it is commutative,
- (ii). it is associative,
- (iii). \mathbb{R} has an identity element with respect to $*$,
- (iv). every $x \in \mathbb{R}$ has an inverse with respect to $*$.

Instructions For (i), compute $x * y$ and $y * x$, and verify whether or not they are equal. For (ii), compute $x * (y * z)$ and $(x * y) * z$, and verify whether or not they are equal. For (iii), first solve the equation $x * e = x$ for e ; if the equation cannot be solved, there is no identity element. If it *can* be solved, it is still necessary to check that $e * x = x * e = x$ for any $x \in \mathbb{R}$. If it checks, then e is an identity element. For (iv), first note that if there is no identity element, there can be no inverses. If there is an identity element e , first solve the equation $x * x' = e$ for x' ; if the equation cannot be solved, x does not have an inverse. If it can be solved, check to make sure that $x * x' = x' * x = e$. If this checks, x' is the inverse of x .

1. $x * y = \sqrt{x^2 + y^2}$
2. $x * y = |x + y|$
3. $x * y = |xy|$
4. $x * y = x - y$
5. $x * y = xy + 1$
6. $x * y = \max\{x, y\}$
7. $x * y = \frac{xy}{x+y+1}$

Solution

1.

<i>Commutative</i>	<i>Associative</i>	<i>Identity</i>	<i>Inverses</i>
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>

- (i). $x * y = \sqrt{x^2 + y^2}$
 $y * x = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2}$ because $+$ is a commutative operation on \mathbb{R}
(Thus $$ is commutative.)*

- (ii). $x * (y * z) = x * \left(\sqrt{y^2 + z^2} \right) = \sqrt{x^2 + \left(\sqrt{y^2 + z^2} \right)^2} = \sqrt{x^2 + y^2 + z^2}$
 $(x * y) * z = \sqrt{(x^2 + y^2)} * z = \sqrt{\left(\sqrt{x^2 + y^2} \right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$
(Thus $$ is associative.)*

- (iii). Solve $x * e = x$ for x : $\sqrt{x^2 + e^2} = x$ therefore $e = 0$.
Check: $x * 0 = \sqrt{x^2 + 0} = \sqrt{x^2} = x$; $0 * x = \sqrt{0 + x^2} = \sqrt{x^2} = x$
Therefore, 0 is the identity element.
($$ has an identity element)*

- (iv). Solve $x * x' = e$ for x' .

$$\begin{aligned}
 x * x' &= e \\
 \sqrt{x^2 + x'^2} &= 0 \\
 x' &= \sqrt{-x^2} \\
 x' &= \pm ix
 \end{aligned}$$

where i is the imaginary unit.

Now check that $x * x' = x' * x = e$.

If $x' = ix$ then

$$\begin{aligned}
x * x' &= \sqrt{x^2 + (xi)^2} \\
&= \sqrt{x^2 + (x^2 \cdot i^2)} \\
&= \sqrt{x^2(1 + i^2)} \\
&= \sqrt{x^2(1 + (-1))} \\
&= \sqrt{x^2 \cdot 0} = 0
\end{aligned}$$

and

$$\begin{aligned}
x' * x &= \sqrt{(xi)^2 + x^2} \\
&= \sqrt{(x^2 \cdot i^2) + x^2} \\
&= \sqrt{(1 + i^2)x^2} \\
&= \sqrt{(1 + (-1))x^2} \\
&= \sqrt{0 \cdot x^2} = 0
\end{aligned}$$

If $x' = -ix$ then

$$\begin{aligned}
x * x' &= \sqrt{x^2 + (-xi)^2} \\
&= \sqrt{x^2 + (xi)^2} && \text{exponent rule: } (-a)^n = a^n, \text{ if } n \text{ is even} \\
&= \sqrt{x^2 + (x^2 \cdot i^2)} \\
&= \sqrt{x^2(1 + i^2)} \\
&= \sqrt{x^2(1 + (-1))} \\
&= \sqrt{x^2 \cdot 0} = 0
\end{aligned}$$

and

$$\begin{aligned}
x' * x &= \sqrt{(-xi)^2 + x^2} \\
&= \sqrt{(xi)^2 + x^2} && \text{exponent rule: } (-a)^n = a^n, \text{ if } n \text{ is even} \\
&= \sqrt{(x^2 \cdot i^2) + x^2} \\
&= \sqrt{(1 + i^2)x^2} \\
&= \sqrt{(1 + (-1))x^2} \\
&= \sqrt{0 \cdot x^2} = 0
\end{aligned}$$

Therefore, $\pm ix$ is the inverse of x .

(Every element has an inverse)

2.

Commutative	Associative	Identity	Inverses
Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input type="checkbox"/> No <input checked="" type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>	Yes <input checked="" type="checkbox"/> No <input type="checkbox"/>

(i).