Exercise 3.3.4

Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

So we want to prove that $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$.

First we assume x is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathscr{P}(A)$	$x \in \mathscr{P}(\mathscr{P}(A))$
$x \in \mathscr{P}(A)$	

Assume x is an arbitrary element of $\mathscr{P}(A)$
    Suppose $x \in \mathscr{P}(A)$
        [proof of $x \in \mathscr{P}(\mathscr{P}(A))$]
    Therefore if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$
Since x was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$

We can rewrite our goal as $x \subseteq \mathscr{P}(A)$ or $\forall y(y \in x \to y \in \mathscr{P}(A))$. So we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathscr{P}(A)$	$y \in \mathscr{P}(A)$
$x \in \mathscr{P}(A)$	
$y \in x$	

Assume x is an arbitrary element of $\mathscr{P}(A)$
    Suppose $x \in \mathscr{P}(A)$
        Suppose y is an arbitrary element of x.
            Suppose $y \in x$.
                [proof of $y \in \mathscr{P}(A)$]
            Therefore if $y \in x \to y \in \mathscr{P}(A)$.
        Since y was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$.
    Therefore if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$
Since x was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$

Now looking at our givens $x \in \mathscr{P}(A)$ means that $x \subseteq A$ or $\forall z(z \in x \to z \in A)$. Using universal instantiation we will plug in y for z and using modus ponens we can conclude that $y \in A$.

Now looking at our other given $A \subseteq \mathscr{P}(A) \to \forall m(m \in A \to m \in \mathscr{P}(A))$. Using universal instantiation we will plug in y for m and using modus ponens we can conclude that $y \in \mathscr{P}(A)$, which was our goal to prove.

Theorem. *Suppose* $A \subseteq \mathscr{P}(A)$. *Then* $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

Proof. Suppose x is an arbitrary element of $\mathscr{P}(A)$ and y is an arbitrary element of x. It follows that $y \in A$. But since $A \subseteq \mathscr{P}(A)$ then it also follows that $y \in \mathscr{P}(A)$. So $y \in x \to y \in \mathscr{P}(A)$ and since y was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$. Therefore, if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$. Since x was arbitrary we can also conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. □

Alternate proof (not sure if this is correct)

Proof. Suppose x is an arbitrary element of $\mathscr{P}(A)$. Then $x \in A$. Since $A \subseteq \mathscr{P}(A)$ and $x \in A$ then $x \subseteq \mathscr{P}(A)$. Therefore, $x \in \mathscr{P}(\mathscr{P}(A))$. □

Exercise 3.3.5

The hypothesis of the theorem proven in exercise 3.3.4 is $A \subseteq \mathscr{P}(A)$.

A

Can you think of a set A for which this hypothesis is true?

The empty set $\varnothing$ is a set for which the hypothesis is true.

$A \subseteq \mathscr{P}(A)$ means $x \in A \to x \in \mathscr{P}(A)$. For $\varnothing$ this would mean that $x \in \varnothing \to x \in \mathscr{P}(\varnothing)$, but by definition there are no elements in $\varnothing$. Therefore $x \in \varnothing$ will always be false and the conditional statement $x \in \varnothing \to x \in \mathscr{P}(\varnothing)$ is always true. Therefore if $\varnothing = A$ then $A \subseteq \mathscr{P}(A)$.

B

Can you think of another?

In exercise 3.3.4 we proved that if $A \subseteq \mathscr{P}(A)$ then $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. Therefore, the set $\{\varnothing, \{\varnothing\}\}$, which is the $\mathscr{P}(A)$ if $A = \varnothing$, is another set for which the hypothesis is true. If we let $B = \mathscr{P}(A) = \{\varnothing, \{\varnothing\}\}$ and replace A in the hypothesis $A \subseteq \mathscr{P}(A)$ with B, then we can conclude that $B \subseteq \mathscr{P}(B)$.

Exercise 3.3.6

Suppose x is a real number.

A

Prove that if $x \neq 1$ then there is a real number y such that $\frac{y+1}{y-2} = x$.

So we want to prove that $(x \neq 1) \to \exists y \left(\frac{y+1}{y+2} = x \right)$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$x \neq 1$	$\exists y \left(\frac{y+1}{y+2} = x \right)$

To prove our goal we need to find a y that makes the equation $\frac{y+1}{y+2} = x$ true. So let's try solving the equation for y.

$$\begin{aligned} \frac{y+1}{y+2} &= x \\ y+1 &= x(y-2) \\ y+1 &= xy - 2x \\ 2x+1 &= xy - y \\ 2x+1 &= y(x-1) \\ y &= \frac{2x+1}{x-1} \end{aligned}$$

We see that this y works because we have $x \neq 1$ as a given.

Theorem. *Suppose $x \neq 1$. Then there is a real number y such that $\frac{y+1}{y-2} = x$.*

Proof. Suppose $x \neq 1$ and $y = \frac{2x+1}{x-1}$. Then

$$\frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{x-1} \cdot \frac{x-1}{3} = x$$

□

B

Prove that if there is a real number y such that $\frac{y+1}{y-2} = x$ then $x \neq 1$.

So we want to prove that $\exists y \left(\frac{y+1}{y-1} = x \right) \to (x \neq 1)$

We assume the antecedent and make the consequent our goal to prove.

Using existential instantiation we assume there is a value y_0 such that $\frac{y+1}{y-1} = x$ is true. From part A above, we know that $\left(\frac{y+1}{y-1} = x \right) \to \left(y = \frac{2x+1}{x-1} \right)$ and so $y_0 = \frac{2x+1}{x-1}$. Since y is a real number, then clearly $x \neq 1$.

Givens	Goals
$\exists y \left(\frac{y+1}{y-1} = x\right)$	$x \neq 1$

Theorem. *If y is a real number and $\frac{y+1}{y-2} = x$ then $x \neq 1$.*

Proof. Suppose y is a real number and $\frac{y+1}{y-2} = x$. It follows that $y = \frac{2x+1}{x-1}$ and since y is real number then $x \neq 1$. $\square$

Exercise 3.3.7

Prove for every real number x, if $x > 2$ then there is a real number y such that $y + \frac{1}{y} = x$.

So we want to prove $\forall x \in \mathbb{R}(x > 2 \to \exists y \in \mathbb{R}(y + \frac{1}{y} = x))$

So we let x be an arbitrary real number, then we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
x is arbitrary real number	$\exists y(y + \frac{1}{y} = x)$
$x > 2$	

Our goal is of the form $\exists y P(y)$ where $P(y)$ is $y + \frac{1}{y} = x$ and our strategy suggests we try to find a y for which $P(y)$ is true. We can do this by solving the equation $y + \frac{1}{y} = x$ for y. We can rewrite this equation as $y^2 - \frac{x}{y} + 1 = 0$ and we see this is a quadratic equation and therefore we can use the quadratic formula to solve for y,

$$y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{x \pm \sqrt{x^2 - 4}}{2}.$$

We note that $\sqrt{x^2 - 4}$ is defined because $x > 2$. We have found two solutions that satisfy our original equation, but we only need one to complete the proof. We will use $\frac{x+\sqrt{x^2-4}}{2}$.

Theorem. *For every real number x, if $x > 2$ then there is a real number y such that $y + \frac{1}{y} = x$.*

Proof. Suppose x and y are real numbers, $x > 2$, and $y = \frac{x+\sqrt{x^2-4}}{2}$. Then

$$
\begin{aligned}
\frac{x+\sqrt{x^2-4}}{2} + \frac{1}{\frac{x+\sqrt{(x^2-4}}{2}} &= \frac{x+\sqrt{x^2-4}}{2} + \frac{2}{x+\sqrt{x^2-4}} \\
&= \frac{2x^2+2(x\sqrt{x^2-4})}{2x+2\sqrt{x^2-4}} \\
&= x
\end{aligned}
$$

□

Exercise 3.3.8

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.

So we want to prove that $A \in \mathcal{F} \rightarrow A \subseteq \cup\mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$A \subseteq \cup\mathcal{F}$

Assume $A \in \mathcal{F}$
    [proof of $A \subseteq \cup\mathcal{F}$]
Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$

Our goal $A \subseteq \cup\mathcal{F}$ can be rewritten as $\forall x(x \in A \rightarrow x \in \cup\mathcal{F})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in \cup\mathcal{F}$
$x \in A$	

Assume $A \in \mathcal{F}$
    Assume x is arbitrary
        Assume $x \in A$
            [proof of $x \in \cup\mathcal{F}$]
        Therefore if $x \in A$ then $x \cup \mathcal{F}$
    Since x was arbitrary we can conclude that $A \subseteq \cup\mathcal{F}$.
Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$

Our new goal can be rewritten as $\exists B \in \mathcal{F}(x \in B)$. From our givens we see that $A \in \mathcal{F}$ and $x \in A$, so we have found a set such that $A \in \mathcal{F}(x \in A)$.

Theorem. *If $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.*

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of A. Then since $x \in A$ and $A \in \mathcal{F}$, it follows that $x \in \cup\mathcal{F}$. Since x was arbitrary we can conclude that $A \subseteq \cup\mathcal{F}$ and therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$. □

Exercise 3.3.9

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cap\mathcal{F} \subseteq A$.

We want to prove that $A \in \mathcal{F} \to \cap\mathcal{F} \subseteq A$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$\cap\mathcal{F} \subseteq A$

Assume $A \in \mathcal{F}$
    [proof of $\cap\mathcal{F} \subseteq A$]
Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$.

We can rewrite our goal as $\forall x(x \in \cap\mathcal{F} \to x \in A)$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in A$
$x \in \cap\mathcal{F}$	

Assume $A \in \mathcal{F}$
    Assume x is arbitrary
        Assume $x \in \cap\mathcal{F}$
            [proof of $x \in A$]
        Therefore, if $x \in \cap\mathcal{F}$ then $x \in A$.
    Since x was arbitrary we can conclude that $\cap\mathcal{F} \subseteq A$.
Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$.

Our given $x \in \cap\mathcal{F}$ can be rewritten as $\forall B \in \mathcal{F}(x \in B)$, therefore if $A \in \mathcal{F}$ then $x \in A$, which was our goal to prove.

Theorem. *If $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cup\mathcal{F} \in A$.*

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of $\cap\mathcal{F}$. Since $A \in \mathcal{F}$ and $x \in \cap\mathcal{F}$ it follows that $x \in A$ and therefore, if $x \in \cap\mathcal{F}$ then $x \in A$. Since x was arbitrary we can conclude that $\cap\mathcal{F} \subseteq A$. Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$. □

Exercise 3.3.10

Suppose that $\mathcal{F}$ is a nonempty family of sets B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$. Prove that $B \subseteq \cap\mathcal{F}$.

We want to prove $\forall A \in \mathcal{F}(B \subseteq A) \rightarrow B \subseteq \cap\mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$B \subseteq \cap\mathcal{F}$

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$
    [proof of $B \subseteq \cap\mathcal{F}$]
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap\mathcal{F}$.

Our goal can be rewritten as $\forall x(x \in B \rightarrow x \in \cap\mathcal{F})$. So we assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$B \in \cap\mathcal{F}$
$x \in B$	

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$
    Suppose x is arbitrary.
        Suppose $x \in B$.
            [proof of $x \in \cap\mathcal{F}$]
        Therefore $x \in B \rightarrow x \in \cap\mathcal{F}$
    Since x was arbitrary we can conclude $B \subseteq \cap\mathcal{F}$
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap\mathcal{F}$.

Our goal can be rewritten as $\forall M \in F(x \in M)$ and so we can assume M is an arbitrary set in $\mathcal{F}$ and make our goal $x \in M$.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$x \in M$
$x \in B$	
$M \in \mathcal{F}$	

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$
    Suppose x is arbitrary.
        Suppose $x \in B$.
            Suppose M is an arbitrary set in $\mathcal{F}$.

[proof of $x \in M$]
Therefore $x \in \cap\mathcal{F}$
Therefore $x \in B \to x \in \cap\mathcal{F}$
Since x was arbitrary we can conclude $B \subseteq \cap\mathcal{F}$
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap\mathcal{F}$.

Using universal instantiation we will plug in M for A in our given $\forall A \in \mathcal{F}(B \subseteq A)$ and conclude that $B \subseteq M$. We can rewrite $B \subseteq M$ as $\forall y(y \in B \to y \in M)$ and using universal instantiation plug in x for y and then use moden ponens to conclude $x \in M$, which was our goal to prove.

Theorem. *If $\mathcal{F}$ is a nonempty family of sets, B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$, then $B \subseteq \cap\mathcal{F}$.*

Proof. Suppose $\forall A \in \mathcal{F}(B \subseteq A)$. Suppose x is an arbitrary member of B and M is an arbitrary set in $\mathcal{F}$. Then it follows that $x \in M$ and since M was arbitrary we can conclude that x is in all sets that are in $\mathcal{F}$ or $x \in \cap\mathcal{F}$. Therefore, if $x \in B$ then $x \in \cap\mathcal{F}$, and since x was arbitrary, we can conclude that $B \subseteq \cap\mathcal{F}$. □

Exercise 3.3.11

Suppose that $\mathcal{F}$ is a family of sets. Prove that if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$.
We want to prove that $\varnothing \in \mathcal{F} \to \cap\mathcal{F} = \varnothing$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\varnothing \in \mathcal{F}$	$\cap\mathcal{F} = \varnothing$

Suppose $\varnothing \in \mathcal{F}$
[proof of $\cap\mathcal{F} = \varnothing$]
Therefore if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$.

We will try a proof by contradiction. So we assume that $\cap\mathcal{F} \neq \varnothing$ and try to find a contradiction.

Givens	Goals
$\varnothing \in \mathcal{F}$	contradiction
$\cap\mathcal{F} \neq \varnothing$	

Our given $\cap\mathcal{F} \neq \varnothing$ means that there is an element that is in all sets in $\mathcal{F}$. However, this contradicts $\varnothing \in \mathcal{F}$ because $\varnothing$ is the set that contains nothing.

Theorem. *If $\mathcal{F}$ is a family of sets and $\varnothing \in \mathcal{F}$, then $\cap\mathcal{F} = \varnothing$.*

Proof. We will prove by contradiction. Suppose $\varnothing \in \mathcal{F}$ and $\cap\mathcal{F} \neq \varnothing$. Since $\cap\mathcal{F} \neq \varnothing$ it follows that there is an element that is within all of the sets that are in $\mathcal{F}$. However, this contradicts $\varnothing \in \mathcal{F}$ because $\varnothing$ is the set that contains nothing. Therefore, if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$. □

Exercise 3.3.12

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cup\mathcal{F} \subseteq \cup\mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  [proof of $\cup\mathcal{F} \subseteq \cup\mathcal{G}$]
So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\cup\mathcal{F} \subseteq \cup\mathcal{G} \rightarrow \forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$ so we assume b is an arbitrary element of $\cup\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$b \in \cup\mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  Let b be an arbitrary element of $\cup\mathcal{F}$
    [proof of $b \in \cup\mathcal{G}$]
  Therefore if $b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G}$
Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$b \in \cup\mathcal{F} \rightarrow \exists M(M \in \mathcal{F} \wedge b \in M)$, so let $M = A_0$ (Existential Instantiation)

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  Let b be an arbitrary element and suppose $b \in \cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ and b is in that set. Let that set $= A_0$

[proof of $b \in \cup\mathcal{G}$]

Therefore if $b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G}$

Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall A(A \in \mathcal{F} \rightarrow A \in \mathcal{G})$. Using universal instantiation we will plug in A_0 for A since then we can use modens ponens to conclude that $A_0 \in \mathcal{G}$.

Givens	Goals
$A_0 \in \mathcal{F} \rightarrow A_0 \in \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Our goal $b \in \cup\mathcal{G} \rightarrow \exists N(N \in \mathcal{G} \wedge b \in N)$, which we can now prove. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$, it follows that $A_0 \in G$. By the definition of $\cup\mathcal{G}$ it follows that $b \in \cup\mathcal{G}$ because $A_0 \in G \wedge b \in A_0$, the latter statement being one of our givens.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.*

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ that contains b. Call this set A_0. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$ it follows that $A_0 \in G$, which implies that $b \in \cup\mathcal{G}$. Therefore if $b \in \cup\mathcal{F}$ then $b \in \cup\mathcal{G}$. Since b was arbitrary we can conclude that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$. This completes the proof.

Exercise 3.3.13

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cap\mathcal{G} \subseteq \cap\mathcal{F}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$

[proof of $\cap\mathcal{G} \subseteq \cap\mathcal{F}$]

So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$\cap\mathcal{G} \subseteq \cap\mathcal{F} \rightarrow \forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$, so we assume b is an arbitrary element of $\cap\mathcal{G}$ and assume the antecedent and make the consequent our goal to prove.

Suppose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cap\mathcal{G}$

[proof of $b \in \cap\mathcal{F}$]

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cap\mathcal{F}$
$b \in \cap\mathcal{G}$	

Therefore if $b \in \cap\mathcal{G} \to b \in \cap\mathcal{F}$
Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$b \in \cap\mathcal{F} \to \forall A(A \in \mathcal{F} \to b \in A)$, so we assume A is an arbitrary element of $\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in A$
$b \in \cap\mathcal{G}$	
$A \in \mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    Let b be an arbitrary element of $\cap\mathcal{G}$
        Suppose A is an arbitrary set in $\mathcal{F}$
            [proof of $b \in A$]
        Therefore if $A \in \mathcal{F} \to b \in A$
        Since A was arbitrary we can conclude $b \in \cap\mathcal{F}$
    Therefore if $b \in \cap\mathcal{G} \to b \in \cap\mathcal{F}$
Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

Now looking at our givens, $\mathcal{F} \subseteq \mathcal{G} \to \forall Z(Z \in \mathcal{F} \to Z \in \mathcal{G})$. Using universal instantiation we will plug in A for Z and using modus ponens we can conclude that $A \in \mathcal{G}$.

Our other given, $b \in \cap\mathcal{G} \to \forall Y(Y \in \mathcal{G} \to b \in Y)$. Using universal instantiation we will plug in A for Y and using modus ponens we can conclude that $b \in A$, which was our goal, and we can now write our proof.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.*
Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cap\mathcal{G}$. Suppose A is an arbitrary element of $\mathcal{F}$, then because $\mathcal{F} \subseteq \mathcal{G}$ then it follows that $A \in \mathcal{G}$. By the definition of $\cap\mathcal{G}$ it follows that $b \in A$ and since A was arbitrary then $b \in \cap\mathcal{F}$. Since b was arbitrary we can conclude $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ and therefore that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$. This completes the proof.

Exercise 3.3.14

Suppose $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.

So we want to prove that $\forall a(a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i\in I} A_i))$

First we assume a is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i\in I} \mathscr{P}(A_i)$	$a \in \mathscr{P}(\bigcup_{i\in I} A_i)$

Assume a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
        [proof of $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$]
    Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our goal we see that $a \in \mathscr{P}(\bigcup_{i\in I} A_i) \to a \subseteq \bigcup_{i\in I} A_i \to \forall z(z \in a \to z \in \bigcup_{i\in I} A_i)$. Therefore we assume z is arbitrary, assume the antecedent, and make the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i\in I} \mathscr{P}(A_i)$	$z \in \bigcup_{i\in I} A_i$
$z \in a$	

Assume a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
        Assume z is arbitrary
            Assume $z \in a$
                [proof of $z \in \bigcup_{i\in I} A_i$]
            Therefore $z \in a \to z \in \bigcup_{i\in I} A_i$
        Since z was arbitrary we can conclude $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
    Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our given we see that $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \{a | \exists i \in I(a \in \mathscr{P}(A_i))\}$. Using existential instantiation we will select an i such that $a \in \mathscr{P}(A_i)$ which implies $a \subseteq A_i$. Since $a \subseteq A_i \to \forall m(m \in a \to m \in A_i)$ and using universal instantiation we will plug in z for m and we get $\forall z(z \in a \to z \in A_i)$ and using modus ponens we can conclude that $z \in A_i$, which implies that $z \in \bigcup_{i\in I} A_i$, which was our goal. We can now right our proof.

Theorem. *Suppose $\{A_i | i \in I\}$ is an indexed family of sets, then $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.*

Proof. Suppose that a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$. We choose an $i \in I$ such that $a \in \mathscr{P}(A_i)$, which implies that $a \subseteq A_i$. Suppose z is an arbitrary element of a, then it follows that $z \in A_i$ and therefore $z \in \bigcup_{i\in I} A_i$. Since z was an arbitrary element of a then $a \subseteq \bigcup_{i\in I} A_i$, and it follows that $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$. Thus we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$. This completes the proof.

Exercise 3.3.15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$. Prove that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$

So we want to prove that $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

First we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i\in I} A_i$	$y \in \bigcap_{i\in I} \mathscr{P}(A_i)$

Suppose y is arbitrary element of $\bigcap_{i\in I} A_i$.
    [proof of $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$]
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Our goal $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$ so we make m an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$. So we make z arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i\in I} A_i$	$z \in A_m$
$z \in y$	

Suppose y is arbitrary element of $\bigcap_{i\in I} A_i$.
    Suppose m is an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$.
        Suppose z is an arbitrary element of y
            [proof of $z \in A_m$]
    Therefore $z \in y \to z \in A_m$ and since z was arbitrary $y \subseteq A_m \to y \in \mathscr{P}(A_m)$ and since m was arbitrary $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Now looking at our given $y \in \bigcap_{i\in I} A_i \to \forall i \in I(y \in A_i)$. Using universal instantiation we plug in m for i and therefore $y \in A_m$ and since $z \in y$ we can conclude $z \in A_m$, which was our goal. Now we can write our proof.

Theorem. *Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$, then $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$.*

Proof. Suppose y is an arbitrary element of $\bigcap_{i\in I} A_i$. Suppose m is an arbitrary member of I and therefore $y \subseteq A_m$ which implies $y \subseteq A_m$. Now suppose z is an arbitrary element of y. Since $y \in \bigcap_{i\in I} A_i$ if we choose an i such that $y \in \bigcap_{m\in I} A_m$ then $y \in A_m$ which implies $z \in A_m$. Therefore if $z \in y$ then $z \in A_m$ and since z was arbitrary then $y \subseteq A_m$ or $y \in \mathscr{P}(A_m)$ and since m was arbitrary then $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$. Since y was arbitrary then $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$. This completes the proof.

Exercise 3.3.16

Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup\mathcal{F} \subseteq B$.

We want to prove $\mathcal{F} \subseteq \mathscr{P}(B) \to \cup\mathcal{F} \subseteq B$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathscr{P}(B)$	$\cup\mathcal{F} \subseteq B$

Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$
    [proof of $\cup\mathcal{F} \subseteq B$
Therefore if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup F \subseteq B$.

$\cup F \subseteq B \to \forall x(x \in \cup\mathcal{F} \to x \in B)$. So we assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathscr{P}(B)$	$x \in B$
$x \in \cup\mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$
    Suppose x is arbitrary
        Suppose $x \in \cup\mathcal{F}$
            [proof of $x \in B$]
        Therefore if $x \in \cup\mathcal{F}$ then $x \in B$
    Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq B$.

Therefore if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup F \subseteq B$.

$x \in \cup\mathcal{F} \to \exists M \in \mathcal{F}(x \in M)$. We use existential instantiation and assume there is a set M in $\mathcal{F}$ and x is in that set.

Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$
    Suppose x is arbitrary
        Suppose M is arbitrary set in $\mathcal{F}$
            $x \in M$
                [proof of $x \in B$]
            Since $x \in M$ and M is a set in $\mathcal{F}$ then $x \in \cup\mathcal{F}$
        Therefore if $x \in \cup\mathcal{F}$ then $x \in B$
    Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq B$.
Therefore if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup F \subseteq B$.

Our given $\mathcal{F} \subseteq \mathscr{P}(B)$ means that $\forall N(N \in \mathcal{F} \to \forall z(z \in N \to z \in B)$. We will use universal instantiation and plug in M for N and x for z and we can conclude that $x \in B$, which was our goal to prove.

Theorem. *Suppose B is a set and $\mathcal{F}$ is a family of sets. If $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup\mathcal{F} \subseteq B$.*

Proof. Suppose x is an arbitrary member of $\cup\mathcal{F}$, which means that x is a member of a set that is in $\mathcal{F}$. Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$, which means that any element that is in a set that is a member of $\mathcal{F}$ is also in the set B. It follows that since x is a member of a set in $\mathcal{F}$ then $x \in B$. Therefore, if $x \in \cup\mathcal{F}$ then $x \in B$ and since x was arbitrary we can conclude $\cup\mathcal{F} \subseteq B$. Therefore, if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup\mathcal{F} \subseteq B$. □

Exercise 3.3.17

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Prove that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We want to prove that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We can rewrite this goal as $\forall x(x \in \cup\mathcal{F} \to x \in \cap\mathcal{G})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal.

Givens	Goals
$x \in \cup\mathcal{F}$	$x \in \cap\mathcal{G}$
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \to y \in B)$	

Suppose x is arbitrary.
    Suppose $x \in \cup\mathcal{F}$.

[proof of $x \in \cap\mathcal{G}$]

Therefore if $x \in \cup\mathcal{F}$ then $x \in \cap\mathcal{G}$

Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We can rewrite our goal as $\forall M \in \mathcal{G}(x \in M)$. We assume M is an arbitrary set in $\mathcal{G}$ and then our goal becomes $x \in M$.

Givens	Goals
$x \in \cup\mathcal{F}$	$x \in M$
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \to y \in B)$	

Suppose x is arbitrary.

Suppose $x \in \cup\mathcal{F}$.

Suppose M is an arbitrary set in $\mathcal{G}$

[proof of $x \in M$]

Since M was arbitrary we can conclude that $x \in \cap\mathcal{G}$

Therefore if $x \in \cup\mathcal{F}$ then $x \in \cap\mathcal{G}$

Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We can rewrite our given $x \in \cup\mathcal{F}$ as $\exists N \in \mathcal{F}(x \in N)$. We use existential instantiation and assume there is a set $N \in \mathcal{F}$ and $x \in N$.

Givens	Goals
$N \in \mathcal{F}$	$x \in M$
$x \in N$	
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \to y \in B)$	

Now we can use universal instantiation to plug in N for A and M for B. Then since $x \in N$ we can use modus ponens to conclude that $x \in M$, which was our goal.

Theorem. *If $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$, then $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.*

Proof. Suppose x is an arbitrary member of $\cup\mathcal{F}$, which means there is a set in $\mathcal{F}$ that contains x. Suppose M is an arbitrary set in $\mathcal{G}$. Then since every set in $\mathcal{F}$ is a subset of every set in $\mathcal{G}$ it follows that $x \in M$. Since M was arbitrary we can conclude that $x \in \cap\mathcal{G}$ and therefore if $x \in \cup F$ then $x \in \cap\mathcal{G}$. Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$. $\square$

Exercise 3.3.18

In this problem all variables range over $\mathbb{Z}$, the set of all integers.

A

Prove that if $a|b$ and $a|c$, then $a|(b+c)$.

We want to prove $(a|b) \wedge (a|c) \rightarrow a|(b+c)$

We assume the antecedent and make the consequent our goal.

Givens	Goals
$a\|b$	$a\|(b+c)$
$a\|c$	

Suppose $a|b$ and $a|c$
[proof of $a|(b+c)$
Therefore if $a|b$ and $a|c$ then $a|(b+c)$.

Our goal means that $\exists x \in \mathbb{Z}(ax = (b+c))$. So we need to find an x that makes this statement true. Our goals can be rewritten as $\exists y \in \mathbb{Z}(ay = b)$ and $\exists w \in \mathbb{Z}(aw = c)$. Using existential instantiation we will assume there is a y and and w that makes both of the previous statement true.

Givens	Goals
$ay = b$	$a\|(b+c)$
$aw = c$	

Suppose $ay = b$ and $aw = c$
[proof of $a|(b+c)$
Therefore if $a|b$ and $a|c$ then $a|(b+c)$.

Adding the two inequalities $ay = b$ and $aw = c$ we have $ay + aw = b + c$ or $a(y+w) = b+c$. Since y and w are integers we can conclude that $a|(b+c)$, which was our goal to prove.

Theorem. *If a, b, and c are integers, $a|b$, and $a|c$, then $a|(b+c)$.*

Proof. Suppose a, b, and c are integers, $a|b$, and $a|c$. Since $a|b$ there must be an integer y such that $ay = b$. Also, since $a|c$ there must be an integer w such that $aw = c$. Adding together the previous two equalities we have $ay + aw = b + c$ or $a(y+w) = b+c$. Since y and w are integers we can conclude that $a|(b+c)$. □

B

Prove that if $ac|bc$ and $c \neq 0$, then $a|b$.

We want to prove $(ac|bc) \wedge (c \neq 0) \rightarrow a|b$.

We assume the antecedent and make the consequent our goal.

Givens	Goals
$ac\|bc$	$a\|b$
$c \neq 0$	

Suppose $ac|bc$ and $c \neq 0$
 [proof of $a|b$]
Therefore $ac|bc$ and $c \neq 0$, then $a|b$.

Our goal means that $\exists x(ax = b)$ and we want to find an x that makes this statement true. Looking at our goals we can rewrite $ac|bc$ as $\exists y(acy = bc)$. Using existential instantiation we will assume there is a y that makes $acy = bc$ true and we can add $acy = bc$ to our givens. Since $c \neq 0$ we can divide both sides of $acy = bc$ by c and we have $ay = b$. Since y is an integer we can conclude that $a|b$, which was our goal to prove.

Theorem. *If a, b, and c are integers, $ac|bc$, and $c \neq 0$, then $a|b$.*

Proof. Suppose a, b, and c are integers, $ac|bc$, and $c \neq 0$. Since $ac|bc$ there must be an integer x such that $acx = bc$. Since $c \neq 0$ we can simplify the previous equation by dividing both sides by c so that $ax = b$. Since x is an integer we can conclude that $a|b$. □

Exercise 3.3.19

A

Prove that for all real numbers x and y there is a real number z such that $x + z = y - z$.

We want to prove that $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R}(x + z = y - z)$.

We let x and y stand for arbitrary real numbers and make $\exists z \in \mathbb{R}(x+z = y-z)$ our goal to prove.

Givens	Goals
x arbitrary	$\exists z \in \mathbb{R}(x + z = y - z)$
y arbitrary	

Suppose x and y are arbitrary real numbers
 [proof of $\exists z \in \mathbb{R}(x + z = y - z)$]
Since x and y are arbitrary we can conclude $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R}(x+z = y-z)$

We want to find a z such that $x + z = y - z$, which suggests we try solving this equation for z

$$\begin{aligned} x + z &= y - z \\ x + 2z &= y \\ z &= \frac{y - x}{2}. \end{aligned}$$

Now we are ready to complete our proof.

Theorem. *For all real numbers x and y there is a real number z such that $x + z = y - z$.*

Proof. Suppose x and y are arbitrary real numbers and $z = \frac{y-x}{2}$. Then

$$\begin{aligned} x + \frac{y - x}{2} &= y - \frac{y - x}{2} \\ \frac{2x + y - x}{2} &= \frac{2y - (y - x}{2} \\ 2x + y - x &= 2y - y + x \\ x(2 - 1) + y &= y(2 - 1) + x \\ x + y &= x + y \end{aligned}$$

□

B

Would the statement in part (A) be correct if "real number" were changed to "integer"? Justify your answer.

No, because there are instances where $z = \frac{x-y}{2}$ would not result in an integer. For example, if $x = 5$ and $y = 2$ then $z = \frac{3}{2}$, which is not an integer. Therefore the statement in part (A) would not be correct.

Exercise 3.3.20

Consider the following theorem:

Theorem. *For every real number x, $x^2 \geq 0$.*

What's wrong with the following proof?

Proof. Suppose not. Then for every real number x, $x^2 < 0$. In particular, plugging in $x = 3$ we would get $9 < 0$, which is clearly false. This contradiction shows that for every number $x, x^2 \geq 0$. □

The sentence "Then for every real number x, $x^2 < 0$" is not correct because if we let $x = 0$ then $0 < 0$ is not true.

3.3.21

Consider the following incorrect theorem:

Incorrect Theorem. *If* $\forall x \in A(x \neq 0)$ *and* $A \subseteq B$ *then* $\forall x \in B(x \neq 0)$.

A

What's wrong with the following proof?

Proof. Let x be an arbitrary element of A. Since $\forall x \in A(x \neq 0)$, we can conclude that $x \neq 0$. Also, since $A \subseteq B$, $x \in B$. Since $x \in B$, $x \neq 0$, and x was arbitrary, we can conclude that $\forall x \in B(x \neq 0)$. $\square$

The last sentence is not correct. $A \subseteq B$ means that all elements in A are in B and since $x \neq 0$ then $0 \notin A$, but this doesn't mean that $0 \notin B$, because there can be elements in B that are not in A.

B

Find a counterexample to the theorem. In other words, find an example of sets A and B for which the hypotheses of the theorem are true but the conclusion is false.

Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$. Then the hypotheses of the theorem are true, specifically $\forall x \in A(x \neq 0)$ and $A \subseteq B$, but the conclusion $\forall x \in B(x \neq 0)$ is false.

3.3.22

Consider the following incorrect theorem:

Incorrect Theorem. $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy^2 = y - x)$.

What's wrong with the following proof of the theorem?

Proof. Let $x = \frac{y}{y^2+1}$. Then

$$y - x = y - \frac{y}{y^2} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1} \cdot y^2 = xy^2.$$

$\square$

In the proof, x is defined in terms of y but y has not been introduced into the proof yet. The theorem should start with "Let $x = ...$ and let y be an arbitrary real number...".

3.3.23

Consider the following incorrect theorem:

Incorrect Theorem *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\cup\mathcal{F}$ and $\cup G$ are disjoint, then so are $\mathcal{F}$ and $\mathcal{G}$.*

A

What's wrong with the following proof of the theorem?

Proof. Suppose $\cup\mathcal{F}$ and $\cup G$ are disjoint. Suppose $\mathcal{F}$ and $\mathcal{G}$ are not disjoint. Then we can choose some set A such that $A \in \mathcal{F}$ and $a \in \mathcal{G}$. Since $A \in \mathcal{F}$, by exercise 8, $A \subseteq \cup\mathcal{F}$, so every element of A is in $\cup\mathcal{F}$. Similarly, since $A \in \mathcal{G}$, every element of A is in $\cup\mathcal{G}$. But then every element of A is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$, and this is impossible since $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint. Thus, we have reached a contradiction, so $\mathcal{F}$ and $\mathcal{G}$ must be disjoint. $\square$

The statement "But then every element of A is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$, and this is impossible since $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint." is not correct. If $A = \{\varnothing\}$ then every element of A, which is $\varnothing$, is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$ and by definition $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, or $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \varnothing$. So there is no contradiction in this case.

B

Find a counterexample to the theorem.

Let $\mathcal{F} = \{\{\varnothing\}, \{1\}\}$ and $G = \{\{\varnothing\}, \{2\}\}$. Then $\cup\mathcal{F} = \{1, \varnothing\}$ and $\cup\mathcal{G} = \{2, \varnothing\}$ and therefore $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, or $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \varnothing$. However, $\mathcal{F}$ and $\mathcal{G}$ are not disjoint because $\mathcal{F} \cap \mathcal{G} = \{\varnothing\}$. (Remember, the empty set $\varnothing$ is the set that has no elements, but the set $\{\varnothing\}$ is a set that contains one element, $\varnothing$.)

3.3.24

Consider the following putative theorem:

Theorem? *For all real numbers x and y, $x^2 + xy - 2y^2 = 0$.*

A

What's wrong with the following proof of the theorem?

Proof. Let x and y be equal to some arbitrary real number r. Then

$$x^2 + xy - 2y^2 = r^2 + r \cdot r - 2r^2 = 0.$$

Since x and y were both arbitrary, this shows that for all real numbers x and y, $x^2 + xy - 2y^2 = 0$.

□

The first sentence of the proof assigns x and y to be the same arbitrary real number, however, it should start with "Let x be an arbitrary real number and let y be an arbitrary real number" or "let x and y be arbitrary real numbers".

B

No, the theorem is not correct. We will provide a counterexample. Let $x = 2$ and let $y = 3$, then

$$2^2 + 2 \cdot 3 - 2 \cdot 3^2 = 4 + 6 - 2 \cdot 9 = 10 - 18 = -8$$

and $-8 \neq 0$ so the theorem is not correct.

3.3.25

Prove that for every real number x there is a real number y such that for every real number z, $yz = (x+z)^2 - (x^2 + z^2)$.

We want to prove $\forall x \exists y \forall z (yz = (x+z)^2 - (x^2 + z^2))$.

We let x be arbitrary and make our goal $\exists y \forall z (yz = (x+z)^2 - (x^2 + z^2))$.

Givens	Goals
x is arbitrary	$\exists y \forall z (yz = (x+z)^2 - (x^2 + z^2))$

Let x be an arbitrary real number
  [proof of $\exists y \forall z (yz = (x+z)^2 - (x^2 + z^2))$]
Therefore $\forall x \exists y \forall z (yz = (x+z)^2 - (x^2 + z^2))$

Now we need to find a y that makes the statement $\forall z (yz = (x+z)^2 - (x^2 + z^2))$ true. This suggests we solve the equation $yz = (x+z)^2 - (x^2 + z^2)$ for y.

$$\begin{aligned}
yz &= (x+z)^2 - (x^2 + z^2) \\
yz &= x^2 + 2xz + z^2 - x^2 - z^2 \\
yz &= 2xz \\
y &= 2x
\end{aligned}$$

The last line above $y = 2x$ works even if $z = 0$ because in that case $yz = 0$ and $2xz = 0$ and there is no need to divide by z because we have $0 = 0$.

Let x be an arbitrary real number
  Let $y = 2x$
    Let z be an arbitrary real number
      [proof of $yz = (x+z)^2 - (x^2+z^2)$]
    Therefore $\forall z(yz = (x+z)^2 - (x^2+z^2))$
  Therefore $\exists y \forall z(yz = (x+z)^2 - (x^2+z^2))$
Therefore $\forall x \exists y \forall z(yz = (x+z)^2 - (x^2+z^2))$

When writing the proof we have to make sure the order we introduce the variables is the same as above (i.e., we introduce x, y, and then z). When we state $y = 2x$ in the proof, we only have defined x up to that point, so both values we choose for x and y must then work for every value of z, or every real number. (See https://github.com/kstratto/How-to-Prove-It/blob/master/How%20to%20Prove%20It%20-%20Chapter%203.pdf.)

Theorem. *For every real number x there is a real number y such that for every real number z, $yz = (x+z)^2 - (x^2+z^2)$.*

Proof. Let x be an arbitrary real number. Let $y = 2x$. Let z be an arbitrary real number. Then

$$2xz = (x+z)^2 - (x^2+z^2) = x^2 + 2xz + z^2 - x^2 - z^2 = 2xz$$

□

Exercise 3.3.26

A

Comparing the various rules for dealing with quantifiers in proofs, you should see a similarity between the rules for goals of the form $\forall x P(x)$ and givens of the form $\exists x P(x)$. What is this similarity? What about the rules for goals of the form $\exists x P(x)$ and givens of the from $\forall x P(x)$?

Rules for goals of the form $\forall P(x)$ and givens of the form $\exists x P(x)$ are similar because the strategy for both of these involve introducing a new variable into the proof. In the case of a goal of the form $\forall P(x)$, say $\forall x \in AP(x)$, a new variable y can be introduced that stands for an arbitrary element of the set A and this new variable can be used like any other given. In the case of a given of the form $\exists x P(x)$, a new variable is also introduced, say x_0, that we assume makes the statement $P(x)$ of true. This new variable x_0 can also now be used as a given. With both of the new variables it is important not to make any other assumptions about them.

Rules for goals of the form $\exists x P(x)$ and givens of the form $\forall x P(x)$ are similar because both of these strategies involve introducing a specific value for x that makes $P(x)$ true instead of just introducing a variable that is assumed to make $P(x)$ true.

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Can you think of a reason why these similarities might be expected?
(Hint: Think about how proof by contradiction works when the goal starts with a quantifier.)

When proving a goal of the form $\forall x P(x)$ by contradiction, we assume $\exists x \neg P(x)$ as a goal. When proving a goal of the form $\exists x P(x)$ by contradiction, we assume $\forall x \neg P(x)$ as a goal.

Maybe the similarities are to be expected because the strategies for each set of givens and goals (e.g., goals of the form $\forall x P(x)$ and givens of the form $\exists x P(x)$) are like inverses. For example, for the statement $\forall x P(x)$ to not be true then the statement $\exists x \neg P(x)$ must be true.