Exercise 3.1.5

Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.

So we want to prove that $(a < b < 0) \to (a^2 > b^2)$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a < b < 0$	$a^2 > b^2$

Suppose $a < b < 0$
  [proof of $a^2 > b^2$]
So if $a < b < 0$ then $a^2 > b^2$

If we multiply the inequality $a < b$ on both sides by the negative number b we have $ab > b^2$ and multiplying $a < b$ on both sides by the negative number a we have $a^2 > ab$. Therefore $a^2 > ab > b^2$ and we have proven our goal and now we can write our proof.

Theorem. *Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.*

Proof. Suppose $a < b < 0$. Multiplying the inequality $a < b$ by the negative number a we can conclude $a^2 > ab$, and, similarly, multiplying $a < b$ by the negative number b we get $ab > b^2$. Therefore, $a^2 > ab > b^2$ and $a^2 > b^2$. Thus, if $a < b < 0$ then $a^2 > b^2$. $\square$

Exercise 3.1.6

Suppose a and b are real numbers. Prove that if $0 < a < b$ then $1/b < 1/a$.

So we want to prove that $(0 < a < b) \to (1/b < 1/a)$.

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$0 < a < b$	$1/b < 1/a$

Suppose $0 < a < b$
  [proof of $1/b < 1/a$]
So if $0 < a < b$ then $1/b < 1/a$.

If we multiply both sides of the inequality $a < b$ by $1/ab$ we see that $1/a < 1/b$, which is our goal.

Theorem. *Suppose a and b are real numbers. If $0 < a < b$ then $1/b < 1/a$.*

Proof. Suppose $0 < a < b$. Multiplying both sides of the inequality $a < b$ by $1/ab$ we can conclude that $1/b < 1/a$. Therefore, if $0 < a < b$ then $1/b < 1/a$. $\square$

Exercise 3.1.7

Suppose that a is a real number. Prove that if $a^3 > a$ then $a^5 > a$. (Hint: One approach is to start by completing the following equation: $a^5 - a = (a^3 - 1) \cdot \underline{?}$.)
So we want to prove that $(a^3) \to (a^5)$.

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a^3 > a$	a^5

Suppose $a^3 > a$
    [proof of $a^5 > a$]
So if $a^3 > a$ then $a^5 > a$.

If we multiply both side of $a^3 - a > 0$ by $a^2 + 1$ we can conclude that $a^5 - a > 0$ or $a^5 > a$, which was our goal.

Theorem. *Suppose a is a real number. If $a^3 > a$ then $a^5 > a$.*

Proof. Suppose $a^3 > a$, then $a^3 - a > 0$. Multiplying both sides of the inequality $a^3 - a > 0$ by $a^2 + 1$ we can conclude $a^5 - a > 0$. Therefore, if $a^3 > a$ then $a^5 > a$. $\square$

Exercise 3.1.8

Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.
So we want to prove that $(x \notin D) \to (x \in B)$.

The contrapositive of the goal is $\neg(x \in B) \to \neg(x \notin D)$, or in other words $x \notin B \to x \in D$. First we assume the antecedent and make the consequent our goal.

Givens	Goals
$A \setminus B \subseteq C \cap D$	$x \in D$
$x \in A$	
$x \notin B$	

Looking at our givens we can rewrite $A \setminus B \subseteq C \cap D$ as $(x \in A \wedge x \notin B) \to (x \in A \wedge x \in D)$. Looking at our other givens $x \in A$ and $x \notin B$ we can conclude that $x \in C \cap D$ and therefore $x \in D$, which was our goal to prove.

Theorem. *Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. If $x \notin D$ then $x \in B$.*

Proof. We will prove the contrapositive. Suppose $x \notin B$. Since $x \in A$ and $x \notin B$ we can conclude that $x \in C \cap D$ and it follows that $x \in D$. Therefore, if $x \notin D$ then $x \in B$. $\square$

Exercise 3.1.9

Suppose a and b are real numbers. Prove that if $a < b$ then $\frac{a+b}{2} < b$.

So we want to prove that $(a < b) \to (\frac{a+b}{2} < b)$.

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a < b$	$\frac{a+b}{2} < b$

If we add b to both sides of $a < b$ we see that $a + b < b + b$ or $a + b < 2b$. Then if we divide both sides of $a + b < 2b$ by 2 we can conclude that $\frac{a+b}{2} < b$, which was our goal to prove.

Theorem. *Suppose a and b are real numbers. If $a < b$ then $\frac{a+b}{2}$.*

Proof. Suppose $a < b$. Adding b to both sides of the inequality $a < b$ and then dividing both sides by 2, we can conclude that $\frac{a+b}{2} < b$. Therefore, if $a < b$ then $\frac{a+b}{2} < b$. $\square$

Exercise 3.1.10

Suppose x is a real number and $x \neq 0$. Prove that if $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$.

So we want to prove that $\left(\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}\right) \to (x \neq 8)$.

The contrapositive of the goal is $\neg(x \neq 8) \to \neg(\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x})$ or in other words $(x = 8) \to (\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x})$. First we assume the antecedent and make the consequent our goal.

Givens	Goals
$x \neq 0$	$\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$
$x = 8$	

If we evaluate the expression $\frac{\sqrt[3]{x}+5}{x^2+6}$ for $x = 8$ we see that $\frac{\sqrt[3]{8}+5}{8^2+6} = \frac{1}{7}$ and $\frac{1}{7} \neq \frac{1}{8}$, therefore $\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$, which was our goal to prove.

Theorem. *Suppose x is a real number and $x \neq 0$. If $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$*

Proof. We will prove the contrapositive. Suppose $x = 8$. Then evaluating the equation $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ for $x = 8$ we see that $\frac{1}{7} \neq \frac{1}{8}$ and therefore if $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$. □

Exercise 3.1.11

Suppose a, b, c, and d are real numbers, $0 < a < b$, and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

So we want to prove that $(ac \geq bd) \rightarrow (c > d)$

The contrapositive of the goal is $\neg(c > d) \rightarrow \neg(ac \geq bd)$ or in other words $(c \leq d) \rightarrow (ac < bd)$. First we assume the antecedent and make the consequent our goal.

Givens	Goals
$c \leq d$	$ac < bd$

If we multiply both side of the inequality $c \leq d$ by a we see that $ac \leq ad$ and multiplying both sides of the inequality $a < b$ by d we see that $ad < bd$. Therefore, $ac \leq ad < bd$ and $ac < ad$, which was our goal to prove.

Theorem. *Suppose a, b, c, and d are real numbers, $0 < a < b$ and $d > 0$. If $ac \geq bd$ then $c > d$.*

Proof. We will prove the contrapositive. Suppose $c \leq d$. Multiplying the inequality $c \leq d$ on both sides by a we have $ac \leq ad$ and multiplying the inequality $a < b$ on both sides by d we have $ad < bd$. It follows that $ac \leq ad < bd$ and $ac < bd$. Therefore, if $ac \geq bd$ then $c > d$. □

Exercise 3.1.12

Suppose x and y are real numbers, and $3x + 2y \leq 5$. Prove that if $x > 1$ then $y < 1$.
So we want to prove that $(x > 1) \rightarrow (y < 1)$

First we assume the antecedent and make the consequent our goal.

Givens	Goals
$x > 1$	$y < 1$

Rearranging the inequality $3x + 2y \leq 5$ we see that $\frac{5-2y}{3} > x$. Our given is $x > 1$ and so we can conclude that $\frac{5-2y}{3} > x > 1$ and $\frac{5-2y}{3} > 1$. Solving the latter inequality for y we have $y < 1$, which was our goal to prove.

Theorem. *Suppose x and y are real numbers and $3x + 2y \leq 5$. If $x > 1$ then $y < 1$.*

Proof. Suppose $3x+2y \leq 5$, then it follows that $\frac{5-2y}{3} > x$. Suppose $x > 1$, then $\frac{5-2y}{3} > x > 1$ and $\frac{5-2y}{3} > 1$. Then it follows that $y < 1$. Therefore, if $x > 1$ then $y < 1$. $\square$

Exercise 3.1.13

Suppose that x and y are real numbers. Prove that if $x^2+y = -3$ and $2x-y = 2$ then $x = -1$.
So we want to prove that $(x^2 + y = -3 \wedge 2x - y = 2) \rightarrow (x = -1)$

First we assume the antecedent and make the consequent our goal.

Givens	Goals
$x^2 + y = -3$	$x = -1$
$2x - y = 2$	

Solving $x^2 + y = -3$ for y we have $y = -3 - x^2$. Substituting $y = -3 - x^2$ into $x^2 + y = -3$ and solving for x we can conclude that $x = -1$, which was our goal prove.

Theorem. *Suppose x and y are real numbers. If $x^2 + y = -3$, and $2x - y = 2$ then $x = -1$.*

Proof. Suppose $x^2 + y = -3$ and $2x - y = 2$. If $x^2 + y = -3$ then it follows that $y = -3 - x^2$. Substituting $y = -3 - x^2$ into the equation $x^2 + y = -3$ we can conclude that $x = -1$. Therefore, if $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$. $\square$

Exercise 3.1.14

Prove the first theorem in Example 3.1.1. (Hint: You might find it useful to apply the theorem from Example 3.1.2.)

The first theorem in Example 3.1.1. is: If $x > 3$ and $y < 2$, then $x^2 - 2y < 5$. The theorem from Example 3.1.2 states: Suppose a and b are real numbers. If $0 < a < b$ then $a^2 < b^2$.

So we want to prove that $(x > 3 \wedge y < 2) \rightarrow (x^2 - 2y < 5)$.

First we assume the antecedent and make the consequent our goal.

Since $0 < 3 < x$ we can apply theorem 3.1.1 and conclude that $x^2 > 9$. Multiplying the inequality $y < 2$ by 2 on both sides we have $2y < 4$. Then adding

Givens	Goals
$x > 3$	$x^2 - 2y > 5$
$y < 2$	

the two inequalities $x^2 > 9$ and $4 > 2y$ we can conclude that $4 + x^2 > 9 + 2y$ and if follows that $x^2 - 2y > 5$, which was our goal to prove.

Theorem. *Suppose $x > 3$ and $y < 2$, then $x^2 - 2y < 5$.*

Proof. Suppose $x > 3$ and $y < 2$. Since $0 < 3 < x$ we can apply theorem 3.1.1 and conclude that $x^2 > 9$. Multiplying the inequality $y < 2$ by 2 on both sides we have $2y < 4$. Then adding the two inequalities $x^2 > 9$ and $4 > 2y$ we can conclude that $4 + x^2 > 9 + 2y$. Therefore, if $x > 3$ and $y < 2$, then $x^2 - 2y < 5$. □

Exercise 3.1.15

a

The theorem has a goal of the form $a \to b$ where a is $\frac{2x-5}{x-4}$ and b is $x = 7$. To prove the theorem we could assume a and prove b is true or prove the contrapositive $\neg b \to \neg a$ and assume $\neg a$ and prove $\neg b$. However the proof given here shows that $b \to a$, which does not suffice to prove the theorem.

b

Theorem. *Suppose x is a real number and $x \neq 4$. If $\frac{2x-5}{x-4} = 3$, then $x = 7$.*

Proof. Suppose $\frac{2x-5}{x-4} = 3$, then if follows that $x = 7$. Therefore if $\frac{2x-5}{x-4} = 3$, then $x = 7$. □

3.1.16

a

The mistake is assuming that since $x \neq 3$ then $x^2 \neq 9$, which is not true because if $x = -3$ then $x^2 = 9$.

b

If $x = -3$ then $-3^2y = 9y$ then $9y = 9y$ and $y = 1$.