## Exercise 3.3.4

Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

So we want to prove that $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$.

First we assume $x$ is arbitrary and make the antecedent a given and the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $A \subseteq \mathscr{P}(A)$ | $x \in \mathscr{P}(\mathscr{P}(A))$ |
| $x \in \mathscr{P}(A)$ | |

Assume $x$ is an arbitrary element of $\mathscr{P}(A)$
  Suppose $x \in \mathscr{P}(A)$
    [ proof of $x \in \mathscr{P}(\mathscr{P}(A))$ ]
  Therefore if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$
Since $x$ was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$

We can rewrite our goal as $x \subseteq \mathscr{P}(A)$ or $\forall y(y \in x \to y \in \mathscr{P}(A))$. So we assume $y$ is arbitrary and make the antecedent a given and the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $A \subseteq \mathscr{P}(A)$ | $y \in \mathscr{P}(A)$ |
| $x \in \mathscr{P}(A)$ | |
| $y \in x$ | |

Assume $x$ is an arbitrary element of $\mathscr{P}(A)$
  Suppose $x \in \mathscr{P}(A)$
    Suppose $y$ is an arbitrary element of $x$.
      Suppose $y \in x$.
        [ proof of $y \in \mathscr{P}(A)$ ]
      Therefore if $y \in x \to y \in \mathscr{P}(A)$.
    Since $y$ was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$.
  Therefore if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$
Since $x$ was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$

Now looking at our givens $x \in \mathscr{P}(A)$ means that $x \subseteq A$ or $\forall z(z \in x \to z \in A)$. Using universal instantiation we will plug in $y$ for $z$ and using modus ponens we can conclude that $y \in A$.

Now looking at our other given $A \subseteq \mathscr{P}(A) \to \forall m(m \in A \to m \in \mathscr{P}(A))$. Using universal instantiation we will plug in $y$ for $m$ and using modus ponens we can conclude that $y \in \mathscr{P}(A)$, which was our goal to prove.

**Theorem.** *Suppose* $A \subseteq \mathscr{P}(A)$. *Then* $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

*Proof.* Suppose $x$ is an arbitrary element of $\mathscr{P}(A)$ and $y$ is an arbitrary element of $x$. It follows that $y \in A$. But since $A \subseteq \mathscr{P}(A)$ then it also follows that $y \in \mathscr{P}(A)$. So $y \in x \to y \in \mathscr{P}(A)$ and since $y$ was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$. Therefore, if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$. Since $x$ was arbitrary we can also conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. $\square$

Alternate proof (not sure if this is correct)

*Proof.* Suppose $x$ is an arbitrary element of $\mathscr{P}(A)$. Then $x \in A$. Since $A \subseteq \mathscr{P}(A)$ and $x \in A$ then $x \subseteq \mathscr{P}(A)$. Therefore, $x \in \mathscr{P}(\mathscr{P}(A))$. $\square$

## Exercise 3.3.5

The hypothesis of the theorem proven in exercise 3.3.4 is $A \subseteq \mathscr{P}(A)$.

### A

Can you think of a set $A$ for which this hypothesis is true?

The empty set $\varnothing$ is a set for which the hypothesis is true.

$A \subseteq \mathscr{P}(A)$ means $x \in A \to x \in \mathscr{P}(A)$. For $\varnothing$ this would mean that $x \in \varnothing \to x \in \mathscr{P}(\varnothing)$, but by definition there are no elements in $\varnothing$. Therefore $x \in \varnothing$ will always be false and the conditional statement $x \in \varnothing \to x \in \mathscr{P}(\varnothing)$ is always true. Therefore if $\varnothing = A$ then $A \subseteq \mathscr{P}(A)$.

### B

Can you think of another?

In exercise 3.3.4 we proved that if $A \subseteq \mathscr{P}(A)$ then $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. Therefore, the set $\{\varnothing, \{\varnothing\}\}$, which is the $\mathscr{P}(A)$ if $A = \varnothing$, is another set for which the hypothesis is true. If we let $B = \mathscr{P}(A) = \{\varnothing, \{\varnothing\}\}$ and replace $A$ in the hypothesis $A \subseteq \mathscr{P}(A)$ with $B$, then we can conclude that $B \subseteq \mathscr{P}(B)$.

## 3.3.6

Suppose $x$ is a real number.

## A

Prove that if $x \neq 1$ then there is a real number $y$ such that $\frac{y+1}{y-2} = x$.

So we want to prove that $(x \neq 1) \to \exists y \left( \frac{y+1}{y+2} = x \right)$

We assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $x \neq 1$ | $\exists y \left( \frac{y+1}{y+2} = x \right)$ |

To prove our goal we need to find a $y$ that makes the equation $\frac{y+1}{y+2} = x$ true. So let's try solving the equation for $y$.

$$\begin{aligned} \frac{y+1}{y+2} &= x \\ y+1 &= x(y-2) \\ y+1 &= xy-2x \\ 2x+1 &= xy-y \\ 2x+1 &= y(x-1) \\ y &= \frac{2x+1}{x-1} \end{aligned}$$

We see that this $y$ works because we have $x \neq 1$ as a given.

**Theorem.** *Suppose $x \neq 1$. Then there is a real number $y$ such that $\frac{y+1}{y-2} = x$.*

*Proof.* Suppose $x \neq 1$ and $y = \frac{2x+1}{x-1}$. Then

$$\frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{x-1} \cdot \frac{x-1}{3} = x$$

□

## B

Prove that if there is a real number $y$ such that $\frac{y+1}{y-2} = x$ then $x \neq 1$.

So we want to prove that $\exists y \left( \frac{y+1}{y-1} = x \right) \to (x \neq 1)$

We assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $\exists y \left( \frac{y+1}{y-1} = x \right)$ | $x \neq 1$ |

Using existential instantiation we assume there is a value $y_0$ such that $\frac{y+1}{y-1} = x$ is true. From part A above, we know that $\left(\frac{y+1}{y-1} = x\right) \rightarrow \left(y = \frac{2x+1}{x-1}\right)$ and so $y_0 = \frac{2x+1}{x-1}$. Since $y$ is a real number, then clearly $x \neq 1$.

**Theorem.** *If $y$ is a real number and $\frac{y+1}{y-2} = x$ then $x \neq 1$.*

*Proof.* Suppose $y$ is a real number and $\frac{y+1}{y-2} = x$. It follows that $y = \frac{2x+1}{x-1}$ and since $y$ is real number then $x \neq 1$. □

## Exercise 3.3.7

Prove for every real number $x$, if $x > 2$ then there is a real number $y$ such that $y + \frac{1}{y} = x$.

## Exercise 3.3.12

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $\mathcal{F} \subseteq \mathcal{G}$ | $\cup\mathcal{F} \subseteq \cup\mathcal{G}$ |

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  [proof of $\cup\mathcal{F} \subseteq \cup\mathcal{G}$ ]
So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\cup\mathcal{F} \subseteq \cup\mathcal{G} \rightarrow \forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$ so we assume $b$ is an arbitrary element of $\cup\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $\mathcal{F} \subseteq \mathcal{G}$ | $b \in \cup\mathcal{G}$ |
| $b \in \cup\mathcal{F}$ | |

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  Let $b$ be an arbitrary element of $\cup\mathcal{F}$
    [proof of $b \in \cup\mathcal{G}$ ]
  Therefore if $b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G}$
Since $b$ was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

| Givens | Goals |
|---|---|
| $\mathcal{F} \subseteq \mathcal{G}$ | $b \in \cup\mathcal{G}$ |
| $A_0 \in \mathcal{F} \wedge b \in A_0$ | |

$b \in \cup\mathcal{F} \to \exists M(M \in \mathcal{F} \wedge b \in M)$, so let $M = A_0$ (Existential Instantiation)
Suppose $\mathcal{F} \subseteq \mathcal{G}$
  Let $b$ be an arbitrary element and suppose $b \in \cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ and $b$ is in that set. Let that set $= A_0$
    [proof of $b \in \cup\mathcal{G}$ ]
  Therefore if $b \in \cup\mathcal{F} \to b \in \cup\mathcal{G}$
Since $b$ was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \to \forall A(A \in \mathcal{F} \to A \in \mathcal{G})$. Using universal instantiation we will plug in $A_0$ for $A$ since then we can use modens ponens to conclude that $A_0 \in \mathcal{G}$.

| Givens | Goals |
|---|---|
| $A_0 \in \mathcal{F} \to A_0 \in \mathcal{G}$ | $b \in \cup\mathcal{G}$ |
| $A_0 \in \mathcal{F} \wedge b \in A_0$ | |

Our goal $b \in \cup\mathcal{G} \to \exists N(N \in \mathcal{G} \wedge b \in N)$, which we can now prove. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$, it follows that $A_0 \in G$. By the definition of $\cup\mathcal{G}$ it follows that $b \in \cup\mathcal{G}$ because $A_0 \in G \wedge b \in A_0$, the latter statement being one of our givens.

**Theorem.** *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.*
*Proof.* Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let $b$ be an arbitrary element of $\cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ that contains $b$. Call this set $A_0$. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$ it follows that $A_0 \in G$, which implies that $b \in \cup\mathcal{G}$. Therefore if $b \in \cup\mathcal{F}$ then $b \in \cup\mathcal{G}$. Since $b$ was arbitrary we can conclude that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$. This completes the proof.

## Exercise 3.3.13

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $\mathcal{F} \subseteq \mathcal{G}$ | $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ |

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  [proof of $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ ]

So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$\cap\mathcal{G} \subseteq \cap\mathcal{F} \rightarrow \forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$, so we assume $b$ is an arbitrary element of $\cap\mathcal{G}$ and assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $\mathcal{F} \subseteq \mathcal{G}$ | $b \in \cap\mathcal{F}$ |
| $b \in \cap\mathcal{G}$ | |

Suppose $\mathcal{F} \subseteq \mathcal{G}$
Let $b$ be an arbitrary element of $\cap\mathcal{G}$
[proof of $b \in \cap\mathcal{F}$ ]
Therefore if $b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F}$
Since $b$ was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$b \in \cap\mathcal{F} \rightarrow \forall A(A \in \mathcal{F} \rightarrow b \in A)$, so we assume $A$ is an arbitrary element of $\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $\mathcal{F} \subseteq \mathcal{G}$ | $b \in A$ |
| $b \in \cap\mathcal{G}$ | |
| $A \in \mathcal{F}$ | |

Suppose $\mathcal{F} \subseteq \mathcal{G}$
Let $b$ be an arbitrary element of $\cap\mathcal{G}$
Suppose $A$ is an arbitrary set in $\mathcal{F}$
[proof of $b \in A$ ]
Therefore if $A \in \mathcal{F} \rightarrow b \in A$
Since A was arbitrary we can conclude $b \in \cap\mathcal{F}$
Therefore if $b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F}$
Since $b$ was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

Now looking at our givens, $\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall Z(Z \in \mathcal{F} \rightarrow Z \in \mathcal{G})$. Using universal instantiation we will plug in $A$ for $Z$ and using modus ponens we can conclude that $A \in \mathcal{G}$.

Our other given, $b \in \cap\mathcal{G} \rightarrow \forall Y(Y \in \mathcal{G} \rightarrow b \in Y)$. Using universal instantiation we will plug in $A$ for $Y$ and using modus ponens we can conclude that $b \in A$, which was our goal, and we can now write our proof.

**Theorem.** *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.*
*Proof.* Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let $b$ be an arbitrary element of $\cap\mathcal{G}$. Suppose $A$ is an

arbitrary element of $\mathcal{F}$, then because $\mathcal{F} \subseteq \mathcal{G}$ then it follows that $A \in \mathcal{G}$. By the definition of $\cap\mathcal{G}$ it follows that $b \in A$ and since $A$ was arbitrary then $b \in \cap\mathcal{F}$. Since $b$ was arbitrary we can conclude $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ and therefore that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$. This completes the proof.

## Exercise 3.3.14

Suppose $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.

So we want to prove that $\forall a(a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i\in I} A_i))$

First we assume $a$ is arbitrary and make the antecedent a given and the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$ | $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$ |

Assume $a$ is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
        [ proof of $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$ ]
    Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since $a$ was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our goal we see that $a \in \mathscr{P}(\bigcup_{i\in I} A_i) \rightarrow a \subseteq \bigcup_{i\in I} A_i \rightarrow \forall z(z \in a \rightarrow z \in \bigcup_{i\in I} A_i)$. Therefore we assume $z$ is arbitrary, assume the antecedent, and make the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$ | $z \in \bigcup_{i\in I} A_i$ |
| $z \in a$ | |

Assume $a$ is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
        Assume z is arbitrary
            Assume $z \in a$
                [ proof of $z \in \bigcup_{i\in I} A_i$ ]
            Therefore $z \in a \rightarrow z \in \bigcup_{i\in I} A_i$
        Since z was arbitrary we can conclude $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
    Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since $a$ was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our given we see that $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \{a | \exists i \in I(a \in \mathscr{P}(A_i))\}$. Using existential instantiation we will select an $i$ such that $a \in \mathscr{P}(A_i)$ which implies $a \subseteq A_i$. Since $a \subseteq A_i \to \forall m(m \in a \to m \in A_i)$ and using universal instantiation we will plug in $z$ for $m$ and we get $\forall z(z \in a \to z \in A_i)$ and using modus ponens we can conclude that $z \in A_i$, which implies that $z \in \bigcup_{i\in I} A_i$, which was our goal. We can now right our proof.

**Theorem.** *Suppose $\{A_i | i \in I\}$ is an indexed family of sets, then $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.*
*Proof.* Suppose that $a$ is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$. We choose an $i \in I$ such that $a \in \mathscr{P}(A_i)$, which implies that $a \subseteq A_i$. Suppose $z$ is an arbitrary element of $a$, then it follows that $z \in A_i$ and therefore $z \in \bigcup_{i\in I} A_i$. Since $z$ was an arbitrary element of $a$ then $a \subseteq \bigcup_{i\in I} A_i$, and it follows that $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$. Thus we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$. This completes the proof.

## 3.3.15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$. Prove that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$

So we want to prove that $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

First we assume $y$ is arbitrary and make the antecedent a given and the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $y \in \bigcap_{i\in I} A_i$ | $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$ |

Suppose $y$ is arbitrary element of $\bigcap_{i\in I} A_i$.
[proof of $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$]
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Our goal $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$ so we make $m$ an arbitrary element of $I$ and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$. So we make $z$ arbitrary and make the antecedent a given and the consequent our goal to prove.

| Givens | Goals |
|---|---|
| $y \in \bigcap_{i\in I} A_i$ | $z \in A_m$ |
| $z \in y$ | |

Suppose $y$ is arbitrary element of $\bigcap_{i\in I} A_i$.
Suppose $m$ is an arbitrary element of $I$ and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$.

Suppose $z$ is an arbitrary element of $y$

[proof of $z \in A_m$]

Therefore $z \in y \to z \in A_m$ and since $z$ was arbitrary $y \subseteq A_m \to y \in \mathscr{P}(A_m)$ and since $m$ was arbitrary $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$

Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i \in I} A_i \to y \in \bigcap_{i \in I} \mathscr{P}(A_i))$.

Now looking at our given $y \in \bigcap_{i \in I} A_i \to \forall i \in I(y \in A_i)$. Using universal instantiation we plug in $m$ for $i$ and therefore $y \in A_m$ and since $z \in y$ we can conclude $z \in A_m$, which was our goal. Now we can write our proof.

**Theorem.** *Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$, then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$.*

*Proof.* Suppose $y$ is an arbitrary element of $\bigcap_{i \in I} A_i$. Suppose $m$ is an arbitrary member of $I$ and therefore $y \subseteq A_m$ which implies $y \subseteq A_m$. Now suppose $z$ is an arbitrary element of $y$. Since $y \in \bigcap_{i \in I} A_i$ if we choose an $i$ such that $y \in \bigcap_{m \in I} A_m$ then $y \in A_m$ which implies $z \in A_m$. Therefore if $z \in y$ then $z \in A_m$ and since $z$ was arbitrary then $y \subseteq A_m$ or $y \in \mathscr{P}(A_m)$ and since $m$ was arbitrary then $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$. Since $y$ was arbitrary then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$. This completes the proof.