

3.4.1

Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall xP(x) \wedge \forall xQ(x)$.

We want to prove $\forall x(P(x) \wedge Q(x)) \iff \forall xP(x) \wedge \forall xQ(x)$.

Theorem. *The statement $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall xP(x) \wedge \forall xQ(x)$.*

Proof. (\rightarrow) Suppose $\forall x(P(x) \wedge Q(x))$. Let y be arbitrary. Since $\forall x(P(x) \wedge Q(x))$ it follows $P(y)$ and $Q(y)$. Since y was arbitrary, we can conclude $\forall xP(x)$ and $\forall xQ(x)$ or $\forall xP(x) \wedge \forall xQ(x)$.

(\leftarrow) Let y be arbitrary. Since $\forall xP(x)$ and $\forall xQ(x)$ then it follows $P(y)$ and $Q(y)$. Since y was arbitrary we can conclude $\forall x(P(x) \wedge Q(x))$. \square

3.4.2

Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Theorem. *If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B$ then $x \in B$ and since $A \subseteq C$ then $x \in C$ or $x \in B \cap C$. Therefore, if $x \in A$ then $x \in B \cap C$ and since x was arbitrary we can conclude $A \subseteq B \cap C$. \square

3.4.3

Suppose $A \subseteq B$. Prove that for every set C , $C \setminus B \subseteq C \setminus A$.

Theorem. *Suppose $A \subseteq B$, then for every set C , $C \setminus B \subseteq C \setminus A$.*

Proof. Suppose $A \subseteq B$ and C is an arbitrary set. Let x be arbitrary and suppose $x \in C \setminus B$, which means $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$, then $x \notin A$, which means that $x \in C \setminus A$. Therefore, if $x \in C \setminus B$ then $x \in C \setminus A$ and since x and C were arbitrary, we can conclude $\forall C(C \setminus B \subseteq C \setminus A)$. \square

3.4.5

Prove that if $A \subseteq B \setminus C$ and $A \neq \emptyset$ then $B \not\subseteq C$.

Theorem. *If $A \subseteq B \setminus C$ and $A \neq \emptyset$ then $B \not\subseteq C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B \setminus C$ then $x \in B$ and $x \notin C$. Since x was arbitrary we can conclude $B \not\subseteq C$. \square

3.4.6

Prove that for any sets A , B , and C , $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ finding a string of equivalences starting with $x \in A \setminus (B \cap C)$ and ending with $x \in (A \setminus B) \cup (A \setminus C)$.

Theorem. for any sets A , B , and C , $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof. Suppose A , B , and C are arbitrary sets. Then

$$\begin{aligned}
 x \in A \setminus (B \cap C) &\text{ iff } x \in A \rightarrow (x \notin B \wedge x \notin C) \\
 &\text{ iff } x \notin A \vee (x \notin B \wedge x \notin C) \\
 &\text{ iff } (x \notin A \vee x \notin B) \wedge (x \notin A \vee x \notin C) \\
 &\text{ iff } (x \in A \rightarrow x \notin B) \vee (x \in A \rightarrow x \notin C) \\
 &\text{ iff } x \in A \setminus B \vee x \in A \setminus C \\
 &\text{ iff } x \in (A \setminus B) \cup (A \setminus C)
 \end{aligned}$$

□

3.4.7

Theorem. For any sets A and B , $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Proof. (\rightarrow) Let M be an arbitrary set and suppose $M \in \mathcal{P}(A \cap B)$. Then $M \subseteq A \cap B$. Let x be arbitrary and suppose $x \in M$. Since $M \subseteq A \cap B$, $x \in A \cap B$ and therefore $x \in A$. Since x was arbitrary, $M \subseteq A$ and therefore $M \in \mathcal{P}(A)$. Similarly, since $M \subseteq A \cap B$, $x \in B$. Since x was arbitrary, $M \subseteq B$ and therefore $M \in \mathcal{P}(B)$. Therefore, $M \in \mathcal{P}(A)$ and $M \in \mathcal{P}(B)$.

(\leftarrow) Now suppose $M \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $M \subseteq A$ and $M \subseteq B$. Suppose $x \in M$. Since $M \subseteq A$ and $M \subseteq B$ then $x \in A \cap B$. Since x was arbitrary, $M \subseteq A \cap B$ and therefore $M \in \mathcal{P}(A \cap B)$. □

3.4.8

Theorem. $A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$

Proof. (\rightarrow) Suppose $A \subseteq B$. Let M be an arbitrary set and suppose $M \in \mathcal{P}(A)$. Then $M \subseteq A$. Now let y be arbitrary and suppose $y \in M$. Since $M \subseteq A$ then $y \in A$, and since $A \subseteq B$ then $y \in B$. Since y was arbitrary, $M \subseteq B$ and therefore $M \in \mathcal{P}(B)$. Since M was arbitrary, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(\leftarrow) Now suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and $y \in A$. Then the set $\{y\}$ is in $\mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $\{y\} \in \mathcal{P}(B)$ and $y \in B$. Since y was arbitrary, $A \subseteq B$. □

3.4.9

Theorem. *If x and y are odd integers, then xy is odd.*

Proof. Suppose x and y are odd integers. This means there is an integer k such that $x = 2k + 1$ and there is an integer j such that $y = 2j + 1$. Therefore, $xy = 2(2kj + k + j) = 4kj + 2k + 2j + 1 = (2k + 1)(2j + 1)$, and since $2kj + k + j$ is an integer, then xy is odd. \square

3.4.10

Theorem. *For every integer n , n^3 is even iff n is even.*

Proof. (\rightarrow) Let n be arbitrary. We will prove the contrapositive. Suppose x is odd, which means there exists an integer k such that $x = 2k + 1$. Therefore, $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Since $4k^3 + 6k^2 + 3k$ is an integer, n^3 is odd. Therefore, if n^3 is even, n is even.

(\leftarrow) Now suppose n is even, which means there exists an integer m such that $n = 2m$. Now $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$ and since $4m^3$ is an integer, n^3 is even. \square

3.4.11

A

The problem is with using the same variable k for defining m as an even integer and n as an odd integer when k may take on different values for n and m .

B

Let $m = 2$ and $n = -3$. Then $n^2 - m^2 = (-3)^2 - 2^2 = 9 - 4 = 5$ and $n + m = -3 + 2 = -1$. Therefore $n^2 - m^2 \neq n + m$.

3.4.12

Theorem. $\forall x \in \mathbb{R} [\exists y \in \mathbb{R} (x + y = xy) \iff x \neq 1]$

Proof. (\rightarrow) We will prove by contradiction. Suppose x is an arbitrary real number and there exists a real number y such that $x + y = xy$. Now suppose $x = 1$. Since $x + y = xy$, then $y = \frac{x}{x-1}$. But this contradicts $x = 1$ because there is no real number y such that $y = x/0$.

(\leftarrow) Now suppose $x \neq 1$ and $y = \frac{x}{x-1}$. Then

$$\begin{aligned}
x + y &= x + \frac{x}{x+1} = \frac{x(x+1) + x}{x+1} \\
&= \frac{x^2 - x + x}{x-1} \\
&= \frac{x^2}{x-1} = xy
\end{aligned}$$

□

3.4.13

Theorem. $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+ [\exists y \in \mathbb{R} (y - x = \frac{y}{x}) \iff x \neq z]$

Proof. (\rightarrow) Let $z = 1$. Let x be an arbitrary real number and suppose $x > 0$. Suppose $y \in \mathbb{R}$ and $y - x = \frac{y}{x}$. Then $y = \frac{x^2}{x-1}$. Now suppose $x = 1$. This contradicts $y \in \mathbb{R}$ and $y = \frac{x^2}{x-1}$. Therefore, $x \neq z$ and since x was arbitrary we can conclude $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+ [\exists y \in \mathbb{R} (y - x = \frac{y}{x}) \rightarrow x \neq z]$.

(\leftarrow) Now suppose $x \neq 1$ and $y = \frac{x^2}{x-1}$. Then

$$\begin{aligned}
y - x &= \frac{x^2}{x-1} - x = \frac{x^2 - x(x-1)}{x-1} \\
&= \frac{x^2 - x + 2 + x}{x-1} = \frac{x}{x-1} = \frac{y}{x}
\end{aligned}$$

□

3.4.14

Theorem. If B is a set and \mathcal{F} is a family of sets, then $\cup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathcal{P}(B))$.

Proof. Let x be arbitrary and suppose $x \in \cup\{A \setminus B \mid A \in \mathcal{F}\}$. This means that there is a set $A \in \mathcal{F}$ such that $x \in A$ and also $x \notin B$. Since $x \in A$ and $x \notin B$, then $A \not\subseteq B$ and $A \notin \mathcal{P}(B)$. Thus there is a set $A \in \mathcal{F}$ such that $x \in A$, and $A \notin \mathcal{P}(B)$, which means that $x \in \cup(\mathcal{F} \setminus \mathcal{P}(B))$. Therefore, if $x \in \cup\{A \setminus B \mid A \in \mathcal{F}\}$ then $x \in \cup(\mathcal{F} \setminus \mathcal{P}(B))$ and since x was arbitrary, we can conclude $\cup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathcal{P}(B))$. □

3.4.15

Theorem. If \mathcal{F} and \mathcal{G} are nonempty families of sets and every element of \mathcal{F} is disjoint from some element of \mathcal{G} , then $\cup\mathcal{F}$ and $\cap\mathcal{G}$ are disjoint.

Proof. Suppose \mathcal{F} and \mathcal{G} are nonempty families of sets and every element of \mathcal{F} is disjoint from some element of \mathcal{G} . We will use proof by contradiction. Now suppose $\cup\mathcal{F}$ and $\cap\mathcal{G}$ are not disjoint. Then there exists a y such that $y \in \cup\mathcal{F}$ and $y \in \cap\mathcal{G}$. Since $y \in \cup\mathcal{F}$ there is a set in \mathcal{F} that contains y and since $y \in \cap\mathcal{G}$, y is in every set in \mathcal{G} . But because every element of \mathcal{F} is disjoint from some element of \mathcal{G} , then there is at least one set in \mathcal{G} that does not contain y . But this contradicts $y \in \cap\mathcal{G}$. Therefore, $(\cup\mathcal{F}) \cap (\cap\mathcal{G}) = \emptyset$. \square

3.4.16

Theorem. For any set A , $A = \cup\mathcal{P}(A)$.

Proof. (\rightarrow) Suppose A is an arbitrary set, x is arbitrary, and $x \in A$. Then there is subset of A that contains x and, by definition, this subset is in $\mathcal{P}(A)$. Therefore, $x \in \cup\mathcal{P}(A)$. Since x was arbitrary $A \subseteq \cup\mathcal{P}(A)$.

(\leftarrow) Now suppose $x \in \cup\mathcal{P}(A)$. This means there is a subset of A that contains x and therefore $x \in A$. Since x was arbitrary we conclude $\cup\mathcal{P}(A) \subseteq A$. Since A was arbitrary, we can conclude for all sets A , $A = \cup\mathcal{P}(A)$. \square

3.4.17

A

Theorem. $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$

Proof. Let x be arbitrary and suppose $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Since $x \in \cup(\mathcal{F} \cap \mathcal{G})$ there is a set in \mathcal{F} and in \mathcal{G} that both contain x . Since there is a set in \mathcal{F} that contains x , then $x \in \cup\mathcal{F}$ and since there is a set in \mathcal{G} that contains x , $x \in \cup\mathcal{G}$. Therefore, $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. Since x was arbitrary, we can conclude $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. \square

B

The mistake is that we can't choose a set A such that $A \in \mathcal{F}$ and $A \in \mathcal{G}$ and $x \in A$. The given $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ means that x is within a set in \mathcal{F} and within a set in \mathcal{G} , but these two sets are not necessarily the same set.

C

Let $\mathcal{F} = \{\{1, 2\}, \{3\}\}$ and $\mathcal{G} = \{\{4, 5\}, \{1\}\}$. Then $\cup(\mathcal{F} \cap \mathcal{G}) = \emptyset$, but $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \{1\}$.

3.4.18

Theorem. Suppose \mathcal{F} and \mathcal{G} are families of sets, then $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G}) \iff \forall A \in \mathcal{F} \forall B \in \mathcal{G} (A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$.

Proof. (\rightarrow) Suppose $(\cup \mathcal{F}) \cap (\cup \mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$. Suppose A is an arbitrary set in \mathcal{F} , B is an arbitrary set in \mathcal{G} , x is arbitrary, and $x \in A \cap B$. Since $x \in A \cap B$ and A is an arbitrary set in \mathcal{F} , then $x \in \cup \mathcal{F}$. Also, since $x \in A \cap B$ and B is an arbitrary set in \mathcal{G} , then $x \in \cup \mathcal{G}$. Therefore $x \in (\cup \mathcal{F}) \cap (\cup \mathcal{G})$ and since $(\cup \mathcal{F}) \cap (\cup \mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$, it follows that $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Therefore, if $x \in (\cup \mathcal{F}) \cap (\cup \mathcal{G}) \rightarrow x \in \cup(\mathcal{F} \cap \mathcal{G})$ and since x , A , and B were arbitrary we can conclude that $\forall A \in \mathcal{F} \forall B \in \mathcal{G} (A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$.

(\leftarrow) Now suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G} (A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ and $x \in (\cup \mathcal{F}) \cap (\cup \mathcal{G})$. Since $x \in (\cup \mathcal{F}) \cap (\cup \mathcal{G})$, then there is a set $M \in \mathcal{F}$ such that $x \in M$ and there is a set $N \in \mathcal{G}$ such that $x \in N$ and it follows that $x \in M \cap N$. Then since $M \in \mathcal{F}$, $N \in \mathcal{G}$, $x \in M \cap N$, and $\forall A \in \mathcal{F} \forall B \in \mathcal{G} (A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ we can conclude that $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Therefore if $\forall A \in \mathcal{F} \forall B \in \mathcal{G} (A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ then $(\cup \mathcal{F}) \cap (\cup \mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$. □

1 3.4.19

Theorem. Suppose \mathcal{F} and \mathcal{G} are families of sets. Prove that $\cup \mathcal{F}$ and $\cup \mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint.

Proof. (\rightarrow) Suppose $\cup \mathcal{F} \cap \cup \mathcal{G} = \emptyset$ □