Exercise 3.3.4

Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

So we want to prove that $\forall x(x \in \mathscr{P}(A) \rightarrow x \in \mathscr{P}(\mathscr{P}(A)))$.

First we assume x is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathscr{P}(A)$	$x \in \mathscr{P}(\mathscr{P}(A))$
$x \in \mathscr{P}(A)$	

Assume x is an arbitrary element of $\mathscr{P}(A)$
    Suppose $x \in \mathscr{P}(A)$
        [proof of $x \in \mathscr{P}(\mathscr{P}(A))$]
    Therefore if $x \in \mathscr{P}(A) \rightarrow x \in \mathscr{P}(\mathscr{P}(A))$
Since x was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \rightarrow x \in \mathscr{P}(\mathscr{P}(A)))$

We can rewrite our goal as $x \subseteq \mathscr{P}(A)$ or $\forall y(y \in x \rightarrow y \in \mathscr{P}(A))$. So we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathscr{P}(A)$	$y \in \mathscr{P}(A)$
$x \in \mathscr{P}(A)$	
$y \in x$	

Assume x is an arbitrary element of $\mathscr{P}(A)$
    Suppose $x \in \mathscr{P}(A)$
        Suppose y is an arbitrary element of x.
            Suppose $y \in x$.
                [proof of $y \in \mathscr{P}(A)$]
            Therefore if $y \in x \rightarrow y \in \mathscr{P}(A)$.
        Since y was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$.
    Therefore if $x \in \mathscr{P}(A) \rightarrow x \in \mathscr{P}(\mathscr{P}(A))$
Since x was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \rightarrow x \in \mathscr{P}(\mathscr{P}(A)))$

Now looking at our givens $x \in \mathscr{P}(A)$ means that $x \subseteq A$ or $\forall z(z \in x \rightarrow z \in A)$. Using universal instantiation we will plug in y for z and using modus ponens we can conclude that $y \in A$.

Now looking at our other given $A \subseteq \mathscr{P}(A) \rightarrow \forall m(m \in A \rightarrow m \in \mathscr{P}(A))$. Using universal instantiation we will plug in y for m and using modus ponens we can conclude that $y \in \mathscr{P}(A)$, which was our goal to prove.

Theorem. *Suppose $A \subseteq \mathscr{P}(A)$. Then $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.*

Proof. Suppose x is an arbitrary element of $\mathscr{P}(A)$ and y is an arbitrary element of x. It follows that $y \in A$. But since $A \subseteq \mathscr{P}(A)$ then it also follows that $y \in \mathscr{P}(A)$. So $y \in x \rightarrow y \in \mathscr{P}(A)$ and since y was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$. Therefore, if $x \in \mathscr{P}(A) \rightarrow x \in \mathscr{P}(\mathscr{P}(A))$. Since x was arbitrary we can also conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. □

Alternate proof (not sure if this is correct)

Proof. Suppose x is an arbitrary element of $\mathscr{P}(A)$. Then $x \in A$. Since $A \subseteq \mathscr{P}(A)$ and $x \in A$ then $x \subseteq \mathscr{P}(A)$. Therefore, $x \in \mathscr{P}(\mathscr{P}(A))$. □

Exercise 3.3.5

The hypothesis of the theorem proven in exercise 3.3.4 is $A \subseteq \mathscr{P}(A)$.

A

Can you think of a set A for which this hypothesis is true?

The empty set $\varnothing$ is a set for which the hypothesis is true.

$A \subseteq \mathscr{P}(A)$ means $x \in A \rightarrow x \in \mathscr{P}(A)$. For $\varnothing$ this would mean that $x \in \varnothing \rightarrow x \in \mathscr{P}(\varnothing)$, but by definition there are no elements in $\varnothing$. Therefore $x \in \varnothing$ will always be false and the conditional statement $x \in \varnothing \rightarrow x \in \mathscr{P}(\varnothing)$ is always true. Therefore if $\varnothing = A$ then $A \subseteq \mathscr{P}(A)$.

B

Can you think of another?

In exercise 3.3.4 we proved that if $A \subseteq \mathscr{P}(A)$ then $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. Therefore, the set $\{\varnothing, \{\varnothing\}\}$, which is the $\mathscr{P}(A)$ if $A = \varnothing$, is another set for which the hypothesis is true. If we let $B = \mathscr{P}(A) = \{\varnothing, \{\varnothing\}\}$ and replace A in the hypothesis $A \subseteq \mathscr{P}(A)$ with B, then we can conclude that $B \subseteq \mathscr{P}(B)$.

Exercise 3.3.6

Suppose x is a real number.

A

Prove that if $x \neq 1$ then there is a real number y such that $\frac{y+1}{y-2} = x$.

So we want to prove that $(x \neq 1) \to \exists y \left(\frac{y+1}{y+2} = x \right)$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$x \neq 1$	$\exists y \left(\frac{y+1}{y+2} = x \right)$

To prove our goal we need to find a y that makes the equation $\frac{y+1}{y+2} = x$ true. So let's try solving the equation for y.

$$\begin{aligned} \frac{y+1}{y+2} &= x \\ y+1 &= x(y-2) \\ y+1 &= xy - 2x \\ 2x+1 &= xy - y \\ 2x+1 &= y(x-1) \\ y &= \frac{2x+1}{x-1} \end{aligned}$$

We see that this y works because we have $x \neq 1$ as a given.

Theorem. *Suppose $x \neq 1$. Then there is a real number y such that $\frac{y+1}{y-2} = x$.*

Proof. Suppose $x \neq 1$ and $y = \frac{2x+1}{x-1}$. Then

$$\frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{x-1} \cdot \frac{x-1}{3} = x$$

□

B

Prove that if there is a real number y such that $\frac{y+1}{y-2} = x$ then $x \neq 1$.

So we want to prove that $\exists y \left(\frac{y+1}{y-1} = x \right) \to (x \neq 1)$

We assume the antecedent and make the consequent our goal to prove.

Using existential instantiation we assume there is a value y_0 such that $\frac{y+1}{y-1} = x$ is true. From part A above, we know that $\left(\frac{y+1}{y-1} = x \right) \to \left(y = \frac{2x+1}{x-1} \right)$ and so $y_0 = \frac{2x+1}{x-1}$. Since y is a real number, then clearly $x \neq 1$.

Givens	Goals
$\exists y\left(\frac{y+1}{y-1}=x\right)$	$x \neq 1$

Theorem. *If y is a real number and $\frac{y+1}{y-2}=x$ then $x \neq 1$.*

Proof. Suppose y is a real number and $\frac{y+1}{y-2}=x$. It follows that $y=\frac{2x+1}{x-1}$ and since y is real number then $x \neq 1$. □

Exercise 3.3.7

Prove for every real number x, if $x>2$ then there is a real number y such that $y+\frac{1}{y}=x$.

So we want to prove $\forall x \in \mathbb{R}(x>2 \rightarrow \exists y \in \mathbb{R}(y+\frac{1}{y}=x))$

So we let x be an arbitrary real number, then we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
x is arbitrary real number	$\exists y(y+\frac{1}{y}=x)$
$x>2$	

Our goal is of the form $\exists y P(y)$ where $P(y)$ is $y+\frac{1}{y}=x$ and our strategy suggests we try to find a y for which $P(y)$ is true. We can do this by solving the equation $y+\frac{1}{y}=x$ for y. We can rewrite this equation as $y^2-\frac{x}{y}+1=0$ and we see this is a quadratic equation and therefore we can use the quadratic formula to solve for y,

$$y=\frac{-(-x)\pm\sqrt{(-x)^2-4\cdot 1\cdot 1}}{2\cdot 1}=\frac{x\pm\sqrt{x^2-4}}{2}.$$

We note that $\sqrt{x^2-4}$ is defined because $x>2$. We have found two solutions that satisfy our original equation, but we only need one to complete the proof. We will use $\frac{x+\sqrt{x^2-4}}{2}$.

Theorem. *For every real number x, if $x>2$ then there is a real number y such that $y+\frac{1}{y}=x$.*

Proof. Suppose x and y are real numbers, $x>2$, and $y=\frac{x+\sqrt{x^2-4}}{2}$. Then

$$\frac{x+\sqrt{x^2-4}}{2} + \frac{1}{\frac{x+\sqrt{(x^2-4}}{2}} = \frac{x+\sqrt{x^2-4}}{2} + \frac{2}{x+\sqrt{x^2-4}}$$
$$= \frac{2x^2+2(x\sqrt{x^2-4})}{2x+2\sqrt{x^2-4}}$$
$$= x$$

□

Exercise 3.3.8

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.

So we want to prove that $A \in \mathcal{F} \to A \subseteq \cup\mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$A \subseteq \cup\mathcal{F}$

Assume $A \in \mathcal{F}$
    [proof of $A \subseteq \cup\mathcal{F}$]
Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$

Our goal $A \subseteq \cup\mathcal{F}$ can be rewritten as $\forall x(x \in A \to x \in \cup\mathcal{F})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in \cup\mathcal{F}$
$x \in A$	

Assume $A \in \mathcal{F}$
    Assume x is arbitrary
        Assume $x \in A$
            [proof of $x \in \cup\mathcal{F}$]
        Therefore if $x \in A$ then $x \cup \mathcal{F}$
    Since x was arbitrary we can conclude that $A \subseteq \cup\mathcal{F}$.
Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$

Our new goal can be rewritten as $\exists B \in \mathcal{F}(x \in B)$. From our givens we see that $A \in \mathcal{F}$ and $x \in A$, so we have found a set such that $A \in \mathcal{F}(x \in A)$.

Theorem. *If $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.*

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of A. Then since $x \in A$ and $A \in \mathcal{F}$, it follows that $x \in \cup\mathcal{F}$. Since x was arbitrary we can conclude that $A \subseteq \cup\mathcal{F}$ and therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$. □

Exercise 3.3.9

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cap\mathcal{F} \subseteq A$.

We want to prove that $A \in \mathcal{F} \to \cap\mathcal{F} \subseteq A$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$\cap\mathcal{F} \subseteq A$

Assume $A \in \mathcal{F}$
    [proof of $\cap\mathcal{F} \subseteq A$]
Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$.

We can rewrite our goal as $\forall x(x \in \cap\mathcal{F} \to x \in A)$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in A$
$x \in \cap\mathcal{F}$	

Assume $A \in \mathcal{F}$
    Assume x is arbitrary
        Assume $x \in \cap\mathcal{F}$
            [proof of $x \in A$]
        Therefore, if $x \in \cap\mathcal{F}$ then $x \in A$.
    Since x was arbitrary we can conclude that $\cap\mathcal{F} \subseteq A$.
Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$.

Our given $x \in \cap\mathcal{F}$ can be rewritten as $\forall B \in \mathcal{F}(x \in B)$, therefore if $A \in \mathcal{F}$ then $x \in A$, which was our goal to prove.

Theorem. *If $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cup\mathcal{F} \in A$.*

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of $\cap\mathcal{F}$. Since $A \in \mathcal{F}$ and $x \in \cap\mathcal{F}$ it follows that $x \in A$ and therefore, if $x \in \cap\mathcal{F}$ then $x \in A$. Since x was arbitrary we can conclude that $\cap\mathcal{F} \subseteq A$. Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$. □

Exercise 3.3.10

Suppose that $\mathcal{F}$ is a nonempty family of sets B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$. Prove that $B \subseteq \cap\mathcal{F}$.

We want to prove $\forall A \in \mathcal{F}(B \subseteq A) \to B \subseteq \cap\mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$B \subseteq \cap\mathcal{F}$

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$
    [proof of $B \subseteq \cap\mathcal{F}$]
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap\mathcal{F}$.

Our goal can be rewritten as $\forall x(x \in B \to x \in \cap\mathcal{F})$. So we assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$B \in \cap\mathcal{F}$
$x \in B$	

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$
    Suppose x is arbitrary.
        Suppose $x \in B$.
            [proof of $x \in \cap\mathcal{F}$]
        Therefore $x \in B \to x \in \cap\mathcal{F}$
    Since x was arbitrary we can conclude $B \subseteq \cap\mathcal{F}$
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap\mathcal{F}$.

Our goal can be rewritten as $\forall M \in F(x \in M)$ and so we can assume M is an arbitrary set in $\mathcal{F}$ and make our goal $x \in M$.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$x \in M$
$x \in B$	
$M \in \mathcal{F}$	

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$
    Suppose x is arbitrary.
        Suppose $x \in B$.
            Suppose M is an arbitrary set in $\mathcal{F}$.

[proof of $x \in M$]
Therefore $x \in \cap\mathcal{F}$
Therefore $x \in B \to x \in \cap\mathcal{F}$
Since x was arbitrary we can conclude $B \subseteq \cap\mathcal{F}$
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap\mathcal{F}$.

Using universal instantiation we will plug in M for A in our given $\forall A \in \mathcal{F}(B \subseteq A)$ and conclude that $B \subseteq M$. We can rewrite $B \subseteq M$ as $\forall y(y \in B \to y \in M)$ and using universal instantiation plug in x for y and then use moden ponens to conclude $x \in M$, which was our goal to prove.

Theorem. *If $\mathcal{F}$ is a nonempty family of sets, B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$, then $B \subseteq \cap\mathcal{F}$.*

Proof. Suppose $\forall A \in \mathcal{F}(B \subseteq A)$. Suppose x is an arbitrary member of B and M is an arbitrary set in $\mathcal{F}$. Then it follows that $x \in M$ and since M was arbitrary we can conclude that x is in all sets that are in $\mathcal{F}$ or $x \in \cap\mathcal{F}$. Therefore, if $x \in B$ then $x \in \cap\mathcal{F}$, and since x was arbitrary, we can conclude that $B \subseteq \cap\mathcal{F}$. □

Exercise 3.3.11

Suppose that $\mathcal{F}$ is a family of sets. Prove that if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$.
We want to prove that $\varnothing \in \mathcal{F} \to \cap\mathcal{F} = \varnothing$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\varnothing \in \mathcal{F}$	$\cap\mathcal{F} = \varnothing$

Suppose $\varnothing \in \mathcal{F}$
[proof of $\cap\mathcal{F} = \varnothing$]
Therefore if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$.

We will try a proof by contradiction. So we assume that $\cap\mathcal{F} \neq \varnothing$ and try to find a contradiction.

Givens	Goals
$\varnothing \in \mathcal{F}$	contradiction
$\cap\mathcal{F} \neq \varnothing$	

Our given $\cap\mathcal{F} \neq \varnothing$ means that there is an element that is in all sets in $\mathcal{F}$. However, this contradicts $\varnothing \in \mathcal{F}$ because $\varnothing$ is the set that contains nothing.

Theorem. *If $\mathcal{F}$ is a family of sets and $\varnothing \in \mathcal{F}$, then $\cap\mathcal{F} = \varnothing$.*

Proof. We will prove by contradiction. Suppose $\varnothing \in \mathcal{F}$ and $\cap\mathcal{F} \neq \varnothing$. Since $\cap\mathcal{F} \neq \varnothing$ it follows that there is an element that is within all of the sets that are in $\mathcal{F}$. However, this contradicts $\varnothing \in \mathcal{F}$ because $\varnothing$ is the set that contains nothing. Therefore, if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$. □

Exercise 3.3.12

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cup\mathcal{F} \subseteq \cup\mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$

[proof of $\cup\mathcal{F} \subseteq \cup\mathcal{G}$]

So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\cup\mathcal{F} \subseteq \cup\mathcal{G} \to \forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$ so we assume b is an arbitrary element of $\cup\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$b \in \cup\mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cup\mathcal{F}$

[proof of $b \in \cup\mathcal{G}$]

Therefore if $b \in \cup\mathcal{F} \to b \in \cup\mathcal{G}$

Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$b \in \cup\mathcal{F} \to \exists M(M \in \mathcal{F} \land b \in M)$, so let $M = A_0$ (Existential Instantiation)

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \land b \in A_0$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element and suppose $b \in \cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ and b is in that set. Let that set $= A_0$

[proof of $b \in \cup\mathcal{G}$]

Therefore if $b \in \cup\mathcal{F} \to b \in \cup\mathcal{G}$

Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \to \forall A(A \in \mathcal{F} \to A \in \mathcal{G})$. Using universal instantiation we will plug in A_0 for A since then we can use modens ponens to conclude that $A_0 \in \mathcal{G}$.

Givens	Goals
$A_0 \in \mathcal{F} \to A_0 \in \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Our goal $b \in \cup\mathcal{G} \to \exists N(N \in \mathcal{G} \wedge b \in N)$, which we can now prove. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$, it follows that $A_0 \in G$. By the definition of $\cup\mathcal{G}$ it follows that $b \in \cup\mathcal{G}$ because $A_0 \in G \wedge b \in A_0$, the latter statement being one of our givens.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.*

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ that contains b. Call this set A_0. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$ it follows that $A_0 \in G$, which implies that $b \in \cup\mathcal{G}$. Therefore if $b \in \cup\mathcal{F}$ then $b \in \cup\mathcal{G}$. Since b was arbitrary we can conclude that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$. This completes the proof.

Exercise 3.3.13

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cap\mathcal{G} \subseteq \cap\mathcal{F}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$

[proof of $\cap\mathcal{G} \subseteq \cap\mathcal{F}$]

So if $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$\cap\mathcal{G} \subseteq \cap\mathcal{F} \to \forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$, so we assume b is an arbitrary element of $\cap\mathcal{G}$ and assume the antecedent and make the consequent our goal to prove.

Suppose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cap\mathcal{G}$

[proof of $b \in \cap\mathcal{F}$]

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cap\mathcal{F}$
$b \in \cap\mathcal{G}$	

Therefore if $b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F}$
Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$b \in \cap\mathcal{F} \rightarrow \forall A(A \in \mathcal{F} \rightarrow b \in A)$, so we assume A is an arbitrary element of $\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in A$
$b \in \cap\mathcal{G}$	
$A \in \mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    Let b be an arbitrary element of $\cap\mathcal{G}$
        Suppose A is an arbitrary set in $\mathcal{F}$
            [proof of $b \in A$]
        Therefore if $A \in \mathcal{F} \rightarrow b \in A$
        Since A was arbitrary we can conclude $b \in \cap\mathcal{F}$
    Therefore if $b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F}$
Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

Now looking at our givens, $\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall Z(Z \in \mathcal{F} \rightarrow Z \in \mathcal{G})$. Using universal instantiation we will plug in A for Z and using modus ponens we can conclude that $A \in \mathcal{G}$.

Our other given, $b \in \cap\mathcal{G} \rightarrow \forall Y(Y \in \mathcal{G} \rightarrow b \in Y)$. Using universal instantiation we will plug in A for Y and using modus ponens we can conclude that $b \in A$, which was our goal, and we can now write our proof.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.*
Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cap\mathcal{G}$. Suppose A is an arbitrary element of $\mathcal{F}$, then because $\mathcal{F} \subseteq \mathcal{G}$ then it follows that $A \in \mathcal{G}$. By the definition of $\cap\mathcal{G}$ it follows that $b \in A$ and since A was arbitrary then $b \in \cap\mathcal{F}$. Since b was arbitrary we can conclude $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ and therefore that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$. This completes the proof.

Exercise 3.3.14

Suppose $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i \in I} A_i)$.

So we want to prove that $\forall a(a \in \bigcup_{i \in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i \in I} A_i))$

First we assume a is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i \in I} \mathscr{P}(A_i)$	$a \in \mathscr{P}(\bigcup_{i \in I} A_i)$

Assume a is an arbitrary element of $\bigcup_{i \in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i \in I} \mathscr{P}(A_i)$
        [proof of $a \in \mathscr{P}(\bigcup_{i \in I} A_i)$]
    Therefore if $a \in \bigcup_{i \in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i \in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i \in I} A_i)$

Looking at our goal we see that $a \in \mathscr{P}(\bigcup_{i \in I} A_i) \to a \subseteq \bigcup_{i \in I} A_i \to \forall z(z \in a \to z \in \bigcup_{i \in I} A_i)$. Therefore we assume z is arbitrary, assume the antecedent, and make the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i \in I} \mathscr{P}(A_i)$	$z \in \bigcup_{i \in I} A_i$
$z \in a$	

Assume a is an arbitrary element of $\bigcup_{i \in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i \in I} \mathscr{P}(A_i)$
        Assume z is arbitrary
            Assume $z \in a$
                [proof of $z \in \bigcup_{i \in I} A_i$]
            Therefore $z \in a \to z \in \bigcup_{i \in I} A_i$
        Since z was arbitrary we can conclude $a \in \mathscr{P}(\bigcup_{i \in I} A_i)$
    Therefore if $a \in \bigcup_{i \in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i \in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i \in I} A_i)$

Looking at our given we see that $a \in \bigcup_{i \in I} \mathscr{P}(A_i) \to a \in \{a | \exists i \in I(a \in \mathscr{P}(A_i))\}$. Using existential instantiation we will select an i such that $a \in \mathscr{P}(A_i)$ which implies $a \subseteq A_i$. Since $a \subseteq A_i \to \forall m(m \in a \to m \in A_i)$ and using universal instantiation we will plug in z for m and we get $\forall z(z \in a \to z \in A_i)$ and using modus ponens we can conclude that $z \in A_i$, which implies that $z \in \bigcup_{i \in I} A_i$, which was our goal. We can now right our proof.

Theorem. *Suppose $\{A_i | i \in I\}$ is an indexed family of sets, then $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i \in I} A_i)$.*

Proof. Suppose that a is an arbitrary element of $\bigcup_{i \in I} \mathscr{P}(A_i)$. We choose an $i \in I$ such that $a \in \mathscr{P}(A_i)$, which implies that $a \subseteq A_i$. Suppose z is an arbitrary element of a, then it follows that $z \in A_i$ and therefore $z \in \bigcup_{i \in I} A_i$. Since z was an arbitrary element of a then $a \subseteq \bigcup_{i \in I} A_i$, and it follows that $a \in \mathscr{P}(\bigcup_{i \in I} A_i)$. Thus we can conclude $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i \in I} A_i)$. This completes the proof.

Exercise 3.3.15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$. Prove that $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$

So we want to prove that $\forall y(y \in \bigcap_{i \in I} A_i \to y \in \bigcap_{i \in I} \mathscr{P}(A_i))$.

First we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i \in I} A_i$	$y \in \bigcap_{i \in I} \mathscr{P}(A_i)$

Suppose y is arbitrary element of $\bigcap_{i \in I} A_i$.
    [proof of $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$]
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i \in I} A_i \to y \in \bigcap_{i \in I} \mathscr{P}(A_i))$.

Our goal $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$ so we make m an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$. So we make z arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i \in I} A_i$	$z \in A_m$
$z \in y$	

Suppose y is arbitrary element of $\bigcap_{i \in I} A_i$.
    Suppose m is an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$.
        Suppose z is an arbitrary element of y
            [proof of $z \in A_m$]
    Therefore $z \in y \to z \in A_m$ and since z was arbitrary $y \subseteq A_m \to y \in \mathscr{P}(A_m)$ and since m was arbitrary $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i \in I} A_i \to y \in \bigcap_{i \in I} \mathscr{P}(A_i))$.

Now looking at our given $y \in \bigcap_{i \in I} A_i \to \forall i \in I(y \in A_i)$. Using universal instantiation we plug in m for i and therefore $y \in A_m$ and since $z \in y$ we can conclude $z \in A_m$, which was our goal. Now we can write our proof.

Theorem. *Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$, then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$.*

Proof. Suppose y is an arbitrary element of $\bigcap_{i \in I} A_i$. Suppose m is an arbitrary member of I and therefore $y \subseteq A_m$ which implies $y \subseteq A_m$. Now suppose z is an arbitrary element of y. Since $y \in \bigcap_{i \in I} A_i$ if we choose an i such that $y \in \bigcap_{m \in I} A_m$ then $y \in A_m$ which implies $z \in A_m$. Therefore if $z \in y$ then $z \in A_m$ and since z was arbitrary then $y \subseteq A_m$ or $y \in \mathscr{P}(A_m)$ and since m was arbitrary then $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$. Since y was arbitrary then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$. This completes the proof.

Exercise 3.3.16

Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup\mathcal{F} \subseteq B$.

We want to prove $\mathcal{F} \subseteq \mathscr{P}(B) \to \cup\mathcal{F} \subseteq B$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathscr{P}(B)$	$\cup\mathcal{F} \subseteq B$

Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$
  [proof of $\cup\mathcal{F} \subseteq B$
Therefore if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup F \subseteq B$.

$\cup F \subseteq B \to \forall x(x \in \cup\mathcal{F} \to x \in B)$. So we assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathscr{P}(B)$	$x \in B$
$x \in \cup\mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$
  Suppose x is arbitrary
    Suppose $x \in \cup\mathcal{F}$
      [proof of $x \in B$]
    Therefore if $x \in \cup\mathcal{F}$ then $x \in B$
  Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq B$.

Therefore if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup F \subseteq B$.

$x \in \cup\mathcal{F} \to \exists M \in \mathcal{F}(x \in M)$. We use existential instantiation and assume there is a set M in $\mathcal{F}$ and x is in that set.

Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$
    Suppose x is arbitrary
        Suppose M is arbitrary set in $\mathcal{F}$
            $x \in M$
                [proof of $x \in B$]
            Since $x \in M$ and M is a set in $\mathcal{F}$ then $x \in \cup\mathcal{F}$
        Therefore if $x \in \cup\mathcal{F}$ then $x \in B$
    Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq B$.
Therefore if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup F \subseteq B$.

Our given $\mathcal{F} \subseteq \mathscr{P}(B)$ means that $\forall N(N \in \mathcal{F} \to \forall z(z \in N \to z \in B)$. We will use universal instantiation and plug in M for N and x for z and we can conclude that $x \in B$, which was our goal to prove.

Theorem. *Suppose B is a set and $\mathcal{F}$ is a family of sets. If $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup\mathcal{F} \subseteq B$.*

Proof. Suppose x is an arbitrary member of $\cup\mathcal{F}$, which means that x is a member of a set that is in $\mathcal{F}$. Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$, which means that any element that is in a set that is a member of $\mathcal{F}$ is also in the set B. It follows that since x is a member of a set in $\mathcal{F}$ then $x \in B$. Therefore, if $x \in \cup\mathcal{F}$ then $x \in B$ and since x was arbitrary we can conclude $\cup\mathcal{F} \subseteq B$. Therefore, if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\cup\mathcal{F} \subseteq B$. □

Exercise 3.3.17

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Prove that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We want to prove that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We can rewrite this goal as $\forall x(x \in \cup\mathcal{F} \to x \in \cap\mathcal{G})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal.

Givens	Goals
$x \in \cup\mathcal{F}$	$x \in \cap\mathcal{G}$
$\forall A \in \mathcal{F}\forall B \in \mathcal{G}(y \in A \to y \in B)$	

Suppose x is arbitrary.
    Suppose $x \in \cup\mathcal{F}$.

        [proof of $x \in \cap\mathcal{G}$]
    Therefore if $x \in \cup\mathcal{F}$ then $x \in \cap\mathcal{G}$
Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We can rewrite our goal as $\forall M \in \mathcal{G}(x \in M)$. We assume M is an arbitrary set in $\mathcal{G}$ and then our goal becomes $x \in M$.

Givens	Goals
$x \in \cup\mathcal{F}$	$x \in M$
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \rightarrow y \in B)$	

Suppose x is arbitrary.
    Suppose $x \in \cup\mathcal{F}$.
        Suppose M is an arbitrary set in $\mathcal{G}$
            [proof of $x \in M$]
        Since M was arbitrary we can conclude that $x \in \cap\mathcal{G}$
    Therefore if $x \in \cup\mathcal{F}$ then $x \in \cap\mathcal{G}$
Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.

We can rewrite our given $x \in \cup\mathcal{F}$ as $\exists N \in \mathcal{F}(x \in N)$. We use existential instantiation and assume there is a set $N \in \mathcal{F}$ and $x \in N$.

Givens	Goals
$N \in \mathcal{F}$	$x \in M$
$x \in N$	
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \rightarrow y \in B)$	

Now we can use universal instantiation to plug in N for A and M for B. Then since $x \in N$ we can use modus ponens to conclude that $x \in M$, which was our goal.

Theorem. *If $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$, then $\cup\mathcal{F} \subseteq \cap\mathcal{G}$.*

Proof. Suppose x is an arbitrary member of $\cup\mathcal{F}$, which means there is a set in $\mathcal{F}$ that contains x. Suppose M is an arbitrary set in $\mathcal{G}$. Then since every set in $\mathcal{F}$ is a subset of every set in $\mathcal{G}$ it follows that $x \in M$. Since M was arbitrary we can conclude that $x \in \cap\mathcal{G}$ and therefore if $x \in \cup F$ then $x \in \cap\mathcal{G}$. Since x was arbitrary we can conclude that $\cup\mathcal{F} \subseteq \cap\mathcal{G}$. $\square$

Exercise 3.3.18

In this problem all variables range over $\mathbb{Z}$, the set of all integers.

A

Prove that if $a|b$ and $a|c$, then $a|(b+c)$.

We want to prove $(a|b) \wedge (a|c) \rightarrow a|(b+c)$

We assume the antecedent and make the consequent our goal.

Givens	Goals
$a\|b$	$a\|(b+c)$
$a\|c$	

Suppose $a|b$ and $a|c$
[proof of $a|(b+c)$
Therefore if $a|b$ and $a|c$ then $a|(b+c)$.

Our goal means that $\exists x \in \mathbb{Z}(ax = (b+c))$. So we need to find an x that makes this statement true. Our goals can be rewritten as $\exists y \in \mathbb{Z}(ay = b)$ and $\exists w \in \mathbb{Z}(aw = c)$. Using existential instantiation we will assume there is a y and and w that makes both of the previous statement true.

Givens	Goals
$ay = b$	$a\|(b+c)$
$aw = c$	

Suppose $ay = b$ and $aw = c$
[proof of $a|(b+c)$
Therefore if $a|b$ and $a|c$ then $a|(b+c)$.

Adding the two inequalities $ay = b$ and $aw = c$ we have $ay + aw = b + c$ or $a(y + w) = b + c$. Since y and w are integers we can conclude that $a|(b + c)$, which was our goal to prove.

Theorem. *If a, b, and c are integers, $a|b$, and $a|c$, then $a|(b+c)$.*

Proof. Suppose a, b, and c are integers, $a|b$, and $a|c$. Since $a|b$ there must be an integer y such that $ay = b$. Also, since $a|c$ there must be an integer w such that $aw = c$. Adding together the previous two equalities we have $ay + aw = b + c$ or $a(y + w) = b + c$. Since y and w are integers we can conclude that $a|(b+c)$. $\square$

B

Prove that if $ac|bc$ and $c \neq 0$, then $a|b$.

We want to prove $(ac|bc) \wedge (c \neq 0) \rightarrow a|b$.

We assume the antecedent and make the consequent our goal.

Givens	Goals
$ac\|bc$	$a\|b$
$c \neq 0$	

Suppose $ac|bc$ and $c \neq 0$
    [proof of $a|b$]
Therefore $ac|bc$ and $c \neq 0$, then $a|b$.

Our goal means that $\exists x(ax = b)$ and we want to find an x that makes this statement true. Looking at our goals we can rewrite $ac|bc$ as $\exists y(acy = bc)$. Using existential instantiation we will assume there is a y that makes $acy = bc$ true and we can add $acy = bc$ to our givens. Since $c \neq 0$ we can divide both sides of $acy = bc$ by c and we have $ay = b$. Since y is an integer we can conclude that $a|b$, which was our goal to prove.

Theorem. *If a, b, and c are integers, $ac|bc$, and $c \neq 0$, then $a|b$.*

Proof. Suppose a, b, and c are integers, $ac|bc$, and $c \neq 0$. Since $ac|bc$ there must be an integer x such that $acx = bc$. Since $c \neq 0$ we can simplify the previous equation by dividing both sides by c so that $ax = b$. Since x is an integer we can conclude that $a|b$. □

Exercise 3.3.19

A

Prove that for all real numbers x and y there is a real number z such that $x + z = y - z$.

We want to prove that $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R}(x + z = y - z)$.

We let x and y stand for arbitrary real numbers and make $\exists z \in \mathbb{R}(x+z = y-z)$ our goal to prove.

Givens	Goals
x arbitrary	$\exists z \in \mathbb{R}(x + z = y - z)$
y arbitrary	

Suppose x and y are arbitrary real numbers
    [proof of $\exists z \in \mathbb{R}(x + z = y - z)$]
Since x and y are arbitrary we can conclude $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R}(x+z = y-z)$

We want to find a z such that $x+z=y-z$, which suggests we try solving this equation for z

$$\begin{aligned} x+z &= y-z \\ x+2z &= y \\ z &= \frac{y-x}{2}. \end{aligned}$$

Now we are ready to complete our proof.

Theorem. *For all real numbers x and y there is a real number z such that $x+z=y-z$.*

Proof. Suppose x and y are arbitrary real numbers and $z=\frac{y-x}{2}$. Then

$$\begin{aligned} x+\frac{y-x}{2} &= y-\frac{y-x}{2} \\ \frac{2x+y-x}{2} &= \frac{2y-(y-x}{2} \\ 2x+y-x &= 2y-y+x \\ x(2-1)+y &= y(2-1)+x \\ x+y &= x+y \end{aligned}$$

□

B

Would the statement in part (A) be correct if "real number" were changed to "integer"? Justify your answer.

No, because there are instances where $z=\frac{x-y}{2}$ would not result in an integer. For example, if $x=5$ and $y=2$ then $z=\frac{3}{2}$, which is not an integer. Therefore the statement in part (A) would not be correct.

Exercise 3.3.20

Consider the following theorem:

Theorem. *For every real number x, $x^2 \geq 0$.*

What's wrong with the following proof?

Proof. Suppose not. Then for every real number x, $x^2<0$. In particular, plugging in $x=3$ we would get $9<0$, which is clearly false. This contradiction shows that for every number $x, x^2 \geq 0$. □

The sentence "Then for every real number x, $x^2<0$" is not correct because if we let $x=0$ then $0<0$ is not true.

3.3.21

Consider the following incorrect theorem:

Incorrect Theorem. *If* $\forall x \in A(x \neq 0)$ *and* $A \subseteq B$ *then* $\forall x \in B(x \neq 0)$.

A

What's wrong with the following proof?

Proof. Let x be an arbitrary element of A. Since $\forall x \in A(x \neq 0)$, we can conclude that $x \neq 0$. Also, since $A \subseteq B$, $x \in B$. Since $x \in B$, $x \neq 0$, and x was arbitrary, we can conclude that $\forall x \in B(x \neq 0)$. □

The last sentence is not correct. $A \subseteq B$ means that all elements in A are in B and since $x \neq 0$ then $0 \notin A$, but this doesn't mean that $0 \notin B$, because there can be elements in B that are not in A.

B

Find a counterexample to the theorem. In other words, find an example of sets A and B for which the hypotheses of the theorem are true but the conclusion is false.

Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$. Then the hypotheses of the theorem are true, specifically $\forall x \in A(x \neq 0)$ and $A \subseteq B$, but the conclusion $\forall x \in B(x \neq 0)$ is false.

3.3.22

Consider the following incorrect theorem:

Incorrect Theorem. $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy^2 = y - x)$.

What's wrong with the following proof of the theorem?

Proof. Let $x = \frac{y}{y^2+1}$. Then

$$y - x = y - \frac{y}{y^2} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1} \cdot y^2 = xy^2.$$

□

In the proof, x is defined in terms of y but y has not been introduced into the proof yet. The theorem should start with "Let $x =$ .... and let y be an arbitrary real number...".