Exercise 3.1.5

Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.

So we want to prove that $(a < b < 0) \to (a^2 > b^2)$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a < b < 0$	$a^2 > b^2$

Suppose $a < b < 0$
    [proof of $a^2 > b^2$]
So if $a < b < 0$ then $a^2 > b^2$

If we multiply the inequality $a < b$ on both sides by the negative number b we have $ab > b^2$ and multiplying $a < b$ on both sides by the negative number a we have $a^2 > ab$. Therefore $a^2 > ab > b^2$ and we have proven our goal and now we can write our proof.

Theorem. *Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.*

Proof. Suppose $a < b < 0$. Multiplying the inequality $a < b$ by the negative number a we can conclude $a^2 > ab$, and, similarly, multiplying $a < b$ by the negative number b we get $ab > b^2$. Therefore, $a^2 > ab > b^2$ and $a^2 > b^2$. Thus, if $a < b < 0$ then $a^2 > b^2$. $\square$

Exercise 3.1.6

Suppose a and b are real numbers. Prove that if $0 < a < b$ then $1/b < 1/a$.

So we want to prove that $(0 < a < b) \to (1/b < 1/a)$.

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$0 < a < b$	$1/b < 1/a$

Suppose $0 < a < b$
    [proof of $1/b < 1/a$]
So if $0 < a < b$ then $1/b < 1/a$.

If we multiply both sides of the inequality $a < b$ by $1/ab$ we see that $1/a < 1/b$, which is our goal.

Theorem. *Suppose a and b are real numbers. If $0 < a < b$ then $1/b < 1/a$.*

Proof. Suppose $0 < a < b$. Multiplying both sides of the inequality $a < b$ by $1/ab$ we can conclude that $1/b < 1/a$. Therefore, if $0 < a < b$ then $1/b < 1/a$. $\square$

Exercise 3.1.7

Suppose that a is a real number. Prove that if $a^3 > a$ then $a^5 > a$. (Hint: One approach is to start by completing the following equation: $a^5 - a = (a^3 - 1) \cdot \underline{?}$.)
So we want to prove that $(a^3) \to (a^5)$.

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a^3 > a$	a^5

Suppose $a^3 > a$
  [proof of $a^5 > a$]
So if $a^3 > a$ then $a^5 > a$.

If we multiply both side of $a^3 - a > 0$ by $a^2 + 1$ we can conclude that $a^5 - a > 0$ or $a^5 > a$, which was our goal.

Theorem. *Suppose a is a real number. If $a^3 > a$ then $a^5 > a$.*

Proof. Suppose $a^3 > a$, then $a^3 - a > 0$. Multiplying both sides of the inequality $a^3 - a > 0$ by $a^2 + 1$ we can conclude $a^5 - a > 0$. Therefore, if $a^3 > a$ then $a^5 > a$. $\square$

Exercise 3.1.8

Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.
So we want to prove that $(x \notin D) \to (x \in B)$.

The contrapositive of the goal is $\neg(x \in B) \to \neg(x \notin D)$, or in other words $x \notin B \to x \in D$. First we assume the antecedent and make the consequent our goal.

Givens	Goals
$A \setminus B \subseteq C \cap D$	$x \in D$
$x \in A$	
$x \notin B$	

Looking at our givens we can rewrite $A \setminus B \subseteq C \cap D$ as $(x \in A \wedge x \notin B) \to (x \in A \wedge x \in D)$. Looking at our other givens $x \in A$ and $x \notin B$ we can conclude that $x \in C \cap D$ and therefore $x \in D$, which was our goal to prove.

Theorem. *Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. If $x \notin D$ then $x \in B$.*

Proof. We will prove the contrapositive. Suppose $x \notin B$. Since $x \in A$ and $x \notin B$ we can conclude that $x \in C \cap D$ and it follows that $x \in D$. Therefore, if $x \notin D$ then $x \in B$. $\square$

Exercise 3.1.9

Suppose a and b are real numbers. Prove that if $a < b$ then $\frac{a+b}{2} < b$.

So we want to prove that $(a < b) \to (\frac{a+b}{2} < b)$.

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a < b$	$\frac{a+b}{2} < b$

If we add b to both sides of $a < b$ we see that $a + b < b + b$ or $a + b < 2b$. Then if we divide both sides of $a + b < 2b$ by 2 we can conclude that $\frac{a+b}{2} < b$, which was our goal to prove.

Theorem. *Suppose a and b are real numbers. If $a < b$ then $\frac{a+b}{2}$.*

Proof. Suppose $a < b$. Adding b to both sides of the inequality $a < b$ and then dividing both sides by 2, we can conclude that $\frac{a+b}{2} < b$. Therefore, if $a < b$ then $\frac{a+b}{2} < b$. $\square$

Exercise 3.1.10

Suppose x is a real number and $x \neq 0$. Prove that if $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$.

So we want to prove that $\left(\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}\right) \to (x \neq 8)$.

The contrapositive of the goal is $\neg(x \neq 8) \to \neg(\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x})$ or in other words $(x = 8) \to (\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x})$. First we assume the antecedent and make the consequent our goal.

Givens	Goals
$x \neq 0$	$\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$
$x = 8$	

If we evaluate the expression $\frac{\sqrt[3]{x}+5}{x^2+6}$ for $x = 8$ we see that $\frac{\sqrt[3]{8}+5}{8^2+6} = \frac{1}{7}$ and $\frac{1}{7} \neq \frac{1}{8}$, therefore $\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$, which was our goal to prove.

Theorem. *Suppose x is a real number and $x \neq 0$. If $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$*

Proof. We will prove the contrapositive. Suppose $x = 8$. Then evaluating the equation $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ for $x = 8$ we see that $\frac{1}{7} \neq \frac{1}{8}$ and therefore if $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$. $\square$