3.5.1

Suppose A, B, and C are sets.

Theorem. $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

Proof. Let x be arbitrary and suppose $x \in A \cap (B \cup C)$. Thus $x \in A$ and $x \in B$ or $x \in C$. If $x \in C$ then $x \in (A \cap B) \cup C$. In the case where $x \in B$ it follows that $x \in A \cap B$ and therefore $x \in (A \cap B) \cup C$. Since x was arbitrary we can conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.2

Suppose A, B, and C are sets.

Theorem. $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

Proof. Let x be arbitrary and suppose $x \in (A \cup B) \setminus C$. Thus $x \notin C$ and $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \setminus C)$. If $x \in B$ then if follows that $x \in B \setminus C$ and therefore $x \in A \cup (B \setminus C)$. Since x was arbitrary we can conclude $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.3

Suppose A and B are sets.

Theorem. $A \setminus (A \setminus B) = A \cap B$

Proof. Let x be arbitrary and suppose $x \in A \setminus (A \setminus B)$. Then

$$\begin{aligned} x \in A \setminus (A \setminus B) &\text{ iff } x \in A \wedge x \notin A \setminus B \\ &\text{ iff } x \in A \wedge \neg(x \in A \wedge x \notin B) \\ &\text{ iff } x \in A \wedge (x \notin A \vee x \in B) \\ &\text{ iff } (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \\ &\text{ iff } x \in A \wedge x \in B \\ &\text{ iff } x \in (A \cap B) \end{aligned}$$

□

3.5.4

Theorem. *If $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ then $A \subseteq B$.*

Proof. Suppose $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Let x be arbitrary and suppose $x \in A$. Thus $x \in A \cup C$ and it follows that $x \in B \cup C$. Now if $x \in B \cup C$ then either $x \in B$ or $x \in C$. If $x \in B$ then since x was arbitrary we can conclude $A \subseteq B$. In the case that $x \in C$, then $x \in A \cap C$ and it follows that $x \in B \cap C$. Therefore $x \in C$ and $x \in B$. Thus, if $x \in A$ then $x \in B$ and since x was arbitrary we can conclude $A \subseteq B$. □

3.5.5

Suppose A and B are sets.

Theorem. *If $A \triangle B \subseteq A$ then $B \subseteq A$.*

Proof. Suppose $A \triangle B \subseteq A$. We will prove by contradiction. Let x be arbitrary and suppose $x \in B$ and $x \notin A$. Since $x \in B$ and $x \notin A$ then $x \in A \triangle B$. Since $A \triangle B \subseteq A$, then $x \in A$. But this contradicts $x \notin A$. Therefore, if $x \in B$ then $x \in A$ and since x was arbitrary we can conclude that $B \subseteq A$. $\square$

3.5.6

Suppose A, B, and C are sets.

Theorem. $A \cup C \subseteq B \cup C \iff A \setminus C \subseteq B \setminus C$.

Proof. $(\rightarrow)$ Suppose A, B, and C are sets. Suppose $(A \cup C) \subseteq (B \cup C)$. Let x be arbitrary and suppose $c \in A \setminus C$, which means $x \in A$ and $x \notin C$. Since $x \in A$, then $x \in A \cup C$ and therefore $x \in B \cup C$. This means $x \in B$ or $x \in C$ and since $x \notin C$, it must be that $x \in B$. Now since $x \in B$ and $x \notin C$ then $x \in B \setminus C$. Therefore, if $x \in A \setminus C$ then $x \in B \setminus C$ and since x was arbitrary we can conclude if $A \cup C \subseteq B \cup C$ then $A \setminus C \subseteq B \setminus C$.

$(\leftarrow)$ Now suppose $A \setminus C \subseteq B \setminus C$. Let x be arbitrary and suppose $x \in A \cup C$, which means $x \in A$ or $x \in C$. If $x \in C$ then $x \in B \cup C$ and since x was arbitrary then $A \cup C \subseteq B \cup C$. In the case that $x \in A$, since $A \setminus C \subseteq B \setminus C$ then $x \in B$. Therefore, $x \in B \cup C$ and since x was arbitrary then $A \cup C \subseteq B \cup C$. $\square$

3.5.7

Theorem. *For any sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$*

Proof. Let A and B be arbitrary sets. Let M be arbitrary and suppose $M \in \mathscr{P}(A) \cup \mathscr{P}(B)$. Thus $M \in \mathscr{P}(A)$ or $M \in \mathscr{P}(B)$, which means $M \subseteq A$ or $M \subseteq B$. In the case where $M \subseteq A$, let x be an arbitrary member of M and it follows that $x \in A$. Since $x \in A$ then $x \in A \cup B$ and because x was arbitrary we can conclude $M \subseteq A \cup B$ and therefore $M \in \mathscr{P}(A \cup B)$. In the case where $M \subseteq B$, let x be an arbitrary member of M and it follows that $x \in B$. Since $x \in B$ then $x \in A \cup B$ and because x was arbitrary we can conclude $M \subseteq A \cup B$ and therefore $M \in \mathscr{P}(A \cup B)$. $\square$

3.5.8

Theorem. *For any sets A and B, if $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.*

Proof. We will prove the contrapositive. Since we proved that $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$ in exercise 3.5.7, we must show that $\mathscr{P}(A \cup B) \nsubseteq \mathscr{P}(A) \cup \mathscr{P}(B)$ to prove our goal that $\mathscr{P}(A) \cup \mathscr{P}(B) \neq \mathscr{P}(A \cup B)$. Let A and B be arbitrary sets and suppose $A \nsubseteq B$ and $B \nsubseteq A$. This means there is an element $x \in A \setminus B$ and an element $y \in B \setminus A$. Since $x \in A$ and $y \in B$ then both x and y are in $A \cup B$ and therefore the set $\{x, y\}$ is in $\mathscr{P}(A \cup B)$ but not in $\mathscr{P}(A)$ or $\mathscr{P}(B)$. Thus $\mathscr{P}(A \cup B) \nsubseteq \mathscr{P}(A) \cup \mathscr{P}(B)$. □

3.5.9

Theorem. *Suppose x and y are real numbers and $x \neq 0$. Then $y + 1/x = 1 + y/x$ iff either $x = 1$ or $y = 1$.*

Proof. ($\rightarrow$) Suppose that $y + 1/x = 1 + y/x$. Now if $y = 1$ then we have proven our goal. So now assume $y \neq 1$ and $y + 1/x = 1 + y/x$, then it follows that $x = 1$.

($\leftarrow$) Now suppose $x = 1$ or $y = 1$. In the case that $x = 1$ we have

$$y + \frac{1}{x} = y + \frac{1}{1} = y + 1 = 1 + \frac{y}{1} = 1 + \frac{y}{x}$$

In the case that $y = 1$ we have

$$y + \frac{1}{x} = 1 + \frac{1}{x} = 1 + \frac{y}{x}$$

□

3.5.10

Theorem. *For every real number x, if $|x - 3| > 3$ then $x^2 > 6x$.*

Proof. Suppose that x is an arbitrary real number and that $|x - 3| > 3$. Then either $x - 3 \geq 0$ or $x - 3 < 0$. In the case that $x - 3 \geq 0$, then $|x - 3| = x - 3$ and therefore $|x - 3| > 3 = x - 3 > 3$. Solving for x, we have $x > 6$ and then multiplying both sides by x we have $x^2 > 6x$. In the case that $x - 3 < 0$, then $|x - 3| = 3 - x$ and therefore $3 - x > 3$. Solving for x we have $x < 0$. Multiplying both sides of $x < 0$ by $6 - x$ we have $6x - x^2 < 0$ and therefore $x^2 > 6x$. □

3.5.11

Theorem. *For every real number x, $|2x - 6| > x$ iff $|x - 4| > 2$.*

Proof. ($\rightarrow$) Let x be an arbitrary real number and suppose $|2x - 6| > x$. Our goal $|x - 4| > 2$ means that either $x - 4 > 2$ or $4 - x > 2$. Since $|2x - 6| > 2$ then either $2x - 6 > x$ or $6 - 2x > x$. If $2x - 6 > x$ then it follows that $x - 4 > 2$. Now if $6 - 2x > x$ then if follows that $4 - x > 2$.

($\leftarrow$) Now suppose $|x-4|>2$. Our goal $|2x-6|>x$ means that either $2x-6>x$ or $6-2x>x$. Since $|x-4|>2$ then either $x-4>2$ or $4-x>2$. If $x-4>2$ then it follows that $2x-6>x$. In the case that $4-x>2$ then it follows that $6-2x>x$. $\square$

3.5.12

Theorem. *For all real numbers a and b, $|a|\leq b$ if and only if $-b\leq a\leq b$.*

Proof. ($\rightarrow$) Suppose a and b are arbitrary real numbers and that $|a|\leq b$. There are two cases to consider: $a\geq 0$ and $a<0$. If $a\geq 0$ then $|a|=a\leq b$. It follows that $-b\leq -a$ and since $a\geq 0$ then $-a\leq a$. Therefore, $-b\leq -a\leq a\leq b$ and $-b\leq a\leq b$. Now in the case that $a<0$ then $|a|=-a\leq b$. It follows that $-b\leq a$ and since $a<0$ then $-a>a$ or $a<-a$. Therefore $-b\leq a<-a\leq b$ and $-b\leq a\leq b$.

($\leftarrow$) Now suppose $-b\leq a\leq b$ and therefore $a\leq b$. Now we must prove that $-a\leq b$ to complete the proof. If we subtract a from both sides of $-b\leq a$ and add b to both sides we have $-a\leq b$. $\square$

3.5.13

Theorem. *For every integer x, x^2+x is even.*

Proof. Let x be an arbitrary integer. There are two cases to consider: x is even or x is odd. If x is even then there exists an integer k such that $x=2k$. Plugging in $2k$ for x in x^2+x we have $x^2+x=(2k)^2+2k=4k^2+2k=2(2k^2+k)$. Since $2k^2+k$ is an integer then x^2+x is even. In the case that x is odd there is a j such that $x=2j+1$. Plugging in $2j+1$ for x in x^2+x we have $x^2+x=(2j+1)^2+(2j+1)=(4j^2+4j+1)+(2j+1)=4j^2+6j+2=2(2j^2+3j+1)$. Since $2j^2+3j+1$ is an integer, x^2+x is even. $\square$

3.5.14

Theorem. *For every integer x, the remainder when x^4 is divided by 8 is either 0 or 1.*

Proof. Suppose x is an integer and there exists an integer k such that $8k=x^4$. Since x is an integer, x is either even or odd. If x is even then there exists an integer m such that $x=2m$. Then $8k=(2m)^4=16m^4$ and $k=2m^4$ r 0. In the case that x is odd, then there exists an integer m such that $x=2m+1$. Then $8k=(2m+1)^4=16x^4+32x^3+24x^2+8x+1$ and $k=2x^4+4x^3+3x^2+x$ r 1. Therefore, when x^4 is divided by 8 the remainder is either 0 or 1. $\square$

3.5.15

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets.

Theorem. $\cup(\mathcal{F} \cup \mathcal{G}) = (\cup\mathcal{F}) \cup (\cup\mathcal{G})$

Proof. ($\rightarrow$) Suppose $x \in \cup(\mathcal{F} \cup \mathcal{G})$, which means there is a set in $\mathcal{F} \cup \mathcal{G}$ that contains x. Thus the set that contains x is in $\mathcal{F}$ or $\mathcal{G}$. If the set that contains x is in $\mathcal{F}$ then $x \in \cup\mathcal{F}$ and $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$. In the case that the set that contains x is in $\mathcal{G}$, then $x \in \cup\mathcal{G}$ and $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$

($\leftarrow$) Now suppose $x \in (\cup F) \cup (\cup G)$, which means there is a set in $\mathcal{F}$ that contains x or a set in $\mathcal{G}$ that contains x. If there is a set in $\mathcal{F}$ that contains x, and this same set is in $\mathcal{F} \cup \mathcal{G}$. Thus there is a set in $\mathcal{F} \cup \mathcal{G}$ that contains x. In the case that there is a set in $\mathcal{G}$ that contains x, then this set is in $\mathcal{F} \cup \mathcal{G}$. Thus there is a set in $\mathcal{F} \cup \mathcal{G}$ that contains x. Therefore $x \in \cup(\mathcal{F} \cup \mathcal{G})$.

□

Alternate proof?

Proof.

$$\begin{aligned} x \in \cup(\mathcal{F} \cup \mathcal{G}) &\text{ iff} \\ \exists M \in \mathcal{F} \cup \mathcal{G}(x \in M) &\text{ iff} \\ \exists M \in \mathcal{F}(x \in M) \vee \exists M \in \mathcal{G}(x \in M) &\text{ iff} \\ x \in \cup\mathcal{F} \vee x \in \cup\mathcal{G} &\text{ iff} \\ x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G}) \end{aligned}$$

□

3.5.16

Suppose $\mathcal{F}$ is a nonempty family of sets and B is a set.

A

Theorem. $B \cup (\cup\mathcal{F}) \subseteq \cup(\mathcal{F} \cup \{B\})$

Proof. ($\rightarrow$) Suppose x is arbitrary and $x \in B \cup (\cup\mathcal{F})$. Then $x \in B$ or $x \in \cup\mathcal{F}$. If $x \in B$ then because $B \in \mathcal{F} \cup \{B\}$, it follows that $x \in \cup(\mathcal{F} \cup \{B\})$. In the case that $x \in \cup\mathcal{F}$, there is a set $M \in \mathcal{F}$ such that $x \in M$. Since $M \in \mathcal{F}$ then $M \in \mathcal{F} \cup \{B\}$ and therefore $x \in \cup(\mathcal{F} \cup \{B\})$.

($\leftarrow$) Now suppose $x \in \cup(\mathcal{F} \cup \{B\})$. Then there is a set M such that $x \in M$ and $M \in (\mathcal{F} \cup \{B\})$, which means $M \in \mathcal{F}$ or $M \in \{B\}$. If $M \in \mathcal{F}$ then it follows that $x \in \cup\mathcal{F}$ and thus $x \in B \cup (\cup\mathcal{F})$. In the case that $M \in \{B\}$ then it follows that $x \in B$ and thus $x \in B \cup (\cup\mathcal{F})$. □

B

Theorem. $B \cup (\cap\mathcal{F}) = \bigcap_{A\in\mathcal{F}}(B \cup A)$

Proof. ($\rightarrow$) Let x be arbitrary and suppose $x \in B \cup (\cap\mathcal{F})$. Then $x \in B$ or $x \in \cap\mathcal{F}$. If $x \in B$, then $x \in B \cup A$ for any set A and thus $x \in \bigcap_{A\in\mathcal{F}}(B \cup A)$. In the case that $x \in \cap\mathcal{F}$, then x is in every set $A \in \mathcal{F}$ and so $x \in \bigcap_{A\in\mathcal{F}} A$. Therefore $x \in \bigcap_{A\in\mathcal{F}}(B \cup A)$. Since x was arbitrary then $B \cup (\cap\mathcal{F}) \subseteq \bigcap_{A\in\mathcal{F}}(B \cup A)$.

($\leftarrow$) Now suppose $x \in \bigcap_{A\in\mathcal{F}}(B \cup A)$. Thus $x \in B$ or $x \in A$ for all $A \in \mathcal{F}$. If $x \in B$ then $x \in B \cup (\cap\mathcal{F})$. If $x \in A$ for all $A \in \mathcal{F}$ then $x \in \cap\mathcal{F}$ and therefore $x \in B \cup (\cap\mathcal{F})$. Since x was arbitrary then $\bigcap_{A\in\mathcal{F}}(B \cup A) \subseteq B \cup (\cap\mathcal{F})$. $\square$

C

Theorem. $B \cap (\cap\mathcal{F}) = \bigcap_{A\in\mathcal{F}}(B \cap A)$

Proof. ($\rightarrow$) Let x be arbitrary and suppose $x \in B \cap (\cap\mathcal{F})$, which means $x \in B$ and for all $A \in \mathcal{F}$, $x \in A$. Thus $x \in A \cap B$ and since $x \in A$ for all $A \in \mathcal{F}$, then $x \in \bigcap_{A\in\mathcal{F}}(A \cap B)$. Since x was arbitrary, we conclude $B \cap (\cap\mathcal{F}) \subseteq \bigcap_{A\in\mathcal{F}}(B \cap A)$.

($\leftarrow$) Now suppose $x \in \bigcap_{A\in\mathcal{F}}(A \cap B)$, which means for all $A \in \mathcal{F}$, $x \in A \cap B$. Therefore $xinB$ and for all $A \in \mathcal{F}$, $x \in A$ and thus $x \in \cap\mathcal{F}$. Since x was arbitrary we conclude $\bigcap_{A\in\mathcal{F}}(B \cap A) \subseteq B \cap (\cap\mathcal{F})$. $\square$