3.6.2

Theorem. *There is a unique* $x \in \mathbb{R}$ *such that for all* $y \in \mathbb{R}$, $xy + x - 4 = 4y$.

Proof. First we prove existence. Let $x = 4$ and suppose y is an arbitrary real number. Then we have $4y + 4 - 4 = 4y + 0 = 4y$, as desired. To prove uniqueness suppose a and b are arbitrary real numbers and that $ay + a - 4 = 4y$ and that $by + b - 4 = 4b$. For $ay + a - 4 = 4y$, let $y = b$ and we have $ab + a - 4 = 4b$. For $by + b - 4 = 4y$, let $y = a$ and we have $ba + b - 4 = 4a$. Now subtracting both sides of $ab + a - 4 = 4b$ from $ba + b - 4 = 4a$ we have

$$\begin{aligned}
ba + b - 4 - (ab + a - 4) &= 4a - 4b \\
ba + b - 4 - ab - a + 4 &= 4a - 4b \\
b - a &= 4a - 4b \\
b + 4b &= 4a + a \\
5b &= 5a \\
b &= a
\end{aligned}$$

Therefore, if $ay + a - 4 = 4y$ and $by + b - 4 = 4y$, then $a = b$.

□

3.6.3

Theorem. $\forall x \in \mathbb{R}[(x \neq 0 \wedge x \neq 1) \implies \exists! y \in \mathbb{R}(y/x = y - x)]$

Proof. Suppose x is an arbitrary real number, $x \neq 0$, and $x \neq 1$. Let $y = x^2/(x-1)$, which is defined because $x \neq 1$. Then

$$\begin{aligned}
\frac{y}{x} = \frac{\frac{x^2}{x-1}}{x} = \frac{x^2}{x-1} \cdot \frac{1}{x} = \frac{x^2}{x(x-1)} &= \frac{x}{x-1} \\
&= \frac{x^2 - x^2 + x}{x-1} \\
&= \frac{x^2 - x(x-1)}{x-1} \\
&= \frac{x^2}{x-1} - \frac{x(x-1)}{x-1} = \frac{x^2}{x-1} - x = y - x
\end{aligned}$$

□

3.6.4

Theorem. $\forall x \in \mathbb{R}(x \neq 0 \implies \exists! y \in \mathbb{R} \forall z \in \mathbb{R}(zy = z/x))$.

Proof. Let x be an arbitrary real number. To prove existence, suppose $x \neq 0$ and $y = 1/x$. Then $zy = z(1/x) = z/x$. To prove uniqueness let a and b be arbitrary real numbers and suppose $za = z/x$ and $zb = z/x$. For $za = z/x$ let $z = b$ and we have $ba = bx$, which can be rearranged as $xba = b$. For $zb = z/x$ let $z = a$ and we have $ab = a/x$, which can be rearranged as $xab = a$. Subtracting both sides of $xab = b$ from $xab = a$ we have $xab - xab = a - b$ and so $a - b = 0$ or $a = b$. □

3.6.5

If $\mathcal{F}$ is a family of sets, then $\cup\mathcal{F} = \{x | \exists A(A \in \mathcal{F} \wedge x \in A)\}$. Define a new set $\cup!\mathcal{F}$ by the formula $\cup!\mathcal{F} = \{x | \exists! A(A \in \mathcal{F} \wedge x \in A)\}$.

(a)

Theorem. $\forall \mathcal{F}(\cup!\mathcal{F} \subseteq \cup F)$

Proof. Suppose $\mathcal{F}$ is and arbitrary family of sets. Let x be arbitrary and suppose $x \in \cup!\mathcal{F}$. This means $\exists! A \in \mathcal{F}(x \in A)$. Since $A \in \mathcal{F}$ and $x \in A$ then we can concluded that $x \in \cup\mathcal{F}$. Since x was arbitrary then $\cup!\mathcal{F} \subseteq \cup\mathcal{F}$, and since $\mathcal{F}$ was arbitrary we can conclude for all $\mathcal{F}$, $\cup!\mathcal{F} \subseteq \cup\mathcal{F}$. □

(b)

Theorem. $\forall \mathcal{F}(\cup!\mathcal{F} = \cup\mathcal{F}$ *iff* $\mathcal{F}$ *is pairwise disjoint*).

Note that pairwise disjoint means that
$\forall A \in \mathcal{F} \forall B \in \mathcal{F}(A \neq B \implies A \cap B = \varnothing)$.

Proof. Let $\mathcal{F}$ be an arbitrary family of sets.
($\rightarrow$) Suppose $\cup!\mathcal{F} = \cup\mathcal{F}$. We will prove the contrapositive. Let A and B be arbitrary, suppose $A \in \mathcal{F}$, $B \in \mathcal{F}$, and A and B are not disjoint. Then there is an element x such that $x \in A \cap B$. Since $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $x \in \cup\mathcal{F}$ and it follows by assumption that $x \in \cup!\mathcal{F}$. Since $x \in \cup!\mathcal{F}$ then there is a unique set $X \in \mathcal{F}$ such that $x \in X$ and so $x \in A = X$ and $x \in B = X$. Therefore $A = B$.

($\leftarrow$) Now suppose that $\mathcal{F}$ is pairwise disjoint. We need to show that $\cup!\mathcal{F} = \cup\mathcal{F}$, which means $\cup!\mathcal{F} \subseteq \cup F$ and $\cup\mathcal{F} \subseteq \cup!\mathcal{F}$.

To see that $\cup!\mathcal{F} \subseteq \cup\mathcal{F}$, let y be arbitrary and suppose $y \in \cup!\mathcal{F}$. Then there is a unique set $Y \in \mathcal{F}$ such that $y \in Y$ and therefore $y \in \cup\mathcal{F}$. Since y was arbitrary, this shows that $\cup!\mathcal{F} \subseteq \cup\mathcal{F}$.

To see that $\cup\mathcal{F} \subseteq \cup!\mathcal{F}$, now suppose $y \in \cup\mathcal{F}$. Then there is a set $Y \in \mathcal{F}$ such that $y \in Y$. To see that Y is unique, suppose there is another set $Z \in \mathcal{F}$ such that $y \in Z$. By assumption, $\mathcal{F}$ is pairwise disjoint and since $y \in Y$ and $y \in Z$, it follows that $Y = Z$. Thus there is a unique set $Y \in \mathcal{F}$ such that $yinY$ and therefore $y \in \cup!\mathcal{F}$. Since y was arbitrary this shows that $\cup\mathcal{F} \subseteq \cup!\mathcal{F}$.

We have shown that $\cup!\mathcal{F} \subseteq \cup F$ and $\cup\mathcal{F} \subseteq \cup!\mathcal{F}$ and therefore $\cup!\mathcal{F} = \cup\mathcal{F}$. □

3.6.6

Let U be any set.

(a)

Theorem. *There is a unique $A \in \mathscr{P}(U)$ such that for every $B \in \mathscr{P}(U)$, $A \cup B = B$.*

Proof. First we prove existence. Let $A = \varnothing$. Let B be arbitrary and suppose $B \in \mathscr{P}(U)$. To see that $A \cup B \subseteq B$, let x be arbitrary and suppose $x \in A \cup B$. Since $A = \varnothing$ then $x \notin A$ and therefore it must be that $x \in B$. Since x was arbitrary then $A \cup B \subseteq B$. Now to see that $B \subseteq A \cup B$, suppose $x \in B$. It follows that $x \in A \cup B$ and therefore $B \subseteq A \cup B$. Since $A \cup B \subseteq B$ and $B \subseteq A \cup B$, then $A \cup B = B$. Since B was arbitrary this shows that there exists an $A \in \mathscr{P}(U)$ such that for every $B \in \mathscr{P}(U), A \cup B = B$.

Too prove that A is unique, suppose C is arbitrary and for all $B \in \mathscr{P}(U)$, $C \cup B = B$. In particular, let $B = \varnothing$, then $C \cup \varnothing = \varnothing$. But we also have that $C \cup \varnothing = C$ and therefore $C = \varnothing = A$. □