3.4.1

Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

We want to prove $\forall x(P(x) \wedge Q(x) \iff \forall x P(x) \wedge \forall x Q(x))$.

Theorem. *The statement $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.*

Proof. ($\rightarrow$) Suppose $\forall x(P(x) \wedge Q(x))$. Let y be arbitrary. Since $\forall x(P(x) \wedge Q(x))$ it follows $P(y)$ and $Q(y)$. Since y was arbitrary, we can conclude $\forall x P(x)$ and $\forall x Q(x)$ or $\forall x P(x) \wedge \forall x Q(x)$.

($\leftarrow$) Let y be arbitrary. Since $\forall x P(x)$ and $\forall x Q(x)$ then it follows $P(y)$ and $Q(y)$. Since y was arbitrary we can conclude $\forall x(P(x) \wedge Q(x))$. $\square$

3.4.2

Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Theorem. *If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B$ then $x \in B$ and since $A \subseteq C$ then $x \in C$ or $x \in B \cap C$. Therefore, if $x \in A$ then $x \in B \cap C$ and since x was arbitrary we can conclude $A \subseteq B \cap C$. $\square$

3.4.3

Suppose $A \subseteq B$. Prove that for every set C, $C \setminus B \subseteq C \setminus A$.

Theorem. *Suppose $A \subseteq B$, then for every set C, $C \setminus B \subseteq C \setminus A$.*

Proof. Suppose $A \subseteq B$ and C is an arbitrary set. Let x be arbitrary and suppose $x \in C \setminus B$, which means $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$, then $x \notin A$, which means that $x \in C \setminus A$. Therefore, if $x \in C \setminus B$ then $x \in C \setminus A$ and since x and C were arbitrary, we can conclude $\forall C(C \setminus B \subseteq C \setminus A)$. $\square$

3.4.5

Prove that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \not\subseteq C$.

Theorem. *If $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \not\subseteq C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B \setminus C$ then $x \in B$ and $x \notin C$. Since x was arbitrary we can conclude $B \not\subseteq C$. $\square$

3.4.6

Prove that for any sets A, B, and C, $A\backslash(B\cap C) = (A\backslash B)\cup(A\backslash C)$ finding a string of equivalences starting with $x \in A\backslash(B\cap C)$ and ending with $x \in (A\backslash B)\cup(A\backslash C)$.

Theorem. *for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.*

Proof. Suppose A, B, and C are arbitrary sets. Then

$$\begin{aligned}
x \in A \setminus (B \cap C) &\text{ iff } x \in A \to (x \notin B \wedge x \notin C) \\
&\text{ iff } x \notin A \vee (x \notin B \wedge x \notin C) \\
&\text{ iff } (x \notin A \vee x \notin B) \wedge (x \notin A \vee x \notin C) \\
&\text{ iff } (x \in A \to x \notin B) \vee (x \in A \to x \notin C) \\
&\text{ iff } x \in A \setminus B \vee x \in A \setminus C \\
&\text{ iff } x \in (A \setminus B) \cup (A \setminus C)
\end{aligned}$$

□

3.4.7

Theorem. *For any sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.*

Proof. ($\to$) Let M be an arbitrary set and suppose $M \in \mathscr{P}(A \cap B)$. Then $M \subseteq A \cap B$. Let x be arbitrary and suppose $x \in M$. Since $M \subseteq A \cap B$, $x \in A \cap B$ and therefore $x \in A$. Since x was arbitrary, $M \subseteq A$ and therefore $M \in \mathscr{P}(A)$. Similarly, since $M \subseteq A \cap B$, $x \in B$. Since x was arbitrary, $M \subseteq B$ and therefore $M \in \mathscr{P}(B)$. Therefore, $M \in \mathscr{P}(A)$ and $M \in \mathscr{P}(B)$.

($\leftarrow$) Now suppose $M \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then $M \subseteq A$ and $M \subseteq B$. Suppose $x \in M$. Since $M \subseteq A$ and $M \subseteq B$ then $x \in A \cap B$. Since x was arbitrary, $M \subseteq A \cap B$ and therefore $M \in \mathscr{P}(A \cap B)$. □

3.4.8

Theorem. $A \subseteq B \iff \mathscr{P}(A) \subseteq \mathscr{P}(B)$

Proof. ($\to$) Suppose $A \subseteq B$. Let M be an arbitrary set and suppose $M \in \mathscr{P}(A)$. Then $M \subseteq A$. Now let y be arbitrary and suppose $y \in M$. Since $M \subseteq A$ then $y \in A$, and since $A \subseteq B$ then $y \in B$. Since y was arbitrary, $M \subseteq B$ and therefore $M \in \mathscr{P}(B)$. Since M was arbitrary, $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

($\leftarrow$) Now suppose $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ and $y \in A$. Then the set $\{y\}$ is in $\mathscr{P}(A)$. Since $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ then $\{y\} \in \mathscr{P}(B)$ and $y \in B$. Since y was arbitrary, $A \subseteq B$. □

3.4.9

Theorem. *If x and y are odd integers, then xy is odd.*

Proof. Suppose x and y are odd integers. This means there is an integer k such that $x = 2k + 1$ and there is an integer j such that $y = 2j + 1$. Therefore, $xy = 2(2kj + k + j) = 4kj + 2k + 2j + 1 = (2k+1)(2j+1)$, and since $2kj + k + j$ is an integer, then xy is odd. $\square$

3.4.10

Theorem. *For every integer n, n^3 is even iff n is even.*

Proof. ($\rightarrow$) Let n be arbitrary. We will prove the contrapositive. Suppose x is odd, which means there exists an integer k such that $x = 2k + 1$. Therefore, $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Since $4k^3 + 6k^2 + 3k$ is an integer, n^3 is odd. Therefore, if n^3 is even, n is even.

($\leftarrow$) Now suppose n is even, which means there exists an integer m such that $n = 2m$. Now $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$ and since $4m^3$ is an integer, n^3 is even. $\square$

3.4.11

A

The problem is with using the same variable k for defining m as an even integer and n as an odd integer when k may take on different values for n and m.

B

Let $m = 2$ and $n = -3$. Then $n^2 - m^2 = (-3)^2 - 2^2 = 9 - 4 = 5$ and $n + m = -3 + 2 = -1$. Therefore $n^2 - m^2 \neq n + m$.

3.4.12

Theorem. $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x + y = xy) \iff x \neq 1]$

Proof. ($\rightarrow$) We will prove by contradiction. Suppose x is an arbitrary real number and there exists a real number y such that $x + y = xy$. Now suppose $x = 1$. Since $x + y = xy$, then $y = \frac{x}{x-1}$. But this contradicts $x = 1$ because there is no real number y such that $y = x/0$.

($\leftarrow$) Now suppose $x \neq 1$ and $y = \frac{x}{x-1}$. Then

$$
\begin{aligned}
x + y = x + \frac{x}{x+1} &= \frac{x(x-1)+x}{x-1} \\
&= \frac{x^2 - x + x}{x-1} \\
&= \frac{x^2}{x-1} = xy
\end{aligned}
$$

□

3.4.13

Theorem. $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = \frac{y}{x}) \iff x \neq z]$

Proof. ($\rightarrow$) Let $z = 1$. Let x be an arbitrary real number and suppose $x > 0$. Suppose $y \in \mathbb{R}$ and $y - x = \frac{y}{x}$. Then $y = \frac{x^2}{x-1}$. Now suppose $x = 1$. This contradicts $y \in \mathbb{R}$ and $y = \frac{x^2}{x-1}$. Therefore, $x \neq z$ and since x was arbitrary we can conclude $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = \frac{y}{x}) \rightarrow x \neq z]$.

($\leftarrow$) Now suppose $x \neq 1$ and $y = \frac{x^2}{x-1}$. Then

$$
\begin{aligned}
y - x = \frac{x^2}{x-1} - x &= \frac{x^2 - x(x-1)}{x-1} \\
&= \frac{x^2 - x + 2 + x}{x-1} = \frac{x}{x-1} = \frac{y}{x}
\end{aligned}
$$

□

3.4.14

Theorem. *If B is a set and $\mathcal{F}$ is a family of sets, then $\cup\{A \setminus B | A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathscr{P}(B))$.*

Proof. Let x be arbitrary and suppose $x \in \cup\{A \setminus B | A \in \mathcal{F}\}$. This means that there is a set $A \in \mathcal{F}$ such that $x \in A$ and also $x \notin B$. Since $x \in A$ and $x \notin B$, then $A \not\subseteq B$ and $A \notin \mathscr{P}(B)$. Thus there is a set $A \in \mathcal{F}$ such that $x \in A$, and $A \notin \mathscr{P}(B)$, which means that $x \in \cup(\mathcal{F} \setminus \mathscr{P}(B))$. Therefore, if $x \in \cup\{A \setminus B | A \in \mathcal{F}\}$ then $x \in \cup(\mathcal{F} \setminus \mathscr{P}(B))$ and since x was arbitrary, we can conclude $\cup\{A \setminus B | A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathscr{P}(B))$. □

3.4.15

Theorem. *If $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$, then $\cup\mathcal{F}$ and $\cap\mathcal{G}$ are disjoint.*

Proof. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$. We will use proof by contradiction. Now suppose $\cup\mathcal{F}$ and $\cap\mathcal{G}$ are not disjoint. Then there exists a y such that $y \in \cup\mathcal{F}$ and $y \in \cap\mathcal{G}$. Since $y \in \cup\mathcal{F}$ there is a set in $\mathcal{F}$ that contains y and since $y \in \cap\mathcal{G}$, y is in every set in $\mathcal{G}$. But because every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$, then there is at least one set in $\mathcal{G}$ that does not contain y. But this contradicts $y \in \cap\mathcal{G}$. Therefore, $(\cup\mathcal{F}) \cap (\cap\mathcal{G}) = \varnothing$. $\square$

3.4.16

Theorem. *For any set A, $A = \cup\mathscr{P}(A)$.*

Proof. $(\rightarrow)$ Suppose A is an arbitrary set, x is arbitrary, and $x \in A$. Then there is subset of A that contains x and, by definition, this subset is in $\mathscr{P}(A)$. Therefore, $x \in \cup\mathscr{P}(A)$. Since x was arbitrary $A \subseteq \mathscr{P}(A)$.

$(\leftarrow)$ Now suppose $x \in \cup\mathscr{P}(A)$. This means there is a subset of A that contains x and therefore $x \in A$. Since x was arbitrary we conclude $\cup\mathscr{P}(A) \subseteq A$. Since A was arbitrary, we can conclude for all sets A, $A = \cup\mathscr{P}(A)$. $\square$

3.4.17

A

Theorem. $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$

Proof. Let x be arbitrary and suppose $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Since $x \in \cup(\mathcal{F} \cap \mathcal{G})$ there is a set in $\mathcal{F}$ and in $\mathcal{G}$ that both contain x. Since there is a set in $\mathcal{F}$ than contains x, then $x \in \cup\mathcal{F}$ and since there is a set in $\mathcal{G}$ that contains x, $x \in \cup\mathcal{G}$. Therefore, $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. Since x was arbitrary, we can conclude $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. $\square$

B

The mistake is that we can't choose a set A such that $A \in \mathcal{F}$ and $A \in \mathcal{G}$ and $x \in A$. The given $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ means that x is within a set in $\mathcal{F}$ and within a set in $\mathcal{G}$, but these two sets are not necessarily the same set.

C

Let $\mathcal{F} = \{\{1, 2\}, \{3\}\}$ and $\mathcal{G} = \{\{4, 5\}, \{1\}\}$. Then $\cup(\mathcal{F} \cap \mathcal{G}) = \varnothing$, but $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \{1\}$.

3.4.18

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets, then* $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G}) \iff \forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$.

Proof. ($\rightarrow$) Suppose $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$. Suppose A is an arbitrary set in $\mathcal{F}$, B is an arbitrary set in $\mathcal{G}$, x is arbitrary, and $x \in A \cap B$. Since $x \in A \cap B$ and A is an arbitrary set in $\mathcal{F}$, then $x \in \cup F$. Also, since $x \in A \cap B$ and B is an arbitrary set in $\mathcal{G}$, then $x \in \cup\mathcal{G}$. Therefore $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ and since $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$, it follows that $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Therefore, if $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G}) \rightarrow x \in \cup(\mathcal{F} \cap \mathcal{G})$ and since x, A, and B were arbitrary we can conclude that $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$.

($\leftarrow$) Now suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ and $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. Since $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$, then there is a set $M \in F$ such that $x \in M$ and there is a set $N \in \mathcal{G}$ such that $x \in N$ and it follows that $x \in M \cap \mathcal{G}$. Then since $M \in \mathcal{F}$, $N \in \mathcal{G}$, $x \in M \cap \mathcal{G}$, and $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ we can conclude that $x \in \cup(\mathcal{F} \cap \mathcal{G}))$. Therefore if $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ then $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$.

$\square$

3.4.19

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Then $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint.*

Proof. ($\rightarrow$) Suppose $\cup\mathcal{F} \cap \cup\mathcal{G} = \varnothing$. We will prove by contradiction. Let A be an arbitrary set in $\mathcal{F}$ and B be an arbitrary set in $\mathcal{G}$. Suppose $x \in A \cap B$, which means $x \in A$, $x \in B$, and $A \cap B \neq \varnothing$. Since $x \in A$ and $A \in \mathcal{F}$ then $x \in \cup\mathcal{F}$ and since $x \in B$ and $B \in \mathcal{G}$ then $x \in \cup\mathcal{G}$. Therefore $x \in \cup\mathcal{F} \cap \cup\mathcal{G}$, but this contradicts $\cup\mathcal{F} \cap \cup\mathcal{G} = \varnothing$. Therefore $A \cap B = \varnothing$ and since A and B were arbitrary we can conclude $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B = \varnothing)$.

($\leftarrow$) Now suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B = \varnothing)$. We will again prove by contradiction. Suppose $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are not disjoint, which means there is an element x that is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$. This means that there is a set in $\mathcal{F}$ that contains x and there is a set in $\mathcal{G}$ that contains x. However, this contradicts our given that every set in $\mathcal{F}$ is disjoint from every set in $\mathcal{G}$. Therefore $\cup\mathcal{F} \cap \cup\mathcal{G} = \varnothing$. $\square$

3.4.20

Theorem. $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$ *iff* $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G}(A \cap B = \varnothing)$

Proof. ($\rightarrow$) Suppose $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. Let A be an arbitrary set in $(\mathcal{F} \setminus \mathcal{G})$ and B be an arbitrary set in $\mathcal{G}$. We will prove by contradiction. Now suppose that A and B are not disjoint, which means there is an element x such that $x \in A$ and $x \in B$. Since $x \in A$ and $A \in (\mathcal{F} \setminus \mathcal{G})$ then $x \in \cup(\mathcal{F} \setminus \mathcal{G})$ and because $\cup(\mathcal{F} \setminus \mathcal{G})$ is a subset of $(\cup\mathcal{F}) \setminus (\cup\mathcal{G})$, then $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. This means that $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$. Since $x \notin \cup\mathcal{G}$ then there is no set in $\mathcal{G}$ that contains x, but this contradicts $x \in B$ and $B \in \mathcal{G}$. Therefore A and B are disjoint and since A and B were arbitrary we can conclude $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G}(A \cap B = \varnothing)$.

$\leftarrow$ Suppose all sets in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G}$ are disjoint. Let x be arbitrary and suppose $x \in \cup(\mathcal{F} \setminus \mathcal{G})$, which means there is a set in $\mathcal{F}$ that contains x and $x \in \cup\mathcal{F}$. Now let B be an arbitrary set in G. Since all sets in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G}$ are disjoint and $x \in \cup\mathcal{F}$, then $x \notin B$ and since B was arbitrary we can conclude $\forall B \in \mathcal{G}(x \notin B)$ or $x \notin \cup\mathcal{G}$. Since $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$, then $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. Therefore if $x \in \cup(\mathcal{F} \setminus \mathcal{G})$ then $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$ and since x was arbitrary we can conclude $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. $\square$