3.5.1

Suppose A, B, and C are sets.

Theorem. $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

Proof. Let x be arbitrary and suppose $x \in A \cap (B \cup C)$. Thus $x \in A$ and $x \in B$ or $x \in C$. If $x \in C$ then $x \in (A \cap B) \cup C$. In the case where $x \in B$ it follows that $x \in A \cap B$ and therefore $x \in (A \cap B) \cup C$. Since x was arbitrary we can conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.2

Suppose A, B, and C are sets.

Theorem. $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

Proof. Let x be arbitrary and suppose $x \in (A \cup B) \setminus C$. Thus $x \notin C$ and $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \setminus C)$. If $x \in B$ then if follows that $x \in B \setminus C$ and therefore $x \in A \cup (B \setminus C)$. Since x was arbitrary we can conclude $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.3

Suppose A and B are sets.

Theorem. $A \setminus (A \setminus B) = A \cap B$

Proof. Let x be arbitrary and suppose $x \in A \setminus (A \setminus B)$. Then

$$\begin{aligned}
x \in A \setminus (A \setminus B) &\text{ iff } x \in A \land x \notin A \setminus B \\
&\text{ iff } x \in A \land \neg(x \in A \land x \notin B) \\
&\text{ iff } x \in A \land (x \notin A \lor x \in B) \\
&\text{ iff } (x \in A \land x \notin A) \lor (x \in A \land x \in B) \\
&\text{ iff } x \in A \land x \in B \\
&\text{ iff } x \in (A \cap B)
\end{aligned}$$

□

3.5.4

Theorem. *If* $A \cap C \subseteq B \cap C$ *and* $A \cup C \subseteq B \cup C$ *then* $A \subseteq B$.

Proof. Suppose $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Let x be arbitrary and suppose $x \in A$. Thus $x \in A \cup C$ and it follows that $x \in B \cup C$. Now if $x \in B \cup C$ then either $x \in B$ or $x \in C$. If $x \in B$ then since x was arbitrary we can conclude $A \subseteq B$. In the case that $x \in C$, then $x \in A \cap C$ and it follows that $x \in B \cap C$. Therefore $x \in C$ and $x \in B$. Thus, if $x \in A$ then $x \in B$ and since x was arbitrary we can conclude $A \subseteq B$. □

3.5.5

Suppose A and B are sets.

Theorem. *If $A \triangle B \subseteq A$ then $B \subseteq A$.*

Proof. Suppose $A \triangle B \subseteq A$. We will prove by contradiction. Let x be arbitrary and suppose $x \in B$ and $x \notin A$. Since $x \in B$ and $x \notin A$ then $x \in A \triangle B$. Since $A \triangle B \subseteq A$, then $x \in A$. But this contradicts $x \notin A$. Therefore, if $x \in B$ then $x \in A$ and since x was arbitrary we can conclude that $B \subseteq A$. $\square$

3.5.6

Suppose A, B, and C are sets.

Theorem. $A \cup C \subseteq B \cup C \iff A \setminus C \subseteq B \setminus C$.

Proof. ($\rightarrow$) Suppose A, B, and C are sets. Suppose $(A \cup C) \subseteq (B \cup C)$. Let x be arbitrary and suppose $c \in A \setminus C$, which means $x \in A$ and $x \notin C$. Since $x \in A$, then $x \in A \cup C$ and therefore $x \in B \cup C$. This means $x \in B$ or $x \in C$ and since $x \notin C$, it must be that $x \in B$. Now since $x \in B$ and $x \notin C$ then $x \in B \setminus C$. Therefore, if $x \in A \setminus C$ then $x \in B \setminus C$ and since x was arbitrary we can conclude if $A \cup C \subseteq B \cup C$ then $A \setminus C \subseteq B \setminus C$.

($\leftarrow$) Now suppose $A \setminus C \subseteq B \setminus C$. Let x be arbitrary and suppose $x \in A \cup C$, which means $x \in A$ or $x \in C$. If $x \in C$ then $x \in B \cup C$ and since x was arbitrary then $A \cup C \subseteq B \cup C$. In the case that $x \in A$, since $A \setminus C \subseteq B \setminus C$ then $x \in B$. Therefore, $x \in B \cup C$ and since x was arbitrary then $A \cup C \subseteq B \cup C$. $\square$

3.5.7

Theorem. *For any sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$*

Proof. Let A and B be arbitrary sets. Let M be arbitrary and suppose $M \in \mathscr{P}(A) \cup \mathscr{P}(B)$. Thus $M \in \mathscr{P}(A)$ or $M \in \mathscr{P}(B)$, which means $M \subseteq A$ or $M \subseteq B$. In the case where $M \subseteq A$, let x be an arbitrary member of M and it follows that $x \in A$. Since $x \in A$ then $x \in A \cup B$ and because x was arbitrary we can conclude $M \subseteq A \cup B$ and therefore $M \in \mathscr{P}(A \cup B)$. In the case where $M \subseteq B$, let x be an arbitrary member of M and it follows that $x \in B$. Since $x \in B$ then $x \in A \cup B$ and because x was arbitrary we can conclude $M \subseteq A \cup B$ and therefore $M \in \mathscr{P}(A \cup B)$. $\square$

3.5.8

Theorem. *For any sets A and B, if $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.*

Proof. We will prove the contrapositive. Since we proved that $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$ in exercise 3.5.7, we must show that $\mathscr{P}(A \cup B) \nsubseteq \mathscr{P}(A) \cup \mathscr{P}(B)$ to prove our goal that $\mathscr{P}(A) \cup \mathscr{P}(B) \neq \mathscr{P}(A \cup B)$. Let A and B be arbitrary sets and suppose $A \nsubseteq B$ and $B \nsubseteq A$. This means there is an element $x \in A \setminus B$ and an element $y \in B \setminus A$. Since $x \in A$ and $y \in B$ then both x and y are in $A \cup B$ and therefore the set $\{x, y\}$ is in $\mathscr{P}(A \cup B)$ but not in $\mathscr{P}(A)$ or $\mathscr{P}(B)$. Thus $\mathscr{P}(A \cup B) \nsubseteq \mathscr{P}(A) \cup \mathscr{P}(B)$. □

3.5.9

Theorem. *Suppose x and y are real numbers and $x \neq 0$. Then $y+1/x = 1+y/x$ iff either $x = 1$ or $y = 1$.*

Proof. ($\rightarrow$) Suppose that $y + 1/x = 1 + y/x$. Now if $y = 1$ then we have proven our goal. So now assume $y \neq 1$ and $y + 1/x = 1 + y/x$, then it follows that $x = 1$.

($\leftarrow$) Now suppose $x = 1$ or $y = 1$. In the case that $x = 1$ we have

$$y + \frac{1}{x} = y + \frac{1}{1} = y + 1 = 1 + \frac{y}{1} = 1 + \frac{y}{x}$$

In the case that $y = 1$ we have

$$y + \frac{1}{x} = 1 + \frac{1}{x} = 1 + \frac{y}{x}$$

□

3.5.10

Theorem. *For every real number x, if $|x - 3| > 3$ then $x^2 > 6x$.*

Proof. Suppose that x is an arbitrary real number and that $|x - 3| > 3$. Then either $x - 3 \geq 0$ or $x - 3 < 0$. In the case that $x - 3 \geq 0$, then $|x - 3| = x - 3$ and therefore $|x - 3| > 3 = x - 3 > 3$. Solving for x, we have $x > 6$ and then multiplying both sides by x we have $x^2 > 6x$. In the case that $x - 3 < 0$, then $|x-3| = 3-x$ and therefore $3-x > 3$. Solving for x we have $x < 0$. Multiplying both sides of $x < 0$ by $6 - x$ we have $6x - x^2 < 0$ and therefore $x^2 > 6x$.

□