

### Exercise 3.3.4

Suppose  $A \subseteq \mathcal{P}(A)$ . Prove that  $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$ .

So we want to prove that  $\forall x(x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A)))$ .

First we assume  $x$  is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathcal{P}(A)$ $x \in \mathcal{P}(A)$	$x \in \mathcal{P}(\mathcal{P}(A))$

Assume  $x$  is an arbitrary element of  $\mathcal{P}(A)$

Suppose  $x \in \mathcal{P}(A)$

[ proof of  $x \in \mathcal{P}(\mathcal{P}(A))$  ]

Therefore if  $x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A))$

Since  $x$  was arbitrary we can conclude  $\forall x(x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A)))$

We can rewrite our goal as  $x \subseteq \mathcal{P}(A)$  or  $\forall y(y \in x \rightarrow y \in \mathcal{P}(A))$ . So we assume  $y$  is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathcal{P}(A)$ $x \in \mathcal{P}(A)$ $y \in x$	$y \in \mathcal{P}(A)$

Assume  $x$  is an arbitrary element of  $\mathcal{P}(A)$

Suppose  $x \in \mathcal{P}(A)$

Suppose  $y$  is an arbitrary element of  $x$ .

Suppose  $y \in x$ .

[ proof of  $y \in \mathcal{P}(A)$  ]

Therefore if  $y \in x \rightarrow y \in \mathcal{P}(A)$ .

Since  $y$  was arbitrary we can conclude that  $x \subseteq \mathcal{P}(A)$ .

Therefore if  $x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A))$

Since  $x$  was arbitrary we can conclude  $\forall x(x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A)))$

Now looking at our givens  $x \in \mathcal{P}(A)$  means that  $x \subseteq A$  or  $\forall z(z \in x \rightarrow z \in A)$ . Using universal instantiation we will plug in  $y$  for  $z$  and using modus ponens we can conclude that  $y \in A$ .

Now looking at our other given  $A \subseteq \mathcal{P}(A) \rightarrow \forall m(m \in A \rightarrow m \in \mathcal{P}(A))$ . Using universal instantiation we will plug in  $y$  for  $m$  and using modus ponens we can conclude that  $y \in \mathcal{P}(A)$ , which was our goal to prove.

**Theorem.** Suppose  $A \subseteq \mathcal{P}(A)$ . Then  $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$ .

*Proof.* Suppose  $x$  is an arbitrary element of  $\mathcal{P}(A)$  and  $y$  is an arbitrary element of  $x$ . It follows that  $y \in A$ . But since  $A \subseteq \mathcal{P}(A)$  then it also follows that  $y \in \mathcal{P}(A)$ . So  $y \in x \rightarrow y \in \mathcal{P}(A)$  and since  $y$  was arbitrary we can conclude that  $x \subseteq \mathcal{P}(A)$ . Therefore, if  $x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A))$ . Since  $x$  was arbitrary we can also conclude that  $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$ .  $\square$

Alternate proof (not sure if this is correct)

*Proof.* Suppose  $x$  is an arbitrary element of  $\mathcal{P}(A)$ . Then  $x \in A$ . Since  $A \subseteq \mathcal{P}(A)$  and  $x \in A$  then  $x \subseteq \mathcal{P}(A)$ . Therefore,  $x \in \mathcal{P}(\mathcal{P}(A))$ .  $\square$

### Exercise 3.3.5

The hypothesis of the theorem proven in exercise 3.3.4 is  $A \subseteq \mathcal{P}(A)$ .

#### A

Can you think of a set  $A$  for which this hypothesis is true?

The empty set  $\emptyset$  is a set for which the hypothesis is true.

$A \subseteq \mathcal{P}(A)$  means  $x \in A \rightarrow x \in \mathcal{P}(A)$ . For  $\emptyset$  this would mean that  $x \in \emptyset \rightarrow x \in \mathcal{P}(\emptyset)$ , but by definition there are no elements in  $\emptyset$ . Therefore  $x \in \emptyset$  will always be false and the conditional statement  $x \in \emptyset \rightarrow x \in \mathcal{P}(\emptyset)$  is always true. Therefore if  $\emptyset = A$  then  $A \subseteq \mathcal{P}(A)$ .

#### B

Can you think of another?

In exercise 3.3.4 we proved that if  $A \subseteq \mathcal{P}(A)$  then  $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$ . Therefore, the set  $\{\emptyset, \{\emptyset\}\}$ , which is the  $\mathcal{P}(A)$  if  $A = \emptyset$ , is another set for which the hypothesis is true. If we let  $B = \mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$  and replace  $A$  in the hypothesis  $A \subseteq \mathcal{P}(A)$  with  $B$ , then we can conclude that  $B \subseteq \mathcal{P}(B)$ .

### 3.3.6

Suppose  $x$  is a real number.

## A

Prove that if  $x \neq 1$  then there is a real number  $y$  such that  $\frac{y+1}{y-2} = x$ .

So we want to prove that  $(x \neq 1) \rightarrow \exists y \left( \frac{y+1}{y-2} = x \right)$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$x \neq 1$	$\exists y \left( \frac{y+1}{y-2} = x \right)$

To prove our goal we need to find a  $y$  that makes the equation  $\frac{y+1}{y-2} = x$  true. So let's try solving the equation for  $y$ .

$$\begin{aligned}
 \frac{y+1}{y-2} &= x \\
 y+1 &= x(y-2) \\
 y+1 &= xy-2x \\
 2x+1 &= xy-y \\
 2x+1 &= y(x-1) \\
 y &= \frac{2x+1}{x-1}
 \end{aligned}$$

We see that this  $y$  works because we have  $x \neq 1$  as a given.

**Theorem.** Suppose  $x \neq 1$ . Then there is a real number  $y$  such that  $\frac{y+1}{y-2} = x$ .

*Proof.* Suppose  $x \neq 1$  and  $y = \frac{2x+1}{x-1}$ . Then

$$\frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{x-1} \cdot \frac{x-1}{3} = x$$

□

## B

Prove that if there is a real number  $y$  such that  $\frac{y+1}{y-2} = x$  then  $x \neq 1$ .

So we want to prove that  $\exists y \left( \frac{y+1}{y-2} = x \right) \rightarrow (x \neq 1)$

We assume the antecedent and make the consequent our goal to prove.

Using existential instantiation we assume there is a value  $y_0$  such that  $\frac{y_0+1}{y_0-2} = x$  is true. From part A above, we know that  $\left( \frac{y+1}{y-2} = x \right) \rightarrow \left( y = \frac{2x+1}{x-1} \right)$  and so  $y_0 = \frac{2x+1}{x-1}$ . Since  $y$  is a real number, then clearly  $x \neq 1$ .

Givens	Goals
$\exists y \left( \frac{y+1}{y-1} = x \right)$	$x \neq 1$

**Theorem.** If  $y$  is a real number and  $\frac{y+1}{y-2} = x$  then  $x \neq 1$ .

*Proof.* Suppose  $y$  is a real number and  $\frac{y+1}{y-2} = x$ . It follows that  $y = \frac{2x+1}{x-1}$  and since  $y$  is real number then  $x \neq 1$ .  $\square$

### Exercise 3.3.7

Prove for every real number  $x$ , if  $x > 2$  then there is a real number  $y$  such that  $y + \frac{1}{y} = x$ .

So we want to prove  $\forall x \in \mathbb{R}(x > 2 \rightarrow \exists y \in \mathbb{R}(y + \frac{1}{y} = x))$

So we let  $x$  be an arbitrary real number, then we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$x$ is arbitrary real number	$\exists y(y + \frac{1}{y} = x)$
$x > 2$	

Our goal is of the form  $\exists y P(y)$  where  $P(y)$  is  $y + \frac{1}{y} = x$  and our strategy suggests we try to find a  $y$  for which  $P(y)$  is true. We can do this by solving the equation  $y + \frac{1}{y} = x$  for  $y$ . We can rewrite this equation as  $y^2 - \frac{x}{y} + 1 = 0$  and we see this is a quadratic equation and therefore we can use the quadratic formula to solve for  $y$ ,

$$y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{x \pm \sqrt{x^2 - 4}}{2}.$$

We note that  $\sqrt{x^2 - 4}$  is defined because  $x > 2$ . We have found two solutions that satisfy our original equation, but we only need one to complete the proof. We will use  $\frac{x + \sqrt{x^2 - 4}}{2}$ .

**Theorem.** For every real number  $x$ , if  $x > 2$  then there is a real number  $y$  such that  $y + \frac{1}{y} = x$ .

*Proof.* Suppose  $x$  and  $y$  are real numbers,  $x > 2$ , and  $y = \frac{x + \sqrt{x^2 - 4}}{2}$ . Then

$$\begin{aligned}
\frac{x + \sqrt{x^2 - 4}}{2} + \frac{1}{\frac{x + \sqrt{x^2 - 4}}{2}} &= \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} \\
&= \frac{2x^2 + 2(x\sqrt{x^2 - 4})}{2x + 2\sqrt{x^2 - 4}} \\
&= x
\end{aligned}$$

□

### Exercise 3.3.8

Prove that if  $\mathcal{F}$  is a family of sets and  $A \in \mathcal{F}$ , then  $A \subseteq \cup \mathcal{F}$ .

So we want to prove that  $A \in \mathcal{F} \rightarrow A \subseteq \cup \mathcal{F}$ .

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$A \subseteq \cup \mathcal{F}$

Assume  $A \in \mathcal{F}$

[ proof of  $A \subseteq \cup \mathcal{F}$  ]

Therefore if  $A \in \mathcal{F}$  then  $A \subseteq \cup \mathcal{F}$

Our goal  $A \subseteq \cup \mathcal{F}$  can be rewritten as  $\forall x(x \in A \rightarrow x \in \cup \mathcal{F})$ . We assume  $x$  is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in \cup \mathcal{F}$
$x \in A$	

Assume  $A \in \mathcal{F}$

Assume  $x$  is arbitrary

Assume  $x \in A$

[ proof of  $x \in \cup \mathcal{F}$  ]

Therefore if  $x \in A$  then  $x \in \cup \mathcal{F}$

Since  $x$  was arbitrary we can conclude that  $A \subseteq \cup \mathcal{F}$ .

Therefore if  $A \in \mathcal{F}$  then  $A \subseteq \cup \mathcal{F}$

Our new goal can be rewritten as  $\exists B \in \mathcal{F}(x \in B)$ . From our givens we see that  $A \in \mathcal{F}$  and  $x \in A$ , so we have found a set such that  $A \in \mathcal{F}(x \in A)$ .

**Theorem.** If  $\mathcal{F}$  is a family of sets and  $A \in \mathcal{F}$ , then  $A \subseteq \cup \mathcal{F}$ .

*Proof.* Assume  $A \in \mathcal{F}$  and  $x$  is an arbitrary member of  $A$ . Then since  $x \in A$  and  $A \in \mathcal{F}$ , it follows that  $x \in \cup \mathcal{F}$ . Since  $x$  was arbitrary we can conclude that  $A \subseteq \cup \mathcal{F}$  and therefore if  $A \in \mathcal{F}$  then  $A \subseteq \cup \mathcal{F}$ .  $\square$

### Exercise 3.3.9

Prove that if  $\mathcal{F}$  is a family of sets and  $A \in \mathcal{F}$ , then  $\cap \mathcal{F} \subseteq A$ .

We want to prove that  $A \in \mathcal{F} \rightarrow \cap \mathcal{F} \subseteq A$ .

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$\cap \mathcal{F} \subseteq A$

Assume  $A \in \mathcal{F}$   
     [proof of  $\cap \mathcal{F} \subseteq A$ ]  
 Therefore, if  $A \in \mathcal{F}$  then  $\cap \mathcal{F} \subseteq A$ .

We can rewrite our goal as  $\forall x(x \in \cap \mathcal{F} \rightarrow x \in A)$ . We assume  $x$  is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in A$
$x \in \cap \mathcal{F}$	

Assume  $A \in \mathcal{F}$   
     Assume  $x$  is arbitrary  
         Assume  $x \in \cap \mathcal{F}$   
             [proof of  $x \in A$ ]  
             Therefore, if  $x \in \cap \mathcal{F}$  then  $x \in A$ .  
         Since  $x$  was arbitrary we can conclude that  $\cap \mathcal{F} \subseteq A$ .  
 Therefore, if  $A \in \mathcal{F}$  then  $\cap \mathcal{F} \subseteq A$ .

Our given  $x \in \cap \mathcal{F}$  can be rewritten as  $\forall B \in \mathcal{F}(x \in B)$ , therefore if  $A \in \mathcal{F}$  then  $x \in A$ , which was our goal to prove.

**Theorem.** If  $\mathcal{F}$  is a family of sets and  $A \in \mathcal{F}$ , then  $\cup \mathcal{F} \in A$ .

*Proof.* Assume  $A \in \mathcal{F}$  and  $x$  is an arbitrary member of  $\cap \mathcal{F}$ . Since  $A \in \mathcal{F}$  and  $x \in \cap \mathcal{F}$  it follows that  $x \in A$  and therefore, if  $x \in \cap \mathcal{F}$  then  $x \in A$ . Since  $x$  was arbitrary we can conclude that  $\cap \mathcal{F} \subseteq A$ . Therefore, if  $A \in \mathcal{F}$  then  $\cap \mathcal{F} \subseteq A$ .  $\square$

### Exercise 3.3.10

Suppose that  $\mathcal{F}$  is a nonempty family of sets  $B$  is a set, and  $\forall A \in \mathcal{F}(B \subseteq A)$ .  
Prove that  $B \subseteq \cap \mathcal{F}$ .

We want to prove  $\forall A \in \mathcal{F}(B \subseteq A) \rightarrow B \subseteq \cap \mathcal{F}$ .

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$B \subseteq \cap \mathcal{F}$

Suppose  $\forall A \in \mathcal{F}(B \subseteq A)$

[proof of  $B \subseteq \cap \mathcal{F}$ ]

Therefore if  $\forall A \in \mathcal{F}(B \subseteq A)$  then  $B \subseteq \cap \mathcal{F}$ .

Our goal can be rewritten as  $\forall x(x \in B \rightarrow x \in \cap \mathcal{F})$ . So we assume  $x$  is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$ $x \in B$	$x \in \cap \mathcal{F}$

Suppose  $\forall A \in \mathcal{F}(B \subseteq A)$

Suppose  $x$  is arbitrary.

Suppose  $x \in B$ .

[proof of  $x \in \cap \mathcal{F}$ ]

Therefore  $x \in B \rightarrow x \in \cap \mathcal{F}$

Since  $x$  was arbitrary we can conclude  $B \subseteq \cap \mathcal{F}$

Therefore if  $\forall A \in \mathcal{F}(B \subseteq A)$  then  $B \subseteq \cap \mathcal{F}$ .

Our goal can be rewritten as  $\forall M \in \mathcal{F}(x \in M)$  and so we can assume  $M$  is an arbitrary set in  $\mathcal{F}$  and make our goal  $x \in M$ .

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$ $x \in B$ $M \in \mathcal{F}$	$x \in M$

Suppose  $\forall A \in \mathcal{F}(B \subseteq A)$

Suppose  $x$  is arbitrary.

Suppose  $x \in B$ .

Suppose  $M$  is an arbitrary set in  $\mathcal{F}$ .

[proof of  $x \in M$ ]  
Therefore  $x \in \cap \mathcal{F}$   
Therefore  $x \in B \rightarrow x \in \cap \mathcal{F}$   
Since  $x$  was arbitrary we can conclude  $B \subseteq \cap \mathcal{F}$   
Therefore if  $\forall A \in \mathcal{F}(B \subseteq A)$  then  $B \subseteq \cap \mathcal{F}$ .

Using universal instantiation we will plug in  $M$  for  $A$  in our given  $\forall A \in \mathcal{F}(B \subseteq A)$  and conclude that  $B \subseteq M$ . We can rewrite  $B \subseteq M$  as  $\forall y(y \in B \rightarrow y \in M)$  and using universal instantiation plug in  $x$  for  $y$  and then use modus ponens to conclude  $x \in M$ , which was our goal to prove.

**Theorem.** *If  $\mathcal{F}$  is a nonempty family of sets,  $B$  is a set, and  $\forall A \in \mathcal{F}(B \subseteq A)$ , then  $B \subseteq \cap \mathcal{F}$ .*

*Proof.* Suppose  $\forall A \in \mathcal{F}(B \subseteq A)$ . Suppose  $x$  is an arbitrary member of  $B$  and  $M$  is an arbitrary set in  $\mathcal{F}$ . Then it follows that  $x \in M$  and since  $M$  was arbitrary we can conclude that  $x$  is in all sets that are in  $\mathcal{F}$  or  $x \in \cap \mathcal{F}$ . Therefore, if  $x \in B$  then  $x \in \cap \mathcal{F}$ , and since  $x$  was arbitrary, we can conclude that  $B \subseteq \cap \mathcal{F}$ .  $\square$

### Exercise 3.3.11

Suppose that  $\mathcal{F}$  is a family of sets. Prove that if  $\emptyset \in \mathcal{F}$  then  $\cap \mathcal{F} = \emptyset$ .  
We want to prove that  $\emptyset \in \mathcal{F} \rightarrow \cap \mathcal{F} = \emptyset$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\emptyset \in \mathcal{F}$	$\cap \mathcal{F} = \emptyset$

Suppose  $\emptyset \in \mathcal{F}$   
[proof of  $\cap \mathcal{F} = \emptyset$ ]  
Therefore if  $\emptyset \in \mathcal{F}$  then  $\cap \mathcal{F} = \emptyset$ .

We will try a proof by contradiction. So we assume that  $\cap \mathcal{F} \neq \emptyset$  and try to find a contradiction.

Givens	Goals
$\emptyset \in \mathcal{F}$	contradiction
$\cap \mathcal{F} \neq \emptyset$	

Our given  $\cap \mathcal{F} \neq \emptyset$  means that there is an element that is in all sets in  $\mathcal{F}$ . However, this contradicts  $\emptyset \in \mathcal{F}$  because  $\emptyset$  is the set that contains nothing.

**Theorem.** *If  $\mathcal{F}$  is a family of sets and  $\emptyset \in \mathcal{F}$ , then  $\cap \mathcal{F} = \emptyset$ .*



*Proof.* We will prove by contradiction. Suppose  $\emptyset \in \mathcal{F}$  and  $\cap \mathcal{F} \neq \emptyset$ . Since  $\cap \mathcal{F} \neq \emptyset$  it follows that there is an element that is within all of the sets that are in  $\mathcal{F}$ . However, this contradicts  $\emptyset \in \mathcal{F}$  because  $\emptyset$  is the set that contains nothing. Therefore, if  $\emptyset \in \mathcal{F}$  then  $\cap \mathcal{F} = \emptyset$ .  $\square$

### Exercise 3.3.12

Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are families of sets. Prove that if  $\mathcal{F} \subseteq \mathcal{G}$  then  $\cup \mathcal{F} \subseteq \cup \mathcal{G}$ .

So we want to prove that  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cup \mathcal{F} \subseteq \cup \mathcal{G}$

Suppose  $\mathcal{F} \subseteq \mathcal{G}$   
 [proof of  $\cup \mathcal{F} \subseteq \cup \mathcal{G}$  ]  
 So if  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

$\cup \mathcal{F} \subseteq \cup \mathcal{G} \rightarrow \forall b(b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G})$  so we assume  $b$  is an arbitrary element of  $\cup \mathcal{F}$  and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$ $b \in \cup \mathcal{F}$	$b \in \cup \mathcal{G}$

Suppose  $\mathcal{F} \subseteq \mathcal{G}$   
 Let  $b$  be an arbitrary element of  $\cup \mathcal{F}$   
 [proof of  $b \in \cup \mathcal{G}$  ]  
 Therefore if  $b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G}$   
 Since  $b$  was arbitrary we can conclude  $\forall b(b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G})$ . So if  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

$b \in \cup \mathcal{F} \rightarrow \exists M(M \in \mathcal{F} \wedge b \in M)$ , so let  $M = A_0$  (Existential Instantiation)

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$ $A_0 \in \mathcal{F} \wedge b \in A_0$	$b \in \cup \mathcal{G}$

Suppose  $\mathcal{F} \subseteq \mathcal{G}$   
 Let  $b$  be an arbitrary element and suppose  $b \in \cup \mathcal{F}$ , which implies there is a set in  $\mathcal{F}$  and  $b$  is in that set. Let that set =  $A_0$

[proof of  $b \in \cup \mathcal{G}$  ]

Therefore if  $b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G}$

Since  $b$  was arbitrary we can conclude  $\forall b(b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G})$ . So if  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall A(A \in \mathcal{F} \rightarrow A \in \mathcal{G})$ . Using universal instantiation we will plug in  $A_0$  for  $A$  since then we can use modens ponens to conclude that  $A_0 \in \mathcal{G}$ .

Givens	Goals
$A_0 \in \mathcal{F} \rightarrow A_0 \in \mathcal{G}$	$b \in \cup \mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Our goal  $b \in \cup \mathcal{G} \rightarrow \exists N(N \in \mathcal{G} \wedge b \in N)$ , which we can now prove. Since  $A_0 \in \mathcal{F}$  and  $\mathcal{F}$  is a subset of  $\mathcal{G}$ , it follows that  $A_0 \in \mathcal{G}$ . By the definition of  $\cup \mathcal{G}$  it follows that  $b \in \cup \mathcal{G}$  because  $A_0 \in \mathcal{G} \wedge b \in A_0$ , the latter statement being one of our givens.

**Theorem.** Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are families of sets. If  $\mathcal{F} \subseteq \mathcal{G}$  then  $\cup \mathcal{F} \subseteq \cup \mathcal{G}$ .

*Proof.* Suppose  $\mathcal{F} \subseteq \mathcal{G}$ . Let  $b$  be an arbitrary element of  $\cup \mathcal{F}$ , which implies there is a set in  $\mathcal{F}$  that contains  $b$ . Call this set  $A_0$ . Since  $A_0 \in \mathcal{F}$  and  $\mathcal{F}$  is a subset of  $\mathcal{G}$  it follows that  $A_0 \in \mathcal{G}$ , which implies that  $b \in \cup \mathcal{G}$ . Therefore if  $b \in \cup \mathcal{F}$  then  $b \in \cup \mathcal{G}$ . Since  $b$  was arbitrary we can conclude that if  $\mathcal{F} \subseteq \mathcal{G}$  then  $\cup \mathcal{F} \subseteq \cup \mathcal{G}$ . This completes the proof.

### Exercise 3.3.13

Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are families of sets. Prove that if  $\mathcal{F} \subseteq \mathcal{G}$  then  $\cap \mathcal{G} \subseteq \cap \mathcal{F}$ .

So we want to prove that  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cap \mathcal{G} \subseteq \cap \mathcal{F}$

Suppose  $\mathcal{F} \subseteq \mathcal{G}$

[proof of  $\cap \mathcal{G} \subseteq \cap \mathcal{F}$  ]

So if  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

$\cap \mathcal{G} \subseteq \cap \mathcal{F} \rightarrow \forall b(b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F})$ , so we assume  $b$  is an arbitrary element of  $\cap \mathcal{G}$  and assume the antecedent and make the consequent our goal to prove.

Suppose  $\mathcal{F} \subseteq \mathcal{G}$

Let  $b$  be an arbitrary element of  $\cap \mathcal{G}$

[proof of  $b \in \cap \mathcal{F}$  ]

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cap \mathcal{F}$
$b \in \cap \mathcal{G}$	

Therefore if  $b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F}$

Since  $b$  was arbitrary we can conclude  $\forall b(b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F})$ . So  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

$b \in \cap \mathcal{F} \rightarrow \forall A(A \in \mathcal{F} \rightarrow b \in A)$ , so we assume  $A$  is an arbitrary element of  $\mathcal{F}$  and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in A$
$b \in \cap \mathcal{G}$	
$A \in \mathcal{F}$	

Suppose  $\mathcal{F} \subseteq \mathcal{G}$

Let  $b$  be an arbitrary element of  $\cap \mathcal{G}$

Suppose  $A$  is an arbitrary set in  $\mathcal{F}$

[proof of  $b \in A$ ]

Therefore if  $A \in \mathcal{F} \rightarrow b \in A$

Since  $A$  was arbitrary we can conclude  $b \in \cap \mathcal{F}$

Therefore if  $b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F}$

Since  $b$  was arbitrary we can conclude  $\forall b(b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F})$ . So  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

Now looking at our givens,  $\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall Z(Z \in \mathcal{F} \rightarrow Z \in \mathcal{G})$ . Using universal instantiation we will plug in  $A$  for  $Z$  and using modus ponens we can conclude that  $A \in \mathcal{G}$ .

Our other given,  $b \in \cap \mathcal{G} \rightarrow \forall Y(Y \in \mathcal{G} \rightarrow b \in Y)$ . Using universal instantiation we will plug in  $A$  for  $Y$  and using modus ponens we can conclude that  $b \in A$ , which was our goal, and we can now write our proof.

**Theorem.** Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are families of sets. If  $\mathcal{F} \subseteq \mathcal{G}$  then  $\cap \mathcal{G} \subseteq \cap \mathcal{F}$ .

*Proof.* Suppose  $\mathcal{F} \subseteq \mathcal{G}$ . Let  $b$  be an arbitrary element of  $\cap \mathcal{G}$ . Suppose  $A$  is an arbitrary element of  $\mathcal{F}$ , then because  $\mathcal{F} \subseteq \mathcal{G}$  then it follows that  $A \in \mathcal{G}$ . By the definition of  $\cap \mathcal{G}$  it follows that  $b \in A$  and since  $A$  was arbitrary then  $b \in \cap \mathcal{F}$ . Since  $b$  was arbitrary we can conclude  $\cap \mathcal{G} \subseteq \cap \mathcal{F}$  and therefore that if  $\mathcal{F} \subseteq \mathcal{G}$  then  $\cap \mathcal{G} \subseteq \cap \mathcal{F}$ . This completes the proof.

### Exercise 3.3.14

Suppose  $\{A_i | i \in I\}$  is an indexed family of sets. Prove that  $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$ .

So we want to prove that  $\forall a(a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \mathcal{P}(\bigcup_{i \in I} A_i))$

First we assume  $a$  is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i \in I} \mathcal{P}(A_i)$	$a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Assume  $a$  is an arbitrary element of  $\bigcup_{i \in I} \mathcal{P}(A_i)$

Suppose  $a \in \bigcup_{i \in I} \mathcal{P}(A_i)$

[ proof of  $a \in \mathcal{P}(\bigcup_{i \in I} A_i)$  ]

Therefore if  $a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Since  $a$  was arbitrary we can conclude  $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$

Looking at our goal we see that  $a \in \mathcal{P}(\bigcup_{i \in I} A_i) \rightarrow a \subseteq \bigcup_{i \in I} A_i \rightarrow \forall z(z \in a \rightarrow z \in \bigcup_{i \in I} A_i)$ . Therefore we assume  $z$  is arbitrary, assume the antecedent, and make the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i \in I} \mathcal{P}(A_i)$ $z \in a$	$z \in \bigcup_{i \in I} A_i$

Assume  $a$  is an arbitrary element of  $\bigcup_{i \in I} \mathcal{P}(A_i)$

Suppose  $a \in \bigcup_{i \in I} \mathcal{P}(A_i)$

Assume  $z$  is arbitrary

Assume  $z \in a$

[ proof of  $z \in \bigcup_{i \in I} A_i$  ]

Therefore  $z \in a \rightarrow z \in \bigcup_{i \in I} A_i$

Since  $z$  was arbitrary we can conclude  $a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Therefore if  $a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Since  $a$  was arbitrary we can conclude  $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$

Looking at our given we see that  $a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \{a | \exists i \in I(a \in \mathcal{P}(A_i))\}$ . Using existential instantiation we will select an  $i$  such that  $a \in \mathcal{P}(A_i)$  which implies  $a \subseteq A_i$ . Since  $a \subseteq A_i \rightarrow \forall m(m \in a \rightarrow m \in A_i)$  and using universal instantiation we will plug in  $z$  for  $m$  and we get  $\forall z(z \in a \rightarrow z \in A_i)$  and using modus ponens we can conclude that  $z \in A_i$ , which implies that  $z \in \bigcup_{i \in I} A_i$ , which was our goal. We can now right our proof.

**Theorem.** Suppose  $\{A_i | i \in I\}$  is an indexed family of sets, then  $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$ .

*Proof.* Suppose that  $a$  is an arbitrary element of  $\bigcup_{i \in I} \mathcal{P}(A_i)$ . We choose an  $i \in I$  such that  $a \in \mathcal{P}(A_i)$ , which implies that  $a \subseteq A_i$ . Suppose  $z$  is an arbitrary element of  $a$ , then it follows that  $z \in A_i$  and therefore  $z \in \bigcup_{i \in I} A_i$ . Since  $z$  was an arbitrary element of  $a$  then  $a \subseteq \bigcup_{i \in I} A_i$ , and it follows that  $a \in \mathcal{P}(\bigcup_{i \in I} A_i)$ . Thus we can conclude  $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$ . This completes the proof.

### Exercise 3.3.15

Suppose  $\{A_i | i \in I\}$  is an indexed family of sets and  $I \neq \emptyset$ . Prove that  $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$

So we want to prove that  $\forall y (y \in \bigcap_{i \in I} A_i \rightarrow y \in \bigcap_{i \in I} \mathcal{P}(A_i))$ .

First we assume  $y$  is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i \in I} A_i$	$y \in \bigcap_{i \in I} \mathcal{P}(A_i)$

Suppose  $y$  is arbitrary element of  $\bigcap_{i \in I} A_i$ .

[proof of  $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$ ]

Since  $y$  was arbitrary we can conclude  $\forall y (y \in \bigcap_{i \in I} A_i \rightarrow y \in \bigcap_{i \in I} \mathcal{P}(A_i))$ .

Our goal  $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$  so we make  $m$  an arbitrary element of  $I$  and therefore  $y \in \mathcal{P}(A_m) \rightarrow y \subseteq A_m \rightarrow \forall z (z \in y \rightarrow z \in A_m)$ . So we make  $z$  arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i \in I} A_i$	$z \in A_m$
$z \in y$	

Suppose  $y$  is arbitrary element of  $\bigcap_{i \in I} A_i$ .

Suppose  $m$  is an arbitrary element of  $I$  and therefore  $y \in \mathcal{P}(A_m) \rightarrow y \subseteq A_m \rightarrow \forall z (z \in y \rightarrow z \in A_m)$ .

Suppose  $z$  is an arbitrary element of  $y$

[proof of  $z \in A_m$ ]

Therefore  $z \in y \rightarrow z \in A_m$  and since  $z$  was arbitrary  $y \subseteq A_m \rightarrow y \in \mathcal{P}(A_m)$

and since  $m$  was arbitrary  $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$

Since  $y$  was arbitrary we can conclude  $\forall y (y \in \bigcap_{i \in I} A_i \rightarrow y \in \bigcap_{i \in I} \mathcal{P}(A_i))$ .

Now looking at our given  $y \in \bigcap_{i \in I} A_i \rightarrow \forall i \in I (y \in A_i)$ . Using universal instantiation we plug in  $m$  for  $i$  and therefore  $y \in A_m$  and since  $z \in y$  we can conclude  $z \in A_m$ , which was our goal. Now we can write our proof.

**Theorem.** Suppose  $\{A_i | i \in I\}$  is an indexed family of sets and  $I \neq \emptyset$ , then  $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$ .

*Proof.* Suppose  $y$  is an arbitrary element of  $\bigcap_{i \in I} A_i$ . Suppose  $m$  is an arbitrary member of  $I$  and therefore  $y \subseteq A_m$  which implies  $y \subseteq A_m$ . Now suppose  $z$  is an arbitrary element of  $y$ . Since  $y \in \bigcap_{i \in I} A_i$  if we choose an  $i$  such that  $y \in \bigcap_{m \in I} A_m$  then  $y \in A_m$  which implies  $z \in A_m$ . Therefore if  $z \in y$  then  $z \in A_m$  and since  $z$  was arbitrary then  $y \subseteq A_m$  or  $y \in \mathcal{P}(A_m)$  and since  $m$  was arbitrary then  $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$ . Since  $y$  was arbitrary then  $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$ . This completes the proof.

### Exercise 3.3.16

Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if  $\mathcal{F} \subseteq \mathcal{P}(B)$  then  $\cup \mathcal{F} \subseteq B$ .

We want to prove  $\mathcal{F} \subseteq \mathcal{P}(B) \rightarrow \cup \mathcal{F} \subseteq B$ .

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{P}(B)$	$\cup \mathcal{F} \subseteq B$

Suppose  $\mathcal{F} \subseteq \mathcal{P}(B)$   
 [proof of  $\cup \mathcal{F} \subseteq B$   
 Therefore if  $\mathcal{F} \subseteq \mathcal{P}(B)$  then  $\cup \mathcal{F} \subseteq B$ .

$\cup \mathcal{F} \subseteq B \rightarrow \forall x (x \in \cup \mathcal{F} \rightarrow x \in B)$ . So we assume  $x$  is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{P}(B)$	$x \in B$
$x \in \cup \mathcal{F}$	

Suppose  $\mathcal{F} \subseteq \mathcal{P}(B)$   
 Suppose  $x$  is arbitrary  
 Suppose  $x \in \cup \mathcal{F}$   
 [proof of  $x \in B$   
 Therefore if  $x \in \cup \mathcal{F}$  then  $x \in B$   
 Since  $x$  was arbitrary we can conclude that  $\cup \mathcal{F} \subseteq B$ .

Therefore if  $\mathcal{F} \subseteq \mathcal{P}(B)$  then  $\cup \mathcal{F} \subseteq B$ .

$x \in \cup \mathcal{F} \rightarrow \exists M \in \mathcal{F}(x \in M)$ . We use existential instantiation and assume there is a set  $M$  in  $\mathcal{F}$  and  $x$  is in that set.

Suppose  $\mathcal{F} \subseteq \mathcal{P}(B)$

Suppose  $x$  is arbitrary

Suppose  $M$  is arbitrary set in  $\mathcal{F}$

$x \in M$

[proof of  $x \in B$ ]

Since  $x \in M$  and  $M$  is a set in  $\mathcal{F}$  then  $x \in \cup \mathcal{F}$

Therefore if  $x \in \cup \mathcal{F}$  then  $x \in B$

Since  $x$  was arbitrary we can conclude that  $\cup \mathcal{F} \subseteq B$ .

Therefore if  $\mathcal{F} \subseteq \mathcal{P}(B)$  then  $\cup \mathcal{F} \subseteq B$ .

Our given  $\mathcal{F} \subseteq \mathcal{P}(B)$  means that  $\forall N(N \in \mathcal{F} \rightarrow \forall z(z \in N \rightarrow z \in B))$ . We will use universal instantiation and plug in  $M$  for  $N$  and  $x$  for  $z$  and we can conclude that  $x \in B$ , which was our goal to prove.

**Theorem.** Suppose  $B$  is a set and  $\mathcal{F}$  is a family of sets. If  $\mathcal{F} \subseteq \mathcal{P}(B)$  then  $\cup \mathcal{F} \subseteq B$ .

*Proof.* Suppose  $x$  is an arbitrary member of  $\cup \mathcal{F}$ , which means that  $x$  is a member of a set that is in  $\mathcal{F}$ . Suppose  $\mathcal{F} \subseteq \mathcal{P}(B)$ , which means that any element that is in a set that is a member of  $\mathcal{F}$  is also in the set  $B$ . It follows that since  $x$  is a member of a set in  $\mathcal{F}$  then  $x \in B$ . Therefore, if  $x \in \cup \mathcal{F}$  then  $x \in B$  and since  $x$  was arbitrary we can conclude  $\cup \mathcal{F} \subseteq B$ . Therefore, if  $\mathcal{F} \subseteq \mathcal{P}(B)$  then  $\cup \mathcal{F} \subseteq B$ .  $\square$

### Exercise 3.3.17

Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are nonempty families of sets, and every element of  $\mathcal{F}$  is a subset of every element of  $\mathcal{G}$ . Prove that  $\cup \mathcal{F} \subseteq \cap \mathcal{G}$ .

We want to prove that  $\cup \mathcal{F} \subseteq \cap \mathcal{G}$ .

We can rewrite this goal as  $\forall x(x \in \cup \mathcal{F} \rightarrow x \in \cap \mathcal{G})$ . We assume  $x$  is arbitrary and then assume the antecedent and make the consequent our goal.

Givens	Goals
$x \in \cup \mathcal{F}$	$x \in \cap \mathcal{G}$
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \subseteq B)$	

Suppose  $x$  is arbitrary.

Suppose  $x \in \cup \mathcal{F}$ .

[proof of  $x \in \cap \mathcal{G}$ ]

Therefore if  $x \in \cup \mathcal{F}$  then  $x \in \cap \mathcal{G}$

Since  $x$  was arbitrary we can conclude that  $\cup \mathcal{F} \subseteq \cap \mathcal{G}$ .

We can rewrite our goal as  $\forall M \in \mathcal{G}(x \in M)$ . We assume  $M$  is an arbitrary set in  $\mathcal{G}$  and then our goal becomes  $x \in M$ .

Givens	Goals
$x \in \cup \mathcal{F}$	$x \in M$
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \rightarrow y \in B)$	

Suppose  $x$  is arbitrary.

Suppose  $x \in \cup \mathcal{F}$ .

Suppose  $M$  is an arbitrary set in  $\mathcal{G}$

[proof of  $x \in M$ ]

Since  $M$  was arbitrary we can conclude that  $x \in \cap \mathcal{G}$

Therefore if  $x \in \cup \mathcal{F}$  then  $x \in \cap \mathcal{G}$

Since  $x$  was arbitrary we can conclude that  $\cup \mathcal{F} \subseteq \cap \mathcal{G}$ .

We can rewrite our given  $x \in \cup \mathcal{F}$  as  $\exists N \in \mathcal{F}(x \in N)$ . We use existential instantiation and assume there is a set  $N \in \mathcal{F}$  and  $x \in N$ .

Givens	Goals
$N \in \mathcal{F}$	$x \in M$
$x \in N$	
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \rightarrow y \in B)$	

Now we can use universal instantiation to plug in  $N$  for  $A$  and  $M$  for  $B$ . Then since  $x \in N$  we can use modus ponens to conclude that  $x \in M$ , which was our goal.

**Theorem.** *If  $\mathcal{F}$  and  $\mathcal{G}$  are nonempty families of sets, and every element of  $\mathcal{F}$  is a subset of every element of  $\mathcal{G}$ , then  $\cup \mathcal{F} \subseteq \cap \mathcal{G}$ .*

*Proof.* Suppose  $x$  is an arbitrary member of  $\cup \mathcal{F}$ , which means there is a set in  $\mathcal{F}$  that contains  $x$ . Suppose  $M$  is an arbitrary set in  $\mathcal{G}$ . Then since every set in  $\mathcal{F}$  is a subset of every set in  $\mathcal{G}$  it follows that  $x \in M$ . Since  $M$  was arbitrary we can conclude that  $x \in \cap \mathcal{G}$  and therefore if  $x \in \cup \mathcal{F}$  then  $x \in \cap \mathcal{G}$ . Since  $x$  was arbitrary we can conclude that  $\cup \mathcal{F} \subseteq \cap \mathcal{G}$ .  $\square$