

### 3.4.1

Use the methods of this chapter to prove that  $\forall x(P(x) \wedge Q(x))$  is equivalent to  $\forall xP(x) \wedge \forall xQ(x)$ .

We want to prove  $\forall x(P(x) \wedge Q(x)) \iff \forall xP(x) \wedge \forall xQ(x)$ .

**Theorem.** *The statement  $\forall x(P(x) \wedge Q(x))$  is equivalent to  $\forall xP(x) \wedge \forall xQ(x)$ .*

*Proof.* ( $\rightarrow$ ) Suppose  $\forall x(P(x) \wedge Q(x))$ . Let  $y$  be arbitrary. Since  $\forall x(P(x) \wedge Q(x))$  it follows  $P(y)$  and  $Q(y)$ . Since  $y$  was arbitrary, we can conclude  $\forall xP(x)$  and  $\forall xQ(x)$  or  $\forall xP(x) \wedge \forall xQ(x)$ .

( $\leftarrow$ ) Let  $y$  be arbitrary. Since  $\forall xP(x)$  and  $\forall xQ(x)$  then it follows  $P(y)$  and  $Q(y)$ . Since  $y$  was arbitrary we can conclude  $\forall x(P(x) \wedge Q(x))$ .  $\square$

### 3.4.2

Prove that if  $A \subseteq B$  and  $A \subseteq C$  then  $A \subseteq B \cap C$ .

**Theorem.** *If  $A \subseteq B$  and  $A \subseteq C$  then  $A \subseteq B \cap C$ .*

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A$ . Since  $A \subseteq B$  then  $x \in B$  and since  $A \subseteq C$  then  $x \in C$  or  $x \in B \cap C$ . Therefore, if  $x \in A$  then  $x \in B \cap C$  and since  $x$  was arbitrary we can conclude  $A \subseteq B \cap C$ .  $\square$

### 3.4.3

Suppose  $A \subseteq B$ . Prove that for every set  $C$ ,  $C \setminus B \subseteq C \setminus A$ .

**Theorem.** *Suppose  $A \subseteq B$ , then for every set  $C$ ,  $C \setminus B \subseteq C \setminus A$ .*

*Proof.* Suppose  $A \subseteq B$  and  $C$  is an arbitrary set. Let  $x$  be arbitrary and suppose  $x \in C \setminus B$ , which means  $x \in C$  and  $x \notin B$ . Since  $x \notin B$  and  $A \subseteq B$ , then  $x \notin A$ , which means that  $x \in C \setminus A$ . Therefore, if  $x \in C \setminus B$  then  $x \in C \setminus A$  and since  $x$  and  $C$  were arbitrary, we can conclude  $\forall C(C \setminus B \subseteq C \setminus A)$ .  $\square$

### 3.4.5

Prove that if  $A \subseteq B \setminus C$  and  $A \neq \emptyset$  then  $B \not\subseteq C$ .

**Theorem.** *If  $A \subseteq B \setminus C$  and  $A \neq \emptyset$  then  $B \not\subseteq C$ .*

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A$ . Since  $A \subseteq B \setminus C$  then  $x \in B$  and  $x \notin C$ . Since  $x$  was arbitrary we can conclude  $B \not\subseteq C$ .  $\square$

### 3.4.6

Prove that for any sets  $A$ ,  $B$ , and  $C$ ,  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$  finding a string of equivalences starting with  $x \in A \setminus (B \cap C)$  and ending with  $x \in (A \setminus B) \cup (A \setminus C)$ .

**Theorem.** *for any sets  $A$ ,  $B$ , and  $C$ ,  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ .*

*Proof.* Suppose  $A$ ,  $B$ , and  $C$  are arbitrary sets. Then

$$\begin{aligned}
 x \in A \setminus (B \cap C) & \text{ iff } x \in A \rightarrow (x \notin B \wedge x \notin C) \\
 & \text{ iff } x \notin A \vee (x \notin B \wedge x \notin C) \\
 & \text{ iff } (x \notin A \vee x \notin B) \wedge (x \notin A \vee x \notin C) \\
 & \text{ iff } (x \in A \rightarrow x \notin B) \vee (x \in A \rightarrow x \notin C) \\
 & \text{ iff } x \in A \setminus B \vee x \in A \setminus C \\
 & \text{ iff } x \in (A \setminus B) \cup (A \setminus C)
 \end{aligned}$$

□

### 3.4.7

**Theorem.** *For any sets  $A$  and  $B$ ,  $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ .*

*Proof.* ( $\rightarrow$ ) Let  $M$  be an arbitrary set and suppose  $M \in \mathcal{P}(A \cap B)$ . Then  $M \subseteq A \cap B$ . Let  $x$  be arbitrary and suppose  $x \in M$ . Since  $M \subseteq A \cap B$ ,  $x \in A \cap B$  and therefore  $x \in A$ . Since  $x$  was arbitrary,  $M \subseteq A$  and therefore  $M \in \mathcal{P}(A)$ . Similarly, since  $M \subseteq A \cap B$ ,  $x \in B$ . Since  $x$  was arbitrary,  $M \subseteq B$  and therefore  $M \in \mathcal{P}(B)$ . Therefore,  $M \in \mathcal{P}(A)$  and  $M \in \mathcal{P}(B)$ .

( $\leftarrow$ ) Now suppose  $M \in \mathcal{P}(A) \cap \mathcal{P}(B)$ . Then  $M \subseteq A$  and  $M \subseteq B$ . Suppose  $x \in M$ . Since  $M \subseteq A$  and  $M \subseteq B$  then  $x \in A \cap B$ . Since  $x$  was arbitrary,  $M \subseteq A \cap B$  and therefore  $M \in \mathcal{P}(A \cap B)$ . □

### 3.4.8

**Theorem.**  $A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$

*Proof.* ( $\rightarrow$ ) Suppose  $A \subseteq B$ . Let  $M$  be an arbitrary set and suppose  $M \in \mathcal{P}(A)$ . Then  $M \subseteq A$ . Now let  $y$  be arbitrary and suppose  $y \in M$ . Since  $M \subseteq A$  then  $y \in A$ , and since  $A \subseteq B$  then  $y \in B$ . Since  $y$  was arbitrary,  $M \subseteq B$  and therefore  $M \in \mathcal{P}(B)$ . Since  $M$  was arbitrary,  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ .

( $\leftarrow$ ) Now suppose  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$  and  $y \in A$ . Then the set  $\{y\}$  is in  $\mathcal{P}(A)$ . Since  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$  then  $\{y\} \in \mathcal{P}(B)$  and  $y \in B$ . Since  $y$  was arbitrary,  $A \subseteq B$ . □

### 3.4.9

**Theorem.** *If  $x$  and  $y$  are odd integers, then  $xy$  is odd.*

*Proof.* Suppose  $x$  and  $y$  are odd integers. This means there is an integer  $k$  such that  $x = 2k + 1$  and there is an integer  $j$  such that  $y = 2j + 1$ . Therefore,  $xy = 2(2kj + k + j) = 4kj + 2k + 2j + 1 = (2k + 1)(2j + 1)$ , and since  $2kj + k + j$  is an integer, then  $xy$  is odd.  $\square$

### 3.4.10

**Theorem.** *For every integer  $n$ ,  $n^3$  is even iff  $n$  is even.*

*Proof.* ( $\rightarrow$ ) Let  $n$  be arbitrary. We will prove the contrapositive. Suppose  $x$  is odd, which means there exists an integer  $k$  such that  $x = 2k + 1$ . Therefore,  $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ . Since  $4k^3 + 6k^2 + 3k$  is an integer,  $n^3$  is odd. Therefore, if  $n^3$  is even,  $n$  is even.

( $\leftarrow$ ) Now suppose  $n$  is even, which means there exists an integer  $m$  such that  $n = 2m$ . Now  $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$  and since  $4m^3$  is an integer,  $n^3$  is even.  $\square$

### 3.4.11

#### A

The problem is with using the same variable  $k$  for defining  $m$  as an even integer and  $n$  as an odd integer when  $k$  may take on different values for  $n$  and  $m$ .

#### B

Let  $m = 2$  and  $n = -3$ . Then  $n^2 - m^2 = (-3)^2 - 2^2 = 9 - 4 = 5$  and  $n + m = -3 + 2 = -1$ . Therefore  $n^2 - m^2 \neq n + m$ .

### 3.4.12

**Theorem.**  $\forall x \in \mathbb{R} [\exists y \in \mathbb{R} (x + y = xy) \iff x \neq 1]$

*Proof.* ( $\rightarrow$ ) We will prove by contradiction. Suppose  $x$  is an arbitrary real number and there exists a real number  $y$  such that  $x + y = xy$ . Now suppose  $x = 1$ . Since  $x + y = xy$ , then  $y = \frac{x}{x-1}$ . But this contradicts  $x = 1$  because there is no real number  $y$  such that  $y = x/0$ .

( $\leftarrow$ ) Now suppose  $x \neq 1$  and  $y = \frac{x}{x-1}$ . Then

$$\begin{aligned}
x + y &= x + \frac{x}{x+1} = \frac{x(x+1) + x}{x+1} \\
&= \frac{x^2 - x + x}{x-1} \\
&= \frac{x^2}{x-1} = xy
\end{aligned}$$

□

### 3.4.13

**Theorem.**  $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+ [\exists y \in \mathbb{R} (y - x = \frac{y}{x}) \iff x \neq z]$

*Proof.* ( $\rightarrow$ ) Let  $z = 1$ . Let  $x$  be an arbitrary real number and suppose  $x > 0$ . Suppose  $y \in \mathbb{R}$  and  $y - x = \frac{y}{x}$ . Then  $y = \frac{x^2}{x-1}$ . Now suppose  $x = 1$ . This contradicts  $y \in \mathbb{R}$  and  $y = \frac{x^2}{x-1}$ . Therefore,  $x \neq z$  and since  $x$  was arbitrary we can conclude  $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+ [\exists y \in \mathbb{R} (y - x = \frac{y}{x}) \rightarrow x \neq z]$ .

( $\leftarrow$ ) Now suppose  $x \neq 1$  and  $y = \frac{x^2}{x-1}$ . Then

$$\begin{aligned}
y - x &= \frac{x^2}{x-1} - x = \frac{x^2 - x(x-1)}{x-1} \\
&= \frac{x^2 - x + 2 + x}{x-1} = \frac{x}{x-1} = \frac{y}{x}
\end{aligned}$$

□

### 3.4.14

**Theorem.** If  $B$  is a set and  $\mathcal{F}$  is a family of sets, then  $\cup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathcal{P}(B))$ .

*Proof.* Let  $x$  be arbitrary and suppose  $x \in \cup\{A \setminus B \mid A \in \mathcal{F}\}$ . This means that there is a set  $A \in \mathcal{F}$  such that  $x \in A$  and also  $x \notin B$ . Since  $x \in A$  and  $x \notin B$ , then  $A \not\subseteq B$  and  $A \notin \mathcal{P}(B)$ . Thus there is a set  $A \in \mathcal{F}$  such that  $x \in A$ , and  $A \notin \mathcal{P}(B)$ , which means that  $x \in \cup(\mathcal{F} \setminus \mathcal{P}(B))$ . Therefore, if  $x \in \cup\{A \setminus B \mid A \in \mathcal{F}\}$  then  $x \in \cup(\mathcal{F} \setminus \mathcal{P}(B))$  and since  $x$  was arbitrary, we can conclude  $\cup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathcal{P}(B))$ . □

### 3.4.15

**Theorem.** If  $\mathcal{F}$  and  $\mathcal{G}$  are nonempty families of sets and every element of  $\mathcal{F}$  is disjoint from some element of  $\mathcal{G}$ , then  $\cup\mathcal{F}$  and  $\cap\mathcal{G}$  are disjoint.

*Proof.* Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are nonempty families of sets and every element of  $\mathcal{F}$  is disjoint from some element of  $\mathcal{G}$ . We will use proof by contradiction. Now suppose  $\cup\mathcal{F}$  and  $\cap\mathcal{G}$  are not disjoint. Then there exists a  $y$  such that  $y \in \cup\mathcal{F}$  and  $y \in \cap\mathcal{G}$ . Since  $y \in \cup\mathcal{F}$  there is a set in  $\mathcal{F}$  that contains  $y$  and since  $y \in \cap\mathcal{G}$ ,  $y$  is in every set in  $\mathcal{G}$ . But because every element of  $\mathcal{F}$  is disjoint from some element of  $\mathcal{G}$ , then there is at least one set in  $\mathcal{G}$  that does not contain  $y$ . But this contradicts  $y \in \cap\mathcal{G}$ . Therefore,  $(\cup\mathcal{F}) \cap (\cap\mathcal{G}) = \emptyset$ .  $\square$

### 3.4.16

**Theorem.** For any set  $A$ ,  $A = \cup\mathcal{P}(A)$ .

*Proof.* ( $\rightarrow$ ) Suppose  $A$  is an arbitrary set,  $x$  is arbitrary, and  $x \in A$ . Then there is subset of  $A$  that contains  $x$  and, by definition, this subset is in  $\mathcal{P}(A)$ . Therefore,  $x \in \cup\mathcal{P}(A)$ . Since  $x$  was arbitrary  $A \subseteq \cup\mathcal{P}(A)$ .

( $\leftarrow$ ) Now suppose  $x \in \cup\mathcal{P}(A)$ . This means there is a subset of  $A$  that contains  $x$  and therefore  $x \in A$ . Since  $x$  was arbitrary we conclude  $\cup\mathcal{P}(A) \subseteq A$ . Since  $A$  was arbitrary, we can conclude for all sets  $A$ ,  $A = \cup\mathcal{P}(A)$ .  $\square$

### 3.4.17

#### A

**Theorem.**  $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in \cup(\mathcal{F} \cap \mathcal{G})$ . Since  $x \in \cup(\mathcal{F} \cap \mathcal{G})$  there is a set in  $\mathcal{F}$  and in  $\mathcal{G}$  that both contain  $x$ . Since there is a set in  $\mathcal{F}$  than contains  $x$ , then  $x \in \cup\mathcal{F}$  and since there is a set in  $\mathcal{G}$  that contains  $x$ ,  $x \in \cup\mathcal{G}$ . Therefore,  $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ . Since  $x$  was arbitrary, we can conclude  $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ .  $\square$

#### B

The mistake is that we can't choose a set  $A$  such that  $A \in \mathcal{F}$  and  $A \in \mathcal{G}$  and  $x \in A$ . The given  $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$  means that  $x$  is within a set in  $\mathcal{F}$  and within a set in  $\mathcal{G}$ , but these two sets are not necessarily the same set.

#### C

Let  $\mathcal{F} = \{\{1, 2\}, \{3\}\}$  and  $\mathcal{G} = \{\{4, 5\}, \{1\}\}$ . Then  $\cup(\mathcal{F} \cap \mathcal{G}) = \emptyset$ , but  $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \{1\}$ .