

Exercise 3.3.4

Suppose $A \subseteq \mathcal{P}(A)$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$.

So we want to prove that $\forall x(x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A)))$.

First we assume x is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathcal{P}(A)$ $x \in \mathcal{P}(A)$	$x \in \mathcal{P}(\mathcal{P}(A))$

Assume x is an arbitrary element of $\mathcal{P}(A)$

Suppose $x \in \mathcal{P}(A)$

[proof of $x \in \mathcal{P}(\mathcal{P}(A))$]

Therefore if $x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A))$

Since x was arbitrary we can conclude $\forall x(x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A)))$

We can rewrite our goal as $x \subseteq \mathcal{P}(A)$ or $\forall y(y \in x \rightarrow y \in \mathcal{P}(A))$. So we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathcal{P}(A)$ $x \in \mathcal{P}(A)$ $y \in x$	$y \in \mathcal{P}(A)$

Assume x is an arbitrary element of $\mathcal{P}(A)$

Suppose $x \in \mathcal{P}(A)$

Suppose y is an arbitrary element of x .

Suppose $y \in x$.

[proof of $y \in \mathcal{P}(A)$]

Therefore if $y \in x \rightarrow y \in \mathcal{P}(A)$.

Since y was arbitrary we can conclude that $x \subseteq \mathcal{P}(A)$.

Therefore if $x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A))$

Since x was arbitrary we can conclude $\forall x(x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A)))$

Now looking at our givens $x \in \mathcal{P}(A)$ means that $x \subseteq A$ or $\forall z(z \in x \rightarrow z \in A)$. Using universal instantiation we will plug in y for z and using modus ponens we can conclude that $y \in A$.

Now looking at our other given $A \subseteq \mathcal{P}(A) \rightarrow \forall m(m \in A \rightarrow m \in \mathcal{P}(A))$. Using universal instantiation we will plug in y for m and using modus ponens we can conclude that $y \in \mathcal{P}(A)$, which was our goal to prove.

Theorem. Suppose $A \subseteq \mathcal{P}(A)$. Then $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$.

Proof. Suppose x is an arbitrary element of $\mathcal{P}(A)$ and y is an arbitrary element of x . It follows that $y \in A$. But since $A \subseteq \mathcal{P}(A)$ then it also follows that $y \in \mathcal{P}(A)$. So $y \in x \rightarrow y \in \mathcal{P}(A)$ and since y was arbitrary we can conclude that $x \subseteq \mathcal{P}(A)$. Therefore, if $x \in \mathcal{P}(A) \rightarrow x \in \mathcal{P}(\mathcal{P}(A))$. Since x was arbitrary we can also conclude that $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$. \square

Alternate proof (not sure if this is correct)

Proof. Suppose x is an arbitrary element of $\mathcal{P}(A)$. Then $x \in A$. Since $A \subseteq \mathcal{P}(A)$ and $x \in A$ then $x \subseteq \mathcal{P}(A)$. Therefore, $x \in \mathcal{P}(\mathcal{P}(A))$. \square

Exercise 3.3.5

The hypothesis of the theorem proven in exercise 3.3.4 is $A \subseteq \mathcal{P}(A)$.

A

Can you think of a set A for which this hypothesis is true?

The empty set \emptyset is a set for which the hypothesis is true.

$A \subseteq \mathcal{P}(A)$ means $x \in A \rightarrow x \in \mathcal{P}(A)$. For \emptyset this would mean that $x \in \emptyset \rightarrow x \in \mathcal{P}(\emptyset)$, but by definition there are no elements in \emptyset . Therefore $x \in \emptyset$ will always be false and the conditional statement $x \in \emptyset \rightarrow x \in \mathcal{P}(\emptyset)$ is always true. Therefore if $\emptyset = A$ then $A \subseteq \mathcal{P}(A)$.

B

Can you think of another?

In exercise 3.3.4 we proved that if $A \subseteq \mathcal{P}(A)$ then $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$. Therefore, the set $\{\emptyset, \{\emptyset\}\}$, which is the $\mathcal{P}(A)$ if $A = \emptyset$, is another set for which the hypothesis is true. If we let $B = \mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$ and replace A in the hypothesis $A \subseteq \mathcal{P}(A)$ with B , then we can conclude that $B \subseteq \mathcal{P}(B)$.

Exercise 3.3.6

Suppose x is a real number.

A

Prove that if $x \neq 1$ then there is a real number y such that $\frac{y+1}{y-2} = x$.

So we want to prove that $(x \neq 1) \rightarrow \exists y \left(\frac{y+1}{y-2} = x \right)$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$x \neq 1$	$\exists y \left(\frac{y+1}{y-2} = x \right)$

To prove our goal we need to find a y that makes the equation $\frac{y+1}{y-2} = x$ true. So let's try solving the equation for y .

$$\begin{aligned}
 \frac{y+1}{y-2} &= x \\
 y+1 &= x(y-2) \\
 y+1 &= xy-2x \\
 2x+1 &= xy-y \\
 2x+1 &= y(x-1) \\
 y &= \frac{2x+1}{x-1}
 \end{aligned}$$

We see that this y works because we have $x \neq 1$ as a given.

Theorem. Suppose $x \neq 1$. Then there is a real number y such that $\frac{y+1}{y-2} = x$.

Proof. Suppose $x \neq 1$ and $y = \frac{2x+1}{x-1}$. Then

$$\frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{x-1} \cdot \frac{x-1}{3} = x$$

□

B

Prove that if there is a real number y such that $\frac{y+1}{y-2} = x$ then $x \neq 1$.

So we want to prove that $\exists y \left(\frac{y+1}{y-2} = x \right) \rightarrow (x \neq 1)$

We assume the antecedent and make the consequent our goal to prove.

Using existential instantiation we assume there is a value y_0 such that $\frac{y_0+1}{y_0-2} = x$ is true. From part A above, we know that $\left(\frac{y+1}{y-2} = x \right) \rightarrow \left(y = \frac{2x+1}{x-1} \right)$ and so $y_0 = \frac{2x+1}{x-1}$. Since y is a real number, then clearly $x \neq 1$.

Givens	Goals
$\exists y \left(\frac{y+1}{y-1} = x \right)$	$x \neq 1$

Theorem. If y is a real number and $\frac{y+1}{y-2} = x$ then $x \neq 1$.

Proof. Suppose y is a real number and $\frac{y+1}{y-2} = x$. It follows that $y = \frac{2x+1}{x-1}$ and since y is real number then $x \neq 1$. \square

Exercise 3.3.7

Prove for every real number x , if $x > 2$ then there is a real number y such that $y + \frac{1}{y} = x$.

So we want to prove $\forall x \in \mathbb{R}(x > 2 \rightarrow \exists y \in \mathbb{R}(y + \frac{1}{y} = x))$

So we let x be an arbitrary real number, then we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
x is arbitrary real number	$\exists y(y + \frac{1}{y} = x)$
$x > 2$	

Our goal is of the form $\exists y P(y)$ where $P(y)$ is $y + \frac{1}{y} = x$ and our strategy suggests we try to find a y for which $P(y)$ is true. We can do this by solving the equation $y + \frac{1}{y} = x$ for y . We can rewrite this equation as $y^2 - \frac{x}{y} + 1 = 0$ and we see this is a quadratic equation and therefore we can use the quadratic formula to solve for y ,

$$y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{x \pm \sqrt{x^2 - 4}}{2}.$$

We note that $\sqrt{x^2 - 4}$ is defined because $x > 2$. We have found two solutions that satisfy our original equation, but we only need one to complete the proof. We will use $\frac{x + \sqrt{x^2 - 4}}{2}$.

Theorem. For every real number x , if $x > 2$ then there is a real number y such that $y + \frac{1}{y} = x$.

Proof. Suppose x and y are real numbers, $x > 2$, and $y = \frac{x + \sqrt{x^2 - 4}}{2}$. Then

$$\begin{aligned}
\frac{x + \sqrt{x^2 - 4}}{2} + \frac{1}{\frac{x + \sqrt{x^2 - 4}}{2}} &= \frac{x + \sqrt{x^2 - 4}}{2} + \frac{2}{x + \sqrt{x^2 - 4}} \\
&= \frac{2x^2 + 2(x\sqrt{x^2 - 4})}{2x + 2\sqrt{x^2 - 4}} \\
&= x
\end{aligned}$$

□

Exercise 3.3.8

Prove that if \mathcal{F} is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup \mathcal{F}$.

So we want to prove that $A \in \mathcal{F} \rightarrow A \subseteq \cup \mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$A \subseteq \cup \mathcal{F}$

Assume $A \in \mathcal{F}$

[proof of $A \subseteq \cup \mathcal{F}$]

Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup \mathcal{F}$

Our goal $A \subseteq \cup \mathcal{F}$ can be rewritten as $\forall x(x \in A \rightarrow x \in \cup \mathcal{F})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in \cup \mathcal{F}$
$x \in A$	

Assume $A \in \mathcal{F}$

Assume x is arbitrary

Assume $x \in A$

[proof of $x \in \cup \mathcal{F}$]

Therefore if $x \in A$ then $x \in \cup \mathcal{F}$

Since x was arbitrary we can conclude that $A \subseteq \cup \mathcal{F}$.

Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup \mathcal{F}$

Our new goal can be rewritten as $\exists B \in \mathcal{F}(x \in B)$. From our givens we see that $A \in \mathcal{F}$ and $x \in A$, so we have found a set such that $A \in \mathcal{F}(x \in A)$.

Theorem. If \mathcal{F} is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup \mathcal{F}$.

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of A . Then since $x \in A$ and $A \in \mathcal{F}$, it follows that $x \in \cup \mathcal{F}$. Since x was arbitrary we can conclude that $A \subseteq \cup \mathcal{F}$ and therefore if $A \in \mathcal{F}$ then $A \subseteq \cup \mathcal{F}$. \square

Exercise 3.3.9

Prove that if \mathcal{F} is a family of sets and $A \in \mathcal{F}$, then $\cap \mathcal{F} \subseteq A$.

We want to prove that $A \in \mathcal{F} \rightarrow \cap \mathcal{F} \subseteq A$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$\cap \mathcal{F} \subseteq A$

Assume $A \in \mathcal{F}$
 [proof of $\cap \mathcal{F} \subseteq A$]
 Therefore, if $A \in \mathcal{F}$ then $\cap \mathcal{F} \subseteq A$.

We can rewrite our goal as $\forall x(x \in \cap \mathcal{F} \rightarrow x \in A)$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in A$
$x \in \cap \mathcal{F}$	

Assume $A \in \mathcal{F}$
 Assume x is arbitrary
 Assume $x \in \cap \mathcal{F}$
 [proof of $x \in A$]
 Therefore, if $x \in \cap \mathcal{F}$ then $x \in A$.
 Since x was arbitrary we can conclude that $\cap \mathcal{F} \subseteq A$.
 Therefore, if $A \in \mathcal{F}$ then $\cap \mathcal{F} \subseteq A$.

Our given $x \in \cap \mathcal{F}$ can be rewritten as $\forall B \in \mathcal{F}(x \in B)$, therefore if $A \in \mathcal{F}$ then $x \in A$, which was our goal to prove.

Theorem. If \mathcal{F} is a family of sets and $A \in \mathcal{F}$, then $\cup \mathcal{F} \in A$.

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of $\cap \mathcal{F}$. Since $A \in \mathcal{F}$ and $x \in \cap \mathcal{F}$ it follows that $x \in A$ and therefore, if $x \in \cap \mathcal{F}$ then $x \in A$. Since x was arbitrary we can conclude that $\cap \mathcal{F} \subseteq A$. Therefore, if $A \in \mathcal{F}$ then $\cap \mathcal{F} \subseteq A$. \square

Exercise 3.3.10

Suppose that \mathcal{F} is a nonempty family of sets B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$.
Prove that $B \subseteq \cap \mathcal{F}$.

We want to prove $\forall A \in \mathcal{F}(B \subseteq A) \rightarrow B \subseteq \cap \mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$	$B \subseteq \cap \mathcal{F}$

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$

[proof of $B \subseteq \cap \mathcal{F}$]

Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap \mathcal{F}$.

Our goal can be rewritten as $\forall x(x \in B \rightarrow x \in \cap \mathcal{F})$. So we assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$ $x \in B$	$x \in \cap \mathcal{F}$

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$

Suppose x is arbitrary.

Suppose $x \in B$.

[proof of $x \in \cap \mathcal{F}$]

Therefore $x \in B \rightarrow x \in \cap \mathcal{F}$

Since x was arbitrary we can conclude $B \subseteq \cap \mathcal{F}$

Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap \mathcal{F}$.

Our goal can be rewritten as $\forall M \in \mathcal{F}(x \in M)$ and so we can assume M is an arbitrary set in \mathcal{F} and make our goal $x \in M$.

Givens	Goals
$\forall A \in \mathcal{F}(B \subseteq A)$ $x \in B$ $M \in \mathcal{F}$	$x \in M$

Suppose $\forall A \in \mathcal{F}(B \subseteq A)$

Suppose x is arbitrary.

Suppose $x \in B$.

Suppose M is an arbitrary set in \mathcal{F} .

[proof of $x \in M$]
Therefore $x \in \cap \mathcal{F}$
Therefore $x \in B \rightarrow x \in \cap \mathcal{F}$
Since x was arbitrary we can conclude $B \subseteq \cap \mathcal{F}$
Therefore if $\forall A \in \mathcal{F}(B \subseteq A)$ then $B \subseteq \cap \mathcal{F}$.

Using universal instantiation we will plug in M for A in our given $\forall A \in \mathcal{F}(B \subseteq A)$ and conclude that $B \subseteq M$. We can rewrite $B \subseteq M$ as $\forall y(y \in B \rightarrow y \in M)$ and using universal instantiation plug in x for y and then use modus ponens to conclude $x \in M$, which was our goal to prove.

Theorem. *If \mathcal{F} is a nonempty family of sets, B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$, then $B \subseteq \cap \mathcal{F}$.*

Proof. Suppose $\forall A \in \mathcal{F}(B \subseteq A)$. Suppose x is an arbitrary member of B and M is an arbitrary set in \mathcal{F} . Then it follows that $x \in M$ and since M was arbitrary we can conclude that x is in all sets that are in \mathcal{F} or $x \in \cap \mathcal{F}$. Therefore, if $x \in B$ then $x \in \cap \mathcal{F}$, and since x was arbitrary, we can conclude that $B \subseteq \cap \mathcal{F}$. \square

Exercise 3.3.11

Suppose that \mathcal{F} is a family of sets. Prove that if $\emptyset \in \mathcal{F}$ then $\cap \mathcal{F} = \emptyset$.
We want to prove that $\emptyset \in \mathcal{F} \rightarrow \cap \mathcal{F} = \emptyset$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\emptyset \in \mathcal{F}$	$\cap \mathcal{F} = \emptyset$

Suppose $\emptyset \in \mathcal{F}$
[proof of $\cap \mathcal{F} = \emptyset$]
Therefore if $\emptyset \in \mathcal{F}$ then $\cap \mathcal{F} = \emptyset$.

We will try a proof by contradiction. So we assume that $\cap \mathcal{F} \neq \emptyset$ and try to find a contradiction.

Givens	Goals
$\emptyset \in \mathcal{F}$	contradiction
$\cap \mathcal{F} \neq \emptyset$	

Our given $\cap \mathcal{F} \neq \emptyset$ means that there is an element that is in all sets in \mathcal{F} . However, this contradicts $\emptyset \in \mathcal{F}$ because \emptyset is the set that contains nothing.

Theorem. *If \mathcal{F} is a family of sets and $\emptyset \in \mathcal{F}$, then $\cap \mathcal{F} = \emptyset$.*

Proof. We will prove by contradiction. Suppose $\emptyset \in \mathcal{F}$ and $\cap \mathcal{F} \neq \emptyset$. Since $\cap \mathcal{F} \neq \emptyset$ it follows that there is an element that is within all of the sets that are in \mathcal{F} . However, this contradicts $\emptyset \in \mathcal{F}$ because \emptyset is the set that contains nothing. Therefore, if $\emptyset \in \mathcal{F}$ then $\cap \mathcal{F} = \emptyset$. \square

Exercise 3.3.12

Suppose \mathcal{F} and \mathcal{G} are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup \mathcal{F} \subseteq \cup \mathcal{G}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cup \mathcal{F} \subseteq \cup \mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
 [proof of $\cup \mathcal{F} \subseteq \cup \mathcal{G}$]
 So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

$\cup \mathcal{F} \subseteq \cup \mathcal{G} \rightarrow \forall b(b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G})$ so we assume b is an arbitrary element of $\cup \mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$ $b \in \cup \mathcal{F}$	$b \in \cup \mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
 Let b be an arbitrary element of $\cup \mathcal{F}$
 [proof of $b \in \cup \mathcal{G}$]
 Therefore if $b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G}$
 Since b was arbitrary we can conclude $\forall b(b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

$b \in \cup \mathcal{F} \rightarrow \exists M(M \in \mathcal{F} \wedge b \in M)$, so let $M = A_0$ (Existential Instantiation)

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$ $A_0 \in \mathcal{F} \wedge b \in A_0$	$b \in \cup \mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
 Let b be an arbitrary element and suppose $b \in \cup \mathcal{F}$, which implies there is a set in \mathcal{F} and b is in that set. Let that set = A_0

[proof of $b \in \cup \mathcal{G}$]

Therefore if $b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G}$

Since b was arbitrary we can conclude $\forall b(b \in \cup \mathcal{F} \rightarrow b \in \cup \mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup \mathcal{F} \subseteq \cup \mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall A(A \in \mathcal{F} \rightarrow A \in \mathcal{G})$. Using universal instantiation we will plug in A_0 for A since then we can use modens ponens to conclude that $A_0 \in \mathcal{G}$.

Givens	Goals
$A_0 \in \mathcal{F} \rightarrow A_0 \in \mathcal{G}$	$b \in \cup \mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Our goal $b \in \cup \mathcal{G} \rightarrow \exists N(N \in \mathcal{G} \wedge b \in N)$, which we can now prove. Since $A_0 \in \mathcal{F}$ and \mathcal{F} is a subset of \mathcal{G} , it follows that $A_0 \in \mathcal{G}$. By the definition of $\cup \mathcal{G}$ it follows that $b \in \cup \mathcal{G}$ because $A_0 \in \mathcal{G} \wedge b \in A_0$, the latter statement being one of our givens.

Theorem. Suppose \mathcal{F} and \mathcal{G} are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup \mathcal{F} \subseteq \cup \mathcal{G}$.

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cup \mathcal{F}$, which implies there is a set in \mathcal{F} that contains b . Call this set A_0 . Since $A_0 \in \mathcal{F}$ and \mathcal{F} is a subset of \mathcal{G} it follows that $A_0 \in \mathcal{G}$, which implies that $b \in \cup \mathcal{G}$. Therefore if $b \in \cup \mathcal{F}$ then $b \in \cup \mathcal{G}$. Since b was arbitrary we can conclude that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup \mathcal{F} \subseteq \cup \mathcal{G}$. This completes the proof.

Exercise 3.3.13

Suppose \mathcal{F} and \mathcal{G} are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap \mathcal{G} \subseteq \cap \mathcal{F}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cap \mathcal{G} \subseteq \cap \mathcal{F}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$

[proof of $\cap \mathcal{G} \subseteq \cap \mathcal{F}$]

So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

$\cap \mathcal{G} \subseteq \cap \mathcal{F} \rightarrow \forall b(b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F})$, so we assume b is an arbitrary element of $\cap \mathcal{G}$ and assume the antecedent and make the consequent our goal to prove.

Suppose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cap \mathcal{G}$

[proof of $b \in \cap \mathcal{F}$]

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cap \mathcal{F}$
$b \in \cap \mathcal{G}$	

Therefore if $b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F}$

Since b was arbitrary we can conclude $\forall b(b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

$b \in \cap \mathcal{F} \rightarrow \forall A(A \in \mathcal{F} \rightarrow b \in A)$, so we assume A is an arbitrary element of \mathcal{F} and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in A$
$b \in \cap \mathcal{G}$	
$A \in \mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cap \mathcal{G}$

Suppose A is an arbitrary set in \mathcal{F}

[proof of $b \in A$]

Therefore if $A \in \mathcal{F} \rightarrow b \in A$

Since A was arbitrary we can conclude $b \in \cap \mathcal{F}$

Therefore if $b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F}$

Since b was arbitrary we can conclude $\forall b(b \in \cap \mathcal{G} \rightarrow b \in \cap \mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap \mathcal{G} \subseteq \cap \mathcal{F}$

Now looking at our givens, $\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall Z(Z \in \mathcal{F} \rightarrow Z \in \mathcal{G})$. Using universal instantiation we will plug in A for Z and using modus ponens we can conclude that $A \in \mathcal{G}$.

Our other given, $b \in \cap \mathcal{G} \rightarrow \forall Y(Y \in \mathcal{G} \rightarrow b \in Y)$. Using universal instantiation we will plug in A for Y and using modus ponens we can conclude that $b \in A$, which was our goal, and we can now write our proof.

Theorem. Suppose \mathcal{F} and \mathcal{G} are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cap \mathcal{G} \subseteq \cap \mathcal{F}$.

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cap \mathcal{G}$. Suppose A is an arbitrary element of \mathcal{F} , then because $\mathcal{F} \subseteq \mathcal{G}$ then it follows that $A \in \mathcal{G}$. By the definition of $\cap \mathcal{G}$ it follows that $b \in A$ and since A was arbitrary then $b \in \cap \mathcal{F}$. Since b was arbitrary we can conclude $\cap \mathcal{G} \subseteq \cap \mathcal{F}$ and therefore that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap \mathcal{G} \subseteq \cap \mathcal{F}$. This completes the proof.

Exercise 3.3.14

Suppose $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$.

So we want to prove that $\forall a (a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \mathcal{P}(\bigcup_{i \in I} A_i))$

First we assume a is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i \in I} \mathcal{P}(A_i)$	$a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Assume a is an arbitrary element of $\bigcup_{i \in I} \mathcal{P}(A_i)$

Suppose $a \in \bigcup_{i \in I} \mathcal{P}(A_i)$

[proof of $a \in \mathcal{P}(\bigcup_{i \in I} A_i)$]

Therefore if $a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Since a was arbitrary we can conclude $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$

Looking at our goal we see that $a \in \mathcal{P}(\bigcup_{i \in I} A_i) \rightarrow a \subseteq \bigcup_{i \in I} A_i \rightarrow \forall z (z \in a \rightarrow z \in \bigcup_{i \in I} A_i)$. Therefore we assume z is arbitrary, assume the antecedent, and make the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i \in I} \mathcal{P}(A_i)$	$z \in \bigcup_{i \in I} A_i$
$z \in a$	

Assume a is an arbitrary element of $\bigcup_{i \in I} \mathcal{P}(A_i)$

Suppose $a \in \bigcup_{i \in I} \mathcal{P}(A_i)$

Assume z is arbitrary

Assume $z \in a$

[proof of $z \in \bigcup_{i \in I} A_i$]

Therefore $z \in a \rightarrow z \in \bigcup_{i \in I} A_i$

Since z was arbitrary we can conclude $a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Therefore if $a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \mathcal{P}(\bigcup_{i \in I} A_i)$

Since a was arbitrary we can conclude $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$

Looking at our given we see that $a \in \bigcup_{i \in I} \mathcal{P}(A_i) \rightarrow a \in \{a | \exists i \in I (a \in \mathcal{P}(A_i))\}$. Using existential instantiation we will select an i such that $a \in \mathcal{P}(A_i)$ which implies $a \subseteq A_i$. Since $a \subseteq A_i \rightarrow \forall m (m \in a \rightarrow m \in A_i)$ and using universal instantiation we will plug in z for m and we get $\forall z (z \in a \rightarrow z \in A_i)$ and using modus ponens we can conclude that $z \in A_i$, which implies that $z \in \bigcup_{i \in I} A_i$, which was our goal. We can now right our proof.

Theorem. Suppose $\{A_i | i \in I\}$ is an indexed family of sets, then $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$.

Proof. Suppose that a is an arbitrary element of $\bigcup_{i \in I} \mathcal{P}(A_i)$. We choose an $i \in I$ such that $a \in \mathcal{P}(A_i)$, which implies that $a \subseteq A_i$. Suppose z is an arbitrary element of a , then it follows that $z \in A_i$ and therefore $z \in \bigcup_{i \in I} A_i$. Since z was an arbitrary element of a then $a \subseteq \bigcup_{i \in I} A_i$, and it follows that $a \in \mathcal{P}(\bigcup_{i \in I} A_i)$. Thus we can conclude $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$. This completes the proof.

Exercise 3.3.15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \emptyset$. Prove that $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$

So we want to prove that $\forall y (y \in \bigcap_{i \in I} A_i \rightarrow y \in \bigcap_{i \in I} \mathcal{P}(A_i))$.

First we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i \in I} A_i$	$y \in \bigcap_{i \in I} \mathcal{P}(A_i)$

Suppose y is arbitrary element of $\bigcap_{i \in I} A_i$.

[proof of $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$]

Since y was arbitrary we can conclude $\forall y (y \in \bigcap_{i \in I} A_i \rightarrow y \in \bigcap_{i \in I} \mathcal{P}(A_i))$.

Our goal $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$ so we make m an arbitrary element of I and therefore $y \in \mathcal{P}(A_m) \rightarrow y \subseteq A_m \rightarrow \forall z (z \in y \rightarrow z \in A_m)$. So we make z arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i \in I} A_i$	$z \in A_m$
$z \in y$	

Suppose y is arbitrary element of $\bigcap_{i \in I} A_i$.

Suppose m is an arbitrary element of I and therefore $y \in \mathcal{P}(A_m) \rightarrow y \subseteq A_m \rightarrow \forall z (z \in y \rightarrow z \in A_m)$.

Suppose z is an arbitrary element of y

[proof of $z \in A_m$]

Therefore $z \in y \rightarrow z \in A_m$ and since z was arbitrary $y \subseteq A_m \rightarrow y \in \mathcal{P}(A_m)$

and since m was arbitrary $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$

Since y was arbitrary we can conclude $\forall y (y \in \bigcap_{i \in I} A_i \rightarrow y \in \bigcap_{i \in I} \mathcal{P}(A_i))$.

Now looking at our given $y \in \bigcap_{i \in I} A_i \rightarrow \forall i \in I (y \in A_i)$. Using universal instantiation we plug in m for i and therefore $y \in A_m$ and since $z \in y$ we can conclude $z \in A_m$, which was our goal. Now we can write our proof.

Theorem. Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \emptyset$, then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$.

Proof. Suppose y is an arbitrary element of $\bigcap_{i \in I} A_i$. Suppose m is an arbitrary member of I and therefore $y \subseteq A_m$ which implies $y \subseteq A_m$. Now suppose z is an arbitrary element of y . Since $y \in \bigcap_{i \in I} A_i$ if we choose an i such that $y \in \bigcap_{m \in I} A_m$ then $y \in A_m$ which implies $z \in A_m$. Therefore if $z \in y$ then $z \in A_m$ and since z was arbitrary then $y \subseteq A_m$ or $y \in \mathcal{P}(A_m)$ and since m was arbitrary then $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$. Since y was arbitrary then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$. This completes the proof.

Exercise 3.3.16

Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\cup \mathcal{F} \subseteq B$.

We want to prove $\mathcal{F} \subseteq \mathcal{P}(B) \rightarrow \cup \mathcal{F} \subseteq B$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{P}(B)$	$\cup \mathcal{F} \subseteq B$

Suppose $\mathcal{F} \subseteq \mathcal{P}(B)$

[proof of $\cup \mathcal{F} \subseteq B$

Therefore if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\cup \mathcal{F} \subseteq B$.

$\cup \mathcal{F} \subseteq B \rightarrow \forall x (x \in \cup \mathcal{F} \rightarrow x \in B)$. So we assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{P}(B)$	$x \in B$
$x \in \cup \mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{P}(B)$

Suppose x is arbitrary

Suppose $x \in \cup \mathcal{F}$

[proof of $x \in B$]

Therefore if $x \in \cup \mathcal{F}$ then $x \in B$

Since x was arbitrary we can conclude that $\cup \mathcal{F} \subseteq B$.

Therefore if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\cup \mathcal{F} \subseteq B$.

$x \in \cup \mathcal{F} \rightarrow \exists M \in \mathcal{F}(x \in M)$. We use existential instantiation and assume there is a set M in \mathcal{F} and x is in that set.

Suppose $\mathcal{F} \subseteq \mathcal{P}(B)$

Suppose x is arbitrary

Suppose M is arbitrary set in \mathcal{F}

$x \in M$

[proof of $x \in B$]

Since $x \in M$ and M is a set in \mathcal{F} then $x \in \cup \mathcal{F}$

Therefore if $x \in \cup \mathcal{F}$ then $x \in B$

Since x was arbitrary we can conclude that $\cup \mathcal{F} \subseteq B$.

Therefore if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\cup \mathcal{F} \subseteq B$.

Our given $\mathcal{F} \subseteq \mathcal{P}(B)$ means that $\forall N(N \in \mathcal{F} \rightarrow \forall z(z \in N \rightarrow z \in B))$. We will use universal instantiation and plug in M for N and x for z and we can conclude that $x \in B$, which was our goal to prove.

Theorem. Suppose B is a set and \mathcal{F} is a family of sets. If $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\cup \mathcal{F} \subseteq B$.

Proof. Suppose x is an arbitrary member of $\cup \mathcal{F}$, which means that x is a member of a set that is in \mathcal{F} . Suppose $\mathcal{F} \subseteq \mathcal{P}(B)$, which means that any element that is in a set that is a member of \mathcal{F} is also in the set B . It follows that since x is a member of a set in \mathcal{F} then $x \in B$. Therefore, if $x \in \cup \mathcal{F}$ then $x \in B$ and since x was arbitrary we can conclude $\cup \mathcal{F} \subseteq B$. Therefore, if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\cup \mathcal{F} \subseteq B$. \square

Exercise 3.3.17

Suppose \mathcal{F} and \mathcal{G} are nonempty families of sets, and every element of \mathcal{F} is a subset of every element of \mathcal{G} . Prove that $\cup \mathcal{F} \subseteq \cap \mathcal{G}$.

We want to prove that $\cup \mathcal{F} \subseteq \cap \mathcal{G}$.

We can rewrite this goal as $\forall x(x \in \cup \mathcal{F} \rightarrow x \in \cap \mathcal{G})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal.

Givens	Goals
$x \in \cup \mathcal{F}$	$x \in \cap \mathcal{G}$
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \subseteq B)$	

Suppose x is arbitrary.

Suppose $x \in \cup \mathcal{F}$.

[proof of $x \in \cap \mathcal{G}$]

Therefore if $x \in \cup \mathcal{F}$ then $x \in \cap \mathcal{G}$

Since x was arbitrary we can conclude that $\cup \mathcal{F} \subseteq \cap \mathcal{G}$.

We can rewrite our goal as $\forall M \in \mathcal{G}(x \in M)$. We assume M is an arbitrary set in \mathcal{G} and then our goal becomes $x \in M$.

Givens	Goals
$x \in \cup \mathcal{F}$	$x \in M$
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \rightarrow y \in B)$	

Suppose x is arbitrary.

Suppose $x \in \cup \mathcal{F}$.

Suppose M is an arbitrary set in \mathcal{G}

[proof of $x \in M$]

Since M was arbitrary we can conclude that $x \in \cap \mathcal{G}$

Therefore if $x \in \cup \mathcal{F}$ then $x \in \cap \mathcal{G}$

Since x was arbitrary we can conclude that $\cup \mathcal{F} \subseteq \cap \mathcal{G}$.

We can rewrite our given $x \in \cup \mathcal{F}$ as $\exists N \in \mathcal{F}(x \in N)$. We use existential instantiation and assume there is a set $N \in \mathcal{F}$ and $x \in N$.

Givens	Goals
$N \in \mathcal{F}$	$x \in M$
$x \in N$	
$M \in \mathcal{G}$	
$\forall A \in \mathcal{F} \forall B \in \mathcal{G}(y \in A \rightarrow y \in B)$	

Now we can use universal instantiation to plug in N for A and M for B . Then since $x \in N$ we can use modus ponens to conclude that $x \in M$, which was our goal.

Theorem. *If \mathcal{F} and \mathcal{G} are nonempty families of sets, and every element of \mathcal{F} is a subset of every element of \mathcal{G} , then $\cup \mathcal{F} \subseteq \cap \mathcal{G}$.*

Proof. Suppose x is an arbitrary member of $\cup \mathcal{F}$, which means there is a set in \mathcal{F} that contains x . Suppose M is an arbitrary set in \mathcal{G} . Then since every set in \mathcal{F} is a subset of every set in \mathcal{G} it follows that $x \in M$. Since M was arbitrary we can conclude that $x \in \cap \mathcal{G}$ and therefore if $x \in \cup \mathcal{F}$ then $x \in \cap \mathcal{G}$. Since x was arbitrary we can conclude that $\cup \mathcal{F} \subseteq \cap \mathcal{G}$. \square

Exercise 3.3.18

In this problem all variables range over \mathbb{Z} , the set of all integers.

A

Prove that if $a|b$ and $a|c$, then $a|(b+c)$.

We want to prove $(a|b) \wedge (a|c) \rightarrow a|(b+c)$

We assume the antecedent and make the consequent our goal.

Givens	Goals
$a b$	$a (b+c)$
$a c$	

Suppose $a|b$ and $a|c$

[proof of $a|(b+c)$

Therefore if $a|b$ and $a|c$ then $a|(b+c)$.

Our goal means that $\exists x \in \mathbb{Z}(ax = (b+c))$. So we need to find an x that makes this statement true. Our goals can be rewritten as $\exists y \in \mathbb{Z}(ay = b)$ and $\exists w \in \mathbb{Z}(aw = c)$. Using existential instantiation we will assume there is a y and w that makes both of the previous statement true.

Givens	Goals
$ay = b$	$a (b+c)$
$aw = c$	

Suppose $ay = b$ and $aw = c$

[proof of $a|(b+c)$

Therefore if $a|b$ and $a|c$ then $a|(b+c)$.

Adding the two inequalities $ay = b$ and $aw = c$ we have $ay + aw = b + c$ or $a(y+w) = b+c$. Since y and w are integers we can conclude that $a|(b+c)$, which was our goal to prove.

Theorem. *If a , b , and c are integers, $a|b$, and $a|c$, then $a|(b+c)$.*

Proof. Suppose a , b , and c are integers, $a|b$, and $a|c$. Since $a|b$ there must be an integer y such that $ay = b$. Also, since $a|c$ there must be an integer w such that $aw = c$. Adding together the previous two equalities we have $ay + aw = b + c$ or $a(y+w) = b+c$. Since y and w are integers we can conclude that $a|(b+c)$. \square

B

Prove that if $ac|bc$ and $c \neq 0$, then $a|b$.

We want to prove $(ac|bc) \wedge (c \neq 0) \rightarrow a|b$.

We assume the antecedent and make the consequent our goal.

Givens	Goals
$ac bc$	$a b$
$c \neq 0$	

Suppose $ac|bc$ and $c \neq 0$

[proof of $a|b$]

Therefore $ac|bc$ and $c \neq 0$, then $a|b$.

Our goal means that $\exists x(ax = b)$ and we want to find an x that makes this statement true. Looking at our goals we can rewrite $ac|bc$ as $\exists y(acy = bc)$. Using existential instantiation we will assume there is a y that makes $acy = bc$ true and we can add $acy = bc$ to our givens. Since $c \neq 0$ we can divide both sides of $acy = bc$ by c and we have $ay = b$. Since y is an integer we can conclude that $a|b$, which was our goal to prove.

Theorem. If a , b , and c are integers, $ac|bc$, and $c \neq 0$, then $a|b$.

Proof. Suppose a , b , and c are integers, $ac|bc$, and $c \neq 0$. Since $ac|bc$ there must be an integer x such that $acx = bc$. Since $c \neq 0$ we can simplify the previous equation by dividing both sides by c so that $ax = b$. Since x is an integer we can conclude that $a|b$. \square

Exercise 3.3.19

A

Prove that for all real numbers x and y there is a real number z such that $x + z = y - z$.

We want to prove that $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R} (x + z = y - z)$.

We let x and y stand for arbitrary real numbers and make $\exists z \in \mathbb{R} (x + z = y - z)$ our goal to prove.

Givens	Goals
x arbitrary	$\exists z \in \mathbb{R} (x + z = y - z)$
y arbitrary	

Suppose x and y are arbitrary real numbers

[proof of $\exists z \in \mathbb{R} (x + z = y - z)$]

Since x and y are arbitrary we can conclude $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R} (x + z = y - z)$

We want to find a z such that $x + z = y - z$, which suggests we try solving this equation for z

$$\begin{aligned}x + z &= y - z \\x + 2z &= y \\z &= \frac{y - x}{2}.\end{aligned}$$

Now we are ready to complete our proof.

Theorem. *For all real numbers x and y there is a real number z such that $x + z = y - z$.*

Proof. Suppose x and y are arbitrary real numbers and $z = \frac{y-x}{2}$. Then

$$\begin{aligned}x + \frac{y-x}{2} &= y - \frac{y-x}{2} \\ \frac{2x + y - x}{2} &= \frac{2y - (y-x)}{2} \\ 2x + y - x &= 2y - y + x \\ x(2-1) + y &= y(2-1) + x \\ x + y &= x + y\end{aligned}$$

□

B

Would the statement in part (A) be correct if "real number" were changed to "integer"? Justify your answer.

No, because there are instances where $z = \frac{x-y}{2}$ would not result in an integer. For example, if $x = 5$ and $y = 2$ then $z = \frac{3}{2}$, which is not an integer. Therefore the statement in part (A) would not be correct.

Exercise 3.3.20

Consider the following theorem:

Theorem. *For every real number x , $x^2 \geq 0$.*

What's wrong with the following proof?

Proof. Suppose not. Then for every real number x , $x^2 < 0$. In particular, plugging in $x = 3$ we would get $9 < 0$, which is clearly false. This contradiction shows that for every number x , $x^2 \geq 0$. □

The sentence "Then for every real number x , $x^2 < 0$ " is not correct because if we let $x = 0$ then $0 < 0$ is not true.

3.3.21

Consider the following incorrect theorem:

Incorrect Theorem. If $\forall x \in A(x \neq 0)$ and $A \subseteq B$ then $\forall x \in B(x \neq 0)$.

A

What's wrong with the following proof?

Proof. Let x be an arbitrary element of A . Since $\forall x \in A(x \neq 0)$, we can conclude that $x \neq 0$. Also, since $A \subseteq B$, $x \in B$. Since $x \in B$, $x \neq 0$, and x was arbitrary, we can conclude that $\forall x \in B(x \neq 0)$. \square

The last sentence is not correct. $A \subseteq B$ means that all elements in A are in B and since $x \neq 0$ then $0 \notin A$, but this doesn't mean that $0 \notin B$, because there can be elements in B that are not in A .

B

Find a counterexample to the theorem. In other words, find an example of sets A and B for which the hypotheses of the theorem are true but the conclusion is false.

Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$. Then the hypotheses of the theorem are true, specifically $\forall x \in A(x \neq 0)$ and $A \subseteq B$, but the conclusion $\forall x \in B(x \neq 0)$ is false.

3.3.22

Consider the following incorrect theorem:

Incorrect Theorem. $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy^2 = y - x)$.

What's wrong with the following proof of the theorem?

Proof. Let $x = \frac{y}{y^2+1}$. Then

$$y - x = y - \frac{y}{y^2+1} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1} \cdot y^2 = xy^2.$$

\square

In the proof, x is defined in terms of y but y has not been introduced into the proof yet. The theorem should start with "Let $x = \dots$ and let y be an arbitrary real number...".

3.3.23

Consider the following incorrect theorem:

Incorrect Theorem *Suppose \mathcal{F} and \mathcal{G} are families of sets. If $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, then so are \mathcal{F} and \mathcal{G} .*

A

What's wrong with the following proof of the theorem?

Proof. Suppose $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint. Suppose \mathcal{F} and \mathcal{G} are not disjoint. Then we can choose some set A such that $A \in \mathcal{F}$ and $a \in \mathcal{G}$. Since $A \in \mathcal{F}$, by exercise 8, $A \subseteq \cup\mathcal{F}$, so every element of A is in $\cup\mathcal{F}$. Similarly, since $A \in \mathcal{G}$, every element of A is in $\cup\mathcal{G}$. But then every element of A is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$, and this is impossible since $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint. Thus, we have reached a contradiction, so \mathcal{F} and \mathcal{G} must be disjoint. \square

The statement “But then every element of A is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$, and this is impossible since $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint.” is not correct. If $A = \{\emptyset\}$ then every element of A , which is \emptyset , is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$ and by definition $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, or $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \emptyset$. So there is no contradiction in this case.

B

Find a counterexample to the theorem.

Let $\mathcal{F} = \{\{\emptyset\}, \{1\}\}$ and $\mathcal{G} = \{\{\emptyset\}, \{2\}\}$. Then $\cup\mathcal{F} = \{1, \emptyset\}$ and $\cup\mathcal{G} = \{2, \emptyset\}$ and therefore $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, or $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \emptyset$. However, \mathcal{F} and \mathcal{G} are not disjoint because $\mathcal{F} \cap \mathcal{G} = \{\emptyset\}$. (Remember, the empty set \emptyset is the set that has no elements, but the set $\{\emptyset\}$ is a set that contains one element, \emptyset .)

3.3.24

Consider the following putative theorem:

Theorem? *For all real numbers x and y , $x^2 + xy - 2y^2 = 0$.*

A

What's wrong with the following proof of the theorem?

Proof. Let x and y be equal to some arbitrary real number r . Then

$$x^2 + xy - 2y^2 = r^2 + r \cdot r - 2r^2 = 0.$$

Since x and y were both arbitrary, this shows that for all real numbers x and y , $x^2 + xy - 2y^2 = 0$.

□

The first sentence of the proof assigns x and y to be the same arbitrary real number, however, it should start with “Let x be an arbitrary real number and let y be an arbitrary real number” or “let x and y be arbitrary real numbers”.

B

No, the theorem is not correct. We will provide a counterexample. Let $x = 2$ and let $y = 3$, then

$$2^2 + 2 \cdot 3 - 2 \cdot 3^2 = 4 + 6 - 2 \cdot 9 = 10 - 18 = -8$$

and $-8 \neq 0$ so the theorem is not correct.

3.3.25

Prove that for every real number x there is a real number y such that for every real number z , $yz = (x + z)^2 - (x^2 + z^2)$.

We want to prove $\forall x \exists y \forall z (yz = (x + z)^2 - (x^2 + z^2))$.

We let x be arbitrary and make our goal $\exists y \forall z (yz = (x + z)^2 - (x^2 + z^2))$.

Givens	Goals
x is arbitrary	$\exists y \forall z (yz = (x + z)^2 - (x^2 + z^2))$

Let x be an arbitrary real number

[proof of $\exists y \forall z (yz = (x + z)^2 - (x^2 + z^2))$]

Therefore $\forall x \exists y \forall z (yz = (x + z)^2 - (x^2 + z^2))$

Now we need to find a y that makes the statement $\forall z (yz = (x + z)^2 - (x^2 + z^2))$ true. This suggests we solve the equation $yz = (x + z)^2 - (x^2 + z^2)$ for y .

$$yz = (x + z)^2 - (x^2 + z^2)$$

$$yz = x^2 + 2xz + z^2 - x^2 - z^2$$

$$yz = 2xz$$

$$y = 2x$$

The last line above $y = 2x$ works even if $z = 0$ because in that case $yz = 0$ and $2xz = 0$ and there is no need to divide by z because we have $0 = 0$.

Let x be an arbitrary real number

Let $y = 2x$

Let z be an arbitrary real number

[proof of $yz = (x + z)^2 - (x^2 + z^2)$]

Therefore $\forall z(yz = (x + z)^2 - (x^2 + z^2))$

Therefore $\exists y \forall z(yz = (x + z)^2 - (x^2 + z^2))$

Therefore $\forall x \exists y \forall z(yz = (x + z)^2 - (x^2 + z^2))$

When writing the proof we have to make sure the order we introduce the variables is the same as above (i.e., we introduce x , y , and then z). When we state $y = 2x$ in the proof, we only have defined x up to that point, so both values we choose for x and y must then work for every value of z , or every real number. (See <https://github.com/kstratto/How-to-Prove-It/blob/master/How%20to%20Prove%20It%20-%20Chapter%203.pdf>.)

Theorem. *For every real number x there is a real number y such that for every real number z , $yz = (x + z)^2 - (x^2 + z^2)$.*

Proof. Let x be an arbitrary real number. Let $y = 2x$. Let z be an arbitrary real number. Then

$$2xz = (x + z)^2 - (x^2 + z^2) = x^2 + 2xz + z^2 - x^2 - z^2 = 2xz$$

□