1 Exercise 3.3.12

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cup\mathcal{F} \subseteq \cup\mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  [proof of $\cup\mathcal{F} \subseteq \cup\mathcal{G}$]
So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\cup\mathcal{F} \subseteq \cup\mathcal{G} \rightarrow \forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$ so we assume b is an arbitrary element of $\cup\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$b \in \cup\mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  Let b be an arbitrary element of $\cup\mathcal{F}$
    [proof of $b \in \cup\mathcal{G}$]
  Therefore if $b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G}$
Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$b \in \cup\mathcal{F} \rightarrow \exists M(M \in \mathcal{F} \wedge b \in M)$, so let $M = A_0$ (Existential Instantiation)

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
  Let b be an arbitrary element and suppose $b \in \cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ and b is in that set. Let that set $= A_0$
    [proof of $b \in \cup\mathcal{G}$]
  Therefore if $b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G}$
Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \rightarrow b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall A(A \in \mathcal{F} \rightarrow A \in \mathcal{G})$. Using universal instantiation we will plug in A_0 for A since then we can use modens ponens to conclude that $A_0 \in \mathcal{G}$.

Givens	Goals
$A_0 \in \mathcal{F} \to A_0 \in \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Our goal $b \in \cup\mathcal{G} \to \exists N(N \in \mathcal{G} \wedge b \in N)$, which we can now prove. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$, it follows that $A_0 \in G$. By the definition of $\cup\mathcal{G}$ it follows that $b \in \cup\mathcal{G}$ because $A_0 \in G \wedge b \in A_0$, the latter statement being one of our givens.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.*
Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ that contains b. Call this set A_0. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$ it follows that $A_0 \in G$, which implies that $b \in \cup\mathcal{G}$. Therefore if $b \in \cup\mathcal{F}$ then $b \in \cup\mathcal{G}$. Since b was arbitrary we can conclude that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$. This completes the proof.

2 Exercise 3.3.13

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cap\mathcal{G} \subseteq \cap\mathcal{F}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    [proof of $\cap\mathcal{G} \subseteq \cap\mathcal{F}$]
So if $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$\cap\mathcal{G} \subseteq \cap\mathcal{F} \to \forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$, so we assume b is an arbitrary element of $\cap\mathcal{G}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cap\mathcal{F}$
$b \in \cap\mathcal{G}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    Let b be an arbitrary element of $\cap\mathcal{G}$
        [proof of $b \in \cap\mathcal{F}$]
    Therefore if $b \in \cap\mathcal{G} \to b \in \cap\mathcal{F}$
Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \to$

$\cap\mathcal{G} \subseteq \cap\mathcal{F}$

$b \in \cap\mathcal{F} \rightarrow \forall A(A \in \mathcal{F} \rightarrow b \in A)$, so we assume A is an arbitrary element of $\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in A$
$b \in \cap\mathcal{G}$	
$A \in \mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    Let b be an arbitrary element of $\cap\mathcal{G}$
        Suppose A is an arbitrary set in $\mathcal{F}$
            [proof of $b \in A$]
        Therefore if $A \in \mathcal{F} \rightarrow b \in A$
        Since A was arbitrary we can conclude $b \in \cap\mathcal{F}$
    Therefore if $b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F}$
Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \rightarrow b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \rightarrow \cap\mathcal{G} \subseteq \cap\mathcal{F}$

Now looking at our givens, $\mathcal{F} \subseteq \mathcal{G} \rightarrow \forall Z(Z \in \mathcal{F} \rightarrow Z \in \mathcal{G})$. Using universal instantiation we will plug in A for Z and using modus ponens we can conclude that $A \in \mathcal{G}$.

Our other given, $b \in \cap\mathcal{G} \rightarrow \forall Y(Y \in \mathcal{G} \rightarrow b \in Y)$. Using universal instantiation we will plug in A for Y and using modus ponens we can conclude that $b \in A$, which was our goal, and we can now write our proof.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.*
Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cap\mathcal{G}$. Suppose A is an arbitrary element of $\mathcal{F}$, then because $\mathcal{F} \subseteq \mathcal{G}$ then it follows that $A \in \mathcal{G}$. By the definition of $\cap\mathcal{G}$ it follows that $b \in A$ and since A was arbitrary then $b \in \cap\mathcal{F}$. Since b was arbitrary we can conclude $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ and therefore that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$. This completes the proof.

3 Exercise 3.3.14

Suppose $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i \in I} A_i)$.

So we want to prove that $\forall a(a \in \bigcup_{i \in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i \in I} A_i))$

First we assume a is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i\in I} \mathscr{P}(A_i)$	$a \in \mathscr{P}(\bigcup_{i\in I} A_i)$

Assume a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
  Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
    [proof of $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$]
  Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our goal we see that $a \in \mathscr{P}(\bigcup_{i\in I} A_i) \to a \subseteq \bigcup_{i\in I} A_i \to \forall z(z \in a \to z \in \bigcup_{i\in I} A_i)$. Therefore we assume z is arbitrary, assume the antecedent, and make the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i\in I} \mathscr{P}(A_i)$	$z \in \bigcup_{i\in I} A_i$
$z \in a$	

Assume a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
  Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
    Assume z is arbitrary
      Assume $z \in a$
        [proof of $z \in \bigcup_{i\in I} A_i$]
      Therefore $z \in a \to z \in \bigcup_{i\in I} A_i$
    Since z was arbitrary we can conclude $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
  Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our given we see that $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \to a \in \{a | \exists i \in I(a \in \mathscr{P}(A_i))\}$. Using existential instantiation we will select an i such that $a \in \mathscr{P}(A_i)$ which implies $a \subseteq A_i$. Since $a \subseteq A_i \to \forall m(m \in a \to m \in A_i)$ and using universal instantiation we will plug in z for m and we get $\forall z(z \in a \to z \in A_i)$ and using modus ponens we can conclude that $z \in A_i$, which implies that $z \in \bigcup_{i\in I} A_i$, which was our goal. We can now right our proof.

Theorem. *Suppose $\{A_i | i \in I\}$ is an indexed family of sets, then $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.*
Proof. Suppose that a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$. We choose an $i \in I$ such that $a \in \mathscr{P}(A_i)$, which implies that $a \subseteq A_i$. Suppose z is an arbitrary element of a, then it follows that $z \in A_i$ and therefore $z \in \bigcup_{i\in I} A_i$. Since z was an arbitrary element of a then $a \subseteq \bigcup_{i\in I} A_i$, and it follows that $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$. Thus we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$. This completes the proof.

4 3.3.15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$. Prove that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$

So we want to prove that $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

First we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i\in I} A_i$	$y \in \bigcap_{i\in I} \mathscr{P}(A_i)$

Suppose y is arbitrary element of $\bigcap_{i\in I} A_i$.
[proof of $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$]
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Our goal $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$ so we make m an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$. So we make z arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i\in I} A_i$	$z \in A_m$
$z \in y$	

Suppose y is arbitrary element of $\bigcap_{i\in I} A_i$.
Suppose m is an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$.
Suppose z is an arbitrary element of y
[proof of $z \in A_m$]
Therefore $z \in y \to z \in A_m$ and since z was arbitrary $y \subseteq A_m \to y \in \mathscr{P}(A_m)$ and since m was arbitrary $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Now looking at our given $y \in \bigcap_{i\in I} A_i \to \forall i \in I(y \in A_i)$. Using universal instantiation we plug in m for i and therefore $y \in A_m$ and since $z \in y$ we can conclude $z \in A_m$, which was our goal. Now we can write our proof.

Theorem. *Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$, then $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$.*
Proof. Suppose y is an arbitrary element of $\bigcap_{i\in I} A_i$. Suppose m is an arbitrary member of I and therefore $y \subseteq A_m$ which implies $y \subseteq A_m$. Now suppose z is an arbitrary element of y. Since $y \in \bigcap_{i\in I} A_i$ if we choose an i such that $y \in \bigcap_{m\in I} A_m$ then $y \in A_m$ which implies $z \in A_m$. Therefore if $z \in y$ then $z \in A_m$

and since z was arbitrary then $y \subseteq A_m$ or $y \in \mathscr{P}(A_m)$ and since m was arbitrary then $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$. Since y was arbitrary then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$. This completes the proof.