

### 3.5.1

Suppose  $A$ ,  $B$ , and  $C$  are sets.

**Theorem.**  $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A \cap (B \cup C)$ . Thus  $x \in A$  and  $x \in B$  or  $x \in C$ . If  $x \in C$  then  $x \in (A \cap B) \cup C$ . In the case where  $x \in B$  it follows that  $x \in A \cap B$  and therefore  $x \in (A \cap B) \cup C$ . Since  $x$  was arbitrary we can conclude that  $A \cap (B \cup C) \subseteq (A \cap B) \cup C$ .  $\square$

### 3.5.2

Suppose  $A$ ,  $B$ , and  $C$  are sets.

**Theorem.**  $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in (A \cup B) \setminus C$ . Thus  $x \notin C$  and  $x \in A$  or  $x \in B$ . If  $x \in A$  then  $x \in A \cup (B \setminus C)$ . If  $x \in B$  then it follows that  $x \in B \setminus C$  and therefore  $x \in A \cup (B \setminus C)$ . Since  $x$  was arbitrary we can conclude  $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$ .  $\square$

### 3.5.3

Suppose  $A$  and  $B$  are sets.

**Theorem.**  $A \setminus (A \setminus B) = A \cap B$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A \setminus (A \setminus B)$ . Then

$$\begin{aligned} x \in A \setminus (A \setminus B) &\text{ iff } x \in A \wedge x \notin A \setminus B \\ &\text{ iff } x \in A \wedge \neg(x \in A \wedge x \notin B) \\ &\text{ iff } x \in A \wedge (x \notin A \vee x \in B) \\ &\text{ iff } (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \\ &\text{ iff } x \in A \wedge x \in B \\ &\text{ iff } x \in (A \cap B) \end{aligned}$$

$\square$

### 3.5.4

**Theorem.** If  $A \cap C \subseteq B \cap C$  and  $A \cup C \subseteq B \cup C$  then  $A \subseteq B$ .

*Proof.* Suppose  $A \cap C \subseteq B \cap C$  and  $A \cup C \subseteq B \cup C$ . Let  $x$  be arbitrary and suppose  $x \in A$ . Thus  $x \in A \cup C$  and it follows that  $x \in B \cup C$ . Now if  $x \in B \cup C$  then either  $x \in B$  or  $x \in C$ . If  $x \in B$  then since  $x$  was arbitrary we can conclude  $A \subseteq B$ . In the case that  $x \in C$ , then  $x \in A \cap C$  and it follows that  $x \in B \cap C$ . Therefore  $x \in C$  and  $x \in B$ . Thus, if  $x \in A$  then  $x \in B$  and since  $x$  was arbitrary we can conclude  $A \subseteq B$ .  $\square$

### 3.5.5

Suppose  $A$  and  $B$  are sets.

**Theorem.** If  $A \triangle B \subseteq A$  then  $B \subseteq A$ .

*Proof.* Suppose  $A \triangle B \subseteq A$ . We will prove by contradiction. Let  $x$  be arbitrary and suppose  $x \in B$  and  $x \notin A$ . Since  $x \in B$  and  $x \notin A$  then  $x \in A \triangle B$ . Since  $A \triangle B \subseteq A$ , then  $x \in A$ . But this contradicts  $x \notin A$ . Therefore, if  $x \in B$  then  $x \in A$  and since  $x$  was arbitrary we can conclude that  $B \subseteq A$ .  $\square$

### 3.5.6

Suppose  $A$ ,  $B$ , and  $C$  are sets.

**Theorem.**  $A \cup C \subseteq B \cup C \iff A \setminus C \subseteq B \setminus C$ .

*Proof.* ( $\rightarrow$ ) Suppose  $A$ ,  $B$ , and  $C$  are sets. Suppose  $(A \cup C) \subseteq (B \cup C)$ . Let  $x$  be arbitrary and suppose  $x \in A \setminus C$ , which means  $x \in A$  and  $x \notin C$ . Since  $x \in A$ , then  $x \in A \cup C$  and therefore  $x \in B \cup C$ . This means  $x \in B$  or  $x \in C$  and since  $x \notin C$ , it must be that  $x \in B$ . Now since  $x \in B$  and  $x \notin C$  then  $x \in B \setminus C$ . Therefore, if  $x \in A \setminus C$  then  $x \in B \setminus C$  and since  $x$  was arbitrary we can conclude if  $A \cup C \subseteq B \cup C$  then  $A \setminus C \subseteq B \setminus C$ .

( $\leftarrow$ ) Now suppose  $A \setminus C \subseteq B \setminus C$ . Let  $x$  be arbitrary and suppose  $x \in A \cup C$ , which means  $x \in A$  or  $x \in C$ . If  $x \in C$  then  $x \in B \cup C$  and since  $x$  was arbitrary then  $A \cup C \subseteq B \cup C$ . In the case that  $x \in A$ , since  $A \setminus C \subseteq B \setminus C$  then  $x \in B$ . Therefore,  $x \in B \cup C$  and since  $x$  was arbitrary then  $A \cup C \subseteq B \cup C$ .  $\square$

### 3.5.7

**Theorem.** For any sets  $A$  and  $B$ ,  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

*Proof.* Let  $A$  and  $B$  be arbitrary sets. Let  $M$  be arbitrary and suppose  $M \in \mathcal{P}(A) \cup \mathcal{P}(B)$ . Thus  $M \in \mathcal{P}(A)$  or  $M \in \mathcal{P}(B)$ , which means  $M \subseteq A$  or  $M \subseteq B$ . In the case where  $M \subseteq A$ , let  $x$  be an arbitrary member of  $M$  and it follows that  $x \in A$ . Since  $x \in A$  then  $x \in A \cup B$  and because  $x$  was arbitrary we can conclude  $M \subseteq A \cup B$  and therefore  $M \in \mathcal{P}(A \cup B)$ . In the case where  $M \subseteq B$ , let  $x$  be an arbitrary member of  $M$  and it follows that  $x \in B$ . Since  $x \in B$  then  $x \in A \cup B$  and because  $x$  was arbitrary we can conclude  $M \subseteq A \cup B$  and therefore  $M \in \mathcal{P}(A \cup B)$ .  $\square$

### 3.5.8

**Theorem.** For any sets  $A$  and  $B$ , if  $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$  then either  $A \subseteq B$  or  $B \subseteq A$ .

*Proof.* We will prove the contrapositive. Since we proved that  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$  in exercise 3.5.7, we must show that  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$  to prove our goal that  $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$ . Let  $A$  and  $B$  be arbitrary sets and suppose  $A \not\subseteq B$  and  $B \not\subseteq A$ . This means there is an element  $x \in A \setminus B$  and an element  $y \in B \setminus A$ . Since  $x \in A$  and  $y \in B$  then both  $x$  and  $y$  are in  $A \cup B$  and therefore the set  $\{x, y\}$  is in  $\mathcal{P}(A \cup B)$  but not in  $\mathcal{P}(A)$  or  $\mathcal{P}(B)$ . Thus  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ .  $\square$

### 3.5.9

**Theorem.** Suppose  $x$  and  $y$  are real numbers and  $x \neq 0$ . Then  $y + 1/x = 1 + y/x$  iff either  $x = 1$  or  $y = 1$ .

*Proof.* ( $\rightarrow$ ) Suppose that  $y + 1/x = 1 + y/x$ . Now if  $y = 1$  then we have proven our goal. So now assume  $y \neq 1$  and  $y + 1/x = 1 + y/x$ , then it follows that  $x = 1$ .

( $\leftarrow$ ) Now suppose  $x = 1$  or  $y = 1$ . In the case that  $x = 1$  we have

$$y + \frac{1}{x} = y + \frac{1}{1} = y + 1 = 1 + \frac{y}{1} = 1 + \frac{y}{x}$$

In the case that  $y = 1$  we have

$$y + \frac{1}{x} = 1 + \frac{1}{x} = 1 + \frac{y}{x}$$

$\square$

### 3.5.10

**Theorem.** For every real number  $x$ , if  $|x - 3| > 3$  then  $x^2 > 6x$ .

*Proof.* Suppose that  $x$  is an arbitrary real number and that  $|x - 3| > 3$ . Then either  $x - 3 \geq 0$  or  $x - 3 < 0$ . In the case that  $x - 3 \geq 0$ , then  $|x - 3| = x - 3$  and therefore  $|x - 3| > 3 = x - 3 > 3$ . Solving for  $x$ , we have  $x > 6$  and then multiplying both sides by  $x$  we have  $x^2 > 6x$ . In the case that  $x - 3 < 0$ , then  $|x - 3| = 3 - x$  and therefore  $3 - x > 3$ . Solving for  $x$  we have  $x < 0$ . Multiplying both sides of  $x < 0$  by  $6 - x$  we have  $6x - x^2 < 0$  and therefore  $x^2 > 6x$ .  $\square$

### 3.5.11

**Theorem.** For every real number  $x$ ,  $|2x - 6| > x$  iff  $|x - 4| > 2$ .

*Proof.* ( $\rightarrow$ ) Let  $x$  be an arbitrary real number and suppose  $|2x - 6| > x$ . Our goal  $|x - 4| > 2$  means that either  $x - 4 > 2$  or  $4 - x > 2$ . Since  $|2x - 6| > 2$  then either  $2x - 6 > x$  or  $6 - 2x > x$ . If  $2x - 6 > x$  then it follows that  $x - 4 > 2$ . Now if  $6 - 2x > x$  then it follows that  $4 - x > 2$ .

( $\leftarrow$ ) Now suppose  $|x - 4| > 2$ . Our goal  $|2x - 6| > x$  means that either  $2x - 6 > x$  or  $6 - 2x > x$ . Since  $|x - 4| > 2$  then either  $x - 4 > 2$  or  $4 - x > 2$ . If  $x - 4 > 2$  then it follows that  $2x - 6 > x$ . In the case that  $4 - x > 2$  then it follows that  $6 - 2x > x$ .  $\square$

### 3.5.12

**Theorem.** For all real numbers  $a$  and  $b$ ,  $|a| \leq b$  if and only if  $-b \leq a \leq b$ .

*Proof.* ( $\rightarrow$ ) Suppose  $a$  and  $b$  are arbitrary real numbers and that  $|a| \leq b$ . There are two cases to consider:  $a \geq 0$  and  $a < 0$ . If  $a \geq 0$  then  $|a| = a \leq b$ . It follows that  $-b \leq -a$  and since  $a \geq 0$  then  $-a \leq a$ . Therefore,  $-b \leq -a \leq a \leq b$  and  $-b \leq a \leq b$ . Now in the case that  $a < 0$  then  $|a| = -a \leq b$ . It follows that  $-b \leq a$  and since  $a < 0$  then  $-a > a$  or  $a < -a$ . Therefore  $-b \leq a < -a \leq b$  and  $-b \leq a \leq b$ .

( $\leftarrow$ ) Now suppose  $-b \leq a \leq b$  and therefore  $a \leq b$ . Now we must prove that  $-a \leq b$  to complete the proof. If we subtract  $a$  from both sides of  $-b \leq a$  and add  $b$  to both sides we have  $-a \leq b$ .  $\square$