3.4.1

Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

We want to prove $\forall x(P(x) \wedge Q(x) \iff \forall x P(x) \wedge \forall x Q(x))$.

Theorem. *The statement $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.*

Proof. $(\rightarrow)$ Suppose $\forall x(P(x) \wedge Q(x))$. Let y be arbitrary. Since $\forall x(P(x) \wedge Q(x))$ it follows $P(y)$ and $Q(y)$. Since y was arbitrary, we can conclude $\forall x P(x)$ and $\forall x Q(x)$ or $\forall x P(x) \wedge \forall x Q(x)$.

$(\leftarrow)$ Let y be arbitrary. Since $\forall x P(x)$ and $\forall x Q(x)$ then it follows $P(y)$ and $Q(y)$. Since y was arbitrary we can conclude $\forall x(P(x) \wedge Q(x))$. □

3.4.2

Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Theorem. *If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B$ then $x \in B$ and since $A \subseteq C$ then $x \in C$ or $x \in B \cap C$. Therefore, if $x \in A$ then $x \in B \cap C$ and since x was arbitrary we can conclude $A \subseteq B \cap C$. □

3.4.3

Suppose $A \subseteq B$. Prove that for every set C, $C \setminus B \subseteq C \setminus A$.

Theorem. *Suppose $A \subseteq B$, then for every set C, $C \setminus B \subseteq C \setminus A$.*

Proof. Suppose $A \subseteq B$ and C is an arbitrary set. Let x be arbitrary and suppose $x \in C \setminus B$, which means $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$, then $x \notin A$, which means that $x \in C \setminus A$. Therefore, if $x \in C \setminus B$ then $x \in C \setminus A$ and since x and C were arbitrary, we can conclude $\forall C(C \setminus B \subseteq C \setminus A)$. □

3.4.5

Prove that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$.

Theorem. *If $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B \setminus C$ then $x \in B$ and $x \notin C$. Since x was arbitrary we can conclude $B \nsubseteq C$. □

3.4.6

Prove that for any sets A, B, and C, $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ finding a string of equivalences starting with $x \in A \backslash (B \cap C)$ and ending with $x \in (A \backslash B) \cup (A \backslash C)$.

Theorem. *for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.*

Proof. Suppose A, B, and C are arbitrary sets. Then

$$\begin{aligned} x \in A \setminus (B \cap C) & \text{ iff } x \in A \to (x \notin B \wedge x \notin C) \\ & \text{ iff } x \notin A \vee (x \notin B \wedge x \notin C) \\ & \text{ iff } (x \notin A \vee x \notin B) \wedge (x \notin A \vee x \notin C) \\ & \text{ iff } (x \in A \to x \notin B) \vee (x \in A \to x \notin C) \\ & \text{ iff } x \in A \setminus B \vee x \in A \setminus C \\ & \text{ iff } x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

□