3.6.2

Theorem. *There is a unique $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $xy + x - 4 = 4y$.*

Proof. First we prove existence. Let $x = 4$ and suppose y is an arbitrary real number. Then we have $4y + 4 - 4 = 4y + 0 = 4y$, as desired. To prove uniqueness suppose a and b are arbitrary real numbers and that $ay + a - 4 = 4y$ and that $by + b - 4 = 4b$. For $ay + a - 4 = 4y$, let $y = b$ and we have $ab + a - 4 = 4b$. For $by + b - 4 = 4y$, let $y = a$ and we have $ba + b - 4 = 4a$. Now subtracting both sides of $ab + a - 4 = 4b$ from $ba + b - 4 = 4a$ we have

$$\begin{aligned}
ba + b - 4 - (ab + a - 4) &= 4a - 4b \\
ba + b - 4 - ab - a + 4 &= 4a - 4b \\
b - a &= 4a - 4b \\
b + 4b &= 4a + a \\
5b &= 5a \\
b &= a
\end{aligned}$$

Therefore, if $ay + a - 4 = 4y$ and $by + b - 4 = 4y$, then $a = b$.

□

3.6.3

Theorem. $\forall x \in \mathbb{R}[(x \neq 0 \wedge x \neq 1) \implies \exists! y \in \mathbb{R}(y/x = y - x)]$

Proof. Suppose x is an arbitrary real number, $x \neq 0$, and $x \neq 1$. Let $y = x^2/(x-1)$, which is defined because $x \neq 1$. Then

$$\begin{aligned}
\frac{y}{x} = \frac{\frac{x^2}{x-1}}{x} = \frac{x^2}{x-1} \cdot \frac{1}{x} = \frac{x^2}{x(x-1)} &= \frac{x}{x-1} \\
&= \frac{x^2 - x^2 + x}{x-1} \\
&= \frac{x^2 - x(x-1)}{x-1} \\
&= \frac{x^2}{x-1} - \frac{x(x-1)}{x-1} = \frac{x^2}{x-1} - x = y - x
\end{aligned}$$

□

3.6.4

Theorem. $\forall x \in \mathbb{R}(x \neq 0 \implies \exists! y \in \mathbb{R} \forall z \in \mathbb{R}(zy = z/x))$.

Proof. Let x be an arbitrary real number. To prove existence, suppose $x \neq 0$ and $y = 1/x$. Then $zy = z(1/x) = z/x$. To prove uniqueness let a and b be arbitrary real numbers and suppose $za = z/x$ and $zb = z/x$. For $za = z/x$ let $z = b$ and we have $ba = bx$, which can be rearranged as $xba = b$. For $zb = z/x$ let $z = a$ and we have $ab = a/x$, which can be rearranged as $xab = a$. Subtracting both sides of $xab = b$ from $xab = a$ we have $xab - xab = a - b$ and so $a - b = 0$ or $a = b$. $\square$

3.6.5

If $\mathcal{F}$ is a family of sets, then $\cup\mathcal{F} = \{x|\exists A(A \in \mathcal{F} \wedge x \in A)\}$. Define a new set $\cup!\mathcal{F}$ by the formula $\cup!\mathcal{F} = \{x|\exists!A(A \in \mathcal{F} \wedge x \in A)\}$.

a)

Theorem. $\forall\mathcal{F}(\cup!\mathcal{F} \subseteq \cup F)$

Proof. Suppose $\mathcal{F}$ is and arbitrary family of sets. Let x be arbitrary and suppose $x \in \cup!\mathcal{F}$. This means $\exists!A \in \mathcal{F}(x \in A)$. Since $A \in \mathcal{F}$ and $x \in A$ then we can concluded that $x \in \cup\mathcal{F}$. Since x was arbitrary then $\cup!\mathcal{F} \subseteq \cup\mathcal{F}$, and since $\mathcal{F}$ was arbitrary we can conclude for all $\mathcal{F}$, $\cup!\mathcal{F} \subseteq \cup\mathcal{F}$. $\square$