

### Exercise 3.1.5

Suppose  $a$  and  $b$  are real numbers. Prove that if  $a < b < 0$  then  $a^2 > b^2$ .

So we want to prove that  $(a < b < 0) \rightarrow (a^2 > b^2)$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a < b < 0$	$a^2 > b^2$

Suppose  $a < b < 0$

[proof of  $a^2 > b^2$ ]

So if  $a < b < 0$  then  $a^2 > b^2$

If we multiply the inequality  $a < b$  on both sides by the negative number  $b$  we have  $ab > b^2$  and multiplying  $a < b$  on both sides by the negative number  $a$  we have  $a^2 > ab$ . Therefore  $a^2 > ab > b^2$  and we have proven our goal and now we can write our proof.

**Theorem.** Suppose  $a$  and  $b$  are real numbers. Prove that if  $a < b < 0$  then  $a^2 > b^2$ .

*Proof.* Suppose  $a < b < 0$ . Multiplying the inequality  $a < b$  by the negative number  $a$  we can conclude  $a^2 > ab$ , and, similarly, multiplying  $a < b$  by the negative number  $b$  we get  $ab > b^2$ . Therefore,  $a^2 > ab > b^2$  and  $a^2 > b^2$ . Thus, if  $a < b < 0$  then  $a^2 > b^2$ .  $\square$

### Exercise 3.1.6

Suppose  $a$  and  $b$  are real numbers. Prove that if  $0 < a < b$  then  $1/b < 1/a$ .

So we want to prove that  $(0 < a < b) \rightarrow (1/b < 1/a)$ .

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$0 < a < b$	$1/b < 1/a$

Suppose  $0 < a < b$

[proof of  $1/b < 1/a$ ]

So if  $0 < a < b$  then  $1/b < 1/a$ .

If we multiply both sides of the inequality  $a < b$  by  $1/ab$  we see that  $1/a < 1/b$ , which is our goal.

**Theorem.** Suppose  $a$  and  $b$  are real numbers. If  $0 < a < b$  then  $1/b < 1/a$ .

*Proof.* Suppose  $0 < a < b$ . Multiplying both sides of the inequality  $a < b$  by  $1/ab$  we can conclude that  $1/b < 1/a$ . Therefore, if  $0 < a < b$  then  $1/b < 1/a$ .  $\square$

### Exercise 3.1.7

Suppose that  $a$  is a real number. Prove that if  $a^3 > a$  then  $a^5 > a$ . (Hint: One approach is to start by completing the following equation:  $a^5 - a = (a^3 - 1) \cdot \underline{\quad}$ .) So we want to prove that  $(a^3) \rightarrow (a^5)$ .

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a^3 > a$	$a^5$

Suppose  $a^3 > a$   
     [proof of  $a^5 > a$ ]  
 So if  $a^3 > a$  then  $a^5 > a$ .

If we multiply both side of  $a^3 - a > 0$  by  $a^2 + 1$  we can conclude that  $a^5 - a > 0$  or  $a^5 > a$ , which was our goal.

**Theorem.** Suppose  $a$  is a real number. If  $a^3 > a$  then  $a^5 > a$ .

*Proof.* Suppose  $a^3 > a$ , then  $a^3 - a > 0$ . Multiplying both sides of the inequality  $a^3 - a > 0$  by  $a^2 + 1$  we can conclude  $a^5 - a > 0$ . Therefore, if  $a^3 > a$  then  $a^5 > a$ .  $\square$

### Exercise 3.1.8

Suppose  $A \setminus B \subseteq C \cap D$  and  $x \in A$ . Prove that if  $x \notin D$  then  $x \in B$ . So we want to prove that  $(x \notin D) \rightarrow (x \in B)$ .

The contrapositive of the goal is  $\neg(x \in B) \rightarrow \neg(x \notin D)$ , or in other words  $x \notin B \rightarrow x \in D$ . First we assume the antecedent and make the consequent our goal.

Givens	Goals
$A \setminus B \subseteq C \cap D$	$x \in D$
$x \in A$	
$x \notin B$	

Looking at our givens we can rewrite  $A \setminus B \subseteq C \cap D$  as  $(x \in A \wedge x \notin B) \rightarrow (x \in C \cap D)$ . Looking at our other givens  $x \in A$  and  $x \notin B$  we can conclude that  $x \in C \cap D$  and therefore  $x \in D$ , which was our goal to prove.

**Theorem.** Suppose  $A \setminus B \subseteq C \cap D$  and  $x \in A$ . If  $x \notin D$  then  $x \in B$ .

*Proof.* We will prove the contrapositive. Suppose  $x \notin B$ . Since  $x \in A$  and  $x \notin B$  we can conclude that  $x \in C \cap D$  and it follows that  $x \in D$ . Therefore, if  $x \notin D$  then  $x \in B$ .  $\square$

### Exercise 3.1.9

Suppose  $a$  and  $b$  are real numbers. Prove that if  $a < b$  then  $\frac{a+b}{2} < b$ .

So we want to prove that  $(a < b) \rightarrow (\frac{a+b}{2} < b)$ .

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$a < b$	$\frac{a+b}{2} < b$

If we add  $b$  to both sides of  $a < b$  we see that  $a + b < b + b$  or  $a + b < 2b$ . Then if we divide both sides of  $a + b < 2b$  by 2 we can conclude that  $\frac{a+b}{2} < b$ , which was our goal to prove.

**Theorem.** Suppose  $a$  and  $b$  are real numbers. If  $a < b$  then  $\frac{a+b}{2}$ .

*Proof.* Suppose  $a < b$ . Adding  $b$  to both sides of the inequality  $a < b$  and then dividing both sides by 2, we can conclude that  $\frac{a+b}{2} < b$ . Therefore, if  $a < b$  then  $\frac{a+b}{2} < b$ .  $\square$

### Exercise 3.1.10

Suppose  $x$  is a real number and  $x \neq 0$ . Prove that if  $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$  then  $x \neq 8$ .

So we want to prove that  $(\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}) \rightarrow (x \neq 8)$ .

The contrapositive of the goal is  $\neg(x \neq 8) \rightarrow \neg(\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x})$  or in other words  $(x = 8) \rightarrow (\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x})$ . First we assume the antecedent and make the consequent our goal.

Givens	Goals
$x \neq 0$	$\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$
$x = 8$	

If we evaluate the expression  $\frac{\sqrt[3]{x}+5}{x^2+6}$  for  $x = 8$  we see that  $\frac{\sqrt[3]{8}+5}{8^2+6} = \frac{1}{7}$  and  $\frac{1}{7} \neq \frac{1}{8}$ , therefore  $\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$ , which was our goal to prove.

**Theorem.** Suppose  $x$  is a real number and  $x \neq 0$ . If  $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$  then  $x \neq 8$

*Proof.* We will prove the contrapositive. Suppose  $x = 8$ . Then evaluating the equation  $\frac{\sqrt[3]{x+5}}{x^2+6} = \frac{1}{x}$  for  $x = 8$  we see that  $\frac{1}{7} \neq \frac{1}{8}$  and therefore if  $\frac{\sqrt[3]{x+5}}{x^2+6} = \frac{1}{x}$  then  $x \neq 8$ .  $\square$

### Exercise 3.1.11

Suppose  $a$ ,  $b$ ,  $c$ , and  $d$  are real numbers,  $0 < a < b$ , and  $d > 0$ . Prove that if  $ac \geq bd$  then  $c > d$ .

So we want to prove that  $(ac \geq bd) \rightarrow (c > d)$

The contrapositive of the goal is  $\neg(c > d) \rightarrow \neg(ac \geq bd)$  or in other words  $(c \leq d) \rightarrow (ac < bd)$ . First we assume the antecedent and make the consequent our goal.

Givens	Goals
$c \leq d$	$ac < bd$

If we multiply both side of the inequality  $c \leq d$  by  $a$  we see that  $ac \leq ad$  and multiplying both sides of the inequality  $a < b$  by  $d$  we see that  $ad < bd$ . Therefore,  $ac \leq ad < bd$  and  $ac < ad$ , which was our goal to prove.

**Theorem.** Suppose  $a$ ,  $b$ ,  $c$ , and  $d$  are real numbers,  $0 < a < b$  and  $d > 0$ . If  $ac \geq bd$  then  $c > d$ .

*Proof.* We will prove the contrapositive. Suppose  $c \leq d$ . Multiplying the inequality  $c \leq d$  on both sides by  $a$  we have  $ac \leq ad$  and multiplying the inequality  $a < b$  on both sides by  $d$  we have  $ad < bd$ . It follows that  $ac \leq ad < bd$  and  $ac < bd$ . Therefore, if  $ac \geq bd$  then  $c > d$ .  $\square$

### Exercise 3.1.12

Suppose  $x$  and  $y$  are real numbers, and  $3x + 2y \leq 5$ . Prove that if  $x > 1$  then  $y < 1$ .