## 3.4.1

Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

We want to prove $\forall x(P(x) \wedge Q(x) \iff \forall x P(x) \wedge \forall x Q(x))$.

**Theorem.** *The statement $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.*

*Proof.* ($\rightarrow$) Suppose $\forall x(P(x) \wedge Q(x))$. Let $y$ be arbitrary. Since $\forall x(P(x) \wedge Q(x))$ it follows $P(y)$ and $Q(y)$. Since $y$ was arbitrary, we can conclude $\forall x P(x)$ and $\forall x Q(x)$ or $\forall x P(x) \wedge \forall x Q(x)$.

($\leftarrow$) Let $y$ be arbitrary. Since $\forall x P(x)$ and $\forall x Q(x)$ then it follows $P(y)$ and $Q(y)$. Since $y$ was arbitrary we can conclude $\forall x(P(x) \wedge Q(x))$. □

## 3.4.2

Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

**Theorem.** *If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.*

*Proof.* Let $x$ be arbitrary and suppose $x \in A$. Since $A \subseteq B$ then $x \in B$ and since $A \subseteq C$ then $x \in C$ or $x \in B \cap C$. Therefore, if $x \in A$ then $x \in B \cap C$ and since $x$ was arbitrary we can conclude $A \subseteq B \cap C$. □

## 3.4.3

Suppose $A \subseteq B$. Prove that for every set $C$, $C \setminus B \subseteq C \setminus A$.

**Theorem.** *Suppose $A \subseteq B$, then for every set $C$, $C \setminus B \subseteq C \setminus A$.*

*Proof.* Suppose $A \subseteq B$ and $C$ is an arbitrary set. Let $x$ be arbitrary and suppose $x \in C \setminus B$, which means $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$, then $x \notin A$, which means that $x \in C \setminus A$. Therefore, if $x \in C \setminus B$ then $x \in C \setminus A$ and since $x$ and $C$ were arbitrary, we can conclude $\forall C(C \setminus B \subseteq C \setminus A)$. □

## 3.4.5

Prove that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$.

**Theorem.** *If $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$.*

*Proof.* Let $x$ be arbitrary and suppose $x \in A$. Since $A \subseteq B \setminus C$ then $x \in B$ and $x \notin C$. Since $x$ was arbitrary we can conclude $B \nsubseteq C$. □

## 3.4.6

Prove that for any sets $A$, $B$, and $C$, $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ finding a string of equivalences starting with $x \in A \backslash (B \cap C)$ and ending with $x \in (A \backslash B) \cup (A \backslash C)$.

**Theorem.** *for any sets $A$, $B$, and $C$, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.*

*Proof.* Suppose $A$, $B$, and $C$ are arbitrary sets. Then

$$\begin{aligned}
x \in A \setminus (B \cap C) &\text{ iff } x \in A \to (x \notin B \wedge x \notin C) \\
&\text{ iff } x \notin A \vee (x \notin B \wedge x \notin C) \\
&\text{ iff } (x \notin A \vee x \notin B) \wedge (x \notin A \vee x \notin C) \\
&\text{ iff } (x \in A \to x \notin B) \vee (x \in A \to x \notin C) \\
&\text{ iff } x \in A \setminus B \vee x \in A \setminus C \\
&\text{ iff } x \in (A \setminus B) \cup (A \setminus C)
\end{aligned}$$

□

## 3.4.7

**Theorem.** *For any sets $A$ and $B$, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.*

*Proof.* ($\rightarrow$) Let $M$ be an arbitrary set and suppose $M \in \mathscr{P}(A \cap B)$. Then $M \subseteq A \cap B$. Let $x$ be arbitrary and suppose $x \in M$. Since $M \subseteq A \cap B$, $x \in A \cap B$ and therefore $x \in A$. Since $x$ was arbitrary, $M \subseteq A$ and therefore $M \in \mathscr{P}(A)$. Similarly, since $M \subseteq A \cap B$, $x \in B$. Since $x$ was arbitrary, $M \subseteq B$ and therefore $M \in \mathscr{P}(B)$. Therefore, $M \in \mathscr{P}(A)$ and $M \in \mathscr{P}(B)$.

($\leftarrow$) Now suppose $M \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then $M \subseteq A$ and $M \subseteq B$. Suppose $x \in M$. Since $M \subseteq A$ and $M \subseteq B$ then $x \in A \cap B$. Since $x$ was arbitrary, $M \subseteq A \cap B$ and therefore $M \in \mathscr{P}(A \cap B)$. □