3.5.1

Suppose A, B, and C are sets.

Theorem. $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

Proof. Let x be arbitrary and suppose $x \in A \cap (B \cup C)$. Thus $x \in A$ and $x \in B$ or $x \in C$. If $x \in C$ then $x \in (A \cap B) \cup C$. In the case where $x \in B$ it follows that $x \in A \cap B$ and therefore $x \in (A \cap B) \cup C$. Since x was arbitrary we can conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.2

Suppose A, B, and C are sets.

Theorem. $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

Proof. Let x be arbitrary and suppose $x \in (A \cup B) \setminus C$. Thus $x \notin C$ and $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cup (B \setminus C)$. If $x \in B$ then if follows that $x \in B \setminus C$ and therefore $x \in A \cup (B \setminus C)$. Since x was arbitrary we can conclude $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.3

Suppose A and B are sets.

Theorem. $A \setminus (A \setminus B) = A \cap B$

Proof. Let x be arbitrary and suppose $x \in A \setminus (A \setminus B)$. Then

$$\begin{aligned}
x \in A \setminus (A \setminus B) &\text{ iff } x \in A \wedge x \notin A \setminus B \\
&\text{ iff } x \in A \wedge \neg(x \in A \wedge x \notin B) \\
&\text{ iff } x \in A \wedge (x \notin A \vee x \in B) \\
&\text{ iff } (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \\
&\text{ iff } x \in A \wedge x \in B \\
&\text{ iff } x \in (A \cap B)
\end{aligned}$$

□

3.5.4

Theorem. *If $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ then $A \subseteq B$.*

Proof. Suppose $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Let x be arbitrary and suppose $x \in A$. Thus $x \in A \cup C$ and it follows that $x \in B \cup C$. Now if $x \in B \cup C$ then either $x \in B$ or $x \in C$. If $x \in B$ then since x was arbitrary we can conclude $A \subseteq B$. In the case that $x \in C$, then $x \in A \cap C$ and it follows that $x \in B \cap C$. Therefore $x \in C$ and $x \in B$. Thus, if $x \in A$ then $x \in B$ and since x was arbitrary we can conclude $A \subseteq B$. □

3.5.5

Suppose A and B are sets.

Theorem. *If $A \triangle B \subseteq A$ then $B \subseteq A$.*

Proof. Suppose $A \triangle B \subseteq A$. We will prove by contradiction. Let x be arbitrary and suppose $x \in B$ and $x \notin A$. Since $x \in B$ and $x \notin A$ then $x \in A \triangle B$. Since $A \triangle B \subseteq A$, then $x \in A$. But this contradicts $x \notin A$. Therefore, if $x \in B$ then $x \in A$ and since x was arbitrary we can conclude that $B \subseteq A$. $\square$

3.5.6

Suppose A, B, and C are sets.

Theorem. $A \cup C \subseteq B \cup C \iff A \setminus C \subseteq B \setminus C$.

Proof. ($\rightarrow$) Suppose A, B, and C are sets. Suppose $(A \cup C) \subseteq (B \cup C)$. Let x be arbitrary and suppose $c \in A \setminus C$, which means $x \in A$ and $x \notin C$. Since $x \in A$, then $x \in A \cup C$ and therefore $x \in B \cup C$. This means $x \in B$ or $x \in C$ and since $x \notin C$, it must be that $x \in B$. Now since $x \in B$ and $x \notin C$ then $x \in B \setminus C$. Therefore, if $x \in A \setminus C$ then $x \in B \setminus C$ and since x was arbitrary we can conclude if $A \cup C \subseteq B \cup C$ then $A \setminus C \subseteq B \setminus C$.

($\leftarrow$) Now suppose $A \setminus C \subseteq B \setminus C$. Let x be arbitrary and suppose $x \in A \cup C$, which means $x \in A$ or $x \in C$. If $x \in C$ then $x \in B \cup C$ and since x was arbitrary then $A \cup C \subseteq B \cup C$. In the case that $x \in A$, since $A \setminus C \subseteq B \setminus C$ then $x \in B$. Therefore, $x \in B \cup C$ and since x was arbitrary then $A \cup C \subseteq B \cup C$. $\square$

3.5.7

Theorem. *For any sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$*

Proof. Let A and B be arbitrary sets. Let M be arbitrary and suppose $M \in \mathscr{P}(A) \cup \mathscr{P}(B)$. Thus $M \in \mathscr{P}(A)$ or $M \in \mathscr{P}(B)$, which means $M \subseteq A$ or $M \subseteq B$. In the case where $M \subseteq A$, let x be an arbitrary member of M and it follows that $x \in A$. Since $x \in A$ then $x \in A \cup B$ and because x was arbitrary we can conclude $M \subseteq A \cup B$ and therefore $M \in \mathscr{P}(A \cup B)$. In the case where $M \subseteq B$, let x be an arbitrary member of M and it follows that $x \in B$. Since $x \in B$ then $x \in A \cup B$ and because x was arbitrary we can conclude $M \subseteq A \cup B$ and therefore $M \in \mathscr{P}(A \cup B)$. $\square$

3.5.8

Theorem. *For any sets A and B, if $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.*

Proof. We will prove the contrapositive. Since we proved that $\mathscr{P}(A)\cup\mathscr{P}(B)\subseteq\mathscr{P}(A\cup B)$ in exercise 3.5.7, we must show that $\mathscr{P}(A\cup B)\nsubseteq\mathscr{P}(A)\cup\mathscr{P}(B)$ to prove our goal that $\mathscr{P}(A)\cup\mathscr{P}(B)\neq\mathscr{P}(A\cup B)$. Let A and B be arbitrary sets and suppose $A\nsubseteq B$ and $B\nsubseteq A$. This means there is an element $x\in A\setminus B$ and an element $y\in B\setminus A$. Since $x\in A$ and $y\in B$ then both x and y are in $A\cup B$ and therefore the set $\{x,y\}$ is in $\mathscr{P}(A\cup B)$ but not in $\mathscr{P}(A)$ or $\mathscr{P}(B)$. Thus $\mathscr{P}(A\cup B)\nsubseteq\mathscr{P}(A)\cup\mathscr{P}(B)$. □

3.5.9

Theorem. *Suppose x and y are real numbers and $x\neq 0$. Then $y+1/x=1+y/x$ iff either $x=1$ or $y=1$.*

Proof. ($\rightarrow$) Suppose that $y+1/x=1+y/x$. Now if $y=1$ then we have proven our goal. So now assume $y\neq 1$ and $y+1/x=1+y/x$, then it follows that $x=1$.

($\leftarrow$) Now suppose $x=1$ or $y=1$. In the case that $x=1$ we have

$$y+\frac{1}{x}=y+\frac{1}{1}=y+1=1+\frac{y}{1}=1+\frac{y}{x}$$

In the case that $y=1$ we have

$$y+\frac{1}{x}=1+\frac{1}{x}=1+\frac{y}{x}$$

□

3.5.10

Theorem. *For every real number x, if $|x-3|>3$ then $x^2>6x$.*

Proof. Suppose that x is an arbitrary real number and that $|x-3|>3$. Then either $x-3\geq 0$ or $x-3<0$. In the case that $x-3\geq 0$, then $|x-3|=x-3$ and therefore $|x-3|>3=x-3>3$. Solving for x, we have $x>6$ and then multiplying both sides by x we have $x^2>6x$. In the case that $x-3<0$, then $|x-3|=3-x$ and therefore $3-x>3$. Solving for x we have $x<0$. Multiplying both sides of $x<0$ by $6-x$ we have $6x-x^2<0$ and therefore $x^2>6x$. □

3.5.11

Theorem. *For every real number x, $|2x-6|>x$ iff $|x-4|>2$.*

Proof. ($\rightarrow$) Let x be an arbitrary real number and suppose $|2x-6|>x$. Our goal $|x-4|>2$ means that either $x-4>2$ or $4-x>2$. Since $|2x-6|>2$ then either $2x-6>x$ or $6-2x>x$. If $2x-6>x$ then it follows that $x-4>2$. Now if $6-2x>x$ then if follows that $4-x>2$.

(←) Now suppose $|x-4|>2$. Our goal $|2x-6|>x$ means that either $2x-6>x$ or $6-2x>x$. Since $|x-4|>2$ then either $x-4>2$ or $4-x>2$. If $x-4>2$ then it follows that $2x-6>x$. In the case that $4-x>2$ then it follows that $6-2x>x$. □

3.5.12

Theorem. *For all real numbers a and b, $|a| \leq b$ if and only if $-b \leq a \leq b$.*

Proof. (→) Suppose a and b are arbitrary real numbers and that $|a| \leq b$. There are two cases to consider: $a \geq 0$ and $a < 0$. If $a \geq 0$ then $|a| = a \leq b$. It follows that $-b \leq -a$ and since $a \geq 0$ then $-a \leq a$. Therefore, $-b \leq -a \leq a \leq b$ and $-b \leq a \leq b$. Now in the case that $a < 0$ then $|a| = -a \leq b$. It follows that $-b \leq a$ and since $a < 0$ then $-a > a$ or $a < -a$. Therefore $-b \leq a < -a \leq b$ and $-b \leq a \leq b$.

(←) Now suppose $-b \leq a \leq b$ and therefore $a \leq b$. Now we must prove that $-a \leq b$ to complete the proof. If we subtract a from both sides of $-b \leq a$ and add b to both sides we have $-a \leq b$. □

3.5.13

Theorem. *For every integer x, $x^2 + x$ is even.*

Proof. Let x be an arbitrary integer. There are two cases to consider: x is even or x is odd. If x is even then there exists an integer k such that $x = 2k$. Plugging in $2k$ for x in $x^2 + x$ we have $x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$. Since $2k^2 + k$ is an integer then $x^2 + x$ is even. In the case that x is odd there is a j such that $x = 2j + 1$. Plugging in $2j + 1$ for x in $x^2 + x$ we have $x^2+x = (2j+1)^2+(2j+1) = (4j^2+4j+1)+(2j+1) = 4j^2+6j+2 = 2(2j^2+3j+1)$. Since $2j^2 + 3j + 1$ is an integer, $x^2 + x$ is even. □

3.5.14

Theorem. *For every integer x, the remainder when x^4 is divided by 8 is either 0 or 1.*

Proof. Suppose x is an integer and there exists an integer k such that $8k = x^4$. Since x is an integer, x is either even or odd. If x is even then there exists an integer m such that $x = 2m$. Then $8k = (2m)^4 = 16m^4$ and $k = 2m^4$ r 0. In the case that x is odd, then there exists an integer m such that $x = 2m + 1$. Then $8k = (2m+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$ and $k = 2x^4 + 4x^3 + 3x^2 + x$ r 1. Therefore, when x^4 is divided by 8 the remainder is either 0 or 1. □

3.5.15

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets.

Theorem. $\cup(\mathcal{F} \cup \mathcal{G}) = (\cup\mathcal{F}) \cup (\cup\mathcal{G})$

Proof. ($\rightarrow$) Suppose $x \in \cup(\mathcal{F} \cup \mathcal{G})$, which means there is a set in $\mathcal{F} \cup \mathcal{G}$ that contains x. Thus the set that contains x is in $\mathcal{F}$ or $\mathcal{G}$. If the set that contains x is in $\mathcal{F}$ then $x \in \cup\mathcal{F}$ and $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$. In the case that the set that contains x is in $\mathcal{G}$, then $x \in \cup\mathcal{G}$ and $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$

($\leftarrow$) Now suppose $x \in (\cup F) \cup (\cup G)$, which means there is a set in $\mathcal{F}$ that contains x or a set in $\mathcal{G}$ that contains x. If there is a set in $\mathcal{F}$ that contains x, and this same set is in $\mathcal{F} \cup \mathcal{G}$. Thus there is a set in $\mathcal{F} \cup \mathcal{G}$ that contains x. In the case that there is a set in $\mathcal{G}$ that contains x, then this set is in $\mathcal{F} \cup \mathcal{G}$. Thus there is a set in $\mathcal{F} \cup \mathcal{G}$ that contains x. Therefore $x \in \cup(\mathcal{F} \cup \mathcal{G})$.

□

Alternate proof?

Proof.

$$\begin{aligned} x \in \cup(\mathcal{F} \cup \mathcal{G}) & \text{ iff} \\ \exists M \in \mathcal{F} \cup \mathcal{G}(x \in M) & \text{ iff} \\ \exists M \in \mathcal{F}(x \in M) \vee \exists M \in \mathcal{G}(x \in M) & \text{ iff} \\ x \in \cup\mathcal{F} \vee x \in \cup\mathcal{G} & \text{ iff} \\ x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G}) & \end{aligned}$$

□

3.5.16

Suppose $\mathcal{F}$ is a nonempty family of sets and B is a set.

A

Theorem. $B \cup (\cup\mathcal{F}) \subseteq \cup(\mathcal{F} \cup \{B\})$

Proof. ($\rightarrow$) Suppose x is arbitrary and $x \in B \cup (\cup\mathcal{F})$. Then $x \in B$ or $x \in \cup\mathcal{F}$. If $x \in B$ then because $B \in \mathcal{F} \cup \{B\}$, it follows that $x \in \cup(\mathcal{F} \cup \{B\})$. In the case that $x \in \cup\mathcal{F}$, there is a set $M \in \mathcal{F}$ such that $x \in M$. Since $M \in \mathcal{F}$ then $M \in \mathcal{F} \cup \{B\}$ and therefore $x \in \cup(\mathcal{F} \cup \{B\})$.

($\leftarrow$) Now suppose $x \in \cup(\mathcal{F} \cup \{B\})$. Then there is a set M such that $x \in M$ and $M \in (\mathcal{F} \cup \{B\})$, which means $M \in \mathcal{F}$ or $M \in \{B\}$. If $M \in \mathcal{F}$ then it follows that $x \in \cup\mathcal{F}$ and thus $x \in B \cup (\cup\mathcal{F})$. In the case that $M \in \{B\}$ then it follows that $x \in B$ and thus $x \in B \cup (\cup\mathcal{F})$. □

B

Theorem. $B \cup (\cap\mathcal{F}) = \bigcap_{A\in\mathcal{F}}(B \cup A)$

Proof. $(\rightarrow)$ Let x be arbitrary and suppose $x \in B \cup (\cap\mathcal{F})$. Then $x \in B$ or $x \in \cap\mathcal{F}$. If $x \in B$, then $x \in B \cup A$ for any set A and thus $x \in \bigcap_{A\in\mathcal{F}}(B \cup A)$. In the case that $x \in \cap\mathcal{F}$, then x is in every set $A \in \mathcal{F}$ and so $x \in \bigcap_{A\in\mathcal{F}} A$. Therefore $x \in \bigcap_{A\in\mathcal{F}}(B \cup A)$. Since x was arbitrary then $B \cup (\cap\mathcal{F}) \subseteq \bigcap_{A\in\mathcal{F}}(B \cup A)$.

$(\leftarrow)$ Now suppose $x \in \bigcap_{A\in\mathcal{F}}(B \cup A)$. Thus $x \in B$ or $x \in A$ for all $A \in \mathcal{F}$. If $x \in B$ then $x \in B \cup (\cap\mathcal{F})$. If $x \in A$ for all $A \in \mathcal{F}$ then $x \in \cap\mathcal{F}$ and therefore $x \in B \cup (\cap\mathcal{F})$. Since x was arbitrary then $\bigcap_{A\in\mathcal{F}}(B \cup A) \subseteq B \cup (\cap\mathcal{F})$. □

C

Theorem. $B \cap (\cap\mathcal{F}) = \bigcap_{A\in\mathcal{F}}(B \cap A)$

Proof. $(\rightarrow)$ Let x be arbitrary and suppose $x \in B \cap (\cap\mathcal{F})$, which means $x \in B$ and for all $A \in \mathcal{F}$, $x \in A$. Thus $x \in A \cap B$ and since $x \in A$ for all $A \in \mathcal{F}$, then $x \in \bigcap_{A\in\mathcal{F}}(A \cap B)$. Since x was arbitrary, we conclude $B \cap (\cap\mathcal{F}) \subseteq \bigcap_{A\in\mathcal{F}}(B \cap A)$.

$(\leftarrow)$ Now suppose $x \in \bigcap_{A\in\mathcal{F}}(A \cap B)$, which means for all $A \in \mathcal{F}$, $x \in A \cap B$. Therefore $x \in B$ and for all $A \in \mathcal{F}$, $x \in A$ and thus $x \in \cap\mathcal{F}$. Since x was arbitrary we conclude $\bigcap_{A\in\mathcal{F}}(B \cap A) \subseteq B \cap (\cap\mathcal{F})$. □

3.5.17

Theorem. *Suppose $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are nonempty families of sets and for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$, $A \cup B \in \mathcal{H}$, then $\cap\mathcal{H}$ is a subset of $(\cap\mathcal{F}) \cup (\cap\mathcal{G})$.*

Proof. Suppose A and B are arbitrary sets, $A \in \mathcal{F}$, $B \in \mathcal{G}$, and $A \cup B \in \mathcal{H}$. Let x be arbitrary and suppose $x \in \cap\mathcal{H}$, which means x is in every set in $\mathcal{H}$. Since $A \cup B \in \mathcal{H}$, it follows that $x \in A$ or $x \in B$. If $x \in A$, then since A is an arbitrary set in $\mathcal{F}$, then $x \in \cap\mathcal{F}$. If $x \in B$, then since B is an arbitrary set in $\mathcal{G}$, then $x \in \cap\mathcal{G}$. Therefore, $x \in (\cap\mathcal{F}) \cup (\cap\mathcal{G})$ and since x was arbitrary we conclude that $\cap\mathcal{H} \subseteq (\cap\mathcal{F}) \cup (\cap\mathcal{G})$. □

3.5.18

Theorem. *Suppose A and B are sets. Then $\forall x(x \in A \triangle B \iff (x \in A \iff x \notin B))$*

Proof. Let x be arbitrary and suppose $x \in A \triangle B$. Then

$$
\begin{aligned}
x \in A \triangle B \text{ iff } & x \in (A \cup B) \setminus (A \cap B) \\
\text{iff } & (x \in A \cup B) \wedge x \notin (A \cap B) \\
\text{iff } & (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) \\
\text{iff } & (x \notin B \implies x \in A) \wedge (x \in A \implies x \notin B) \\
\text{iff } & x \in A \iff x \notin B
\end{aligned}
$$

□

3.5.19

Theorem. *Suppose A, B, and C are sets. Then $A \triangle B$ and C are disjoint if and only if $A \cap C = B \cap C$.*

Proof. ($\rightarrow$) We will prove by contradiction. Suppose $(A \triangle B) \cap C = \varnothing$. Recall that $x \in A \triangle B$ means that $x \in A \setminus B$ or $x \in B \setminus A$. Now suppose $A \cap C \neq B \cap C$, which means $A \cap C \nsubseteq B \cap C$ or $B \cap C \nsubseteq A \cap C$. If $A \cap C \nsubseteq B \cap C$ then there exists an x such that $x \in A \cap C$ and $x \notin B \cap C$. Thus $x \in C$ and $x \in A \setminus B$, which also means $x \in A \triangle B$. However this contradicts our assumption that $(A \triangle B) \cap C = \varnothing$. In the case that $B \cap C \nsubseteq A \cap C$, there exists an x such that $x \in B \cap C$ and $x \notin A \cap C$. Thus $x \in C$ and $x \in B \setminus A$, which also means $x \in A \triangle B$. However this contradicts our assumption that $(A \triangle B) \cap C = \varnothing$.

($\leftarrow$) We will prove by contradiction. Suppose $A \cap C = B \cap C$. Now suppose that $(A \triangle B) \cap C \neq \varnothing$, which means there exists an x such that $x \in (A \setminus B) \cap C$ or $x \in (B \setminus A) \cap C$. If $x \in (A \setminus B) \cap C$, then $x \in A \setminus B$, which means $x \in A$ and $x \notin B$. Since $x \in A$ and $x \in C$, then $x \in A \cap C$. If follows that $x \in B \cap C$ because $A \cap C = B \cap C$, however this contradicts our assumption that $x \notin B$. In the case that $x \in (B \setminus A) \cap C$, $x \in B \setminus A$. Thus $x \in B$ and $x \notin A$. Since $x \in B$ and $x \in C$, then $x \in B \cap C$. It follows that $x \in A \cap C$ because $A \cap C = B \cap C$. However, this contradicts our assumption that $x \notin A$. □

3.5.20

Theorem. *Suppose A, B, and C are sets. Then $A \triangle B \subseteq C$ if and only if $A \cup C = B \cup C$.*

Proof. ($\rightarrow$) Suppose $A \triangle B \subseteq C$. Let x be arbitrary and suppose $x \in A \cup C$. Thus $x \in A$ or $x \in C$. If $x \in C$ then $x \in B \cup C$. In the case that $x \in A$ and $x \notin C$, then it follows that $x \notin A \triangle B$ and thus $x \in A \cap B$. Since $x \in A \cap B$, $x \in B$ and thus $x \in B \cup C$. Now to prove the other direction suppose $x \in B \cup C$. Thus $x \in B$ or $x \in C$. If $x \in C$ then $x \in A \cup C$. In the case that $x \in B$ and $x \notin C$, then it follows that $x \notin A \triangle B$ and thus $x \in A \cap B$. Since $x \in A \cap B$, $x \in A$ and thus $x \in A \cup C$.

(←) Now suppose $A \cup C = B \cup C$ and $x \in A \triangle B$. By the definition of symmetrical difference, $x \in A \setminus B$ or $x \in B \setminus A$. If $x \in A \setminus B$ then $x \in A$ and $x \notin B$. It follows that $x \in A \cup C$ and therefore $x \in B \cup C$. Since $x \in B \cup C$ and $x \notin B$, then $x \in C$. In the case that $x \in B \setminus A$, $x \in B$ and $x \notin A$. It follows that $x \in B \cup C$ and therefore $x \in A \cup C$. Since $x \in A \cup C$ and $x \notin A$, then $x \in C$. Thus if $x \in A \triangle B$ then $x \in C$. □

3.5.21

Theorem. *Suppose A, B, and C are sets. Then $C \subseteq A \triangle B$ if and only if $C \subseteq A \cup B$ and $A \cap B \cap C = \varnothing$.*

Proof. (→) Suppose $C \subseteq A \triangle B$. To show that $C \subseteq A \cup B$, let x be arbitrary and suppose $x \in C$. Since $x \in C$ then $x \in A \triangle B$. By the definition of symmetric difference, $x \in A \cup B$ and $x \notin A \cap B$. Thus if $x \in C$ then $x \in A \cup B$. To show that $A \cap B \cap C = \varnothing$ we will used proof by contradiction. Suppose there is an element y such that $y \in A$, $y \in B$, and $y \in C$. Since $y \in C$ then $y \in A \triangle B$. As noted earlier, if $y \in A \triangle B$ then $y \notin A \cap B$; however, this contradicts our assumption that $y \in A$ and $y \in B$. Thus, it must be that $A \cap B \cap C = \varnothing$.

(←) Now suppose $C \subseteq A \cup B$ and $A \cap B \cap C = \varnothing$. Let x be arbitrary and suppose $x \in C$. It follows that $x \in A \cup B$. Since $x \in C$ and $A \cap B \cap C = \varnothing$, then $x \notin A \cap B$. Now since $x \in A \cup B$ and $x \notin A \cap B$, then $x \in A \triangle B$. □

3.5.22

Suppose A, B, and C are sets.

A

Theorem. $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$

Proof. Suppose x is arbitrary and $x \in A \setminus C$, which means $x \in A$ and $x \notin C$. Now either $x \in B$ or $x \notin B$. If $x \in B$, then it follows that $x \in B \setminus C$ and thus $x \in (A \setminus B) \cup (B \setminus C)$. Therefore if $x \in A \setminus C$ then $x \in (A \setminus B) \cup (B \setminus C)$ and since x was arbitrary we can conclude $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$ □

B

Theorem. $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$

Proof. Suppose x is arbitrary and $x \in A \triangle C$, which means $x \in A \setminus C$ or $x \in C \setminus A$. Also, either $x \in B$ or $x \notin B$. Thus, we have four cases to consider:

Case 1: $x \in A \setminus C$ and $x \in B$. Since $x \in A \setminus C$ then $x \in A$ and $x \notin C$. Therefore $x \in B$ and $x \notin C$ and by the definition of symmetric difference, $x \in B \triangle C$. Therefore if $x \in A \triangle C$ then $x \in (A \triangle B) \cup (B \triangle C)$ and since x was arbitrary we can conclude $A \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.

Case 2: $x \in A \setminus C$ and $x \notin B$. Since $x \in A \setminus C$ then $x \in A$ and $x \notin C$. Therefore, $x \in A$ and $x \notin B$ and therefore $x \in A \triangle B$. Therefore, if $x \in A \triangle C$ then $x \in (A \triangle B) \cup (B \triangle C)$ and since x was arbitrary we can conclude $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

Case 3: $x \in C \setminus A$ and $x \in B$. Since $x \in C \setminus A$ then $x \in C$ and $x \in A$. Therefore $x \in B$ and $x \notin A$, and therefore $x \in A \triangle B$. Therefore, if $x \in A \triangle C$ then $x \in (A \triangle B) \cup (B \triangle C)$ and since x was arbitrary we can conclude that $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

Case 4: $x \in C \setminus A$ and $x \notin B$. Since $x \in C \setminus A$ then $x \in C$ and $x \notin A$. Therefore $x \in C \setminus B$ and thus $x \in B \triangle C$. Therefore, if $x \in A \triangle C$ then $x \in (A \triangle B) \cup (B \triangle C)$ and since x was arbitrary we can conclude that $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$. □

3.5.23

Suppose A, B, and C are sets.

A

Theorem. $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$

Proof. Suppose x is arbitrary and $x \in (A \cup B) \triangle C$. Thus $x \in A \setminus C$ or $x \in C \setminus (A \cup B)$, and we have 4 cases to consider:

Case 1: $x \in A$, $x \notin B$, and $x \notin C$. Since $x \in A$ and $x \notin C$ then $x \in A \setminus C$ and $x \in A \triangle C$. Therefore if $x \in (A \cup B) \triangle C$ then $x \in (A \triangle C) \cup (B \triangle C)$ and since x was arbitrary we can conclude $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.

Case 2: $x \in B$, $x \notin A$, and $x \notin C$. Since $x \in B$ and $x \notin C$ then $x \in B \setminus C$ and $x \in B \triangle C$. Therefore if $x \in (A \cup B) \triangle C$ then $x \in (A \triangle C) \cup (B \triangle C)$ and since x was arbitrary we can conclude $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.

Case 3: $x \in A$, $x \in B$, and $x \notin C$. Since $x \in B$ and $x \notin C$ then $x \in B \setminus C$ and $x \in B \triangle C$. Since $x \in A$ and $x \notin C$ then $x \in A \setminus C$ and $x \in A \triangle C$. Thus $x \in (A \triangle C) \cup (B \triangle C)$. Therefore if $x \in (A \cup B) \triangle C$ then $x \in (A \triangle C) \cup (B \triangle C)$ and since x was arbitrary we can conclude $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.

Case 4: $x \in C$, $x \notin A$, $x \notin B$. Since $x \in C$ and $x \notin A$ then $x \in C \setminus A$ and $x \in A \triangle C$. Since $x \in C$ and $x \notin B$ then $x \in C \setminus B$ and $x \in B \triangle C$. Thus $x \in (A \triangle C) \cup (B \triangle C)$. Therefore if $x \in (A \cup B) \triangle C$ then $x \in (A \triangle C) \cup (B \triangle C$ and since x was arbitrary we can conclude $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$. □

B

Find an example of sets A, B, and C such that $(A \cup B) \triangle C \neq (A \triangle C) \cup (B \triangle C)$

Let $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{1, 5\}$.

Then $A \triangle C = (A \setminus C) \cup (C \setminus A) = \{2\} \cup \{5\} = \{2, 5\}$.

Now $B \triangle C = (B \setminus C) \cup (C \setminus B) = \{1, 3, 4, 5\} \cup \{1, 3, 4, 5\} = \{1, 3, 4, 5\}$.

Therefore $(A \triangle C) \cup (B \triangle C) = \{2, 5\} \cup \{1, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$.

Now $(A \cup B) = \{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\}$.

Therefore $(A \cup B) \triangle C = ((A \cup B) \setminus C) \cup (C \setminus (A \cup B)) = \{2, 3, 4\} \cup \{5\} = \{2, 3, 4, 5\}$.

Since $\{1, 2, 3, 4, 5\} \neq \{2, 3, 4, 5\}$ then $(A \cup B) \triangle C \neq (A \triangle C) \cup (B \triangle C)$.

3.5.24

Suppose A, B, and C are sets.

A

Theorem. *$(A \triangle C) \cap (B \triangle C) \subseteq (A \cap B) \triangle C$.*

Proof. Let x be arbitrary and suppose $x \in (A \triangle C) \cap (B \triangle C)$. Then we have two cases to consider:

Case 1: $x \in A$, $x \in B$, and $x \notin C$. Since $x \in A$ and $x \in B$, then $x \in A \cap B$ and $x \notin C$. Therefore $x \in (A \cap B) \setminus C$ and if $x \in (A \triangle C) \cap (B \triangle C)$ then $x \in (A \cap B) \triangle C$. Since x was arbitrary we can conclude $(A \triangle C) \cap (B \triangle C) \subseteq (A \cap B) \triangle C$.

Case 2: $x \notin A$, $x \notin B$, $x \in C$. Since $x \notin A$ and $x \notin B$, then $x \notin A \cap B$ and since $x \in C$ then $x \in C \setminus (A \cap B)$. Therefore if $x \in (A \triangle C) \cap (B \triangle C)$ then $x \in (A \cap B) \triangle C$ and since x was arbitrary we can conclude $(A \triangle C) \cap (B \triangle C) \subseteq (A \cap B) \triangle C$. □

B

Theorem. *$(A \cap B) \triangle C \subseteq (A \triangle C) \cap (B \triangle C)$*

Proof. Let x be arbitrary and suppose $x \in (A \cap B) \triangle C$. Thus $x \in A \cap B \setminus C$ or $x \in C \setminus A \cap B$.

Case 1: $x \in (A \cap B) \setminus C$. Thus $x \in A \cap B$ and $x \notin C$. Since $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $x \in A$ and $x \notin C$ then $x \in A \setminus C$ and since $x \in B$ and $x \notin C$ then $x \in B \setminus C$. Therefore $x \in A \triangle C$ and $x \in B \triangle C$ and if $x \in (A \cap B) \setminus C$ then $x \in (A \triangle C) \cap (B \triangle C)$. Since x was arbitrary we can conclude $(A \cap B) \triangle C \subseteq (A \triangle C) \cap (B \triangle C)$.

Case 2: $x \in C \setminus (A \cap B)$. Thus $x \in C$ and $x \notin A \cap B$. Since $x \notin A \cap B$ then $x \notin A$ and $x \notin B$. Since $x \in A$ and $x \notin C$ then $x \in A \setminus C$. Also since $x \in B$ and $x \notin C$ then $x \in B \setminus B$. Therefore $x \in A \triangle C$ and $x \in B \triangle C$ and if $x \in C \setminus (A \cap B)$ then $x \in (A \triangle C) \cap (B \triangle C)$. Since x was arbitrary then we can conclude $(A \cap B) \triangle C \subseteq (A \triangle C) \cap (B \triangle C)$. □

3.5.25

Suppose A, B, and C are sets. Consider the sets $(A \setminus B) \triangle C$ and $(A \triangle C) \setminus (B \triangle C)$. Can you prove that either is a subset of the other?

To show that $(A \setminus B) \triangle C$ is not a subset of $(A \triangle C) \setminus (B \triangle C)$ consider the counterexample where $A = \{1,2\}$, $B = \{1,3\}$, and $C = \{3,4\}$. Then $A \setminus B = \{2\}$ and $(A \setminus B) \triangle C = \{3,4\}$. Also, $A \triangle C = \{1,2,3,4\}$, $B \triangle C = \{1,4\}$, and $(A \triangle C) \setminus (B \triangle C) = \{2,3\}$. Therefore $(A \setminus B) \triangle C = \{3,4\} \not\subseteq \{2,3\} = (A \triangle C) \setminus (B \triangle C)$.

We will show that $(A \triangle C) \setminus (B \triangle C) \subseteq (A \setminus B) \triangle C$.

Proof. Suppose x is arbitrary and $x \in (A \triangle C) \setminus (B \triangle C)$, which means $x \in A \triangle C$ and $x \notin B \triangle C$. Consider the two cases, either $x \in A$, $x \notin B$, and $x \notin C$ or $x \notin A$, $x \in B$, and $x \in C$. If $x \in A$, $x \notin B$, and $x \notin C$, then $x \in A$ and $x \notin B$, thus $x \in A \setminus B$. Since $x \in A \setminus B$ and $x \notin C$ then $x \in (A \setminus B) \setminus C$ and therefore $x \in (A \setminus B) \triangle C$. If $x \notin A$, $x \in B$, and $x \in C$, then since $x \notin A$ and $x \in B$ then $x \notin A \setminus B$. Then since $x \in C$, we can conclude $x \in C \setminus (A \setminus B)$ and thus $x \in (A \setminus B) \triangle C$. $\square$

3.5.26

No the proof is not correct because it proves that $0 < x$ or $x < 6$, but we need to show $0 < x$ and $x < 6$, or $0 < x < 6$. The theorem is correct and a proof is given below.

Theorem. *For every real number x, if $|x - 3| < 3$ then $0 < x < 6$.*

Proof. Suppose $|x - 3| < 3$. Then $-3 < x - 3 < 3$ and adding 3 to both sides yields $0 < x < 6$. Therefore, if $|x - 3| < 3$ then $0 < x < 6$. $\square$

3.5.27

Yes, the proof is correct. Some strategies used in the proof are: assume the antecedent and prove the consequent, existential instantiation, proofs involving a disjunction (if a or b and a is false, then b must be true).

3.5.28

Yes, the proof is correct. Some strategies used in the proof are: universal instantiation, existential instantiation, proof by cases.

3.5.29

Theorem. $\forall x P(x) \implies Q(x)$ *then* $\exists x(P(x) \implies Q(x))$.

Proof. Suppose $\forall x(P(x) \implies Q(x))$, which is logically equivalent to $\forall x(\neg P(x) \vee Q(x))$. Let x be arbitrary and then either $\neg P(x)$ or $Q(x)$. If $\neg P(x)$ then we have found an x such that $\neg P(x)$. If $Q(x)$ then we have found an x such that $Q(x)$. Thus we have found an x such that $\neg P(x)$ or $Q(x)$, which is logically equivalent to $P(x) \implies Q(x)$.

$\square$