

### 3.5.1

Suppose  $A$ ,  $B$ , and  $C$  are sets.

**Theorem.**  $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A \cap (B \cup C)$ . Thus  $x \in A$  and  $x \in B$  or  $x \in C$ . If  $x \in C$  then  $x \in (A \cap B) \cup C$ . In the case where  $x \in B$  it follows that  $x \in A \cap B$  and therefore  $x \in (A \cap B) \cup C$ . Since  $x$  was arbitrary we can conclude that  $A \cap (B \cup C) \subseteq (A \cap B) \cup C$ .  $\square$

### 3.5.2

Suppose  $A$ ,  $B$ , and  $C$  are sets.

**Theorem.**  $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in (A \cup B) \setminus C$ . Thus  $x \notin C$  and  $x \in A$  or  $x \in B$ . If  $x \in A$  then  $x \in A \cup (B \setminus C)$ . If  $x \in B$  then it follows that  $x \in B \setminus C$  and therefore  $x \in A \cup (B \setminus C)$ . Since  $x$  was arbitrary we can conclude  $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$ .  $\square$

### 3.5.3

Suppose  $A$  and  $B$  are sets.

**Theorem.**  $A \setminus (A \setminus B) = A \cap B$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A \setminus (A \setminus B)$ . Then

$$\begin{aligned} x \in A \setminus (A \setminus B) &\text{ iff } x \in A \wedge x \notin A \setminus B \\ &\text{ iff } x \in A \wedge \neg(x \in A \wedge x \notin B) \\ &\text{ iff } x \in A \wedge (x \notin A \vee x \in B) \\ &\text{ iff } (x \in A \wedge x \notin A) \vee (x \in A \wedge x \in B) \\ &\text{ iff } x \in A \wedge x \in B \\ &\text{ iff } x \in (A \cap B) \end{aligned}$$

$\square$

### 3.5.4

**Theorem.** If  $A \cap C \subseteq B \cap C$  and  $A \cup C \subseteq B \cup C$  then  $A \subseteq B$ .

*Proof.* Suppose  $A \cap C \subseteq B \cap C$  and  $A \cup C \subseteq B \cup C$ . Let  $x$  be arbitrary and suppose  $x \in A$ . Thus  $x \in A \cup C$  and it follows that  $x \in B \cup C$ . Now if  $x \in B \cup C$  then either  $x \in B$  or  $x \in C$ . If  $x \in B$  then since  $x$  was arbitrary we can conclude  $A \subseteq B$ . In the case that  $x \in C$ , then  $x \in A \cap C$  and it follows that  $x \in B \cap C$ . Therefore  $x \in C$  and  $x \in B$ . Thus, if  $x \in A$  then  $x \in B$  and since  $x$  was arbitrary we can conclude  $A \subseteq B$ .  $\square$

### 3.5.5

Suppose  $A$  and  $B$  are sets.

**Theorem.** If  $A \triangle B \subseteq A$  then  $B \subseteq A$ .

*Proof.* Suppose  $A \triangle B \subseteq A$ . We will prove by contradiction. Let  $x$  be arbitrary and suppose  $x \in B$  and  $x \notin A$ . Since  $x \in B$  and  $x \notin A$  then  $x \in A \triangle B$ . Since  $A \triangle B \subseteq A$ , then  $x \in A$ . But this contradicts  $x \notin A$ . Therefore, if  $x \in B$  then  $x \in A$  and since  $x$  was arbitrary we can conclude that  $B \subseteq A$ .  $\square$

### 3.5.6

Suppose  $A$ ,  $B$ , and  $C$  are sets.

**Theorem.**  $A \cup C \subseteq B \cup C \iff A \setminus C \subseteq B \setminus C$ .

*Proof.* ( $\rightarrow$ ) Suppose  $A$ ,  $B$ , and  $C$  are sets. Suppose  $(A \cup C) \subseteq (B \cup C)$ . Let  $x$  be arbitrary and suppose  $x \in A \setminus C$ , which means  $x \in A$  and  $x \notin C$ . Since  $x \in A$ , then  $x \in A \cup C$  and therefore  $x \in B \cup C$ . This means  $x \in B$  or  $x \in C$  and since  $x \notin C$ , it must be that  $x \in B$ . Now since  $x \in B$  and  $x \notin C$  then  $x \in B \setminus C$ . Therefore, if  $x \in A \setminus C$  then  $x \in B \setminus C$  and since  $x$  was arbitrary we can conclude if  $A \cup C \subseteq B \cup C$  then  $A \setminus C \subseteq B \setminus C$ .

( $\leftarrow$ ) Now suppose  $A \setminus C \subseteq B \setminus C$ . Let  $x$  be arbitrary and suppose  $x \in A \cup C$ , which means  $x \in A$  or  $x \in C$ . If  $x \in C$  then  $x \in B \cup C$  and since  $x$  was arbitrary then  $A \cup C \subseteq B \cup C$ . In the case that  $x \in A$ , since  $A \setminus C \subseteq B \setminus C$  then  $x \in B$ . Therefore,  $x \in B \cup C$  and since  $x$  was arbitrary then  $A \cup C \subseteq B \cup C$ .  $\square$

### 3.5.7

**Theorem.** For any sets  $A$  and  $B$ ,  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

*Proof.* Let  $A$  and  $B$  be arbitrary sets. Let  $M$  be arbitrary and suppose  $M \in \mathcal{P}(A) \cup \mathcal{P}(B)$ . Thus  $M \in \mathcal{P}(A)$  or  $M \in \mathcal{P}(B)$ , which means  $M \subseteq A$  or  $M \subseteq B$ . In the case where  $M \subseteq A$ , let  $x$  be an arbitrary member of  $M$  and it follows that  $x \in A$ . Since  $x \in A$  then  $x \in A \cup B$  and because  $x$  was arbitrary we can conclude  $M \subseteq A \cup B$  and therefore  $M \in \mathcal{P}(A \cup B)$ . In the case where  $M \subseteq B$ , let  $x$  be an arbitrary member of  $M$  and it follows that  $x \in B$ . Since  $x \in B$  then  $x \in A \cup B$  and because  $x$  was arbitrary we can conclude  $M \subseteq A \cup B$  and therefore  $M \in \mathcal{P}(A \cup B)$ .  $\square$

### 3.5.8

**Theorem.** For any sets  $A$  and  $B$ , if  $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$  then either  $A \subseteq B$  or  $B \subseteq A$ .

*Proof.* We will prove the contrapositive. Since we proved that  $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$  in exercise 3.5.7, we must show that  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$  to prove our goal that  $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$ . Let  $A$  and  $B$  be arbitrary sets and suppose  $A \not\subseteq B$  and  $B \not\subseteq A$ . This means there is an element  $x \in A \setminus B$  and an element  $y \in B \setminus A$ . Since  $x \in A$  and  $y \in B$  then both  $x$  and  $y$  are in  $A \cup B$  and therefore the set  $\{x, y\}$  is in  $\mathcal{P}(A \cup B)$  but not in  $\mathcal{P}(A)$  or  $\mathcal{P}(B)$ . Thus  $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ .  $\square$

### 3.5.9

**Theorem.** Suppose  $x$  and  $y$  are real numbers and  $x \neq 0$ . Then  $y + 1/x = 1 + y/x$  iff either  $x = 1$  or  $y = 1$ .

*Proof.* ( $\rightarrow$ ) Suppose that  $y + 1/x = 1 + y/x$ . Now if  $y = 1$  then we have proven our goal. So now assume  $y \neq 1$  and  $y + 1/x = 1 + y/x$ , then it follows that  $x = 1$ .

( $\leftarrow$ ) Now suppose  $x = 1$  or  $y = 1$ . In the case that  $x = 1$  we have

$$y + \frac{1}{x} = y + \frac{1}{1} = y + 1 = 1 + \frac{y}{1} = 1 + \frac{y}{x}$$

In the case that  $y = 1$  we have

$$y + \frac{1}{x} = 1 + \frac{1}{x} = 1 + \frac{y}{x}$$

$\square$

### 3.5.10

**Theorem.** For every real number  $x$ , if  $|x - 3| > 3$  then  $x^2 > 6x$ .

*Proof.* Suppose that  $x$  is an arbitrary real number and that  $|x - 3| > 3$ . Then either  $x - 3 \geq 0$  or  $x - 3 < 0$ . In the case that  $x - 3 \geq 0$ , then  $|x - 3| = x - 3$  and therefore  $|x - 3| > 3 = x - 3 > 3$ . Solving for  $x$ , we have  $x > 6$  and then multiplying both sides by  $x$  we have  $x^2 > 6x$ . In the case that  $x - 3 < 0$ , then  $|x - 3| = 3 - x$  and therefore  $3 - x > 3$ . Solving for  $x$  we have  $x < 0$ . Multiplying both sides of  $x < 0$  by  $6 - x$  we have  $6x - x^2 < 0$  and therefore  $x^2 > 6x$ .  $\square$

### 3.5.11

**Theorem.** For every real number  $x$ ,  $|2x - 6| > x$  iff  $|x - 4| > 2$ .

*Proof.* ( $\rightarrow$ ) Let  $x$  be an arbitrary real number and suppose  $|2x - 6| > x$ . Our goal  $|x - 4| > 2$  means that either  $x - 4 > 2$  or  $4 - x > 2$ . Since  $|2x - 6| > 2$  then either  $2x - 6 > x$  or  $6 - 2x > x$ . If  $2x - 6 > x$  then it follows that  $x - 4 > 2$ . Now if  $6 - 2x > x$  then it follows that  $4 - x > 2$ .

( $\leftarrow$ ) Now suppose  $|x - 4| > 2$ . Our goal  $|2x - 6| > x$  means that either  $2x - 6 > x$  or  $6 - 2x > x$ . Since  $|x - 4| > 2$  then either  $x - 4 > 2$  or  $4 - x > 2$ . If  $x - 4 > 2$  then it follows that  $2x - 6 > x$ . In the case that  $4 - x > 2$  then it follows that  $6 - 2x > x$ .  $\square$

### 3.5.12

**Theorem.** For all real numbers  $a$  and  $b$ ,  $|a| \leq b$  if and only if  $-b \leq a \leq b$ .

*Proof.* ( $\rightarrow$ ) Suppose  $a$  and  $b$  are arbitrary real numbers and that  $|a| \leq b$ . There are two cases to consider:  $a \geq 0$  and  $a < 0$ . If  $a \geq 0$  then  $|a| = a \leq b$ . It follows that  $-b \leq -a$  and since  $a \geq 0$  then  $-a \leq a$ . Therefore,  $-b \leq -a \leq a \leq b$  and  $-b \leq a \leq b$ . Now in the case that  $a < 0$  then  $|a| = -a \leq b$ . It follows that  $-b \leq a$  and since  $a < 0$  then  $-a > a$  or  $a < -a$ . Therefore  $-b \leq a < -a \leq b$  and  $-b \leq a \leq b$ .

( $\leftarrow$ ) Now suppose  $-b \leq a \leq b$  and therefore  $a \leq b$ . Now we must prove that  $-a \leq b$  to complete the proof. If we subtract  $a$  from both sides of  $-b \leq a$  and add  $b$  to both sides we have  $-a \leq b$ .  $\square$

### 3.5.13

**Theorem.** For every integer  $x$ ,  $x^2 + x$  is even.

*Proof.* Let  $x$  be an arbitrary integer. There are two cases to consider:  $x$  is even or  $x$  is odd. If  $x$  is even then there exists an integer  $k$  such that  $x = 2k$ . Plugging in  $2k$  for  $x$  in  $x^2 + x$  we have  $x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$ . Since  $2k^2 + k$  is an integer then  $x^2 + x$  is even. In the case that  $x$  is odd there is a  $j$  such that  $x = 2j + 1$ . Plugging in  $2j + 1$  for  $x$  in  $x^2 + x$  we have  $x^2 + x = (2j+1)^2 + (2j+1) = (4j^2 + 4j + 1) + (2j + 1) = 4j^2 + 6j + 2 = 2(2j^2 + 3j + 1)$ . Since  $2j^2 + 3j + 1$  is an integer,  $x^2 + x$  is even.  $\square$

### 3.5.14

**Theorem.** For every integer  $x$ , the remainder when  $x^4$  is divided by 8 is either 0 or 1.

*Proof.* Suppose  $x$  is an integer and there exists an integer  $k$  such that  $8k = x^4$ . Since  $x$  is an integer,  $x$  is either even or odd. If  $x$  is even then there exists an integer  $m$  such that  $x = 2m$ . Then  $8k = (2m)^4 = 16m^4$  and  $k = 2m^4$  r 0. In the case that  $x$  is odd, then there exists an integer  $m$  such that  $x = 2m + 1$ . Then  $8k = (2m+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$  and  $k = 2x^4 + 4x^3 + 3x^2 + x$  r 1. Therefore, when  $x^4$  is divided by 8 the remainder is either 0 or 1.  $\square$

### 3.5.15

Suppose  $\mathcal{F}$  and  $\mathcal{G}$  are nonempty families of sets.

**Theorem.**  $\cup(\mathcal{F} \cup \mathcal{G}) = (\cup\mathcal{F}) \cup (\cup\mathcal{G})$

*Proof.* ( $\rightarrow$ ) Suppose  $x \in \cup(\mathcal{F} \cup \mathcal{G})$ , which means there is a set in  $\mathcal{F} \cup \mathcal{G}$  that contains  $x$ . Thus the set that contains  $x$  is in  $\mathcal{F}$  or  $\mathcal{G}$ . If the set that contains  $x$  is in  $\mathcal{F}$  then  $x \in \cup\mathcal{F}$  and  $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$ . In the case that the set that contains  $x$  is in  $\mathcal{G}$ , then  $x \in \cup\mathcal{G}$  and  $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$ .

( $\leftarrow$ ) Now suppose  $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$ , which means there is a set in  $\mathcal{F}$  that contains  $x$  or a set in  $\mathcal{G}$  that contains  $x$ . If there is a set in  $\mathcal{F}$  that contains  $x$ , and this same set is in  $\mathcal{F} \cup \mathcal{G}$ . Thus there is a set in  $\mathcal{F} \cup \mathcal{G}$  that contains  $x$ . In the case that there is a set in  $\mathcal{G}$  that contains  $x$ , then this set is in  $\mathcal{F} \cup \mathcal{G}$ . Thus there is a set in  $\mathcal{F} \cup \mathcal{G}$  that contains  $x$ . Therefore  $x \in \cup(\mathcal{F} \cup \mathcal{G})$ . □

Alternate proof?

*Proof.*

$$\begin{aligned} x \in \cup(\mathcal{F} \cup \mathcal{G}) &\text{ iff} \\ \exists M \in \mathcal{F} \cup \mathcal{G} (x \in M) &\text{ iff} \\ \exists M \in \mathcal{F} (x \in M) \vee \exists M \in \mathcal{G} (x \in M) &\text{ iff} \\ x \in \cup\mathcal{F} \vee x \in \cup\mathcal{G} &\text{ iff} \\ x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G}) & \end{aligned}$$

□

### 3.5.16

Suppose  $\mathcal{F}$  is a nonempty family of sets and  $B$  is a set.

**A**

**Theorem.**  $B \cup (\cup\mathcal{F}) \subseteq \cup(\mathcal{F} \cup \{B\})$

*Proof.* ( $\rightarrow$ ) Suppose  $x$  is arbitrary and  $x \in B \cup (\cup\mathcal{F})$ . Then  $x \in B$  or  $x \in \cup\mathcal{F}$ . If  $x \in B$  then because  $B \in \mathcal{F} \cup \{B\}$ , it follows that  $x \in \cup(\mathcal{F} \cup \{B\})$ . In the case that  $x \in \cup\mathcal{F}$ , there is a set  $M \in \mathcal{F}$  such that  $x \in M$ . Since  $M \in \mathcal{F}$  then  $M \in \mathcal{F} \cup \{B\}$  and therefore  $x \in \cup(\mathcal{F} \cup \{B\})$ .

( $\leftarrow$ ) Now suppose  $x \in \cup(\mathcal{F} \cup \{B\})$ . Then there is a set  $M$  such that  $x \in M$  and  $M \in (\mathcal{F} \cup \{B\})$ , which means  $M \in \mathcal{F}$  or  $M \in \{B\}$ . If  $M \in \mathcal{F}$  then it follows that  $x \in \cup\mathcal{F}$  and thus  $x \in B \cup (\cup\mathcal{F})$ . In the case that  $M \in \{B\}$  then it follows that  $x \in B$  and thus  $x \in B \cup (\cup\mathcal{F})$ . □

## B

**Theorem.**  $B \cup (\cap \mathcal{F}) = \cap_{A \in \mathcal{F}} (B \cup A)$

*Proof.* ( $\rightarrow$ ) Let  $x$  be arbitrary and suppose  $x \in B \cup (\cap \mathcal{F})$ . Then  $x \in B$  or  $x \in \cap \mathcal{F}$ . If  $x \in B$ , then  $x \in B \cup A$  for any set  $A$  and thus  $x \in \cap_{A \in \mathcal{F}} (B \cup A)$ . In the case that  $x \in \cap \mathcal{F}$ , then  $x$  is in every set  $A \in \mathcal{F}$  and so  $x \in \cap_{A \in \mathcal{F}} A$ . Therefore  $x \in \cap_{A \in \mathcal{F}} (B \cup A)$ . Since  $x$  was arbitrary then  $B \cup (\cap \mathcal{F}) \subseteq \cap_{A \in \mathcal{F}} (B \cup A)$ .

( $\leftarrow$ ) Now suppose  $x \in \cap_{A \in \mathcal{F}} (B \cup A)$ . Thus  $x \in B$  or  $x \in A$  for all  $A \in \mathcal{F}$ . If  $x \in B$  then  $x \in B \cup (\cap \mathcal{F})$ . If  $x \in A$  for all  $A \in \mathcal{F}$  then  $x \in \cap \mathcal{F}$  and therefore  $x \in B \cup (\cap \mathcal{F})$ . Since  $x$  was arbitrary then  $\cap_{A \in \mathcal{F}} (B \cup A) \subseteq B \cup (\cap \mathcal{F})$ .  $\square$

## C

**Theorem.**  $B \cap (\cap \mathcal{F}) = \cap_{A \in \mathcal{F}} (B \cap A)$

*Proof.* ( $\rightarrow$ ) Let  $x$  be arbitrary and suppose  $x \in B \cap (\cap \mathcal{F})$ , which means  $x \in B$  and for all  $A \in \mathcal{F}$ ,  $x \in A$ . Thus  $x \in A \cap B$  and since  $x \in A$  for all  $A \in \mathcal{F}$ , then  $x \in \cap_{A \in \mathcal{F}} (A \cap B)$ . Since  $x$  was arbitrary, we conclude  $B \cap (\cap \mathcal{F}) \subseteq \cap_{A \in \mathcal{F}} (B \cap A)$ .

( $\leftarrow$ ) Now suppose  $x \in \cap_{A \in \mathcal{F}} (B \cap A)$ , which means for all  $A \in \mathcal{F}$ ,  $x \in A \cap B$ . Therefore  $x \in B$  and for all  $A \in \mathcal{F}$ ,  $x \in A$  and thus  $x \in \cap \mathcal{F}$ . Since  $x$  was arbitrary we conclude  $\cap_{A \in \mathcal{F}} (B \cap A) \subseteq B \cap (\cap \mathcal{F})$ .  $\square$

### 3.5.17

**Theorem.** Suppose  $\mathcal{F}$ ,  $\mathcal{G}$ , and  $\mathcal{H}$  are nonempty families of sets and for every  $A \in \mathcal{F}$  and every  $B \in \mathcal{G}$ ,  $A \cup B \in \mathcal{H}$ , then  $\cap \mathcal{H}$  is a subset of  $(\cap \mathcal{F}) \cup (\cap \mathcal{G})$ .

*Proof.* Suppose  $A$  and  $B$  are arbitrary sets,  $A \in \mathcal{F}$ ,  $B \in \mathcal{G}$ , and  $A \cup B \in \mathcal{H}$ . Let  $x$  be arbitrary and suppose  $x \in \cap \mathcal{H}$ , which means  $x$  is in every set in  $\mathcal{H}$ . Since  $A \cup B \in \mathcal{H}$ , it follows that  $x \in A$  or  $x \in B$ . If  $x \in A$ , then since  $A$  is an arbitrary set in  $\mathcal{F}$ , then  $x \in \cap \mathcal{F}$ . If  $x \in B$ , then since  $B$  is an arbitrary set in  $\mathcal{G}$ , then  $x \in \cap \mathcal{G}$ . Therefore,  $x \in (\cap \mathcal{F}) \cup (\cap \mathcal{G})$  and since  $x$  was arbitrary we conclude that  $\cap \mathcal{H} \subseteq (\cap \mathcal{F}) \cup (\cap \mathcal{G})$ .  $\square$

### 3.5.18

**Theorem.** Suppose  $A$  and  $B$  are sets. Then  $\forall x (x \in A \Delta B \iff (x \in A \iff x \notin B))$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in A \Delta B$ . Then

$$\begin{aligned}
x \in A \Delta B & \text{ iff } x \in (A \cup B) \setminus (A \cap B) \\
& \text{ iff } (x \in A \cup B) \wedge x \notin (A \cap B) \\
& \text{ iff } (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) \\
& \text{ iff } (x \notin B \implies x \in A) \wedge (x \in A \implies x \notin B) \\
& \text{ iff } x \in A \iff x \notin B
\end{aligned}$$

□

### 3.5.19

**Theorem.** Suppose  $A$ ,  $B$ , and  $C$  are sets. Then  $A \Delta B$  and  $C$  are disjoint if and only if  $A \cap C = B \cap C$ .

*Proof.* ( $\rightarrow$ ) We will prove by contradiction. Suppose  $(A \Delta B) \cap C = \emptyset$ . Recall that  $x \in A \Delta B$  means that  $x \in A \setminus B$  or  $x \in B \setminus A$ . Now suppose  $A \cap C \neq B \cap C$ , which means  $A \cap C \not\subseteq B \cap C$  or  $B \cap C \not\subseteq A \cap C$ . If  $A \cap C \not\subseteq B \cap C$  then there exists an  $x$  such that  $x \in A \cap C$  and  $x \notin B \cap C$ . Thus  $x \in C$  and  $x \in A \setminus B$ , which also means  $x \in A \Delta B$ . However this contradicts our assumption that  $(A \Delta B) \cap C = \emptyset$ . In the case that  $B \cap C \not\subseteq A \cap C$ , there exists an  $x$  such that  $x \in B \cap C$  and  $x \notin A \cap C$ . Thus  $x \in C$  and  $x \in B \setminus A$ , which also means  $x \in A \Delta B$ . However this contradicts our assumption that  $(A \Delta B) \cap C = \emptyset$ .

( $\leftarrow$ ) We will prove by contradiction. Suppose  $A \cap C = B \cap C$ . Now suppose that  $(A \Delta B) \cap C \neq \emptyset$ , which means there exists an  $x$  such that  $x \in (A \setminus B) \cap C$  or  $x \in (B \setminus A) \cap C$ . If  $x \in (A \setminus B) \cap C$ , then  $x \in A \setminus B$ , which means  $x \in A$  and  $x \notin B$ . Since  $x \in A$  and  $x \in C$ , then  $x \in A \cap C$ . It follows that  $x \in B \cap C$  because  $A \cap C = B \cap C$ , however this contradicts our assumption that  $x \notin B$ . In the case that  $x \in (B \setminus A) \cap C$ ,  $x \in B \setminus A$ . Thus  $x \in B$  and  $x \notin A$ . Since  $x \in B$  and  $x \in C$ , then  $x \in B \cap C$ . It follows that  $x \in A \cap C$  because  $A \cap C = B \cap C$ . However, this contradicts our assumption that  $x \notin A$ . □

### 3.5.20

**Theorem.** Suppose  $A$ ,  $B$ , and  $C$  are sets. Then  $A \Delta B \subseteq C$  if and only if  $A \cup C = B \cup C$ .

*Proof.* ( $\rightarrow$ ) Suppose  $A \Delta B \subseteq C$ . Let  $x$  be arbitrary and suppose  $x \in A \cup C$ . Thus  $x \in A$  or  $x \in C$ . If  $x \in C$  then  $x \in B \cup C$ . In the case that  $x \in A$  and  $x \notin C$ , then it follows that  $x \notin A \Delta B$  and thus  $x \in A \cap B$ . Since  $x \in A \cap B$ ,  $x \in B$  and thus  $x \in B \cup C$ . Now to prove the other direction suppose  $x \in B \cup C$ . Thus  $x \in B$  or  $x \in C$ . If  $x \in C$  then  $x \in A \cup C$ . In the case that  $x \in B$  and  $x \notin C$ , then it follows that  $x \notin A \Delta B$  and thus  $x \in A \cap B$ . Since  $x \in A \cap B$ ,  $x \in A$  and thus  $x \in A \cup C$ .

( $\leftarrow$ ) Now suppose  $A \cup C = B \cup C$  and  $x \in A \Delta B$ . By the definition of symmetrical difference,  $x \in A \setminus B$  or  $x \in B \setminus A$ . If  $x \in A \setminus B$  then  $x \in A$  and  $x \notin B$ . It follows that  $x \in A \cup C$  and therefore  $x \in B \cup C$ . Since  $x \in B \cup C$  and  $x \notin B$ , then  $x \in C$ . In the case that  $x \in B \setminus A$ ,  $x \in B$  and  $x \notin A$ . It follows that  $x \in B \cup C$  and therefore  $x \in A \cup C$ . Since  $x \in A \cup C$  and  $x \notin A$ , then  $x \in C$ . Thus if  $x \in A \Delta B$  then  $x \in C$ .  $\square$

### 3.5.21

**Theorem.** Suppose  $A$ ,  $B$ , and  $C$  are sets. Then  $C \subseteq A \Delta B$  if and only if  $C \subseteq A \cup B$  and  $A \cap B \cap C = \emptyset$ .

*Proof.* ( $\rightarrow$ ) Suppose  $C \subseteq A \Delta B$ . To show that  $C \subseteq A \cup B$ , let  $x$  be arbitrary and suppose  $x \in C$ . Since  $x \in C$  then  $x \in A \Delta B$ . By the definition of symmetric difference,  $x \in A \cup B$  and  $x \notin A \cap B$ . Thus if  $x \in C$  then  $x \in A \cup B$ . To show that  $A \cap B \cap C = \emptyset$  we will use proof by contradiction. Suppose there is an element  $y$  such that  $y \in A$ ,  $y \in B$ , and  $y \in C$ . Since  $y \in C$  then  $y \in A \Delta B$ . As noted earlier, if  $y \in A \Delta B$  then  $y \notin A \cap B$ ; however, this contradicts our assumption that  $y \in A$  and  $y \in B$ . Thus, it must be that  $A \cap B \cap C = \emptyset$ .

( $\leftarrow$ ) Now suppose  $C \subseteq A \cup B$  and  $A \cap B \cap C = \emptyset$ . Let  $x$  be arbitrary and suppose  $x \in C$ . It follows that  $x \in A \cup B$ . Since  $x \in C$  and  $A \cap B \cap C = \emptyset$ , then  $x \notin A \cap B$ . Now since  $x \in A \cup B$  and  $x \notin A \cap B$ , then  $x \in A \Delta B$ .  $\square$

### 3.5.22

Suppose  $A$ ,  $B$ , and  $C$  are sets.

#### A

**Theorem.**  $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$

*Proof.* Suppose  $x$  is arbitrary and  $x \in A \setminus C$ , which means  $x \in A$  and  $x \notin C$ . Now either  $x \in B$  or  $x \notin B$ . If  $x \in B$ , then it follows that  $x \in B \setminus C$  and thus  $x \in (A \setminus B) \cup (B \setminus C)$ . Therefore if  $x \in A \setminus C$  then  $x \in (A \setminus B) \cup (B \setminus C)$  and since  $x$  was arbitrary we can conclude  $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$ .  $\square$

#### B

**Theorem.**  $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$

*Proof.* Suppose  $x$  is arbitrary and  $x \in A \Delta C$ , which means  $x \in A \setminus C$  or  $x \in C \setminus A$ . Also, either  $x \in B$  or  $x \notin B$ . Thus, we have four cases to consider:

Case 1:  $x \in A \setminus C$  and  $x \in B$ . Since  $x \in A \setminus C$  then  $x \in A$  and  $x \notin C$ . Therefore  $x \in B$  and  $x \notin C$  and by the definition of symmetric difference,  $x \in B \Delta C$ . Therefore if  $x \in A \Delta C$  then  $x \in (A \Delta B) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude  $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$ .



Case 2:  $x \in A \setminus C$  and  $x \notin B$ . Since  $x \in A \setminus C$  then  $x \in A$  and  $x \notin C$ . Therefore,  $x \in A$  and  $x \notin B$  and therefore  $x \in A \Delta B$ . Therefore, if  $x \in A \Delta C$  then  $x \in (A \Delta B) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude  $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$ .

Case 3:  $x \in C \setminus A$  and  $x \in B$ . Since  $x \in C \setminus A$  then  $x \in C$  and  $x \notin A$ . Therefore  $x \in B$  and  $x \notin A$ , and therefore  $x \in A \Delta B$ . Therefore, if  $x \in A \Delta C$  then  $x \in (A \Delta B) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude that  $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$ .

Case 4:  $x \in C \setminus A$  and  $x \notin B$ . Since  $x \in C \setminus A$  then  $x \in C$  and  $x \notin A$ . Therefore  $x \in C \setminus B$  and thus  $x \in B \Delta C$ . Therefore, if  $x \in A \Delta C$  then  $x \in (A \Delta B) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude that  $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$ .  $\square$

### 3.5.23

Suppose  $A$ ,  $B$ , and  $C$  are sets.

#### A

**Theorem.**  $(A \cup B) \Delta C \subseteq (A \Delta C) \cup (B \Delta C)$

*Proof.* Suppose  $x$  is arbitrary and  $x \in (A \cup B) \Delta C$ . Thus  $x \in A \setminus C$  or  $x \in C \setminus (A \cup B)$ , and we have 4 cases to consider:

Case 1:  $x \in A$ ,  $x \notin B$ , and  $x \notin C$ . Since  $x \in A$  and  $x \notin C$  then  $x \in A \setminus C$  and  $x \in A \Delta C$ . Therefore if  $x \in (A \cup B) \Delta C$  then  $x \in (A \Delta C) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude  $(A \cup B) \Delta C \subseteq (A \Delta C) \cup (B \Delta C)$ .

Case 2:  $x \in B$ ,  $x \notin A$ , and  $x \notin C$ . Since  $x \in B$  and  $x \notin C$  then  $x \in B \setminus C$  and  $x \in B \Delta C$ . Therefore if  $x \in (A \cup B) \Delta C$  then  $x \in (A \Delta C) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude  $(A \cup B) \Delta C \subseteq (A \Delta C) \cup (B \Delta C)$ .

Case 3:  $x \in A$ ,  $x \in B$ , and  $x \notin C$ . Since  $x \in B$  and  $x \notin C$  then  $x \in B \setminus C$  and  $x \in B \Delta C$ . Since  $x \in A$  and  $x \notin C$  then  $x \in A \setminus C$  and  $x \in A \Delta C$ . Thus  $x \in (A \Delta C) \cup (B \Delta C)$ . Therefore if  $x \in (A \cup B) \Delta C$  then  $x \in (A \Delta C) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude  $(A \cup B) \Delta C \subseteq (A \Delta C) \cup (B \Delta C)$ .

Case 4:  $x \in C$ ,  $x \notin A$ ,  $x \notin B$ . Since  $x \in C$  and  $x \notin A$  then  $x \in C \setminus A$  and  $x \in A \Delta C$ . Since  $x \in C$  and  $x \notin B$  then  $x \in C \setminus B$  and  $x \in B \Delta C$ . Thus  $x \in (A \Delta C) \cup (B \Delta C)$ . Therefore if  $x \in (A \cup B) \Delta C$  then  $x \in (A \Delta C) \cup (B \Delta C)$  and since  $x$  was arbitrary we can conclude  $(A \cup B) \Delta C \subseteq (A \Delta C) \cup (B \Delta C)$ .  $\square$

#### B

Find an example of sets  $A$ ,  $B$ , and  $C$  such that  $(A \cup B) \Delta C \neq (A \Delta C) \cup (B \Delta C)$

Let  $A = \{1, 2\}$ ,  $B = \{3, 4\}$  and  $C = \{1, 5\}$ .

Then  $A \Delta C = (A \setminus C) \cup (C \setminus A) = \{2\} \cup \{5\} = \{2, 5\}$ .

Now  $B \Delta C = (B \setminus C) \cup (C \setminus B) = \{1, 3, 4, 5\} \cup \{1, 3, 4, 5\} = \{1, 3, 4, 5\}$ .

Therefore  $(A \Delta C) \cup (B \Delta C) = \{2, 5\} \cup \{1, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$ .

Now  $(A \cup B) = \{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\}$ .

Therefore  $(A \cup B) \Delta C = ((A \cup B) \setminus C) \cup (C \setminus (A \cup B)) = \{2, 3, 4\} \cup \{5\} = \{2, 3, 4, 5\}$ .

Since  $\{1, 2, 3, 4, 5\} \neq \{2, 3, 4, 5\}$  then  $(A \cup B) \Delta C \neq (A \Delta C) \cup (B \Delta C)$ .

### 3.5.24

Suppose  $A$ ,  $B$ , and  $C$  are sets.

#### A

**Theorem.**  $(A \Delta C) \cap (B \Delta C) \subseteq (A \cap B) \Delta C$ .

*Proof.* Let  $x$  be arbitrary and suppose  $x \in (A \Delta C) \cap (B \Delta C)$ . Then we have two cases to consider:

Case 1:  $x \in A$ ,  $x \in B$ , and  $x \notin C$ . Since  $x \in A$  and  $x \in B$ , then  $x \in A \cap B$  and  $x \notin C$ . Therefore  $x \in (A \cap B) \setminus C$  and if  $x \in (A \Delta C) \cap (B \Delta C)$  then  $x \in (A \cap B) \Delta C$ . Since  $x$  was arbitrary we can conclude  $(A \Delta C) \cap (B \Delta C) \subseteq (A \cap B) \Delta C$ .

Case 2:  $x \notin A$ ,  $x \notin B$ ,  $x \in C$ . Since  $x \notin A$  and  $x \notin B$ , then  $x \notin A \cap B$  and since  $x \in C$  then  $x \in C \setminus (A \cap B)$ . Therefore if  $x \in (A \Delta C) \cap (B \Delta C)$  then  $x \in (A \cap B) \Delta C$  and since  $x$  was arbitrary we can conclude  $(A \Delta C) \cap (B \Delta C) \subseteq (A \cap B) \Delta C$ .  $\square$

#### B

**Theorem.**  $(A \cap B) \Delta C \subseteq (A \Delta C) \cap (B \Delta C)$

*Proof.* Let  $x$  be arbitrary and suppose  $x \in (A \cap B) \Delta C$ . Thus  $x \in A \cap B \setminus C$  or  $x \in C \setminus A \cap B$ .

Case 1:  $x \in (A \cap B) \setminus C$ . Thus  $x \in A \cap B$  and  $x \notin C$ . Since  $x \in A \cap B$  then  $x \in A$  and  $x \in B$ . Since  $x \in A$  and  $x \notin C$  then  $x \in A \setminus C$  and since  $x \in B$  and  $x \notin C$  then  $x \in B \setminus C$ . Therefore  $x \in A \Delta C$  and  $x \in B \Delta C$  and if  $x \in (A \cap B) \setminus C$  then  $x \in (A \Delta C) \cap (B \Delta C)$ . Since  $x$  was arbitrary we can conclude  $(A \cap B) \Delta C \subseteq (A \Delta C) \cap (B \Delta C)$ .

Case 2:  $x \in C \setminus (A \cap B)$ . Thus  $x \in C$  and  $x \notin A \cap B$ . Since  $x \notin A \cap B$  then  $x \notin A$  and  $x \notin B$ . Since  $x \in A$  and  $x \notin C$  then  $x \in A \setminus C$ . Also since  $x \in B$  and  $x \notin C$  then  $x \in B \setminus C$ . Therefore  $x \in A \Delta C$  and  $x \in B \Delta C$  and if  $x \in C \setminus (A \cap B)$  then  $x \in (A \Delta C) \cap (B \Delta C)$ . Since  $x$  was arbitrary then we can conclude  $(A \cap B) \Delta C \subseteq (A \Delta C) \cap (B \Delta C)$ .  $\square$