Exercise 3.3.4

Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

So we want to prove that $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$.

First we assume x is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathscr{P}(A)$	$x \in \mathscr{P}(\mathscr{P}(A))$
$x \in \mathscr{P}(A)$	

Assume x is an arbitrary element of $\mathscr{P}(A)$
  Suppose $x \in \mathscr{P}(A)$
    [proof of $x \in \mathscr{P}(\mathscr{P}(A))$]
  Therefore if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$
Since x was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$

We can rewrite our goal as $x \subseteq \mathscr{P}(A)$ or $\forall y(y \in x \to y \in \mathscr{P}(A))$. So we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$A \subseteq \mathscr{P}(A)$	$y \in \mathscr{P}(A)$
$x \in \mathscr{P}(A)$	
$y \in x$	

Assume x is an arbitrary element of $\mathscr{P}(A)$
  Suppose $x \in \mathscr{P}(A)$
    Suppose y is an arbitrary element of x.
      Suppose $y \in x$.
        [proof of $y \in \mathscr{P}(A)$]
      Therefore if $y \in x \to y \in \mathscr{P}(A)$.
    Since y was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$.
  Therefore if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$
Since x was arbitrary we can conclude $\forall x(x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A)))$

Now looking at our givens $x \in \mathscr{P}(A)$ means that $x \subseteq A$ or $\forall z(z \in x \to z \in A)$. Using universal instantiation we will plug in y for z and using modus ponens we can conclude that $y \in A$.

Now looking at our other given $A \subseteq \mathscr{P}(A) \to \forall m(m \in A \to m \in \mathscr{P}(A))$. Using universal instantiation we will plug in y for m and using modus ponens we can conclude that $y \in \mathscr{P}(A)$, which was our goal to prove.

Theorem. *Suppose* $A \subseteq \mathscr{P}(A)$*. Then* $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$*.*

Proof. Suppose x is an arbitrary element of $\mathscr{P}(A)$ and y is an arbitrary element of x. It follows that $y \in A$. But since $A \subseteq \mathscr{P}(A)$ then it also follows that $y \in \mathscr{P}(A)$. So $y \in x \to y \in \mathscr{P}(A)$ and since y was arbitrary we can conclude that $x \subseteq \mathscr{P}(A)$. Therefore, if $x \in \mathscr{P}(A) \to x \in \mathscr{P}(\mathscr{P}(A))$. Since x was arbitrary we can also conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. □

Alternate proof (not sure if this is correct)

Proof. Suppose x is an arbitrary element of $\mathscr{P}(A)$. Then $x \in A$. Since $A \subseteq \mathscr{P}(A)$ and $x \in A$ then $x \subseteq \mathscr{P}(A)$. Therefore, $x \in \mathscr{P}(\mathscr{P}(A))$. □

Exercise 3.3.5

The hypothesis of the theorem proven in exercise 3.3.4 is $A \subseteq \mathscr{P}(A)$.

A

Can you think of a set A for which this hypothesis is true?

The empty set $\varnothing$ is a set for which the hypothesis is true.

$A \subseteq \mathscr{P}(A)$ means $x \in A \to x \in \mathscr{P}(A)$. For $\varnothing$ this would mean that $x \in \varnothing \to x \in \mathscr{P}(\varnothing)$, but by definition there are no elements in $\varnothing$. Therefore $x \in \varnothing$ will always be false and the conditional statement $x \in \varnothing \to x \in \mathscr{P}(\varnothing)$ is always true. Therefore if $\varnothing = A$ then $A \subseteq \mathscr{P}(A)$.

B

Can you think of another?

In exercise 3.3.4 we proved that if $A \subseteq \mathscr{P}(A)$ then $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. Therefore, the set $\{\varnothing, \{\varnothing\}\}$, which is the $\mathscr{P}(A)$ if $A = \varnothing$, is another set for which the hypothesis is true. If we let $B = \mathscr{P}(A) = \{\varnothing, \{\varnothing\}\}$ and replace A in the hypothesis $A \subseteq \mathscr{P}(A)$ with B, then we can conclude that $B \subseteq \mathscr{P}(B)$.

3.3.6

Suppose x is a real number.

A

Prove that if $x \neq 1$ then there is a real number y such that $\frac{y+1}{y-2} = x$.

So we want to prove that $(x \neq 1) \to \exists y \left(\frac{y+1}{y+2} = x \right)$

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$x \neq 1$	$\exists y \left(\frac{y+1}{y+2} = x \right)$

To prove our goal we need to find a y that makes the equation $\frac{y+1}{y+2} = x$ true. So let's try solving the equation for y.

$$\begin{aligned}
\frac{y+1}{y+2} &= x \\
y+1 &= x(y-2) \\
y+1 &= xy - 2x \\
2x+1 &= xy - y \\
2x+1 &= y(x-1) \\
y &= \frac{2x+1}{x-1}
\end{aligned}$$

We see that this y works because we have $x \neq 1$ as a given.

Theorem. *Suppose $x \neq 1$. Then there is a real number y such that $\frac{y+1}{y-2} = x$.*

Proof. Suppose $x \neq 1$ and $y = \frac{2x+1}{x-1}$. Then

$$\frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{3x}{x-1} \cdot \frac{x-1}{3} = x$$

□

B

Prove that if there is a real number y such that $\frac{y+1}{y-2} = x$ then $x \neq 1$.

So we want to prove that $\exists y \left(\frac{y+1}{y-1} = x \right) \to (x \neq 1)$

We assume the antecedent and make the consequent our goal to prove.

Using existential instantiation we assume there is a value y_0 such that $\frac{y+1}{y-1} = x$ is true. From part A above, we know that $\left(\frac{y+1}{y-1} = x \right) \to \left(y = \frac{2x+1}{x-1} \right)$ and so $y_0 = \frac{2x+1}{x-1}$. Since y is a real number, then clearly $x \neq 1$.

Givens	Goals
$\exists y\left(\frac{y+1}{y-1}=x\right)$	$x \neq 1$

Theorem. *If y is a real number and $\frac{y+1}{y-2}=x$ then $x \neq 1$.*

Proof. Suppose y is a real number and $\frac{y+1}{y-2}=x$. It follows that $y=\frac{2x+1}{x-1}$ and since y is real number then $x \neq 1$. □

Exercise 3.3.7

Prove for every real number x, if $x > 2$ then there is a real number y such that $y+\frac{1}{y}=x$.

So we want to prove $\forall x \in \mathbb{R}(x > 2 \rightarrow \exists y \in \mathbb{R}(y+\frac{1}{y}=x))$

So we let x be an arbitrary real number, then we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
x is arbitrary real number	$\exists y(y+\frac{1}{y}=x)$
$x > 2$	

Our goal is of the form $\exists y P(y)$ where $P(y)$ is $y+\frac{1}{y}=x$ and our strategy suggests we try to find a y for which $P(y)$ is true. We can do this by solving the equation $y+\frac{1}{y}=x$ for y. We can rewrite this equation as $y^2-\frac{x}{y}+1=0$ and we see this is a quadratic equation and therefore we can use the quadratic formula to solve for y,

$$y=\frac{-(-x)\pm\sqrt{(-x)^2-4\cdot 1\cdot 1}}{2\cdot 1}=\frac{x\pm\sqrt{x^2-4}}{2}.$$

We note that $\sqrt{x^2-4}$ is defined because $x > 2$. We have found two solutions that satisfy our original equation, but we only need one to complete the proof. We will use $\frac{x+\sqrt{x^2-4}}{2}$.

Theorem. *For every real number x, if $x > 2$ then there is a real number y such that $y+\frac{1}{y}=x$.*

Proof. Suppose x and y are real numbers, $x > 2$, and $y=\frac{x+\sqrt{x^2-4}}{2}$. Then

$$
\begin{aligned}
\frac{x+\sqrt{x^2-4}}{2}+\frac{1}{\frac{x+\sqrt{(x^2-4}}{2}} &= \frac{x+\sqrt{x^2-4}}{2}+\frac{2}{x+\sqrt{x^2-4}} \\
&= \frac{2x^2+2(x\sqrt{x^2-4})}{2x+2\sqrt{x^2-4}} \\
&= x
\end{aligned}
$$

□

Exercise 3.3.8

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.

So we want to prove that $A \in \mathcal{F} \to A \subseteq \cup\mathcal{F}$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$A \subseteq \cup\mathcal{F}$

Assume $A \in \mathcal{F}$
    [proof of $A \subseteq \cup\mathcal{F}$]
Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$

Our goal $A \subseteq \cup\mathcal{F}$ can be rewritten as $\forall x(x \in A \to x \in \cup\mathcal{F})$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in \cup\mathcal{F}$
$x \in A$	

Assume $A \in \mathcal{F}$
    Assume x is arbitrary
        Assume $x \in A$
            [proof of $x \in \cup\mathcal{F}$]
        Therefore if $x \in A$ then $x \cup \mathcal{F}$
    Since x was arbitrary we can conclude that $A \subseteq \cup\mathcal{F}$.
Therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$

Our new goal can be rewritten as $\exists B \in \mathcal{F}(x \in B)$. From our givens we see that $A \in \mathcal{F}$ and $x \in A$, so we have found a set such that $A \in \mathcal{F}(x \in A)$.

Theorem. *If $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \cup\mathcal{F}$.*

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of A. Then since $x \in A$ and $A \in \mathcal{F}$, it follows that $x \in \cup\mathcal{F}$. Since x was arbitrary we can conclude that $A \subseteq \cup\mathcal{F}$ and therefore if $A \in \mathcal{F}$ then $A \subseteq \cup\mathcal{F}$. $\square$

3.3.9

Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cap\mathcal{F} \subseteq A$.

We want to prove that $A \in \mathcal{F} \to \cap\mathcal{F} \subseteq A$.

We assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$\cap\mathcal{F} \subseteq A$

Assume $A \in \mathcal{F}$
    [proof of $\cap\mathcal{F} \subseteq A$]
Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$.

We can rewrite our goal as $\forall x(x \in \cap\mathcal{F} \to x \in A)$. We assume x is arbitrary and then assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$A \in \mathcal{F}$	$x \in A$
$x \in \cap\mathcal{F}$	

Assume $A \in \mathcal{F}$
    Assume x is arbitrary
        Assume $x \in \cap\mathcal{F}$
            [proof of $x \in A$]
        Therefore, if $x \in \cap\mathcal{F}$ then $x \in A$.
    Since x was arbitrary we can conclude that $\cap\mathcal{F} \subseteq A$.
Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$.

Our given $x \in \cap\mathcal{F}$ can be rewritten as $\forall B \in \mathcal{F}(x \in B)$, therefore if $A \in \mathcal{F}$ then $x \in A$, which was our goal to prove.

Theorem. *If $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\cup\mathcal{F} \in A$.*

Proof. Assume $A \in \mathcal{F}$ and x is an arbitrary member of $\cap\mathcal{F}$. Since $A \in \mathcal{F}$ and $x \in \cap\mathcal{F}$ it follows that $x \in A$ and therefore, if $x \in \cap\mathcal{F}$ then $x \in A$. Since x was arbitrary we can conclude that $\cap\mathcal{F} \subseteq A$. Therefore, if $A \in \mathcal{F}$ then $\cap\mathcal{F} \subseteq A$. $\square$

Exercise 3.3.10

Suppose that $\mathcal{F}$ is a family of sets. Prove that if $\varnothing \in \mathcal{F}$ then $\cap\mathcal{F} = \varnothing$.

Exercise 3.3.12

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cup\mathcal{F} \subseteq \cup\mathcal{G}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    [proof of $\cup\mathcal{F} \subseteq \cup\mathcal{G}$]
So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\cup\mathcal{F} \subseteq \cup\mathcal{G} \to \forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$ so we assume b is an arbitrary element of $\cup\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$b \in \cup\mathcal{F}$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    Let b be an arbitrary element of $\cup\mathcal{F}$
        [proof of $b \in \cup\mathcal{G}$]
    Therefore if $b \in \cup\mathcal{F} \to b \in \cup\mathcal{G}$
Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$b \in \cup\mathcal{F} \to \exists M(M \in \mathcal{F} \wedge b \in M)$, so let $M = A_0$ (Existential Instantiation)

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Suppose $\mathcal{F} \subseteq \mathcal{G}$
    Let b be an arbitrary element and suppose $b \in \cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ and b is in that set. Let that set $= A_0$
        [proof of $b \in \cup\mathcal{G}$]
    Therefore if $b \in \cup\mathcal{F} \to b \in \cup\mathcal{G}$

Since b was arbitrary we can conclude $\forall b(b \in \cup\mathcal{F} \to b \in \cup\mathcal{G})$. So if $\mathcal{F} \subseteq \mathcal{G} \to \cup\mathcal{F} \subseteq \cup\mathcal{G}$

$\mathcal{F} \subseteq \mathcal{G} \to \forall A(A \in \mathcal{F} \to A \in \mathcal{G})$. Using universal instantiation we will plug in A_0 for A since then we can use modens ponens to conclude that $A_0 \in \mathcal{G}$.

Givens	Goals
$A_0 \in \mathcal{F} \to A_0 \in \mathcal{G}$	$b \in \cup\mathcal{G}$
$A_0 \in \mathcal{F} \wedge b \in A_0$	

Our goal $b \in \cup\mathcal{G} \to \exists N(N \in \mathcal{G} \wedge b \in N)$, which we can now prove. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$, it follows that $A_0 \in G$. By the definition of $\cup\mathcal{G}$ it follows that $b \in \cup\mathcal{G}$ because $A_0 \in G \wedge b \in A_0$, the latter statement being one of our givens.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.*
Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cup\mathcal{F}$, which implies there is a set in $\mathcal{F}$ that contains b. Call this set A_0. Since $A_0 \in \mathcal{F}$ and $\mathcal{F}$ is a subset of $\mathcal{G}$ it follows that $A_0 \in G$, which implies that $b \in \cup\mathcal{G}$. Therefore if $b \in \cup\mathcal{F}$ then $b \in \cup\mathcal{G}$. Since b was arbitrary we can conclude that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$. This completes the proof.

Exercise 3.3.13

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

So we want to prove that $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

First we assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$\cap\mathcal{G} \subseteq \cap\mathcal{F}$

Suppose $\mathcal{F} \subseteq \mathcal{G}$
[proof of $\cap\mathcal{G} \subseteq \cap\mathcal{F}$]
So if $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$\cap\mathcal{G} \subseteq \cap\mathcal{F} \to \forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$, so we assume b is an arbitrary element of $\cap\mathcal{G}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in \cap\mathcal{F}$
$b \in \cap\mathcal{G}$	

Suspose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cap\mathcal{G}$

[proof of $b \in \cap\mathcal{F}$]

Therefore if $b \in \cap\mathcal{G} \to b \in \cap\mathcal{F}$

Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

$b \in \cap\mathcal{F} \to \forall A(A \in \mathcal{F} \to b \in A)$, so we assume A is an arbitrary element of $\mathcal{F}$ and assume the antecedent and make the consequent our goal to prove.

Givens	Goals
$\mathcal{F} \subseteq \mathcal{G}$	$b \in A$
$b \in \cap\mathcal{G}$	
$A \in \mathcal{F}$	

Suspose $\mathcal{F} \subseteq \mathcal{G}$

Let b be an arbitrary element of $\cap\mathcal{G}$

Suppose A is an arbitrary set in $\mathcal{F}$

[proof of $b \in A$]

Therefore if $A \in \mathcal{F} \to b \in A$

Since A was arbitrary we can conclude $b \in \cap\mathcal{F}$

Therefore if $b \in \cap\mathcal{G} \to b \in \cap\mathcal{F}$

Since b was arbitrary we can conclude $\forall b(b \in \cap\mathcal{G} \to b \in \cap\mathcal{F})$. So $\mathcal{F} \subseteq \mathcal{G} \to \cap\mathcal{G} \subseteq \cap\mathcal{F}$

Now looking at our givens, $\mathcal{F} \subseteq \mathcal{G} \to \forall Z(Z \in \mathcal{F} \to Z \in \mathcal{G})$. Using universal instantiation we will plug in A for Z and using modus ponens we can conclude that $A \in \mathcal{G}$.

Our other given, $b \in \cap\mathcal{G} \to \forall Y(Y \in \mathcal{G} \to b \in Y)$. Using universal instantiation we will plug in A for Y and using modus ponens we can conclude that $b \in A$, which was our goal, and we can now write our proof.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.*
Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let b be an arbitrary element of $\cap\mathcal{G}$. Suppose A is an arbitrary element of $\mathcal{F}$, then because $\mathcal{F} \subseteq \mathcal{G}$ then it follows that $A \in \mathcal{G}$. By the definition of $\cap\mathcal{G}$ it follows that $b \in A$ and since A was arbitrary then $b \in \cap\mathcal{F}$. Since b was arbitrary we can conclude $\cap\mathcal{G} \subseteq \cap\mathcal{F}$ and therefore that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cap\mathcal{G} \subseteq \cap\mathcal{F}$. This completes the proof.

Exercise 3.3.14

Suppose $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.

So we want to prove that $\forall a(a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i\in I} A_i))$

First we assume a is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i\in I} \mathscr{P}(A_i)$	$a \in \mathscr{P}(\bigcup_{i\in I} A_i)$

Assume a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
        [proof of $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$]
    Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our goal we see that $a \in \mathscr{P}(\bigcup_{i\in I} A_i) \rightarrow a \subseteq \bigcup_{i\in I} A_i \rightarrow \forall z(z \in a \rightarrow z \in \bigcup_{i\in I} A_i)$. Therefore we assume z is arbitrary, assume the antecedent, and make the consequent our goal to prove.

Givens	Goals
$a \in \bigcup_{i\in I} \mathscr{P}(A_i)$	$z \in \bigcup_{i\in I} A_i$
$z \in a$	

Assume a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$
    Suppose $a \in \bigcup_{i\in I} \mathscr{P}(A_i)$
        Assume z is arbitrary
            Assume $z \in a$
                [proof of $z \in \bigcup_{i\in I} A_i$]
            Therefore $z \in a \rightarrow z \in \bigcup_{i\in I} A_i$
        Since z was arbitrary we can conclude $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
    Therefore if $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \mathscr{P}(\bigcup_{i\in I} A_i)$
Since a was arbitrary we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$

Looking at our given we see that $a \in \bigcup_{i\in I} \mathscr{P}(A_i) \rightarrow a \in \{a|\exists i \in I(a \in \mathscr{P}(A_i))\}$. Using existential instantiation we will select an i such that $a \in \mathscr{P}(A_i)$ which implies $a \subseteq A_i$. Since $a \subseteq A_i \rightarrow \forall m(m \in a \rightarrow m \in A_i)$ and using universal instantiation we will plug in z for m and we get $\forall z(z \in a \rightarrow z \in A_i)$ and using modus ponens we can conclude that $z \in A_i$, which implies that $z \in \bigcup_{i\in I} A_i$, which was our goal. We can now right our proof.

Theorem. *Suppose $\{A_i|i \in I\}$ is an indexed family of sets, then $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$.*
Proof. Suppose that a is an arbitrary element of $\bigcup_{i\in I} \mathscr{P}(A_i)$. We choose an $i \in I$ such that $a \in \mathscr{P}(A_i)$, which implies that $a \subseteq A_i$. Suppose z is an arbitrary

element of a, then it follows that $z \in A_i$ and therefore $z \in \bigcup_{i\in I} A_i$. Since z was an arbitrary element of a then $a \subseteq \bigcup_{i\in I} A_i$, and it follows that $a \in \mathscr{P}(\bigcup_{i\in I} A_i)$. Thus we can conclude $\bigcup_{i\in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\bigcup_{i\in I} A_i)$. This completes the proof.

3.3.15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \varnothing$. Prove that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$

So we want to prove that $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

First we assume y is arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i\in I} A_i$	$y \in \bigcap_{i\in I} \mathscr{P}(A_i)$

Suppose y is arbitrary element of $\bigcap_{i\in I} A_i$.
    [proof of $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$]
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Our goal $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$ so we make m an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$. So we make z arbitrary and make the antecedent a given and the consequent our goal to prove.

Givens	Goals
$y \in \bigcap_{i\in I} A_i$	$z \in A_m$
$z \in y$	

Suppose y is arbitrary element of $\bigcap_{i\in I} A_i$.
    Suppose m is an arbitrary element of I and therefore $y \in \mathscr{P}(A_m) \to y \subseteq A_m \to \forall z(z \in y \to z \in A_m)$.
        Suppose z is an arbitrary element of y
            [proof of $z \in A_m$]
    Therefore $z \in y \to z \in A_m$ and since z was arbitrary $y \subseteq A_m \to y \in \mathscr{P}(A_m)$ and since m was arbitrary $y \in \bigcap_{i\in I} \mathscr{P}(A_i)$
Since y was arbitrary we can conclude $\forall y(y \in \bigcap_{i\in I} A_i \to y \in \bigcap_{i\in I} \mathscr{P}(A_i))$.

Now looking at our given $y \in \bigcap_{i\in I} A_i \to \forall i \in I(y \in A_i)$. Using universal instantiation we plug in m for i and therefore $y \in A_m$ and since $z \in y$ we can conclude $z \in A_m$, which was our goal. Now we can write our proof.

Theorem. *Suppose* $\{A_i | i \in I\}$ *is an indexed family of sets and* $I \neq \varnothing$*, then* $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$.

Proof. Suppose y is an arbitrary element of $\bigcap_{i \in I} A_i$. Suppose m is an arbitrary member of I and therefore $y \subseteq A_m$ which implies $y \subseteq A_m$. Now suppose z is an arbitrary element of y. Since $y \in \bigcap_{i \in I} A_i$ if we choose an i such that $y \in \bigcap_{m \in I} A_m$ then $y \in A_m$ which implies $z \in A_m$. Therefore if $z \in y$ then $z \in A_m$ and since z was arbitrary then $y \subseteq A_m$ or $y \in \mathscr{P}(A_m)$ and since m was arbitrary then $y \in \bigcap_{i \in I} \mathscr{P}(A_i)$. Since y was arbitrary then $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathscr{P}(A_i)$. This completes the proof.