3.4.1

Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

We want to prove $\forall x(P(x) \wedge Q(x) \iff \forall x P(x) \wedge \forall x Q(x))$.

Theorem. *The statement $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.*

Proof. ($\rightarrow$) Suppose $\forall x(P(x) \wedge Q(x))$. Let y be arbitrary. Since $\forall x(P(x) \wedge Q(x))$ it follows $P(y)$ and $Q(y)$. Since y was arbitrary, we can conclude $\forall x P(x)$ and $\forall x Q(x)$ or $\forall x P(x) \wedge \forall x Q(x)$.

($\leftarrow$) Let y be arbitrary. Since $\forall x P(x)$ and $\forall x Q(x)$ then it follows $P(y)$ and $Q(y)$. Since y was arbitrary we can conclude $\forall x(P(x) \wedge Q(x))$. $\square$

3.4.2

Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Theorem. *If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B$ then $x \in B$ and since $A \subseteq C$ then $x \in C$ or $x \in B \cap C$. Therefore, if $x \in A$ then $x \in B \cap C$ and since x was arbitrary we can conclude $A \subseteq B \cap C$. $\square$

3.4.3

Suppose $A \subseteq B$. Prove that for every set C, $C \setminus B \subseteq C \setminus A$.

Theorem. *Suppose $A \subseteq B$, then for every set C, $C \setminus B \subseteq C \setminus A$.*

Proof. Suppose $A \subseteq B$ and C is an arbitrary set. Let x be arbitrary and suppose $x \in C \setminus B$, which means $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$, then $x \notin A$, which means that $x \in C \setminus A$. Therefore, if $x \in C \setminus B$ then $x \in C \setminus A$ and since x and C were arbitrary, we can conclude $\forall C(C \setminus B \subseteq C \setminus A)$. $\square$

3.4.5

Prove that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \not\subseteq C$.

Theorem. *If $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \not\subseteq C$.*

Proof. Let x be arbitrary and suppose $x \in A$. Since $A \subseteq B \setminus C$ then $x \in B$ and $x \notin C$. Since x was arbitrary we can conclude $B \not\subseteq C$. $\square$

3.4.6

Prove that for any sets A, B, and C, $A\backslash(B\cap C) = (A\backslash B)\cup(A\backslash C)$ finding a string of equivalences starting with $x \in A\backslash(B\cap C)$ and ending with $x \in (A\backslash B)\cup(A\backslash C)$.

Theorem. *for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.*

Proof. Suppose A, B, and C are arbitrary sets. Then

$$\begin{aligned} x \in A \setminus (B \cap C) &\text{ iff } x \in A \to (x \notin B \wedge x \notin C) \\ &\text{ iff } x \notin A \vee (x \notin B \wedge x \notin C) \\ &\text{ iff } (x \notin A \vee x \notin B) \wedge (x \notin A \vee x \notin C) \\ &\text{ iff } (x \in A \to x \notin B) \vee (x \in A \to x \notin C) \\ &\text{ iff } x \in A \setminus B \vee x \in A \setminus C \\ &\text{ iff } x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

□

3.4.7

Theorem. *For any sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.*

Proof. ($\to$) Let M be an arbitrary set and suppose $M \in \mathscr{P}(A \cap B)$. Then $M \subseteq A \cap B$. Let x be arbitrary and suppose $x \in M$. Since $M \subseteq A \cap B$, $x \in A \cap B$ and therefore $x \in A$. Since x was arbitrary, $M \subseteq A$ and therefore $M \in \mathscr{P}(A)$. Similarly, since $M \subseteq A \cap B$, $x \in B$. Since x was arbitrary, $M \subseteq B$ and therefore $M \in \mathscr{P}(B)$. Therefore, $M \in \mathscr{P}(A)$ and $M \in \mathscr{P}(B)$.

($\leftarrow$) Now suppose $M \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then $M \subseteq A$ and $M \subseteq B$. Suppose $x \in M$. Since $M \subseteq A$ and $M \subseteq B$ then $x \in A \cap B$. Since x was arbitrary, $M \subseteq A \cap B$ and therefore $M \in \mathscr{P}(A \cap B)$. □

3.4.8

Theorem. $A \subseteq B \iff \mathscr{P}(A) \subseteq \mathscr{P}(B)$

Proof. ($\to$) Suppose $A \subseteq B$. Let M be an arbitrary set and suppose $M \in \mathscr{P}(A)$. Then $M \subseteq A$. Now let y be arbitrary and suppose $y \in M$. Since $M \subseteq A$ then $y \in A$, and since $A \subseteq B$ then $y \in B$. Since y was arbitrary, $M \subseteq B$ and therefore $M \in \mathscr{P}(B)$. Since M was arbitrary, $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

($\leftarrow$) Now suppose $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ and $y \in A$. Then the set $\{y\}$ is in $\mathscr{P}(A)$. Since $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ then $\{y\} \in \mathscr{P}(B)$ and $y \in B$. Since y was arbitrary, $A \subseteq B$. □

3.4.9

Theorem. *If x and y are odd integers, then xy is odd.*

Proof. Suppose x and y are odd integers. This means there is an integer k such that $x = 2k + 1$ and there is an integer j such that $y = 2j + 1$. Therefore, $xy = 2(2kj + k + j) = 4kj + 2k + 2j + 1 = (2k + 1)(2j + 1)$, and since $2kj + k + j$ is an integer, then xy is odd. $\square$

3.4.10

Theorem. *For every integer n, n^3 is even iff n is even.*

Proof. ($\rightarrow$) Let n be arbitrary. We will prove the contrapositive. Suppose x is odd, which means there exists an integer k such that $x = 2k + 1$. Therefore, $n^3 = (2k+1)^3 = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$. Since $4k^3+6k^2+3k$ is an integer, n^3 is odd. Therefore, if n^3 is even, n is even.

($\leftarrow$) Now suppose n is even, which means there exists an integer m such that $n = 2m$. Now $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$ and since $4m^3$ is an integer, n^3 is even. $\square$

3.4.11

A

The problem is with using the same variable k for defining m as an even integer and n as an odd integer when k may take on different values for n and m.

B

Let $m = 2$ and $n = -3$. Then $n^2 - m^2 = (-3)^2 - 2^2 = 9 - 4 = 5$ and $n + m = -3 + 2 = -1$. Therefore $n^2 - m^2 \neq n + m$.

3.4.12

Theorem. $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x + y = xy) \iff x \neq 1]$

Proof. ($\rightarrow$) We will prove by contradiction. Suppose x is an arbitrary real number and there exists a real number y such that $x + y = xy$. Now suppose $x = 1$. Since $x + y = xy$, then $y = \frac{x}{x-1}$. But this contradicts $x = 1$ because there is no real number y such that $y = x/0$.

($\leftarrow$) Now suppose $x \neq 1$ and $y = \frac{x}{x-1}$. Then

$$
\begin{aligned}
x + y = x + \frac{x}{x+1} &= \frac{x(x-1)+x}{x-1} \\
&= \frac{x^2 - x + x}{x-1} \\
&= \frac{x^2}{x-1} = xy
\end{aligned}
$$

□

3.4.13

Theorem. $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = \frac{y}{x}) \iff x \neq z]$

Proof. ($\rightarrow$) Let $z = 1$. Let x be an arbitrary real number and suppose $x > 0$. Suppose $y \in \mathbb{R}$ and $y - x = \frac{y}{x}$. Then $y = \frac{x^2}{x-1}$. Now suppose $x = 1$. This contradicts $y \in \mathbb{R}$ and $y = \frac{x^2}{x-1}$. Therefore, $x \neq z$ and since x was arbitrary we can conclude $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = \frac{y}{x}) \rightarrow x \neq z]$.

($\leftarrow$) Now suppose $x \neq 1$ and $y = \frac{x^2}{x-1}$. Then

$$
\begin{aligned}
y - x = \frac{x^2}{x-1} - x &= \frac{x^2 - x(x-1)}{x-1} \\
&= \frac{x^2 - x + 2 + x}{x-1} = \frac{x}{x-1} = \frac{y}{x}
\end{aligned}
$$

□

3.4.14

Theorem. *If B is a set and $\mathcal{F}$ is a family of sets, then $\cup\{A \setminus B | A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathscr{P}(B))$.*

Proof. Let x be arbitrary and suppose $x \in \cup\{A \setminus B | A \in \mathcal{F}\}$. This means that there is a set $A \in \mathcal{F}$ such that $x \in A$ and also $x \notin B$. Since $x \in A$ and $x \notin B$, then $A \not\subseteq B$ and $A \notin \mathscr{P}(B)$. Thus there is a set $A \in \mathcal{F}$ such that $x \in A$, and $A \notin \mathscr{P}(B)$, which means that $x \in \cup(\mathcal{F} \setminus \mathscr{P}(B))$. Therefore, if $x \in \cup\{A \setminus B | A \in \mathcal{F}\}$ then $x \in \cup(\mathcal{F} \setminus \mathscr{P}(B))$ and since x was arbitrary, we can conclude $\cup\{A \setminus B | A \in \mathcal{F}\} \subseteq \cup(\mathcal{F} \setminus \mathscr{P}(B))$. □

3.4.15

Theorem. *If $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$, then $\cup\mathcal{F}$ and $\cap\mathcal{G}$ are disjoint.*

Proof. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$. We will use proof by contradiction. Now suppose $\cup\mathcal{F}$ and $\cap\mathcal{G}$ are not disjoint. Then there exists a y such that $y \in \cup\mathcal{F}$ and $y \in \cap\mathcal{G}$. Since $y \in \cup\mathcal{F}$ there is a set in $\mathcal{F}$ that contains y and since $y \in \cap\mathcal{G}$, y is in every set in $\mathcal{G}$. But because every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$, then there is at least one set in $\mathcal{G}$ that does not contain y. But this contradicts $y \in \cap\mathcal{G}$. Therefore, $(\cup\mathcal{F}) \cap (\cap\mathcal{G}) = \varnothing$. □

3.4.16

Theorem. *For any set A, $A = \cup\mathscr{P}(A)$.*

Proof. $(\to)$ Suppose A is an arbitrary set, x is arbitrary, and $x \in A$. Then there is subset of A that contains x and, by definition, this subset is in $\mathscr{P}(A)$. Therefore, $x \in \cup\mathscr{P}(A)$. Since x was arbitrary $A \subseteq \mathscr{P}(A)$.

$(\leftarrow)$ Now suppose $x \in \cup\mathscr{P}(A)$. This means there is a subset of A that contains x and therefore $x \in A$. Since x was arbitrary we conclude $\cup\mathscr{P}(A) \subseteq A$. Since A was arbitrary, we can conclude for all sets A, $A = \cup\mathscr{P}(A)$. □

3.4.17

A

Theorem. $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$

Proof. Let x be arbitrary and suppose $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Since $x \in \cup(\mathcal{F} \cap \mathcal{G})$ there is a set in $\mathcal{F}$ and in $\mathcal{G}$ that both contain x. Since there is a set in $\mathcal{F}$ than contains x, then $x \in \cup\mathcal{F}$ and since there is a set in $\mathcal{G}$ that contains x, $x \in \cup\mathcal{G}$. Therefore, $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. Since x was arbitrary, we can conclude $\cup(\mathcal{F} \cap \mathcal{G}) \subseteq (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. □

B

The mistake is that we can't choose a set A such that $A \in \mathcal{F}$ and $A \in \mathcal{G}$ and $x \in A$. The given $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ means that x is within a set in $\mathcal{F}$ and within a set in $\mathcal{G}$, but these two sets are not necessarily the same set.

C

Let $\mathcal{F} = \{\{1, 2\}, \{3\}\}$ and $\mathcal{G} = \{\{4, 5\}, \{1\}\}$. Then $\cup(\mathcal{F} \cap \mathcal{G}) = \varnothing$, but $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) = \{1\}$.

3.4.18

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets, then* $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G}) \iff \forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$.

Proof. ($\to$) Suppose $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$. Suppose A is an arbitrary set in $\mathcal{F}$, B is an arbitrary set in $\mathcal{G}$, x is arbitrary, and $x \in A \cap B$. Since $x \in A \cap B$ and A is an arbitrary set in $\mathcal{F}$, then $x \in \cup F$. Also, since $x \in A \cap B$ and B is an arbitrary set in $\mathcal{G}$, then $x \in \cup\mathcal{G}$. Therefore $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$ and since $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$, it follows that $x \in \cup(\mathcal{F} \cap \mathcal{G})$. Therefore, if $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G}) \to x \in \cup(\mathcal{F} \cap \mathcal{G})$ and since x, A, and B were arbitrary we can conclude that $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$.

($\leftarrow$) Now suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ and $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$. Since $x \in (\cup\mathcal{F}) \cap (\cup\mathcal{G})$, then there is a set $M \in F$ such that $x \in M$ and there is a set $N \in \mathcal{G}$ such that $x \in N$ and it follows that $x \in M \cap \mathcal{G}$. Then since $M \in \mathcal{F}$, $N \in \mathcal{G}$, $x \in M \cap \mathcal{G}$, and $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ we can conclude that $x \in \cup(\mathcal{F} \cap \mathcal{G}))$. Therefore if $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \cup(\mathcal{F} \cap \mathcal{G}))$ then $(\cup\mathcal{F}) \cap (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \cap \mathcal{G})$.

□

3.4.19

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Then $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint.*

Proof. ($\to$) Suppose $\cup\mathcal{F} \cap \cup\mathcal{G} = \varnothing$. We will prove by contradiction. Let A be an arbitrary set in $\mathcal{F}$ and B be an arbitrary set in $\mathcal{G}$. Suppose $x \in A \cap B$, which means $x \in A$, $x \in B$, and $A \cap B \neq \varnothing$. Since $x \in A$ and $A \in \mathcal{F}$ then $x \in \cup\mathcal{F}$ and since $x \in B$ and $B \in \mathcal{G}$ then $x \in \cup\mathcal{G}$. Therefore $x \in \cup\mathcal{F} \cap \cup\mathcal{G}$, but this contradicts $\cup\mathcal{F} \cap \cup\mathcal{G} = \varnothing$. Therefore $A \cap B = \varnothing$ and since A and B were arbitrary we can conclude $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B = \varnothing)$.

($\leftarrow$) Now suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B = \varnothing)$. We will again prove by contradiction. Suppose $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are not disjoint, which means there is an element x that is in both $\cup\mathcal{F}$ and $\cup\mathcal{G}$. This means that there is a set in $\mathcal{F}$ that contains x and there is a set in $\mathcal{G}$ that contains x. However, this contradicts our given that every set in $\mathcal{F}$ is disjoint from every set in $\mathcal{G}$. Therefore $\cup\mathcal{F} \cap \cup\mathcal{G} = \varnothing$. □

3.4.20

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

A

Theorem. $(\cup\mathcal{F}) \setminus (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \setminus \mathcal{G})$

Proof. Let x be arbitrary and suppose $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$, which means $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$. Since $x \in \cup\mathcal{F}$ there exists a set within $\mathcal{F}$ that contains x. Since $x \notin \cup\mathcal{G}$ there is no set in $\mathcal{G}$ that contains x. Since there is a set in $\mathcal{F}$ that contains x and that set is not in $\mathcal{G}$, then $x \in \cup(\mathcal{F} \setminus \mathcal{G})$. Since x was arbitrary we can conclude $(\cup\mathcal{F}) \setminus (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \setminus \mathcal{G})$. □

B

"Since $x \in A$ and $A \notin \mathcal{G}$, $x \notin \cup\mathcal{G}$" is not true. Although $x \in A$ and $A \notin \mathcal{G}$, this does not mean $x \notin \cup\mathcal{G}$ because x could be in another set in G and would therefore be in $\cup\mathcal{G}$.

C

Theorem. $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$ *iff* $\forall A \in (\mathcal{F} \setminus \mathcal{G})\forall B \in \mathcal{G}(A \cap B = \varnothing)$

Proof. ($\rightarrow$) Suppose $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. Let A be an arbitrary set in $(\mathcal{F} \setminus \mathcal{G})$ and B be an arbitrary set in $\mathcal{G}$. We will prove by contradiction. Now suppose that A and B are not disjoint, which means there is an element x such that $x \in A$ and $x \in B$. Since $x \in A$ and $A \in (\mathcal{F} \setminus \mathcal{G})$ then $x \in \cup(\mathcal{F} \setminus \mathcal{G})$ and because $\cup(\mathcal{F} \setminus \mathcal{G})$ is a subset of $(\cup\mathcal{F}) \setminus (\cup\mathcal{G})$, then $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. This means that $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$. Since $x \notin \cup\mathcal{G}$ then there is no set in $\mathcal{G}$ that contains x, but this contradicts $x \in B$ and $B \in \mathcal{G}$. Therefore A and B are disjoint and since A and B were arbitrary we can conclude $\forall A \in (\mathcal{F} \setminus \mathcal{G})\forall B \in \mathcal{G}(A \cap B = \varnothing)$.

$\leftarrow$ Suppose all sets in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G}$ are disjoint. Let x be arbitrary and suppose $x \in \cup(\mathcal{F} \setminus \mathcal{G})$, which means there is a set in $\mathcal{F}$ that contains x and $x \in \cup\mathcal{F}$. Now let B be an arbitrary set in G. Since all sets in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G}$ are disjoint and $x \in \cup\mathcal{F}$, then $x \notin B$ and since B was arbitrary we can conclude $\forall B \in \mathcal{G}(x \notin B)$ or $x \notin \cup\mathcal{G}$. Since $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$, then $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. Therefore if $x \in \cup(\mathcal{F} \setminus \mathcal{G})$ then $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$ and since x was arbitrary we can conclude $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. $\square$

D

Find an example of families of sets $\mathcal{F}$ and $\mathcal{G}$ for which $\cup(\mathcal{F} \setminus \mathcal{G}) \neq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$.
$\mathcal{F} = \{\{1\}, \{2, 5\}\}$ and $\mathcal{G} = \{\{2\}, \{10\}\}$
$\cup(\mathcal{F} \setminus \mathcal{G}) = \{1, 2, 5\}$
$(\cup\mathcal{F}) \setminus (\cup\mathcal{G}) = \{1, 2, 5\} \setminus \{2, 10\} = \{1, 5\}$
$\{1, 2, 5\} \neq \{1, 5\}$

3.4.21

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\cup\mathcal{F} \nsubseteq \cup\mathcal{G}$ then there is some $A \in \mathcal{F}$ such that for all $B \in \mathcal{G}$, $A \nsubseteq B$.*

Proof. Suppose $\cup\mathcal{F} \nsubseteq \cup\mathcal{G}$. This means there is an element x that is in $\cup\mathcal{F}$ and not in $\cup\mathcal{G}$. Since $x \in \cup\mathcal{F}$ then there is a set in $\mathcal{F}$ that contains x and since $x \notin \cup\mathcal{G}$ there is no set in $\mathcal{G}$ that contains x. Therefore there is a set in $\mathcal{F}$ that is not a subset of any set in $\mathcal{G}$ and we can conclude $\exists A \in \mathcal{F}\forall B \in \mathcal{G}(A \nsubseteq B)$. $\square$

3.4.22

A

1. Prove goal of the form $\forall x P(x)$
2. Assume antecedent and prove consequent
3. existential instantiation
4. use a given of the form $P \wedge Q$
5. prove goal of the form $P \wedge Q$
6. Prove goal of the form $P \iff Q$ by proving $P \to Q$ and $Q \to P$.

B

Theorem. $B \setminus (\bigcup_{i\in I} A_i) = \bigcap_{i\in I}(B \setminus A_i)$

Proof.

$$\begin{aligned} x \in B \setminus (\bigcup_{i\in I} A_i) &= x \in B \wedge x \notin \bigcup_{i\in I} A_i \\ &= x \in b \wedge \neg\exists i \in I(x \in A_i) \\ &= x \in b \wedge \forall i \in I \neg(x \in A_i) \\ &= x \in b \wedge \forall i \in I(x \notin A_i) \\ &= \forall i \in I(x \in B \wedge x \notin A_i) \\ &= x \in \bigcap_{i\in I}(B \setminus A_i) \end{aligned}$$

□

C

Theorem. $B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I}(B \setminus A_i)$

Proof.

$$\begin{aligned} x \in B \setminus \bigcap_{i\in I} A_i &= x \in B \wedge \neg(\forall i \in I(x \in A_i)) \\ &= x \in B \wedge \exists i \in I \neg(x \in A_i) \\ &= x \in B \wedge \exists i \in I(x \notin A_i) \\ &= \exists i \in I(x \in B \wedge x \notin A_i) \\ &= x \in \bigcup_{i\in I}(B \setminus A_i) \end{aligned}$$

□

3.4.23

Suppose $\{A_i | i \in I\}$ and $\{B_i | i \in I\}$ are indexed families of sets and $I \neq \varnothing$.

A

Theorem. $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq (\bigcup_{i\in I} A_i) \setminus (\bigcup_{i\in I} B_i)$

Proof. Suppose i is arbitrary and that $x \in \bigcup_i (A_i \setminus B_i)$. This means we can choose an i, say $i = 0$, such that $x \in A_0$ and $x \notin B_0$. Since $x \in A_0$ then x is in $\bigcup_{i\in I} A_i$ and since $x \notin B_0$ then $x \notin \bigcap_{i\in I} B_i$. Therefore $x \in (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i)$ and since i was arbitrary we can conclude $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq (\bigcup_{i\in I} A_i) \setminus (\bigcup_{i\in I} B_i)$. □

B

Find an example for which $\bigcup_{i\in I}(A_i \setminus B_i) \neq (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i)$

$B_1 = \{1,2\}, B_2 = \{3,4\}, A_1 = \{1,2\}, A_2 = \{2,5\}$

$\bigcup_{i\in I} A_i = \bigcup(\{1,2\},\{3,4\}) = \{1,2,3,4\}$

$\bigcap_{i\in I} B_i = \bigcap(\{1,2\},\{2,5\}) = \{2\}$

$(\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i) = \{1,2,3,4\} \setminus \{2\} = \{1,3,4\}$

$\bigcup_{i\in I}(A_i \setminus B_i) = \bigcup(\{1,2\} \setminus \{1,2\}, \{2,5\} \setminus \{3,4\}) = \bigcup(\varnothing, \{2,5\}) = \{2,5\}$

$(\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i) = \{1,3,4\} \neq \{2,5\} = \bigcup_{i\in I}(A_i \setminus B_i)$

3.4.24

Suppose $\{A_i | i \in I\}$ and $\{B_i | i \in I\}$ are families of sets.

A

Theorem. $\bigcup_{i\in I}(A_i \cap B_i) \subseteq (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$

Proof. Let x be arbitrary and suppose $x \in \bigcup_{i\in I}(A_i \cap B_i)$, which means we can choose an i, say $i = 0$, such that $x \in A_0 \cap B_0$. If $x \in A_0 \cap B_0$ then $x \in A_0$ and $x \in B_0$. Since $x \in A_0$ there exists an $i \in I$ such that $x \in A_i$ or $x \in \bigcup_{i\in I} A_i$. Using a similar argument we can conclude that $x \in \bigcup_{i\in I} B_i$. Since x was arbitrary we can conclude that $x \in (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$. □

B

Find an example where $\bigcup_{i\in I}(A_i \cap B_i) \neq (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$.

Since we already proved that $\bigcup_{i\in I}(A_i \cap B_i) \subseteq (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$, we must find an example where $(\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i) \nsubseteq \bigcup_{i\in I}(A_i \cap B_i)$.

Let $A_1 = \{1\}$,
$A_2 = \{2\}$,
$B_1 = \{3\}$,
and $B_2\{1\}$.

Then $\bigcup_{i \in I} A_i = \{1, 2\}$ and $\bigcup_{i \in I} = \{1, 3\}$ and therefore
$(\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i) = \{1, 2\} \cap \{1, 3\} = \{1\}$.

Also, $A_1 \cap B_1 = \{1\} \cap \{3\} = \varnothing$ and $A_2 \cap B_2 = \{2\} \cap \{1\} = \varnothing$ and therefore
$\bigcup_{i \in I}(A_i \cap B_i) = \varnothing$.

$(\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i) = \{1\} \neq \varnothing = \bigcup_{i \in I}(A_i \cap B_i)$

3.4.25

Theorem. *For all integers a and b there is an integer c such that $a|c$ and $b|c$.*

Proof. Let a and b be arbitrary integers. Let $c = ab$ and note that since a and b are both integers then c is also an integer. Since $a|c$ there exists an integer k such that $ak = c = ab$. Similarly, since $b|c$ there exists an integer j such that $bj = c = ab$. If we let $k = b$ then $ab = ab$ and if we let $j = a$ then $ba = ab$. Since a and b were arbitrary we can conclude that for all integers a and b there exists an integer c such that $a|c$ and $b|c$. □

3.4.26

A

Theorem. *For every integer n, $15|n$ iff $3|n$ and $5|n$.*

Proof. ($\rightarrow$) Let n be an arbitrary integer. Suppose $15|n$, which means there exists an integer k such that $15k = n$. Therefore, $5(3k) = n$ and since $3k$ is an integer we can conclude $5|n$. Also since $15k = n$, then $3(5k) = n$ and since $5k$ is an integer we can conclude $3|n$. Therefore, $3|n$ and $5|n$.

($\leftarrow$) Now suppose $3|n$ and $5|n$. This means there is an integer j such that $3j = n$ and there is another integer k such that $5k = n$. Therefore, $15(2k - j) = 30k - 15j = 6n - 5n = n$. Since $2k - j$ is an integer we can conclude that $15|n$. □

B

Consider the case where $n = 30$ and it is true that $6|n$ and $10|n$ but 60 does not divide n. Therefore it is not true that for every integer n, $60|n$ iff $6|n$ and $10|n$.