10 Inferences Involving Two Populations

10.5 Inferences Concerning the Ratio of Variances Using Two Independent Samples

When comparing two populations, we naturally compare their two most fundamental distribution characteristics, their “center” and their “spread,” by comparing their means and standard deviations.

We have learned, in two of the previous sections, how to use the *t*-distribution to make inferences comparing two population means with either dependent or independent samples.

These procedures were intended to be used with normal populations, but they work quite well even when the populations are not exactly normally distributed.

The next logical step in comparing two populations is to compare their standard deviations, the most often used measure of spread.

However, sampling distributions that deal with sample standard deviations (or variances) are very sensitive to slight departures from the assumptions.

Therefore, the only inference procedure to be presented here will be the **hypothesis test for the equality of standard deviations (or variances)** for two normal populations.

Example 15 – *Writing Hypotheses for the Equality of Variances*

State the null and alternative hypotheses to be used for comparing the variances of the two soft drink bottling machines.

Solution:

There are several equivalent ways to express the null and alternative hypotheses, but because the test procedure uses the ratio of variances, the recommended convention is to express the null and alternative hypotheses as ratios of the population variances.

Furthermore, it is recommended that the “larger” or “expected to be larger” variance be the numerator.

Example 15 – *Solution*

cont'd

The concern of the soft drink company is that the new modern machine (*m*) will result in a larger standard deviation in the amounts of fill than its present machine (*p*); $\sigma_m > \sigma_p$ or equivalently $\sigma_m^2 > \sigma_p^2$, which becomes $\frac{\sigma_m^2}{\sigma_p^2} > 1$.

We want to test the manufacturer's claim (the null hypothesis) against the company's concern (the alternative hypothesis):

$$H_o: \frac{\sigma_m^2}{\sigma_p^2} = 1$$ (*m* is no more variable)

$$H_a: \frac{\sigma_m^2}{\sigma_p^2} > 1$$ (*m* is more variable)

Inferences about the ratio of variances for two normally distributed populations use the ***F*-distribution**.

The *F*-distribution, similar to Student's *t*-distribution and the χ^2-distribution, is a family of probability distributions.

Each *F*-distribution is identified by two numbers of degrees of freedom, one for each of the two samples involved.

Before continuing with the details of the hypothesis-testing procedure, let's learn about the *F*-distribution.

Properties of the *F*-distribution

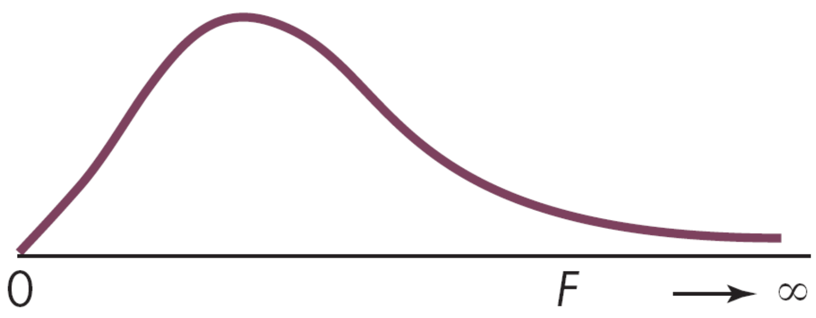
1. *F* is nonnegative; it is zero or positive.

2. *F* is nonsymmetrical; it is skewed to the right.

3. *F* is distributed so as to form a family of distributions; there is a separate distribution for each pair of numbers of degrees of freedom.

For inferences discussed in this section, the number of degrees of freedom for each sample is $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$.

Each different combination of degrees of freedom results in a different *F*-distribution, and each *F*-distribution looks approximately like the distribution shown in Figure 10.2



F-Distribution

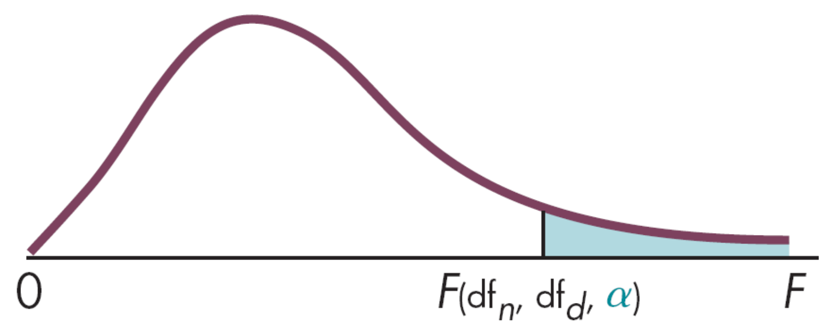
Figure 10.2

Distribution Calculator

The critical values for the *F*-distribution are identified using three values:

- df_n, the degrees of freedom associated with the sample whose variance is in the numerator of the calculated *F*
- df_d, the degrees of freedom associated with the sample whose variance is in the denominator
- α, the area under the distribution curve to the right of the critical value being sought

Therefore, the symbolic name for a critical value of F will be $F(\text{df}_n, \text{df}_d, \alpha)$, as shown in Figure 10.3.



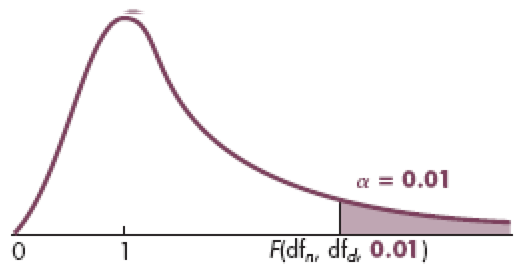
A Critical Value of F

Figure 10.3

Because it takes three values to identify a single critical value of F, making tables for F is not as simple as it was with previously studied distributions.

The tables presented in this textbook are organized so as to have a different table for each different value of α, the "area to the right."

Table 9A in Appendix B shows the critical values for $F(\mathrm{df}_n, \mathrm{df}_d, \alpha)$,when $\alpha = 0.05$; Table 9B gives the critical values when $\alpha = 0.025$; Table 9C gives the values when $\alpha = 0.01$.



Degrees of Freedom for Numerator

Degrees of Freedom for Denominator	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	10,000
1	4052.	5000.	5403.	5625.	5764.	5859.	5928.	5981.	6022.	6056.	6106.	6157.	6209.	6235.	6261.	6287.	6313.	6339.	6366.
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.3	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.01
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
10,000	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.19	2.04	1.88	1.79	1.70	1.59	1.48	1.33	1.05

Example 16 – *Finding Critical F-values*

Find $F(5, 8, 0.05)$, the critical F-value for samples of size 6 and size 9 with 5% of the area in the right-hand tail.

Solution:

Using Table 9A ($\alpha = 0.05$), find the intersection of column df = 8 (for the numerator) and row df = 8 (for the denominator) and read the value: $F(5, 8, 0.05)$ = **3.69**. See the accompanying partial table.

Portion of Table 9A ($\alpha = 0.05$)

		df for Numerator				
		…	5	…	8	
df for Denominator	⋮					
	5				**4.82**	$F(8, 5, 0.05)$ = **4.82**
	⋮					
	8		**3.69**			$F(5, 8, 0.05)$ = **3.69**

Example 16 – *Solution*

cont'd

Notice that $F(8, 5, 0.05)$ is **4.82**. The degrees of freedom associated with the numerator and with the denominator must be kept in the correct order; 3.69 is different from 4.82.

Check some other pairs to verify that interchanging the degrees of freedom numbers will result in different F-values.

Use of the *F*-distribution has a condition

Assumptions for inferences about the ratio of two variance: The samples are randomly selected from normally distributed populations, and the two samples are selected in an independent manner.

Test Statistic for Equality of Variances

$$F\bigstar = \frac{s_n^2}{s_d^2}, \text{ with } df_n = n_n - 1 \text{ and } df_d = n_d - 1 \qquad \textbf{(10.16)}$$

The sample variances are assigned to the numerator and denominator in the order established by the null and alternative hypotheses for one-tailed tests.

The calculated ratio, $F\bigstar$, will have an *F*-distribution with $df_n = n_n - 1$ (numerator) and $df_d = n_d - 1$ (denominator) when the assumptions are met and the null hypothesis is true.

We are ready to use *F* to complete a hypothesis test about the ratio of two population variances.

Example 17 – *One-tailed Hypothesis Test for the Equality of Variances*

We know that our soft drink bottling company was to make a decision about the equality of the variances of amounts of fill between its present machine and a modern, high-speed machine.

Does the sample information in Table 10.6 present sufficient evidence to reject the null hypothesis (the manufacturer's claim) that the modern, high-speed bottle-filling machine fills bottles with no greater variance than the company's present machine?

Sample	n	s^2
Present machine (p)	22	0.0008
Modern, high-speed machine (m)	25	0.0018

Sample Information on Variances of Fills

Table 10.6

Assume the amounts of fill are normally distributed for both machines, and complete the test using $\alpha = 0.01$.

Solution*:*

Step 1 **a. Parameter of interest:** $\frac{\sigma_m^2}{\sigma_p^2}$, the ratio of the variances in the amounts of fill placed in bottles for the modern machine versus the company's present machine

b. Statement of hypotheses:

$$H_o: \frac{\sigma_m^2}{\sigma_p^2} = 1 \ (\leq)$$ (*m* is no more variable)

$$H_a: \frac{\sigma_m^2}{\sigma_p^2} > 1$$ (*m* is more variable)

Example 17 – *Solution*

cont'd

Note

When the "expected to be larger" variance is in the numerator for a one-tailed test, the alternative hypothesis states, "The ratio of the variances is greater than 1."

Step 2 **a. Assumptions:** The sampled populations are normally distributed (given in the statement of the problem), and the samples are independently selected (drawn from two separate populations).

b. Test statistic: The *F*-distribution with the ratio of the sample variances and formula (10.16)

c. Level of significance: $\alpha = 0.01$

Example 17 – *Solution*

cont'd

Step 3 **a. Sample information:** See Table 10.6.

Sample	n	s^2
Present machine (p)	22	0.0008
Modern, high-speed machine (m)	25	0.0018

Sample Information on Variances of Fills

Table 10.6

b. Calculated test statistic:

Using formula (10.16), we have

$$F\star = \frac{s_m^2}{s_p^2}: \qquad F\star = \frac{0.0018}{0.0008} = \mathbf{2.25}$$

Example 17 – *Solution*

cont'd

The number of degrees of freedom for the numerator is df_n = 24 (or 25 – 1) because the sample from the modern, high-speed machine is associated with the numerator, as specified by the null hypothesis.

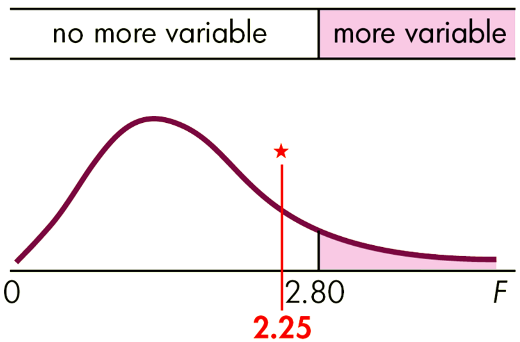
Also, df_d = 21 because the sample associated with the denominator has size 22.

Example 17 – *Solution*

Step 4 Probability Distribution:

Classical:

a. The critical region is the right-hand tail because H_a expresses concern for values related to “more than.” $df_n = 24$ and $df_d = 21$. The critical value is obtained from Table 9C: $F(24, 21, 0.01) =$ **2.80**.



b. $F^{\bigstar}$ is not in the critical region, as shown in **red** in the figure.

Example 17 – *Solution*

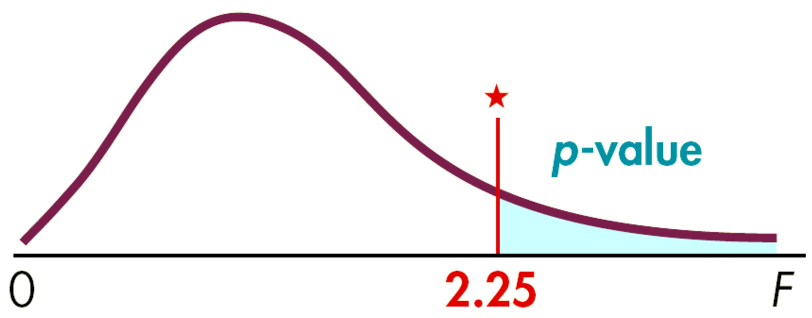
cont'd

Step 4 Probability Distribution:

***p*-Value:**

a. Use the right-hand tail because *Ha* expresses concern for values related to "more than."

$\mathbf{P} = P(F\bigstar > 2.25$, with $df_n = 24$ and $df_d = 21$), as shown in the figure.



Example 17 – *Solution*

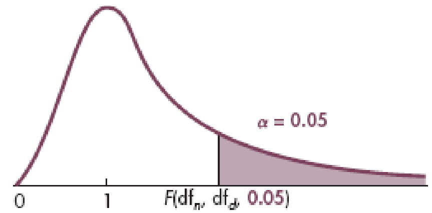
cont'd

To find the *p*-value, you have two options:

1. Use Tables 9A and 9B (Appendix B) to place bounds on the *p*-value: **0.025 < P < 0.05 (see the next slide)**

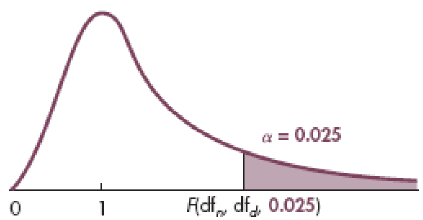
2. Use a computer or calculator to find the *p*-value: **P = 0.0323.**

b. The *p*-value is not smaller than the level of significance, α(0.01).



Degrees of Freedom for Numerator

Degrees of Freedom for Denominator	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	10,000
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39



Degrees of Freedom for Numerator

Degrees of Freedom for Denominator	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	10,000
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48

Example 17 – *Solution*

cont'd

Step 5. **a. Decision:** Fail to reject H_o.

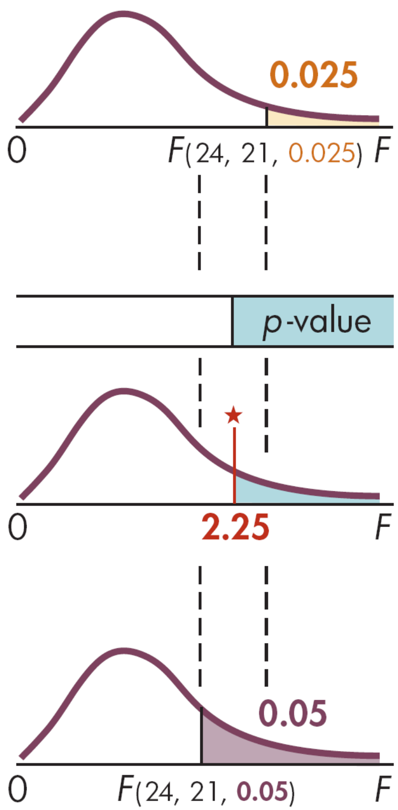
b. Conclusion: At the 0.01 level of significance, the samples do not present sufficient evidence to indicate an increase in variance with the new machine.

Calculating the *p*-value when using the *F*-distribution

Method 1: Use Table 9 in Appendix B to place bounds on the p-value. Using Tables 9A, 9B, and 9C in Appendix B to estimate the *p*-value is very limited.

However, for Example 17, the *p*-value can be estimated. By inspecting Tables 9A and 9B, you will find that $F(24,21,0.025) = 2.37$ and $F(24,21,0.005) = 2.05$.

$F\bigstar = 2.25$ is between the values 2.37 and 2.05; therefore, the *p*-value is between 0.025 and 0.05: $\mathbf{0.025 < P < 0.05}$. (See figure)



Method 2: If you are doing the hypothesis test with the aid of a computer or calculator, most likely it will calculate the *p*-value for you, or you may use the cumulative probability distribution commands.

Critical *F*-Values for One- and Two-Tailed Tests

Critical *F*-Values for One- and Two-Tailed Tests

The tables of critical values for the F-distribution give only the right-hand critical values. This will not be a problem because the right-hand critical value is the only critical value that will be needed.

You can adjust the numerator–denominator order so that all the “activity” is in the right-hand tail. There are two cases: one-tailed tests and two-tailed tests.

One-tailed tests: Arrange the null and alternative hypotheses so that the alternative is always “greater than.” The $F\bigstar$-value is calculated using the same order as specified in the null hypothesis.

Critical *F*-Values for One- and Two-Tailed Tests

Two-tailed tests: When the value of $F^{\star}$ is calculated, always use the sample with the larger variance for the numerator; this will make $F^{\star}$ greater than 1 and place it in the right-hand tail of the distribution. Thus, you will need only the critical value for the right-hand tail.

All hypothesis tests about two variances can be formulated and completed in a way that both the critical value of $F^{\star}$ and the calculated value of *F* will be in the right-hand tail of the distribution.

Critical *F*-Values for One- and Two-Tailed Tests

Since Tables 9A, 9B, and 9C contain only critical values for the right-hand tail, this will be convenient and you will never need critical values for the left-hand tail. The following two examples will demonstrate how this is accomplished.

Example 19 – *Two-Tailed Hypothesis Test for the Equality of Variances*

Find $F\bigstar$ and the critical values for the following hypothesis test so that only the right-hand critical value is needed. Use $\alpha = 0.05$ and the sample information $n_1 = 10$, $n_2 = 8$, $s_1 = 5.4$ and $s_2 = 3.8$.

$$H_0: \sigma_2^2 = \sigma_1^2 \quad \text{or} \quad \frac{\sigma_1^2}{\sigma_2^2} = 1$$

$$H_a: \sigma_2^2 \neq \sigma_1^2 \quad \text{or} \quad \frac{\sigma_1^2}{\sigma_2^2} \neq 1$$

Example 19 – *Solution*

When the alternative hypothesis is two-tailed (≠), the calculated $F\star$ can be either $F\star = \frac{s_1^2}{s_2^2}$ or $F\star = \frac{s_2^2}{s_1^2}$. The choice is ours; we need only make sure that we keep df_n and df_d in the correct order.

We make the choice by looking at the sample information and using the sample with the larger standard deviation or variance as the numerator.

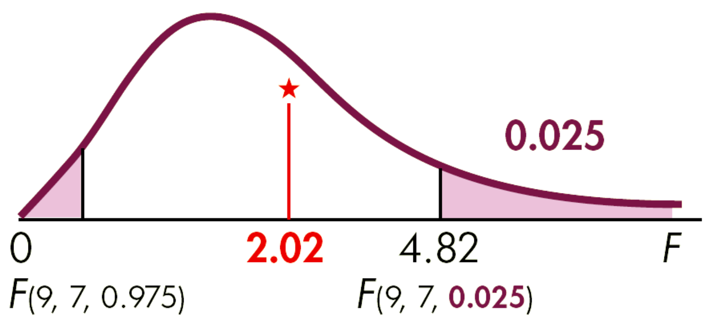
Therefore, in this illustration,

$$F\star = \frac{s_1^2}{s_2^2} = \frac{5.4^2}{3.8^2} = \frac{29.16}{14.44} = \mathbf{2.02}$$

Example 19 – *Solution*

cont'd

The critical values for this test are left tail, $F(9, 7, 0.975)$, and right tail, $F(9, 7, 0.025)$, as shown in the figure.



Since we chose the sample with the larger standard deviation (or variance) for the numerator, the value of F★will be greater than 1 and will be in the right-hand tail; therefore, only the right-hand critical value is needed. (All critical values for left-hand tails will be values between 0 and 1.)