MATH 1700

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Chapter 8A

Department of Mathematical and Statistical Sciences

CHAPTER 8A

- **Inference about the value of the population mean**
  - **Estimating the value of a population parameter, and**
  - **Testing a hypothesis.**
- **Point estimate for a parameter**
- **Interval estimate**
  - **Level of confidence** $(1-\alpha)$
- **Confidence interval**
  - **Maximum error of Estimate**
  - **Required Sample size for a specific level of confidence,** $1-\alpha$

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- The central limit theorem gave us some very important information about the sampling distribution of sample means (SDSM).
- Specifically, it stated that in many realistic cases (when the random sample is large enough) a distribution of sample means is normally or approximately normally distributed about the mean of the population.
- We are now ready to turn this situation around to the case in which the population mean is not known.
- We will draw one sample, calculate its mean value, and then make an inference about the value of the population mean based on the sample's mean value.

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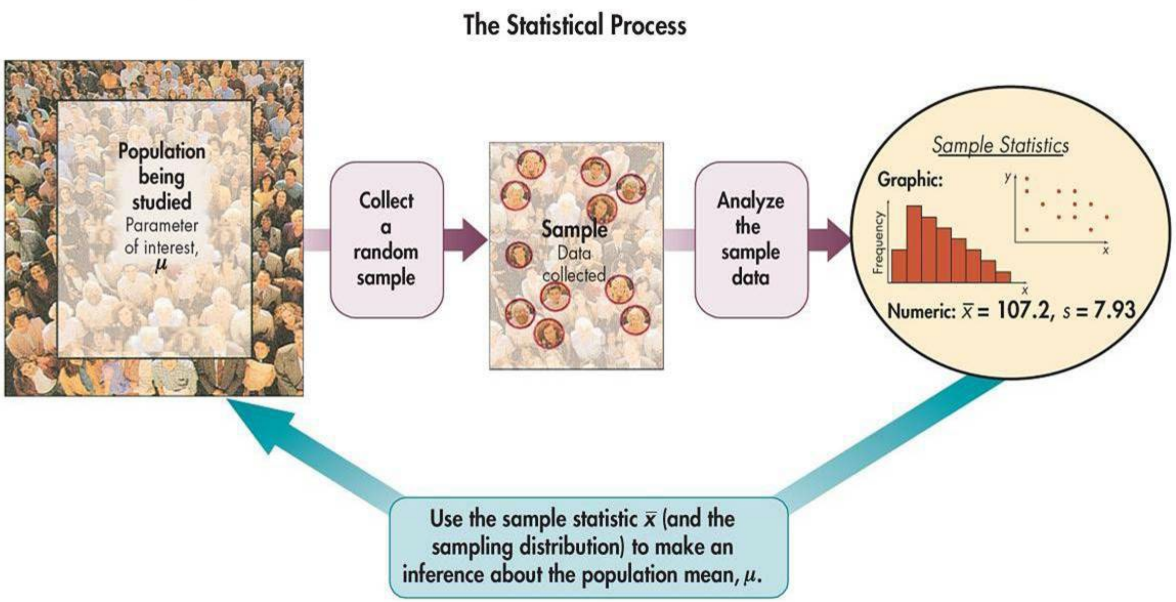
- The objective of inferential statistics is to use the information contained in the sample data to increase our knowledge of the sampled population.
- We will learn about making two types of inferences:
  1. estimating the value of a population parameter and
  2. testing a hypothesis.
- The sampling distribution of sample means (SDSM) is the key to making these inferences



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- **In this chapter, we deal with questions about the population mean using two methods that assume the value of the population standard deviation is a known quantity.**
  - **This assumption is seldom realized in real-life problems, but it will make our first look at the techniques of inference much simpler.**

- **The sampling distribution of sample means (SDSM) and the central limit theorem (CLT) provide the information needed to describe how close the point estimate, $\bar{x}$ is expected to be to the population mean, μ.**

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- Starting with the concept of **estimation**, let's consider a company that manufactures **rivets** for use in building **aircraft**.

- One characteristic of extreme importance is the "**shearing strength**" of each rivet.
- The company's engineers must monitor production to be certain that the shearing strength of the rivets meets the **required specs**.
- To accomplish this, they take a sample and determine the mean shearing strength of the **sample**.
- Based on this sample information, the company can **estimate** the mean shearing strength for **all the rivets** it is manufacturing.

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- **Notes**
  1. Shearing strength is the force required to break a material in a "cutting" action. Obviously, the manufacturer is not going to test all rivets because the test destroys each rivet tested.
     - ✓ Therefore, samples are tested and the information about each sample must be used to make inferences about the population of all such rivets.
  2. Throughout Chapter 8 we will treat the standard deviation, σ, as a known, or given, quantity and concentrate on learning the procedures for making statistical inferences about the population mean, μ.
     - ✓ Therefore, to continue the explanation of statistical inferences, we will assume σ = 18 for the specific rivets described in our example.
- A random sample of 36 rivets is selected, and each rivet is tested for shearing strength.
- The resulting sample mean is $\bar{x} = 924.23\ lb$. Based on this sample, we say, "We believe the mean shearing strength of all such rivets is $924.23\ lb$."

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- **Point estimate for a parameter:** A single number designed to **estimate** a quantitative **parameter of a population**, usually the value of the corresponding **sample statistic**.
  - That is, the **sample mean**, $\bar{x}$, is the **point estimate** (single-number value) for the mean, μ, of the sampled **population**.
  - **Sample means** vary in value and form a sampling distribution in which not all samples result in values equal to the **population mean**.
  - Therefore, we should not expect this sample of 36 rivets to produce a **point estimate** (sample mean) that is **exactly equal** to the mean μ of the sampled population.
  - We should, however, expect the **point estimate** to be **fairly close** in value to the **population mean**.

- **Unbiased statistic:** A **sample statistic** whose sampling distribution has a mean value equal to the value of the **population parameter** being estimated. A statistic that is not unbiased is a **biased statistic**.

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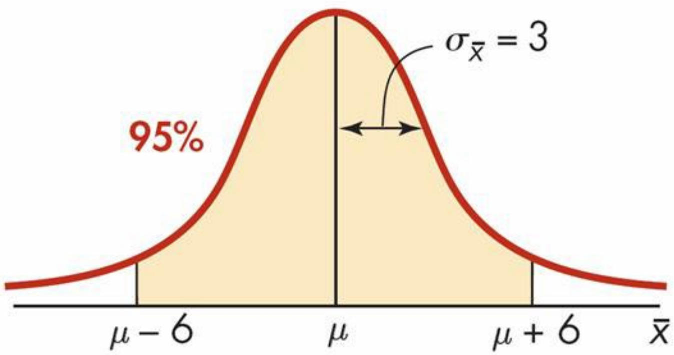
- **Therefore, we should anticipate that 95% of all random samples selected from a population with unknown mean μ and standard deviation $\sigma = 18$ will have means $\bar{x}$ between**
  - $\mu - 2\sigma_{\bar{x}}$ **and** $\mu + 2\sigma_{\bar{x}}$
  - $\mu - 2(\sigma/\sqrt{n})$ **and** $\mu + 2(\sigma/\sqrt{n})$
  - $\mu - 2\left(18/\sqrt{36}\right)$ **and** $\mu + 2\left(18/\sqrt{36}\right)$
  - $\mu - 6$ **and** $\mu + 6$

- **This suggests that 95% of all random samples of size 36 selected from the population of rivets should have a mean $\bar{x}$ between $\mu - 6$ and $\mu + 6$.**



or expressed algebraically:
$P(\mu - 6 < \bar{x} < \mu + 6) = \mathbf{0.95}$

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- Now let's put all of this information together in the form of a *confidence interval*.

- **Interval estimate:** An interval bounded by two values and used to estimate the value of a population parameter. The values that bound this interval are statistics calculated from the sample that is being used as the basis for the estimation.

- **Level of confidence** $(1 - \alpha)$**:** The portion of all interval estimates that include the parameter being estimated.

- **Confidence interval:** An interval estimate with a specified level of confidence.

ESTIMATION OF MEAN μ (σ KNOWN)

- **The assumption for estimating mean μ using a known σ: The sampling distribution of $\bar{x}$ has a normal distribution.**

The sampling distribution of sample means $\bar{x}$ is distributed about a mean equal to μ with a standard error equal to $\sigma/\sqrt{n}$; and (1) if the randomly sampled population is normally distributed, then $\bar{x}$ is normally distributed for all sample sizes, or (2) if the randomly sampled population is not normally distributed, then $\bar{x}$ is approximately normally distributed for sufficiently large sample sizes.

- **Therefore, we can satisfy the required assumption by either**
  1. **knowing that the sampled population is normally distributed or**
  2. **using a random sample that contains a sufficiently large amount of data.**

ESTIMATION OF MEAN μ (σ KNOWN)

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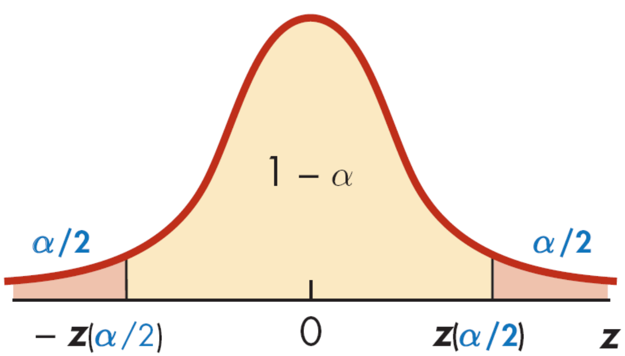
- **The $1-\alpha$ confidence interval for the estimation of mean μ is**

Confidence Interval for Mean

$$\bar{x} - z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right) \text{ to } \bar{x} + z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right) \quad (8.1)$$

- **Here are the parts of the confidence interval formula:**
  1. $\bar{x}$ **is the point estimate and the center point of the confidence interval.**
  2. $z(\alpha/2)$ **is the confidence coefficient. It is the number of multiples of the standard error needed to formulate an interval estimate of the correct width to have a level of confidence of $1-\alpha$.**



Confidence Coefficient $z(\alpha/2)$

ESTIMATION OF MEAN μ (σ KNOWN)

- **Confidence interval for Mean:** $\bar{x} - z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$ **to** $\bar{x} + z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$
- **Further parts of the confidence interval formula:**
  3. $\sigma/\sqrt{n}$ **is the standard error of the mean, or the standard deviation of the sampling distribution of sample means.**
  4. $z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$ **is one-half the width of the confidence interval (the product of the confidence coefficient and the standard error) and is called the maximum error of estimate,** E**.**
  5. $\bar{x} - z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$ **is called the lower confidence limit (LCL), and**
     $\bar{x} + z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$ **is called the upper confidence limit (UCL) for the CI.**
- **The estimation procedure is organized into a five-step process that will take into account all of the preceding information and produce both the point estimate and the confidence interval.**

ESTIMATION OF MEAN μ (σ KNOWN)

- **The Confidence Interval: A Five-step Procedure**
- **Step 1. The Set-Up:**
  - Describe the population parameter of interest.
- **Step 2. The Confidence Interval Criteria:**
  a. Check the assumptions.
  b. Identify the probability distribution and the formula to be used.
  c. State the level of confidence, $1 - \alpha$.
- **Step 3. The Sample Evidence:**
  - Collect the sample information.
- **Step 4. The Confidence Interval:**
  a. Determine the confidence coefficient: $z(\alpha/2)$.
  b. Find the maximum error of estimate: $E = z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$.
  c. Find the lower and upper confidence limits: $\bar{x} - E$ to $\bar{x} + E$.
- **Step 5. The Results:**
  - State the confidence interval.

EXAMPLE 4 – DEMONSTRATING THE MEANING OF A CONFIDENCE INTERVAL

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x	$P(x)$
0	$^1/_{10}$
1	$^1/_{10}$
2	$^1/_{10}$
3	$^1/_{10}$
4	$^1/_{10}$
5	$^1/_{10}$
6	$^1/_{10}$
7	$^1/_{10}$
8	$^1/_{10}$
9	$^1/_{10}$

- **Let's use computer simulation, to draw a sample of 40 single-digit numbers.**
  - $P(x) = \frac{1}{10}$, **for** $x = 0,1,2, \dots, 8,9$
  - **It can be shown that**
    - ✓ $\mu = \sum_{x=0}^{9} xP(x) = 4.5$ **and**
    - ✓ $\sigma^2 = \sum_{x=0}^{9}[x^2 P(x)] - \mu^2 = 8.25$

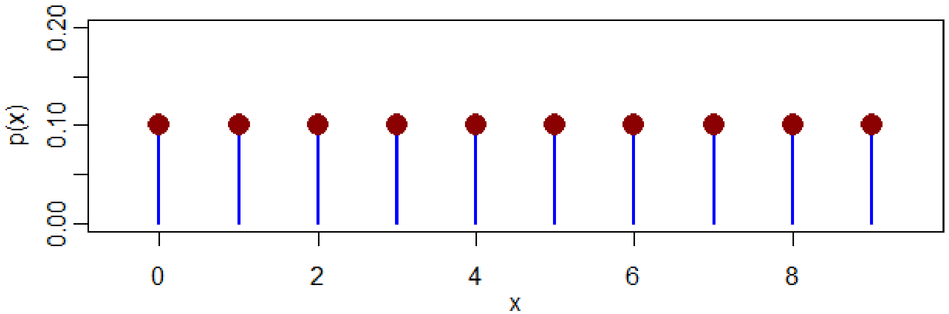
- **Here is the sample we have:**

2	8	2	1	5	5	4	0	9	1
0	4	6	1	5	1	1	3	8	0
3	6	8	4	8	6	8	9	5	0
1	4	1	2	1	7	1	7	9	3

  - **Let's construct the 90% confidence interval for the mean and check if the resulting interval contain the expected value of μ, 4.5?**
  - **If we were to select another sample of 40 single-digit numbers, would we get the same result?**

EXAMPLE 4 – DEMONSTRATING THE MEANING OF A CONFIDENCE INTERVAL

- First we need to address the assumptions; if the assumptions are not satisfied, we cannot expect the 90% and the 10% to occur. We know:
  - The distribution of single-digit random numbers is rectangular (definitely not normal),



  - the distribution of single-digit random numbers is symmetrical about their mean,
  - the $\bar{x}$ distribution for very small samples ($n = 5$) is a distribution that appeared to be approximately normal.
  - Therefore, it seems reasonable to assume that $n = 40$ is large enough for the CLT to apply.

EXAMPLE 4 – DEMONSTRATING THE MEANING OF A CONFIDENCE INTERVAL

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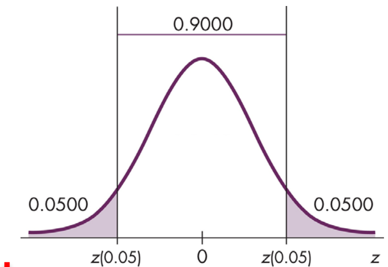
- **The sample was:**

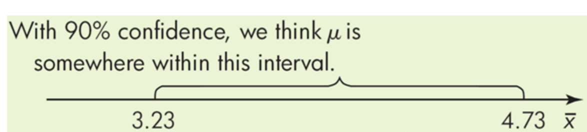
2	8	2	1	5	5	4	0	9	1
0	4	6	1	5	1	1	3	8	0
3	6	8	4	8	6	8	9	5	0
1	4	1	2	1	7	1	7	9	3

- **The sample statistics are:**
  - $n = 40, \sum x = 159$, **therefore** $\bar{x} = \frac{\sum x}{n} = 3.98$.
  - **Remember:** $\sigma^2 = 8.25$
  - $1 - \alpha = 0.9 \implies \alpha/2 = 0.05$
  - $z(\alpha/2) = 1.65$



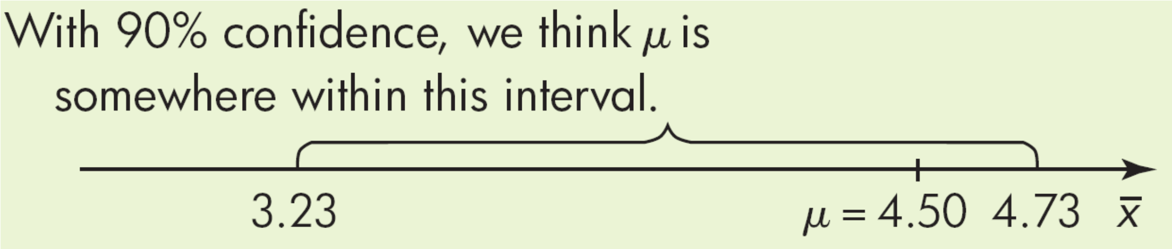
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9485	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890

- **The** 90% **confidence interval:**
  - $\bar{x} \pm z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$:
  - $3.98 \pm (1.65)\left(\frac{\sqrt{8.25}}{\sqrt{40}}\right)$:
  - 3.98 ± 0.75
  - **From** $3.98 - 0.75 = 3.23$ **to** $3.98 + 0.75 = 4.73$ **is the** 90% **confidence interval for** μ



EXAMPLE 4 – DEMONSTRATING THE MEANING OF A CONFIDENCE INTERVAL

- **The expected value for the mean, 4.5, does fall within the bounds of the confidence interval for this sample.**



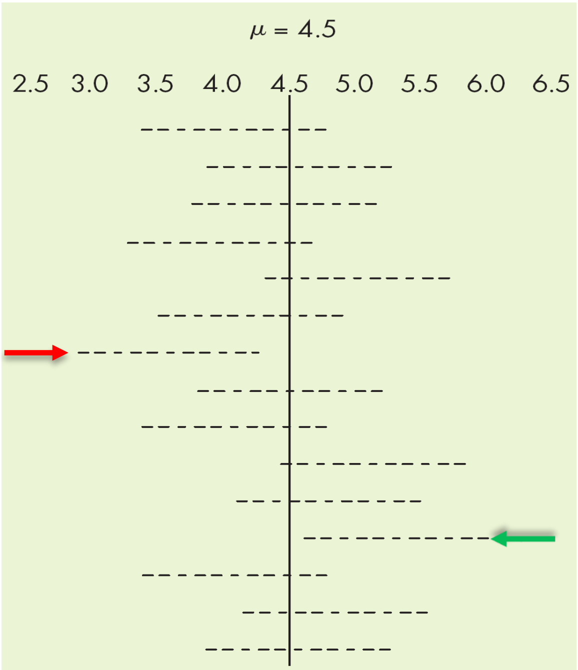
The 90% Confidence Interval

- **Let's now select 14 more random samples using computer simulation, each of size 40.**
- **Would the expected value for μ—namely, 4.5—be contained in all of them?**
  - **Think about the definition of "level of confidence"; it says that in the long run, 90% of the samples will result in bounds that contain μ.**
  - **In other words, 10% of the samples will NOT contain μ. Let's see what happens.**

EXAMPLE 4 – DEMONSTRATING THE MEANING OF A CONFIDENCE INTERVAL

- **Next table lists the mean from the first sample and the means obtained from the 14 additional random samples of size 40.**

Sample Number	Sample Mean, $\bar{x}$	90% Confidence Interval Estimate for μ
1	3.98	3.23 to 4.73
2	4.64	3.89 to 5.39
3	4.56	3.81 to 5.31
4	3.96	3.21 to 4.71
5	5.12	4.37 to 5.87
6	4.24	3.49 to 4.99
7	3.44	2.69 to 4.19
8	4.60	3.85 to 5.35
9	4.08	3.33 to 4.83
10	5.20	4.45 to 5.95
11	4.88	4.13 to 5.63
12	5.36	4.61 to 6.11
13	4.18	3.43 to 4.93
14	4.90	4.15 to 5.65
15	4.48	3.73 to 5.23



- **We see that 86.7% (13 of the 15) of the intervals contain μ and 2 of the 15 samples (sample 7 and sample 12) do not contain μ.**
- **However, in the long run, we should expect approximately $1 - \alpha = 0.90$ (or 90%) of the samples to result in bounds that contain 4.5 and approximately 10% that do not contain 4.5.**
- [**Confidence Interval Applet**]

SAMPLE SIZE

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- The confidence interval has two basic characteristics that determine its quality: its level of confidence and its width.
- It is preferable for the interval to have a high level of confidence and be precise (narrow) at the same time.
  - The higher the level of confidence, the more likely the interval is to contain the parameter, and the narrower the interval, the more precise the estimation.
- Remember that, the $(1-\alpha)$-level confidence interval for μ is
  - from $\bar{x} - z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$ to $\bar{x} + z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$
- The maximum error part of the confidence interval formula specifies the relationship involved.

Maximum Error of Estimate

$$E = z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right) \tag{8.2}$$

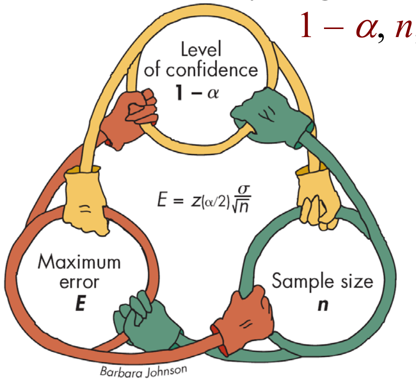
SAMPLE SIZE

- $$E = z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$$

- **This formula has four components:**
  1. the **maximum error**, E, half of the width of the confidence interval;
  2. the **confidence coefficient**, $z(\alpha/2)$, which is determined by the level of confidence;
  3. the **sample size**, n; and
  4. the **standard deviation**, σ. The standard deviation σ is not a concern in this discussion because it is a constant (the standard deviation of a population does not change in value).

- **Notes:**
  - Increasing the **level of confidence** will make the confidence coefficient larger and thereby require either the **maximum error** to increase or the **sample size** to increase;
  - decreasing the **maximum error** will require the **level of confidence** to decrease or the **sample size** to increase; and
  - decreasing the **sample size** will force **the maximum error** to become larger or the **level of confidence** to decrease.

The "Three-Way Tug-of-War" between $1 - \alpha$, n, and E



SAMPLE SIZE

- $$E = z(\alpha/2)\left(\frac{\sigma}{\sqrt{n}}\right)$$
- **The statistician's job is to "balance" the level of confidence, the sample size, and the maximum error so that an acceptable interval results.**
  - This is done by solving the maximum error formula, E, for sample size, n.

Sample Size

$$n = \left(\frac{z(\alpha/2)\cdot\sigma}{E}\right)^2 \tag{8.3}$$

- **Notes**
  - When we solve for the sample size n, it is customary to round up to the next larger integer, no matter what fraction (or decimal) results.
  - If the maximum error is expressed as a multiple of the standard deviation σ, then the actual value of σ is not needed in order to calculate the sample size.

EXAMPLE 7 – DETERMINING THE SAMPLE SIZE WITHOUT A KNOWN VALUE OF SIGMA (σ)

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- **Find the sample size needed to estimate the population mean to within $\frac{1}{5}$ of a standard deviation with 99% confidence.**

- **Solution:**
  - $1 - \alpha = 0.99, E = \frac{\sigma}{5}$
  - $z(\alpha/2) = 2.58$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1094	0.1075	0.1057	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1563	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952

- **Now you are ready to use the sample size formula (8.3):**

$$n = \left(\frac{z(\alpha/2)\cdot\sigma}{E}\right)^2 : \quad n = \left(\frac{(2.58)\cdot\sigma}{\sigma/5}\right)^2$$

$$= \left(\frac{(2.58\sigma)(5)}{\sigma}\right)^2$$

$$= [(2.58)(5)]^2 = (12.90)^2 = 166.4$$

- $n = \mathbf{167}$

QUESTIONS?

- **ANY QUESTION?**