Forecasting Functional Time Series using Rfssa: An R Package for Functional Singular Spectrum Analysis

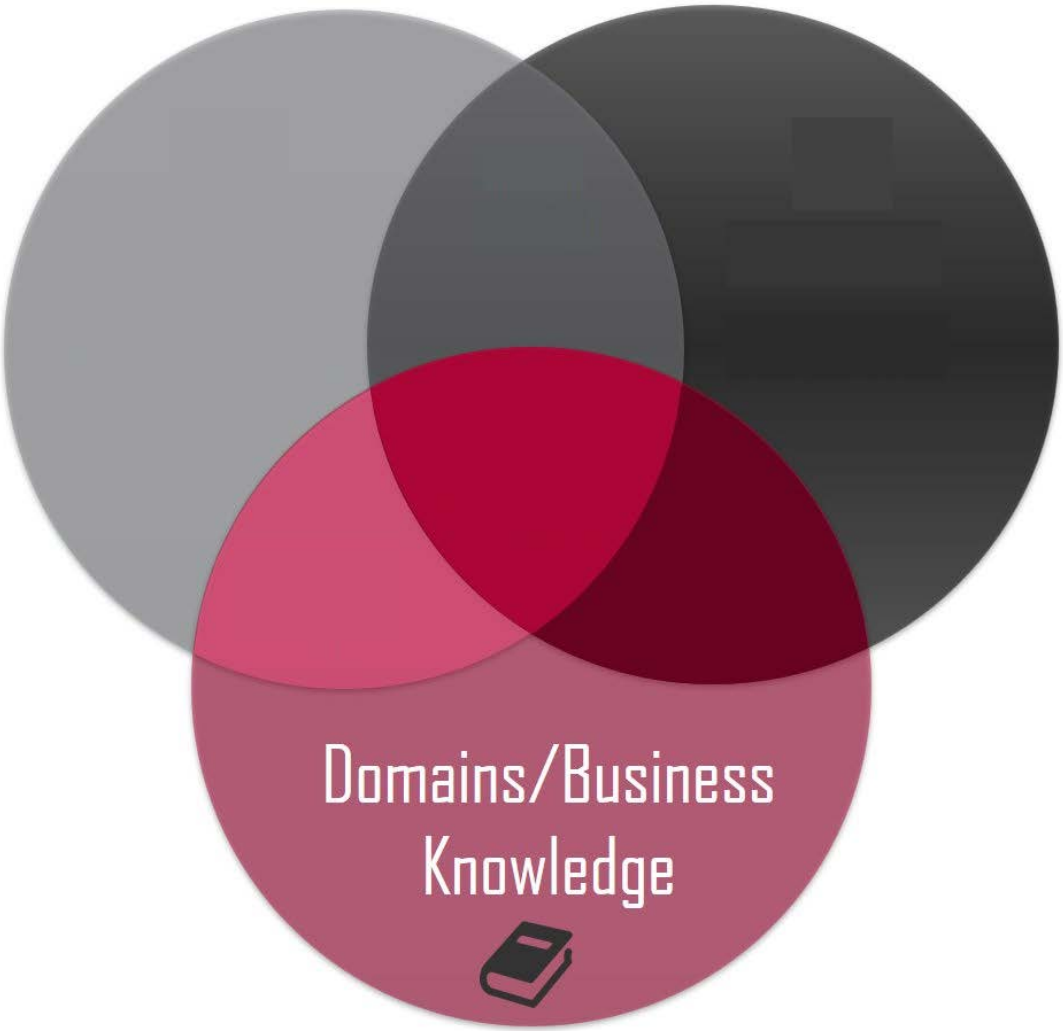
Mehdi Maadooliat
Associate Professor
mehdi.maadooliat@mu.edu

Colloquium Talk
Department of Mathematics
University of Wisconsin-Whitewater

December 3, 2024

1. Introduction (Remote Sensing Data)
2. Functional Time Series (FTS) and Multivariate FTS (MFTS)
3. Singular Spectrum Analysis (SSA) Quick Review
4. Functional Singular Spectrum Analysis (FSSA)
5. Discussion and Conclusion

What is Data Science?



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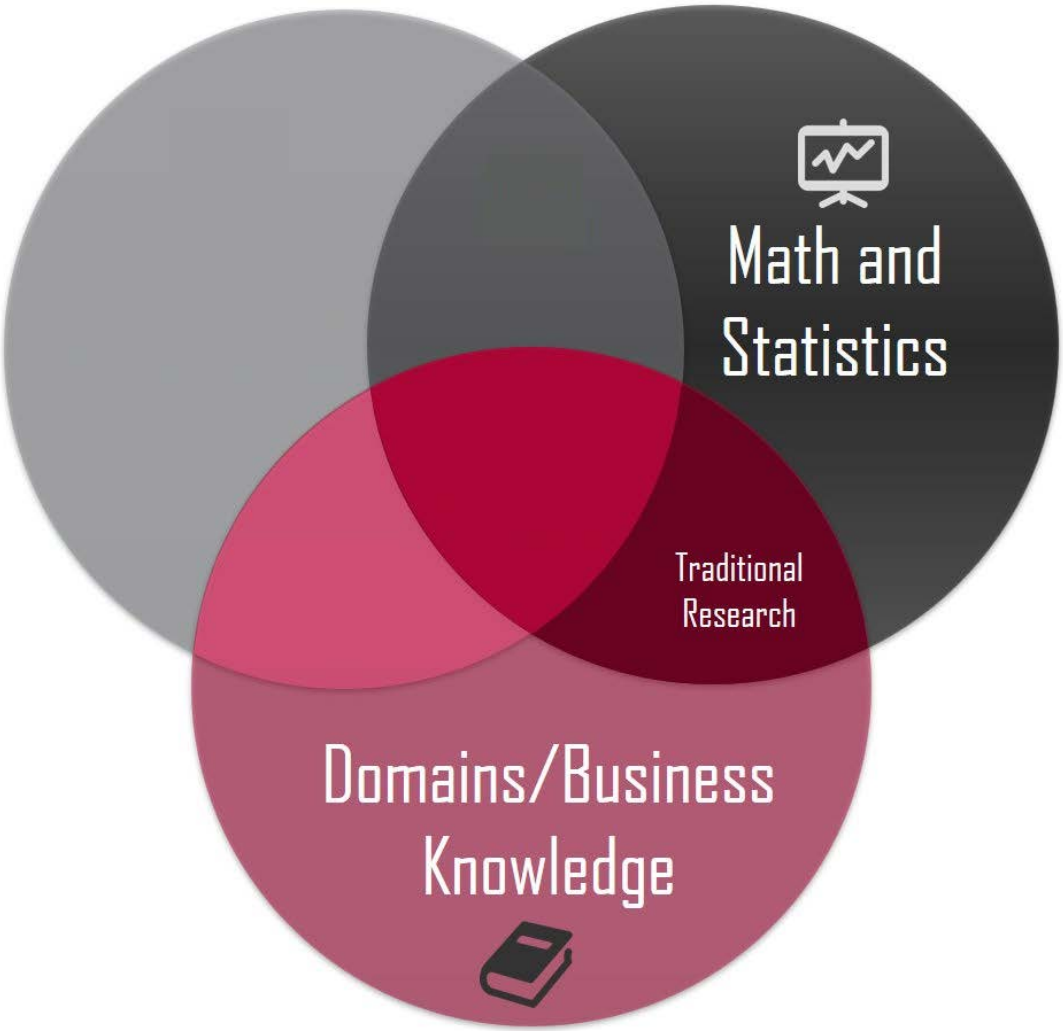

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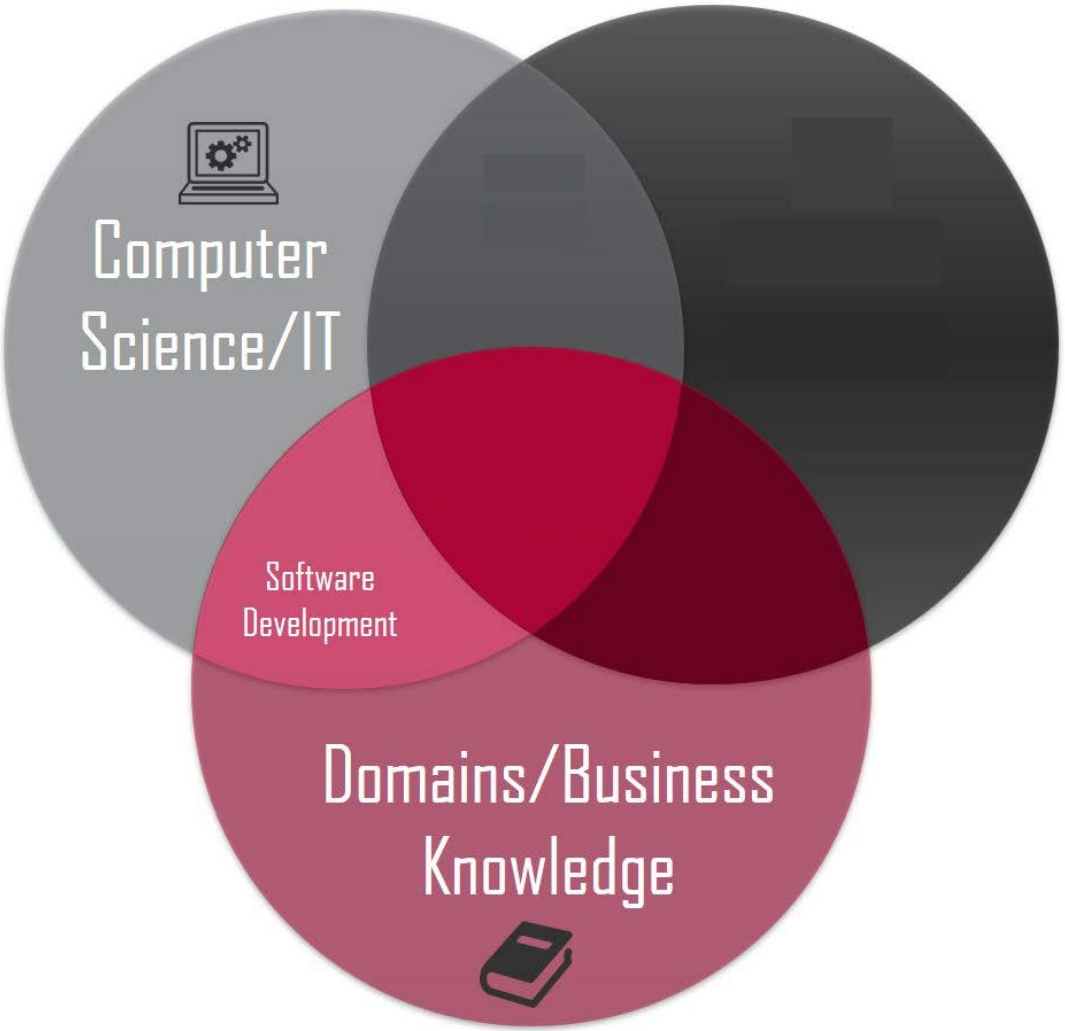

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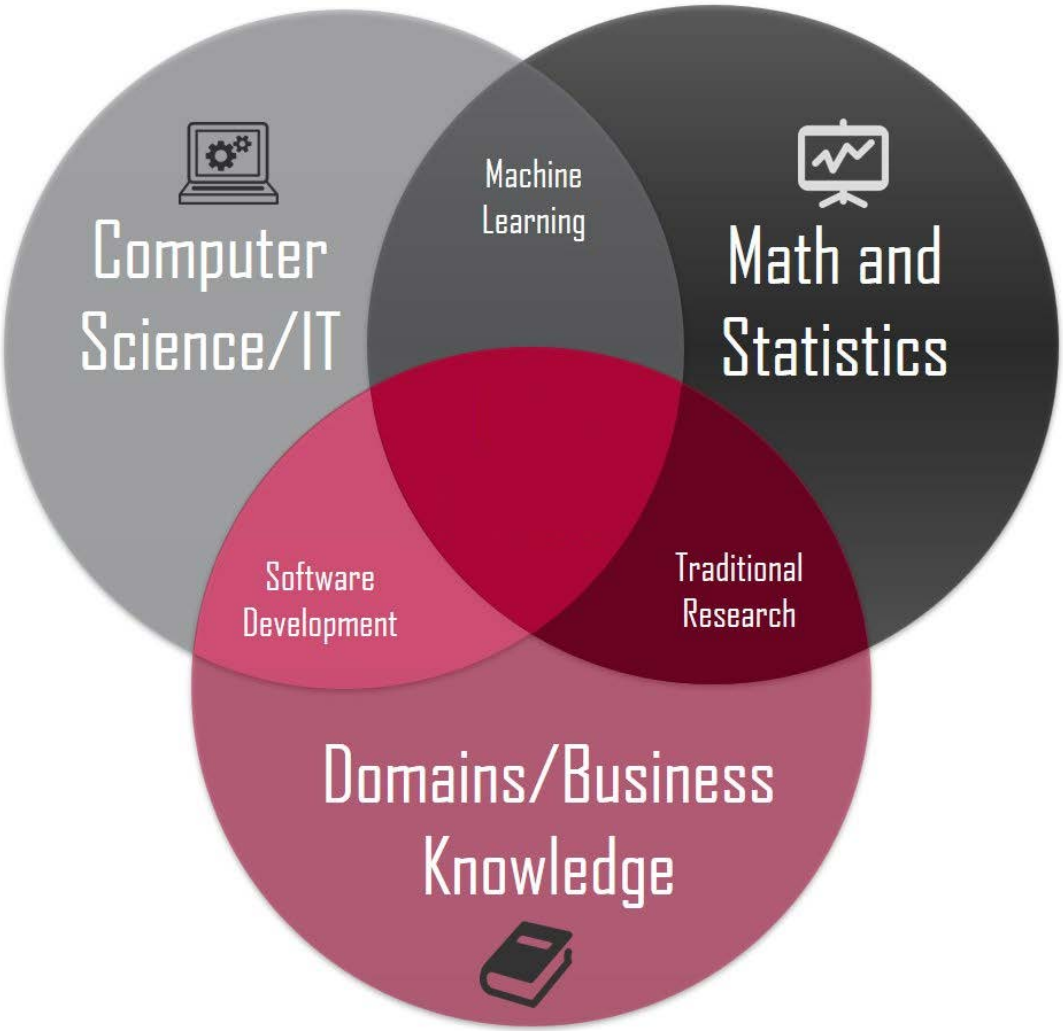

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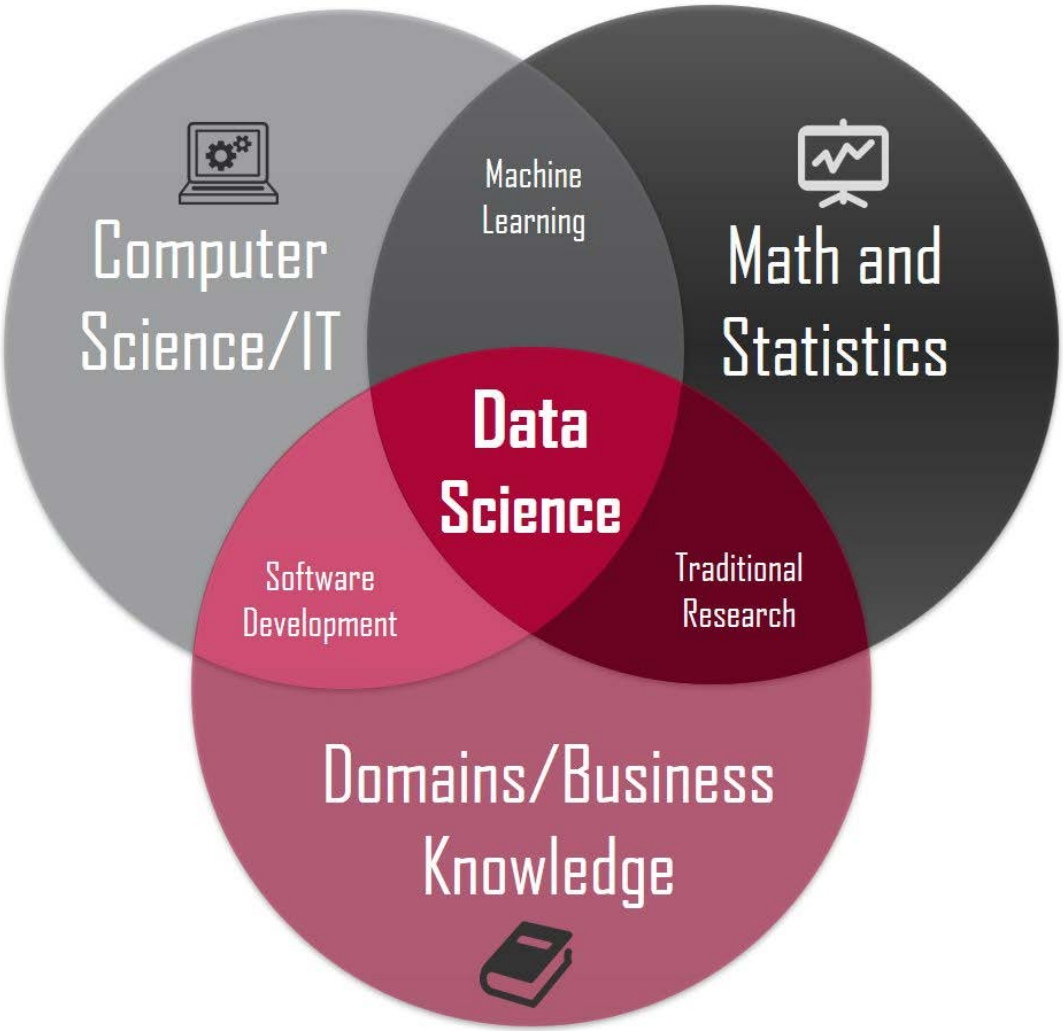
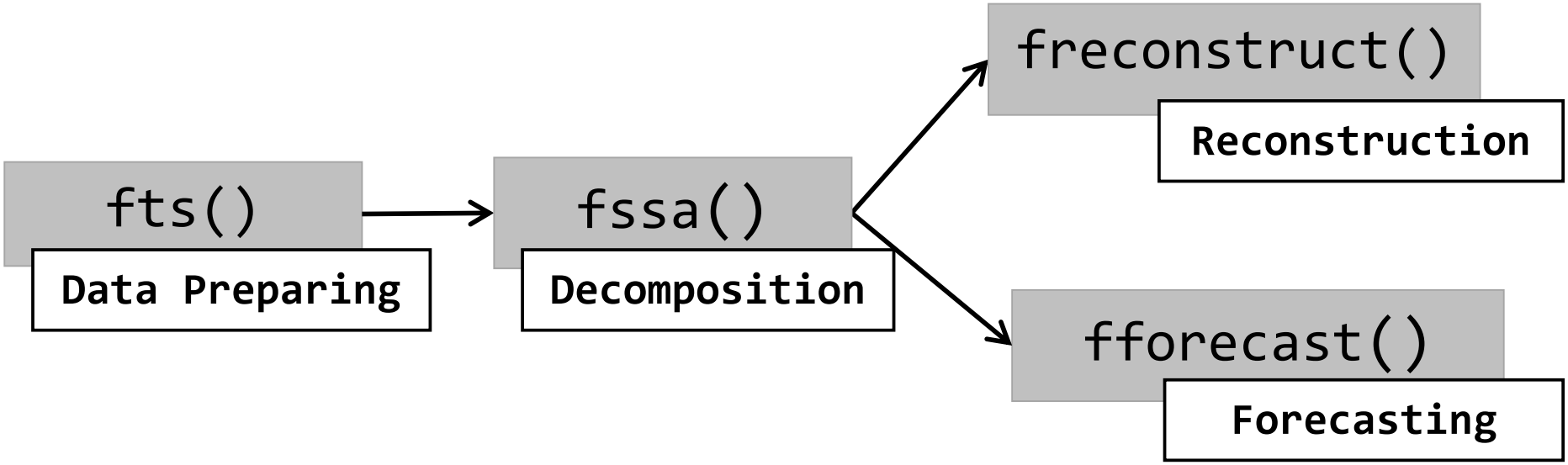


What do I want to cover today?

- **Rfssa** package in CRAN



Remote Sensing Data

- Useful for tracking changes in vegetation in low population areas (Panuju et al., 2012).
- NASA's MODerate-resolution Imaging Spectroradiometer (MODIS) satellite provides images of different regions (Tuck et al., 2014).
- Normalized Difference Vegetation Index (NDVI) is a type of data value recorded in the images which is bounded between zero and one (Tuck et al., 2014).

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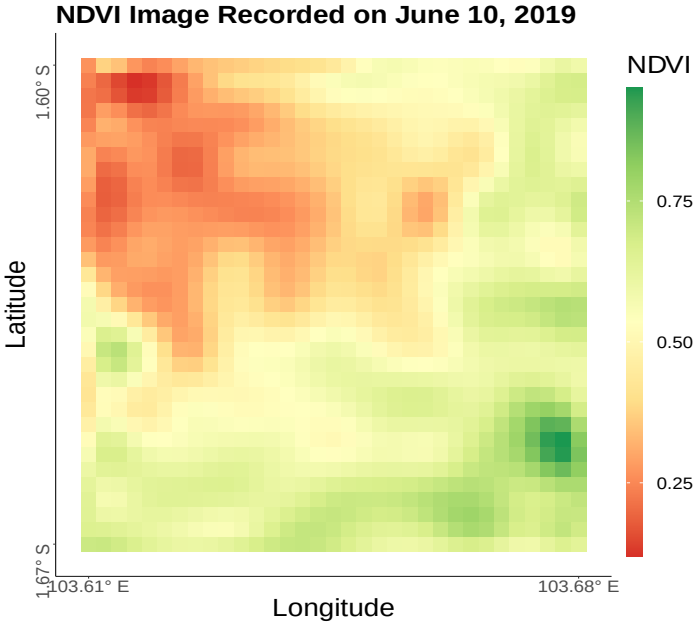

Figure: Remote Sensing Data Courtesy of modis.ornl.gov

- Resolution of images is 250 meters.

RSD cont.

- Many studies have used spectral NDVI data to remotely track changes in vegetation over time (Lambin, 1999).
- The maximum of NDVI values has been used to form a time series that was analyzed using an X12-ARIMA model to identify trend and seasonal changes in woody vegetation (Panuju et al., 2012).
- Temporal average and temporal variability of NDVI values have also been used as explanatory variables for the number of different types of vegetation present in many regions (Tuck et al., 2014).

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Functional Time Series (FTS)

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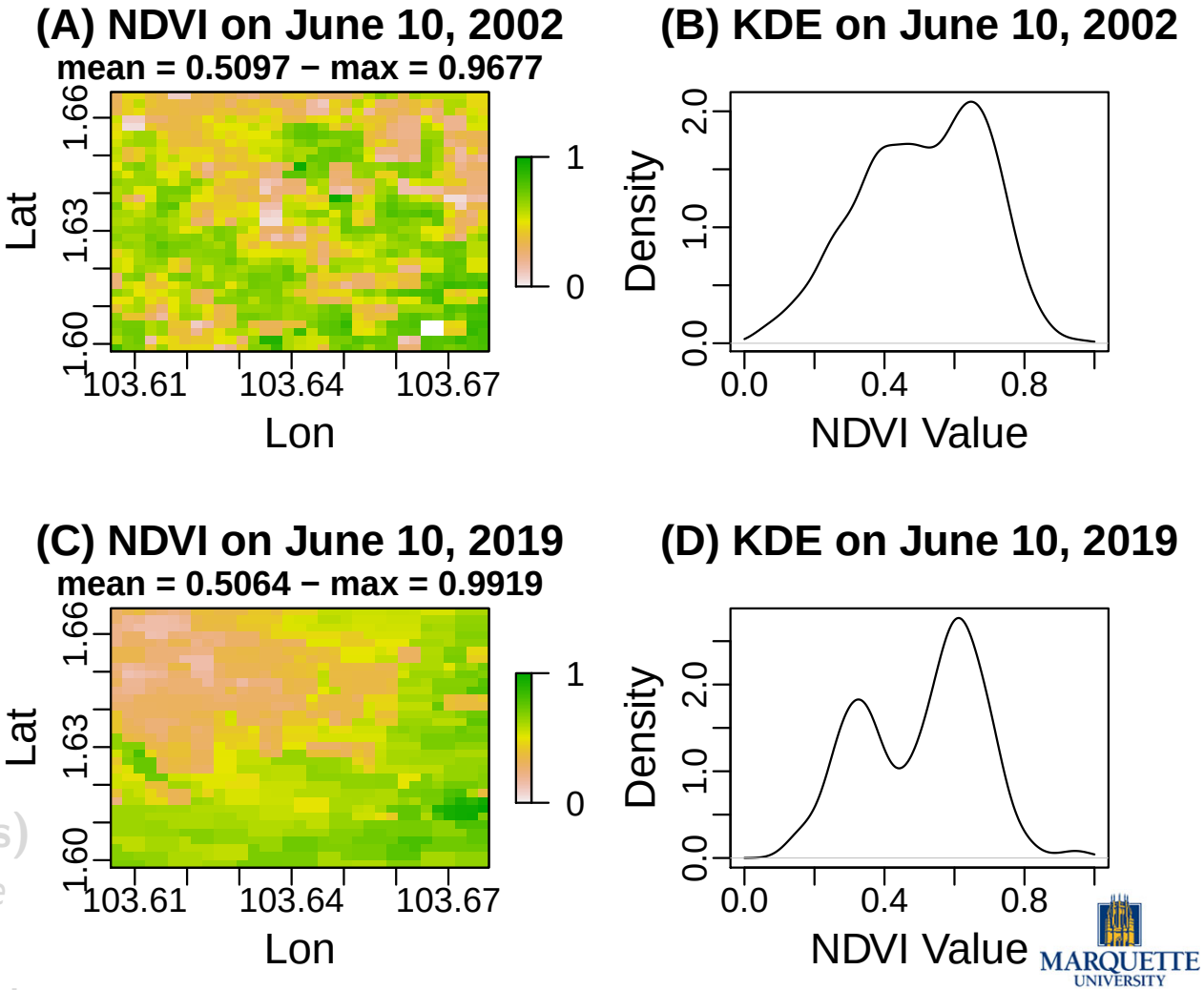

Figure: Densities of NDVI

- Two **images (or densities)** might have similar *sample means* but *different distributions* and as such, images (or densities) are more *informative*.

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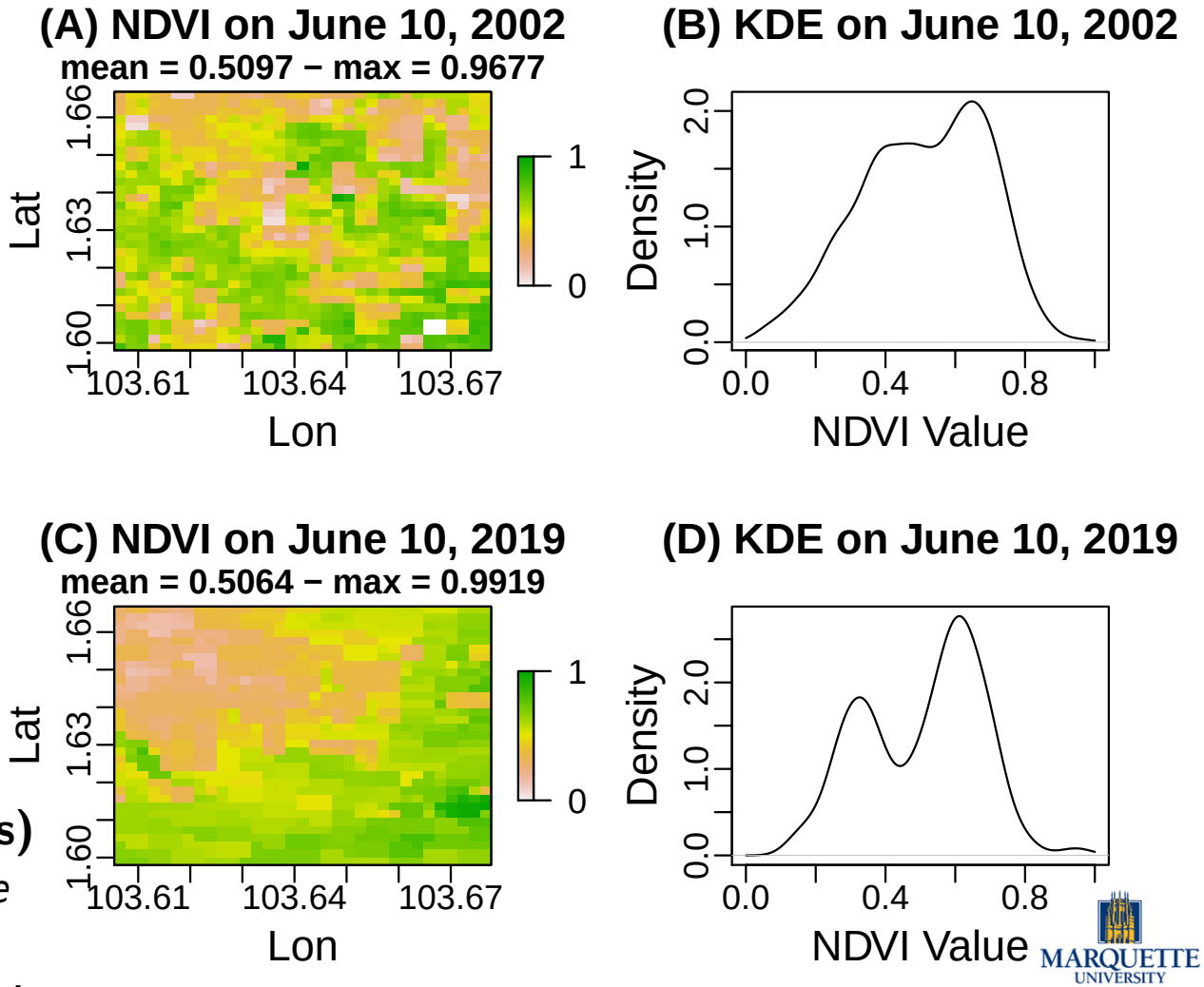

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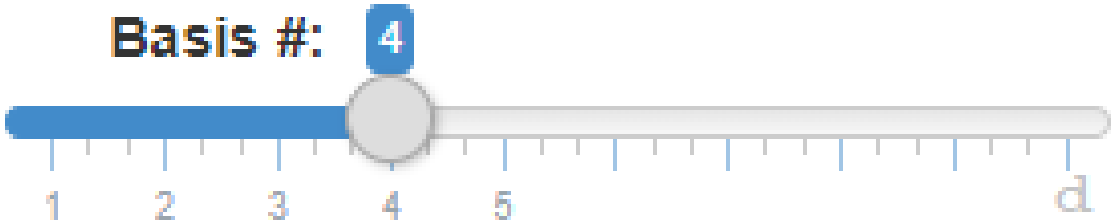
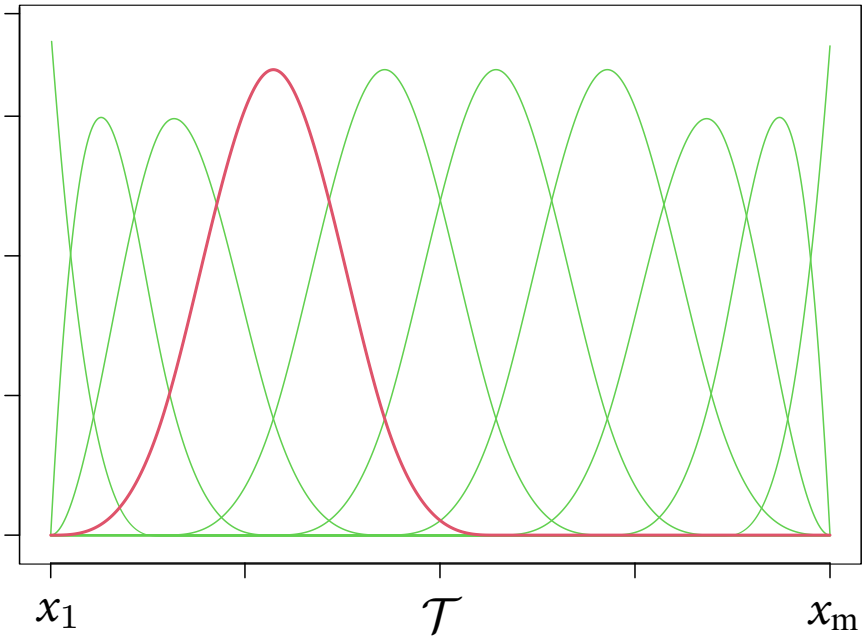
How to represent the FTS as an S4 Object?





As `Basis` object:

`@grid:`	`vector`	$(x_1, x_2, \ldots, x_m)$
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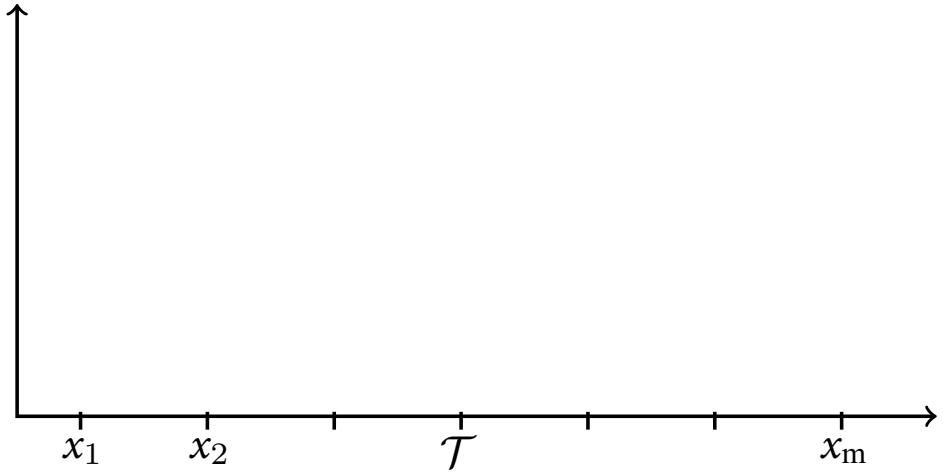

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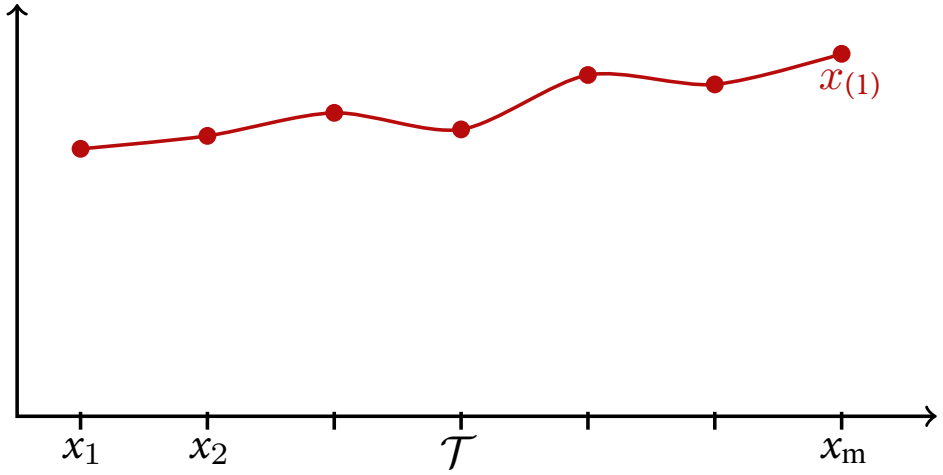

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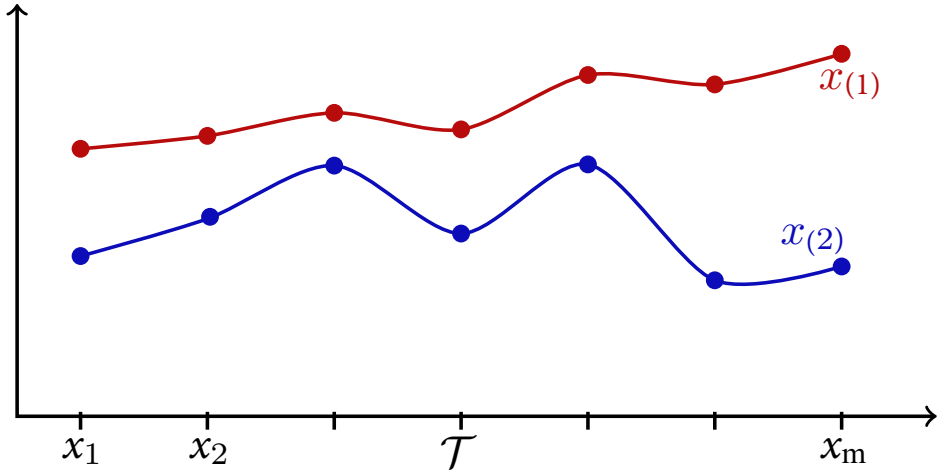

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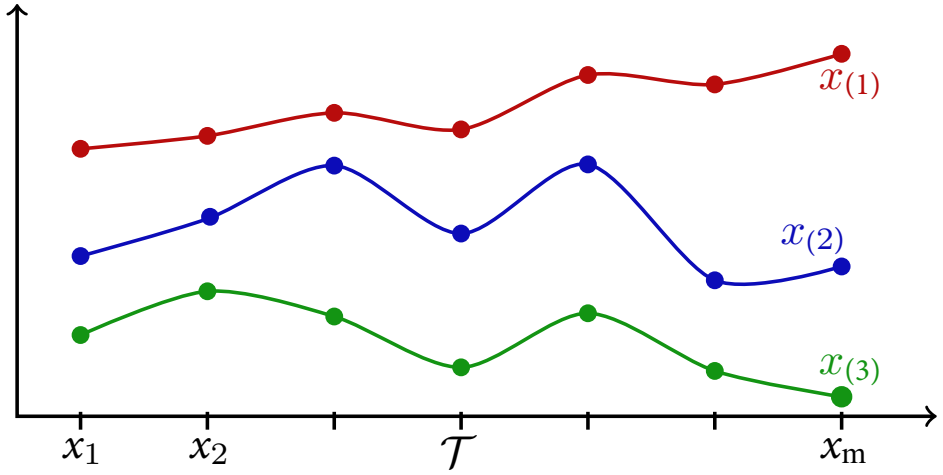

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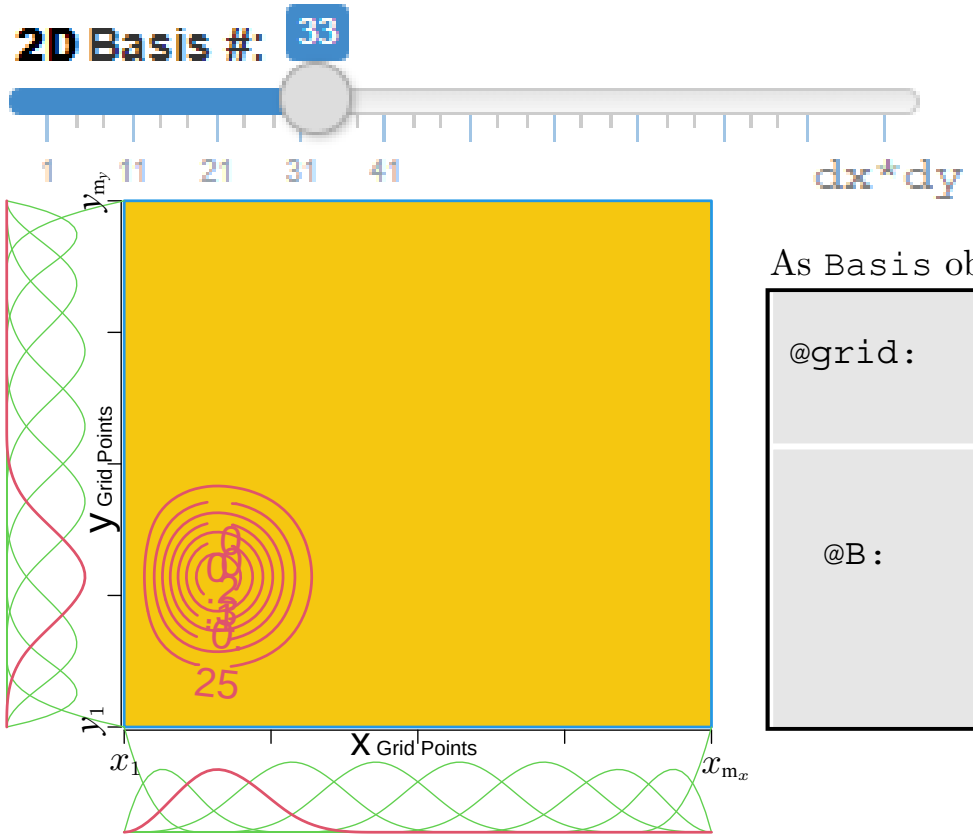

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`@B:`	`matrix`	$\begin{pmatrix} b_{11} & \ldots & b_{dx,1} & \ldots & b_{đ1} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ b_{1\tilde{m}} & \ldots & b_{dx,\tilde{m}} & \ldots & b_{đ\tilde{m}} \end{pmatrix}$ $đ = d_x * d_y$

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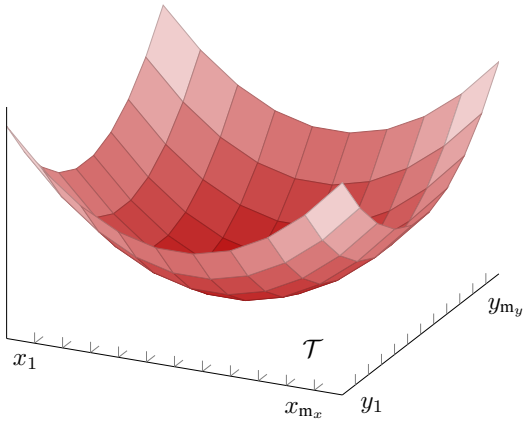

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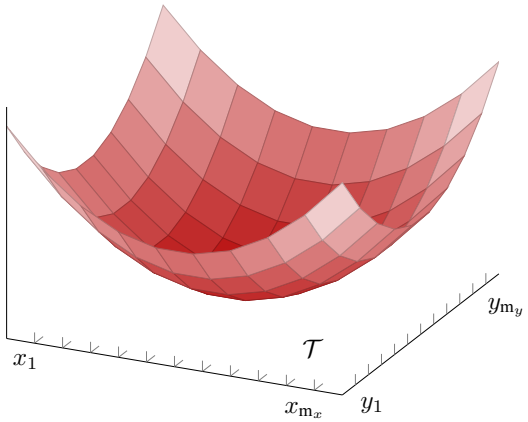

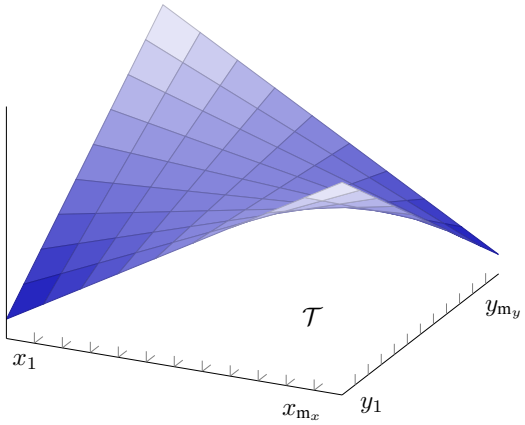

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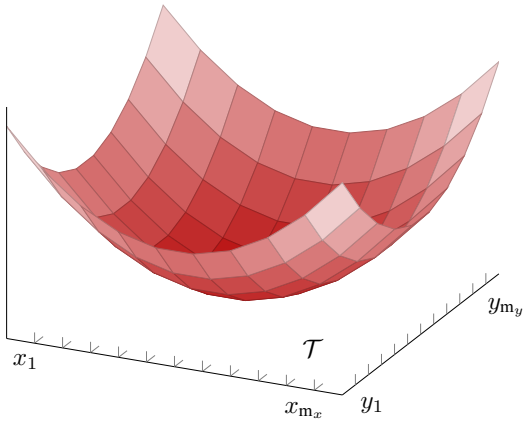

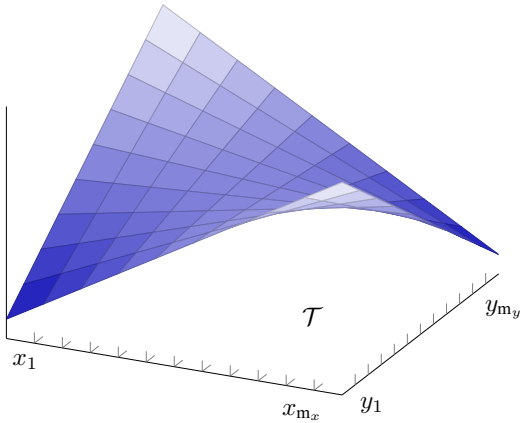

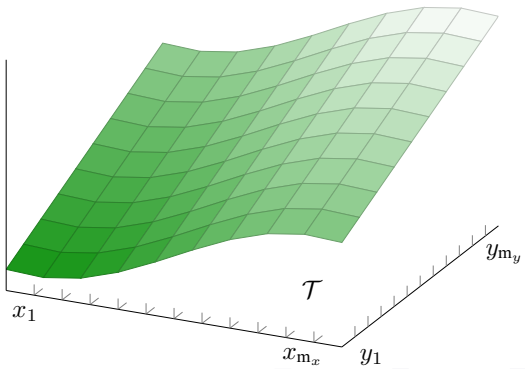

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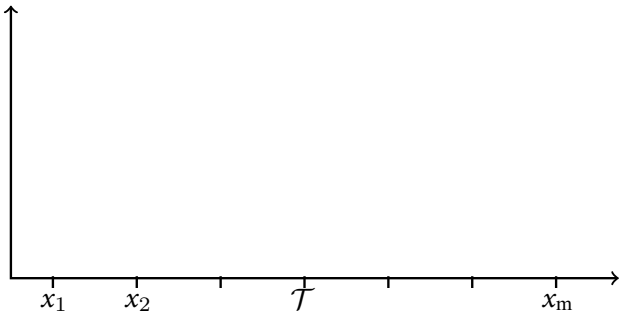

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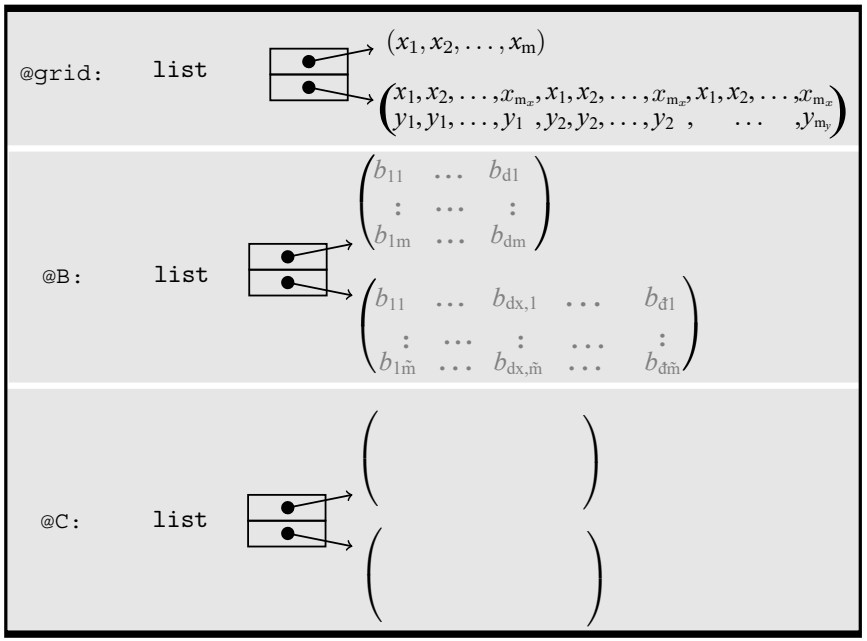

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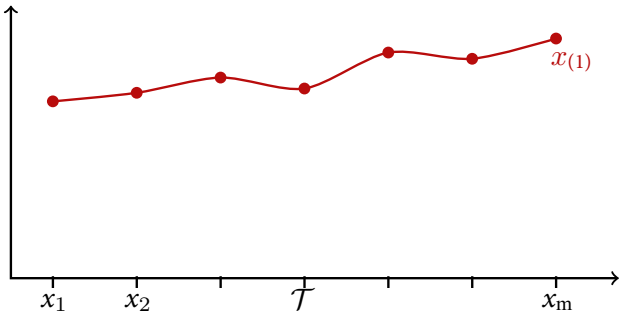

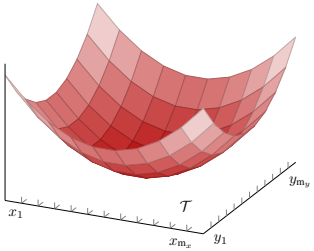

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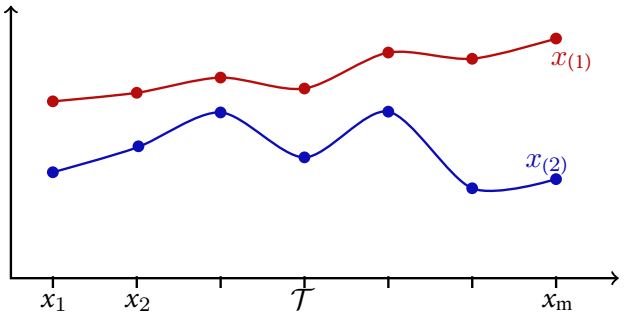

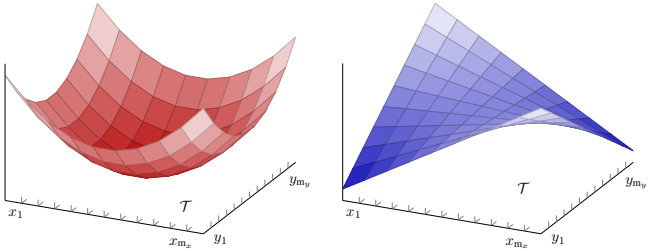

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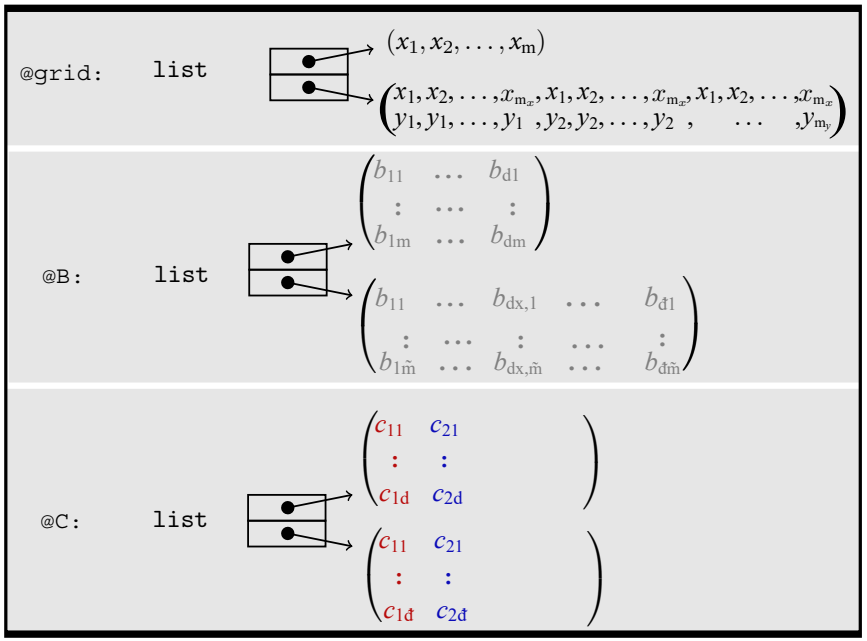

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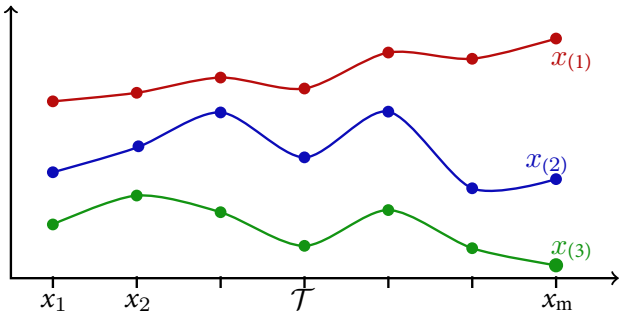

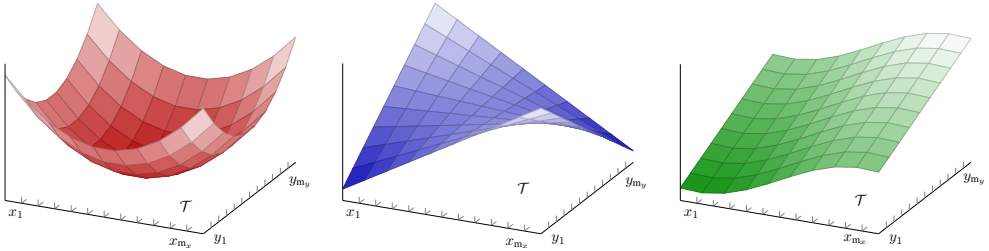

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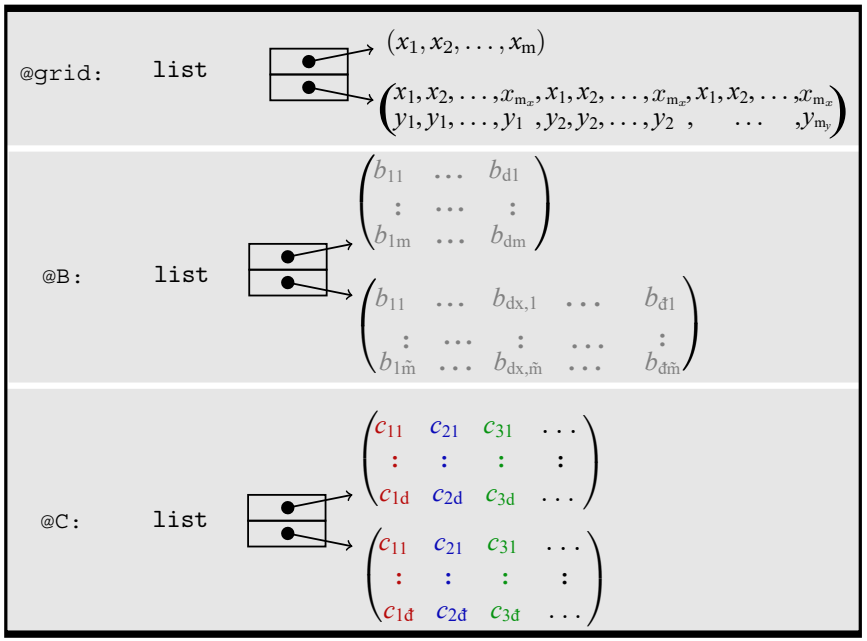
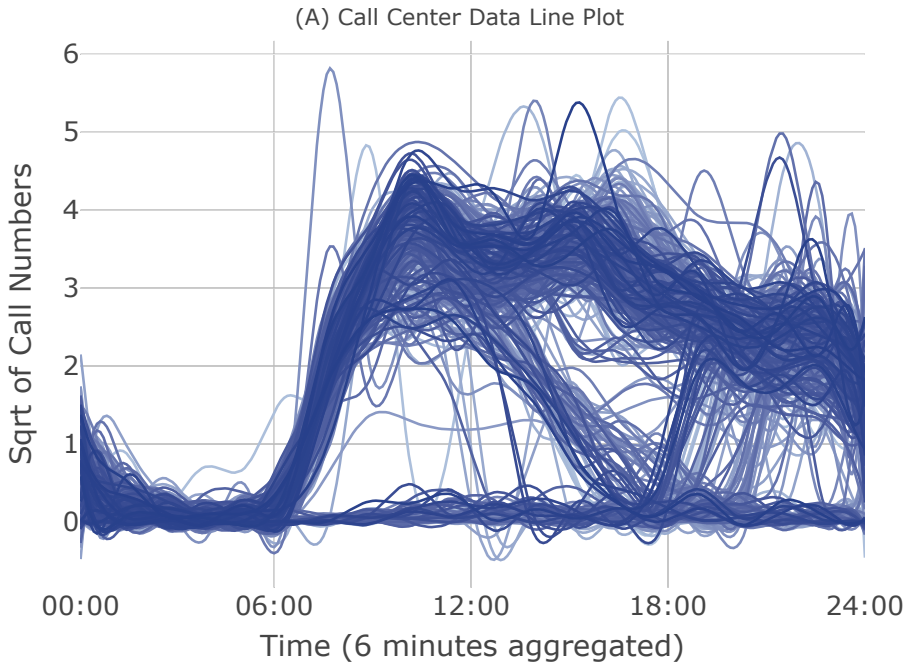


Another FTS Example: Call Center Data

- Record a curve everyday that gives the number of calls, in a six minute interval, to a call center



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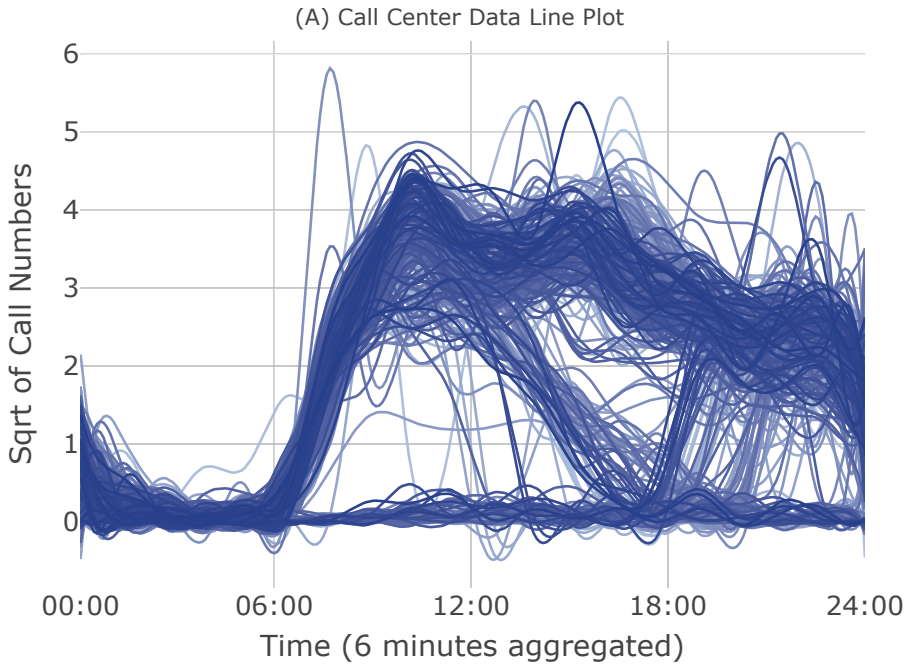

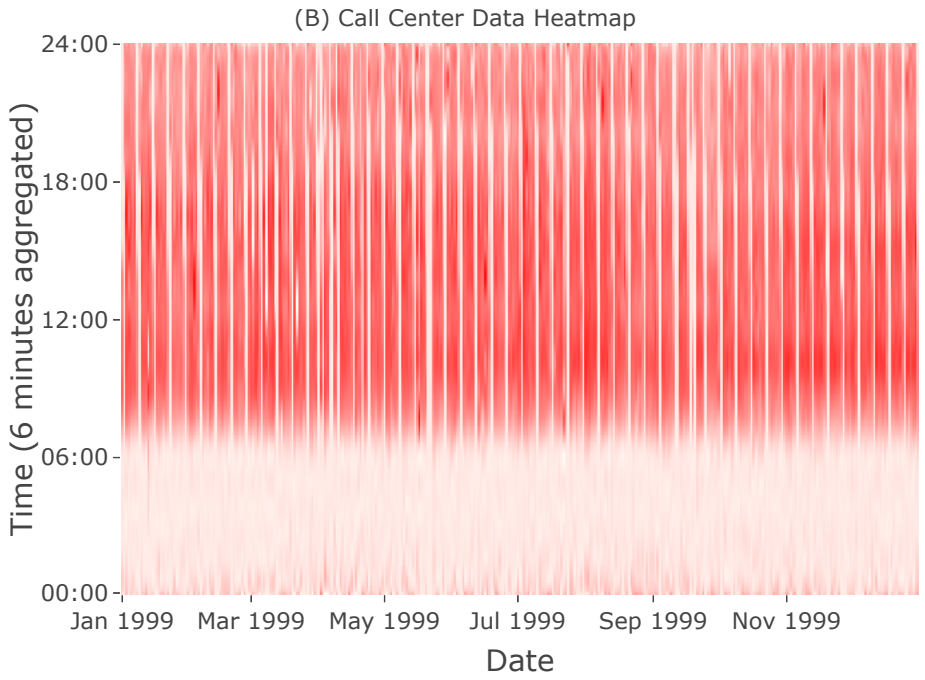

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Singular Spectrum Analysis Applied to NDVI Data

- Singular spectrum analysis (SSA) is a nonparametric, exploratory technique that can be used to identify trend and periodic components present in time series data (Golyandina et al., 2001).

- The algorithm has four steps of:
  1. **Embedding** - Form a trajectory matrix from time series data.
  2. **Decomposition** - Use singular value decomposition (SVD) to decompose the trajectory matrix into rank one elementary matrices.
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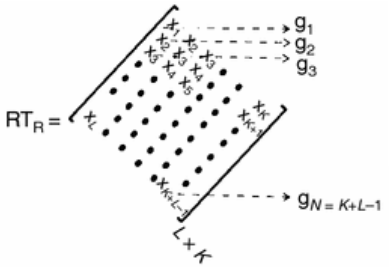

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Rfssa R Package Main Functionalities

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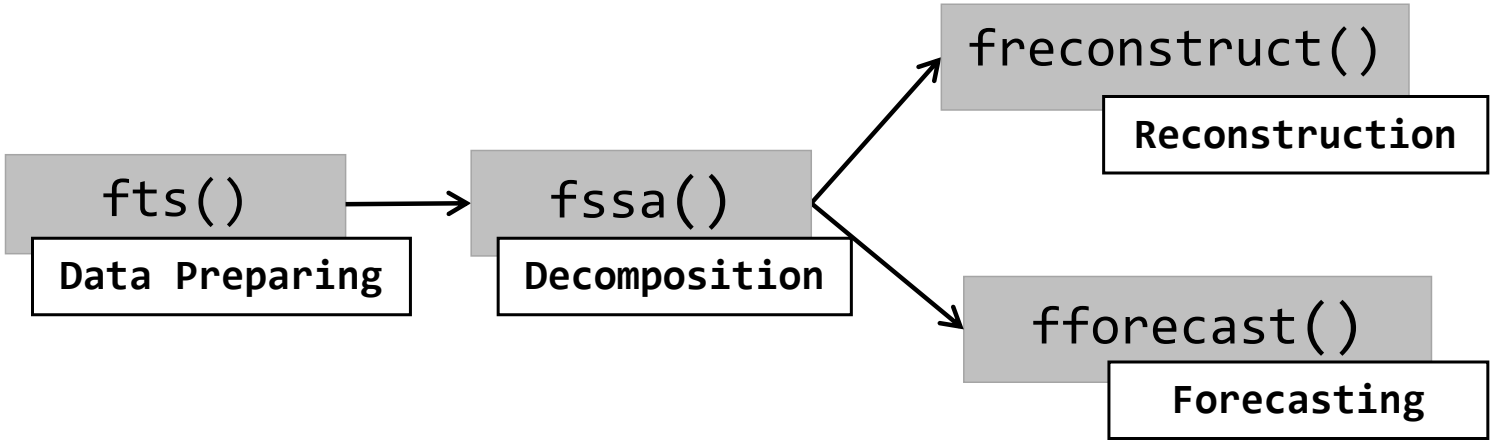

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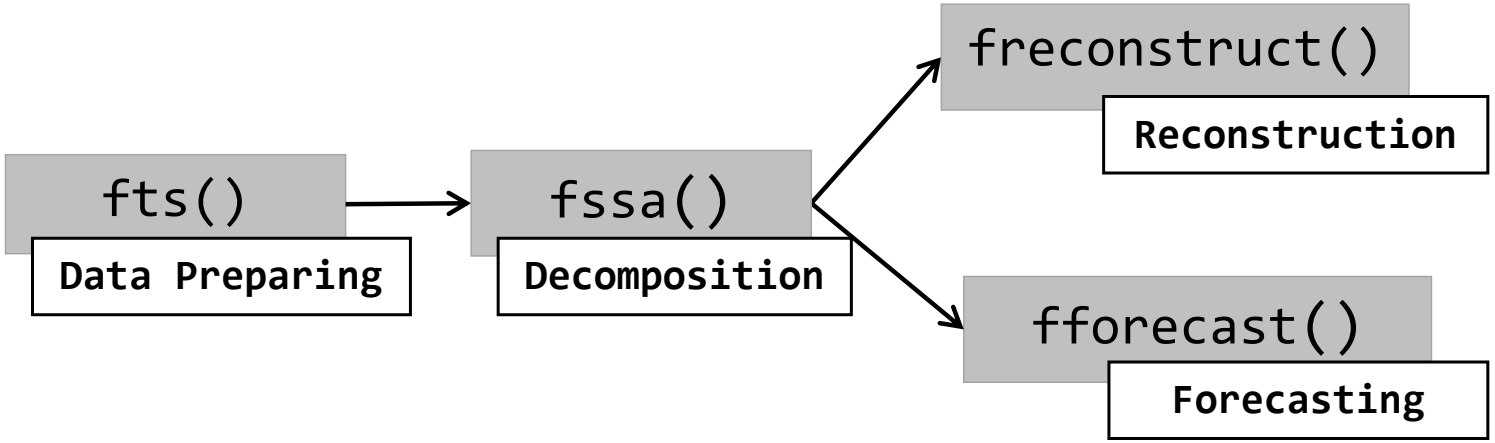

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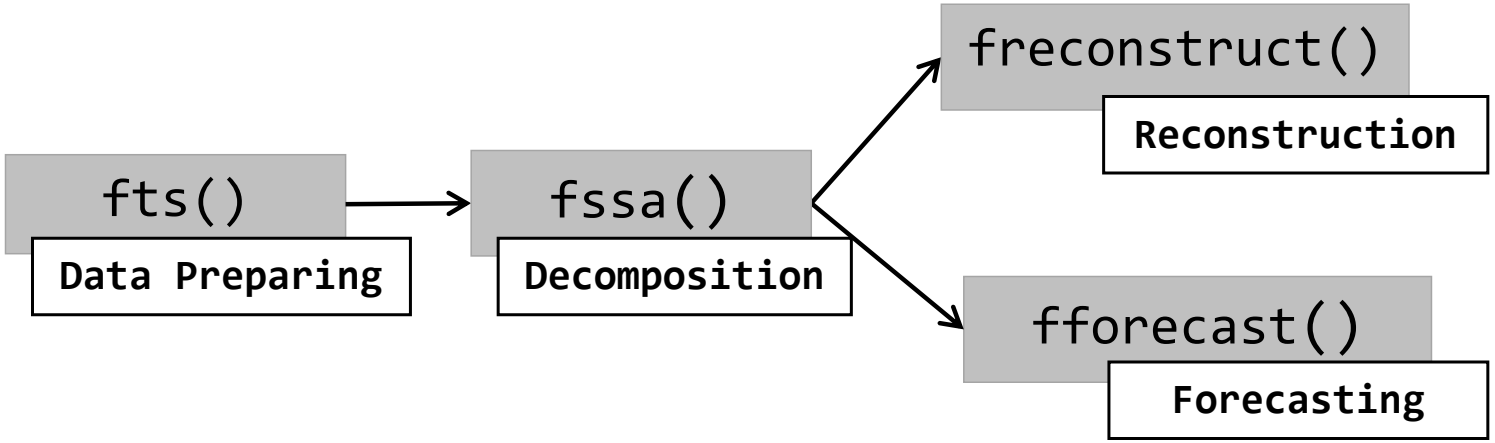

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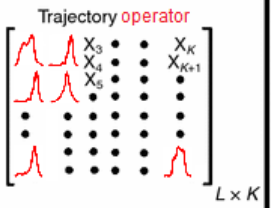

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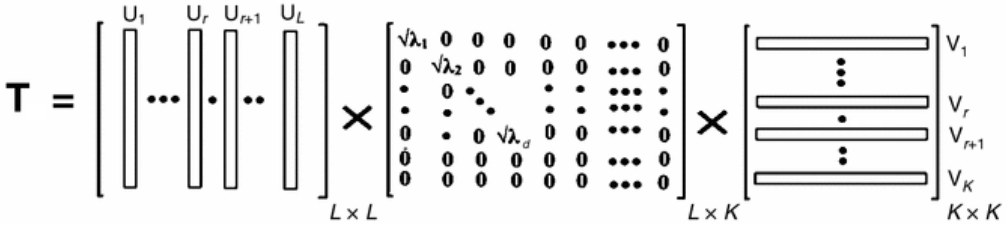

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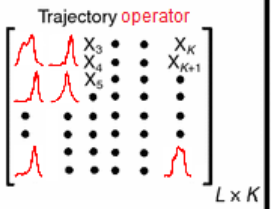

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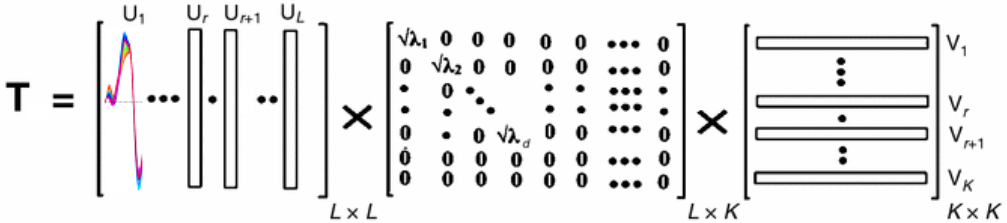

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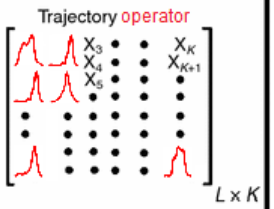

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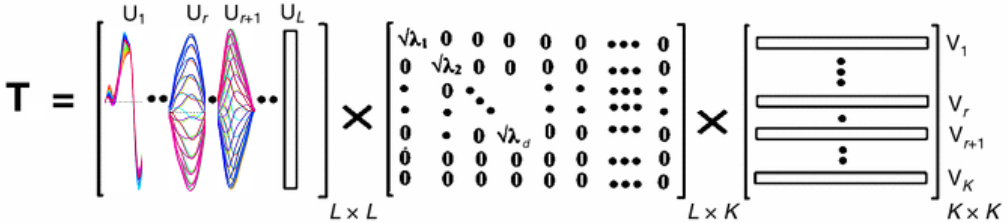

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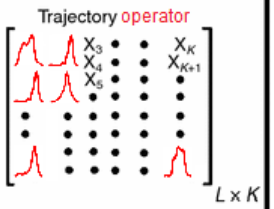

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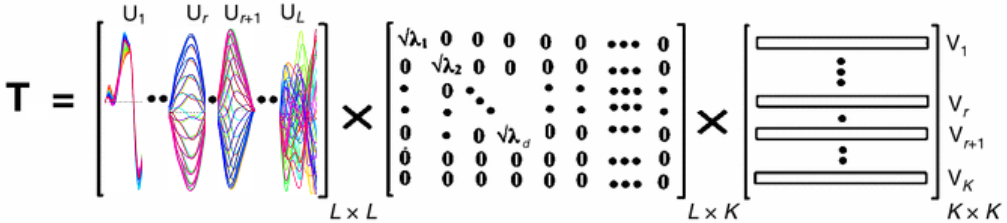

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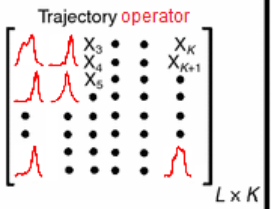

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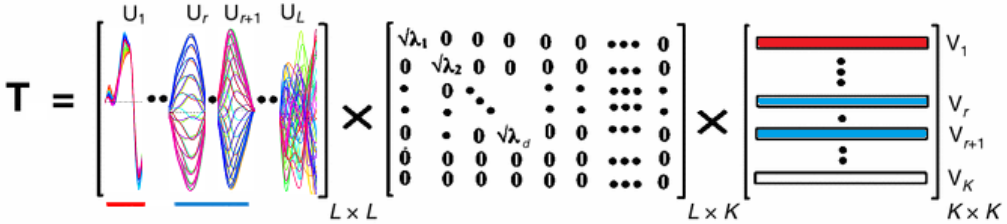

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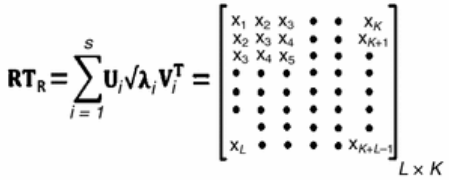

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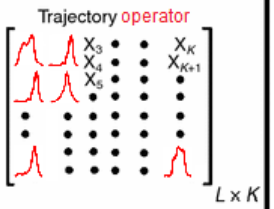

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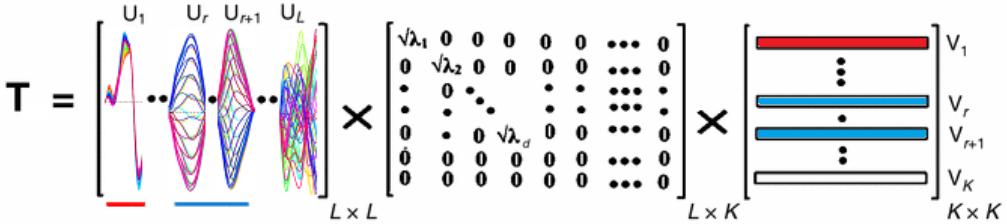

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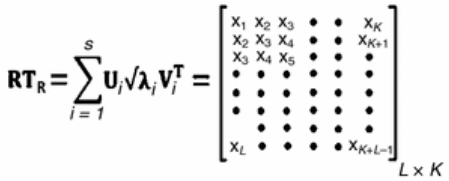

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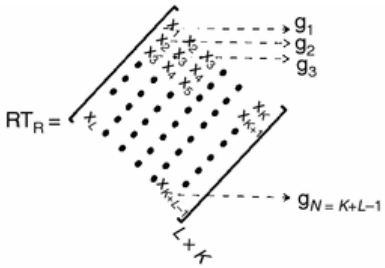

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FSSA Theory

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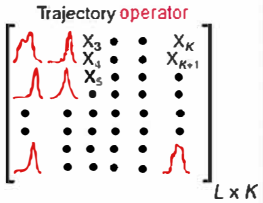
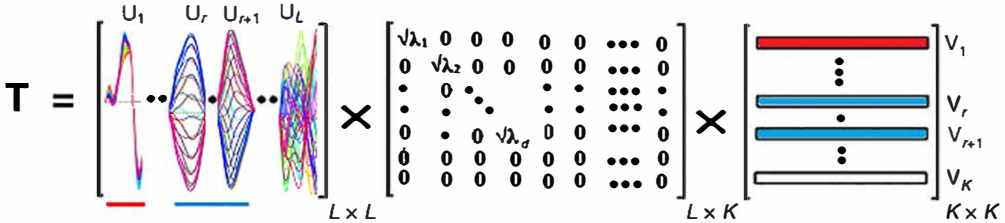
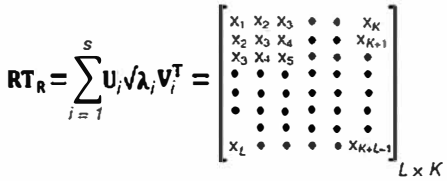


2. Decompose the trajectory operator by functional Singular Value Decomposition (FSVD)

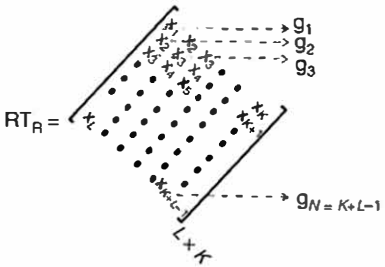


3. Grouping the eigentriples for filtered signal reconstruction



4. Averaging the elements of reconstructed trajectory operator along anti diagonals to get filtered data series represented by

$Y_f = \{g_1, g_2, \ldots g_K, \ldots g_N\}$



U=fssa(Y,L)

U=fssa(Y,L)

freconstruct(U,groups)

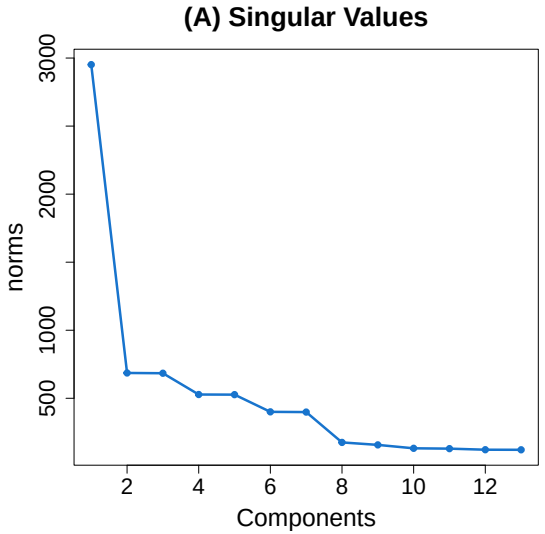
freconstruct(U,groups)

FSSA Theory

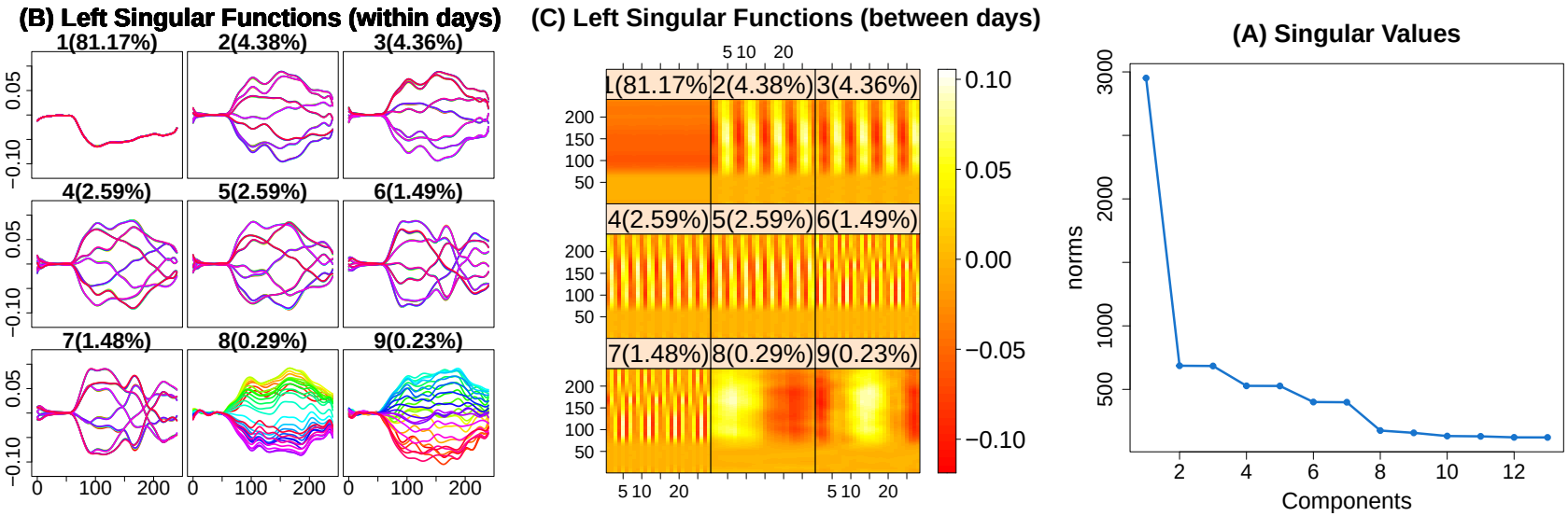
A summary of written functions in the Rfssa package

Function	Descriptions	Main arguments	Returns
`fts`	Create FTS/MFTS objects	`X`: discretely sampled data, `B`: basis specifications, and `grid`: domain of variables.	An object of class `fts`.
`fssa`	Performs the decomposition (including embedding and FSVD steps) stage for FTS/MFTS data.	An object of class `fts` and window length L.	An object of class `fssa`.
`freconstruct`	Performs the reconstruction (including grouping and Hankelization steps) stage.	An object of class `fssa` and list of numeric vectors includes indices of elementary components of a group.	An object of class `fssa`.
`fforecast`	Performs FSSA R-forecasting or FSSA V-forecasting.	An object of class `fssa`, list of numeric vectors includes indices of elementary components of a group used for reconstruction and forecasting and forecast horizon h.	A named list of objects of class `fts` that are reconstructed as according to the specified groups and a numeric vector of eigenvalues.

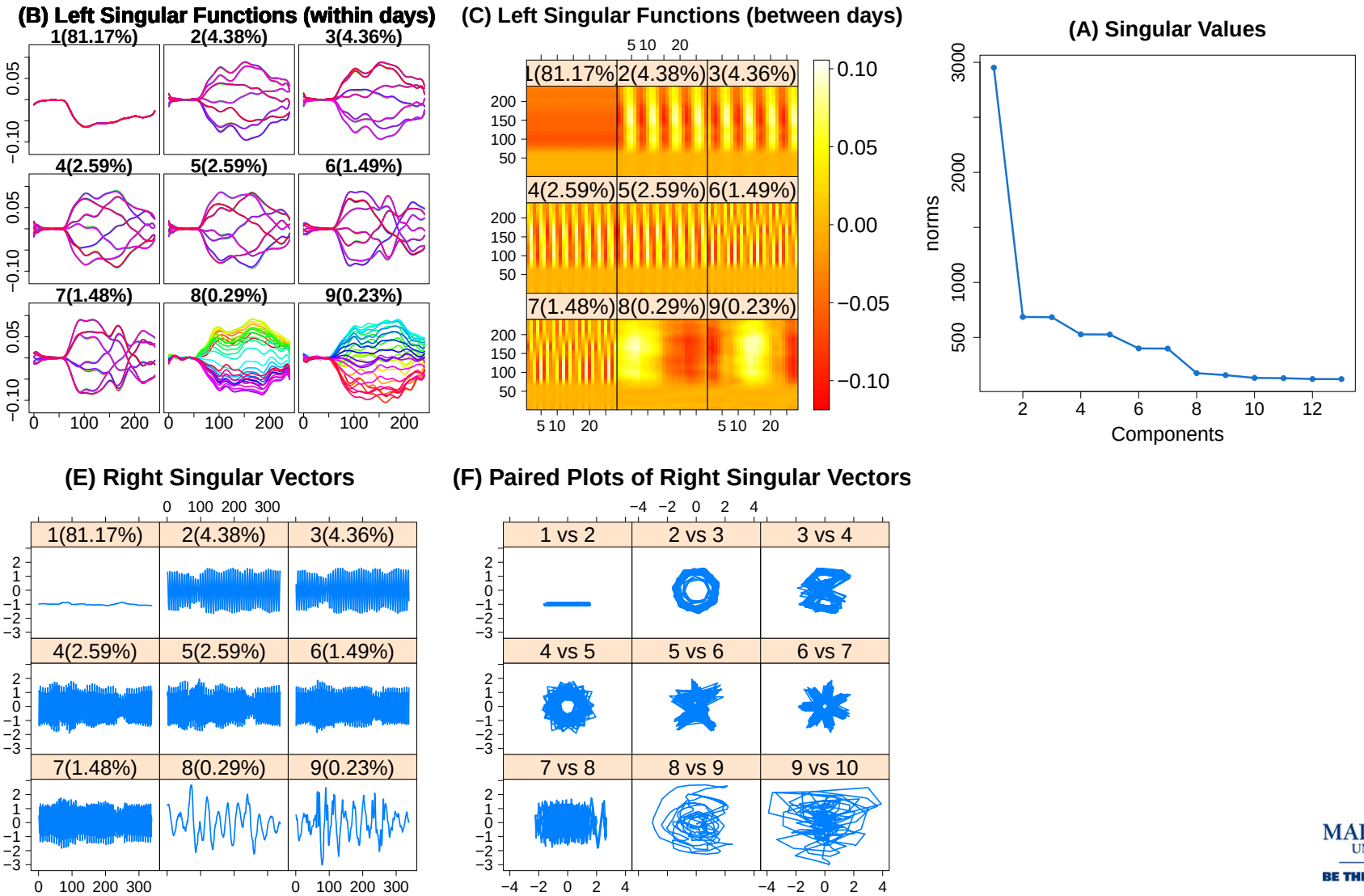
FSSA **Decomposition**: Plotting Methods for the CC Data



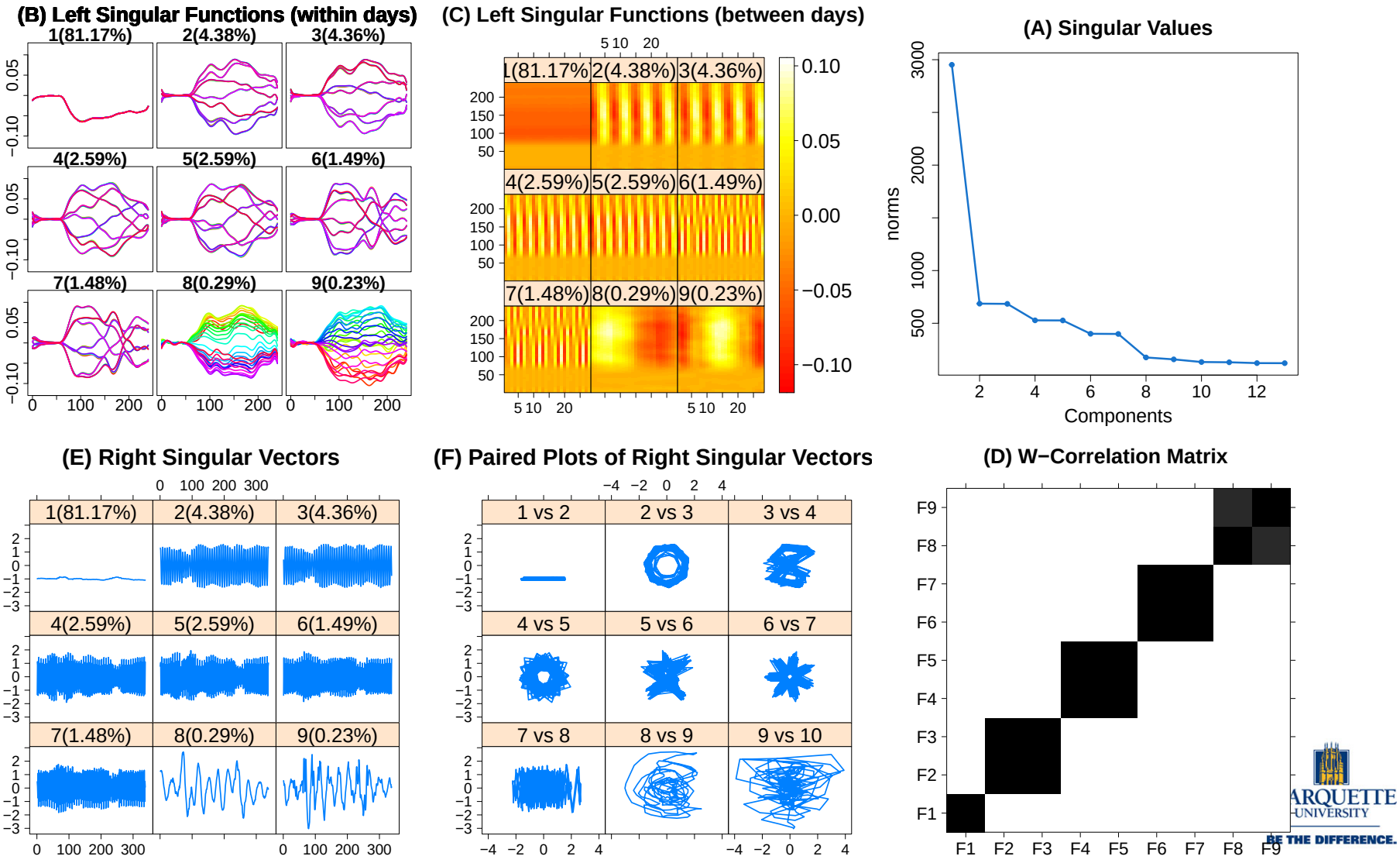
FSSA **Decomposition**: Plotting Methods for the CC Data



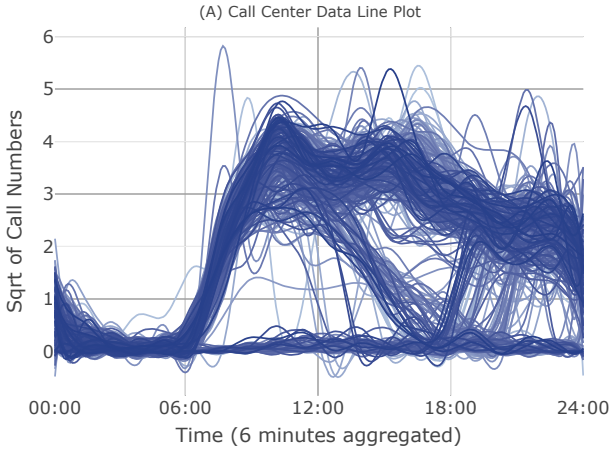
FSSA **Decomposition**: Plotting Methods for the CC Data



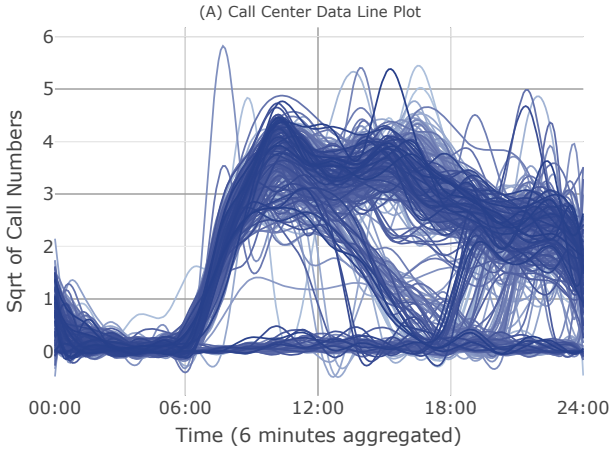
FSSA **Decomposition**: Plotting Methods for the CC Data

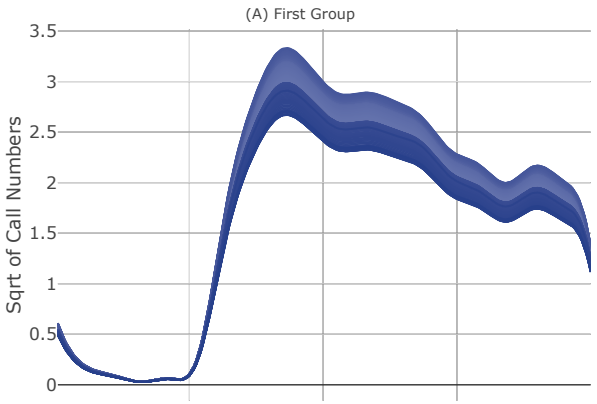


FSSA **Reconstruction**: Results for the Call Center Data

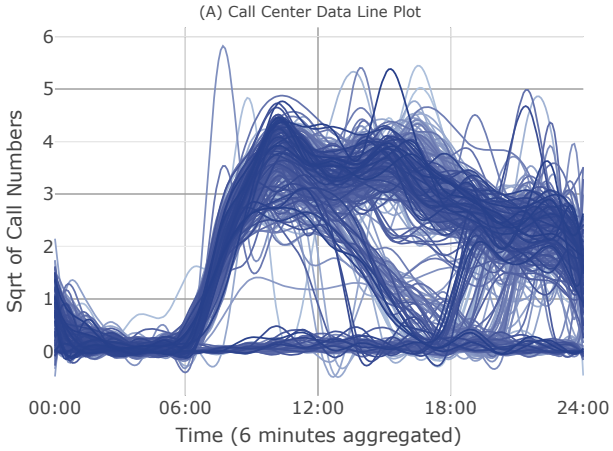


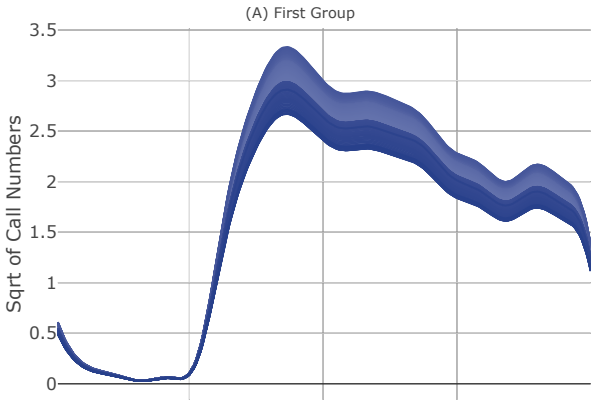
FSSA **Reconstruction**: Results for the Call Center Data

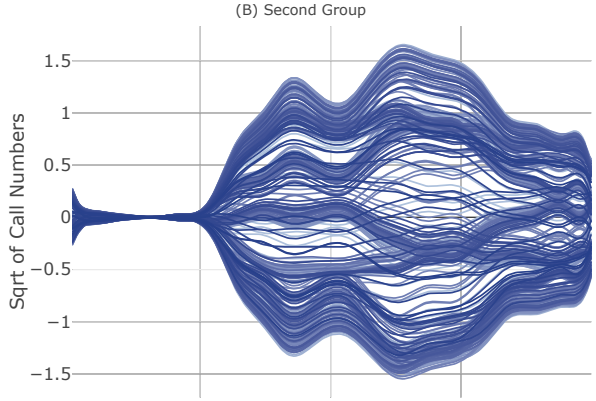


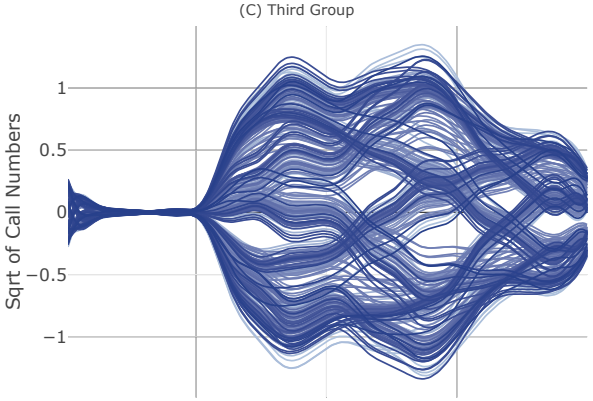


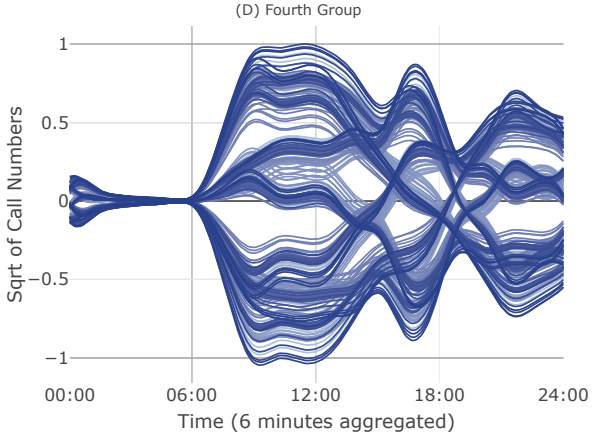
FSSA **Reconstruction**: Results for the Call Center Data



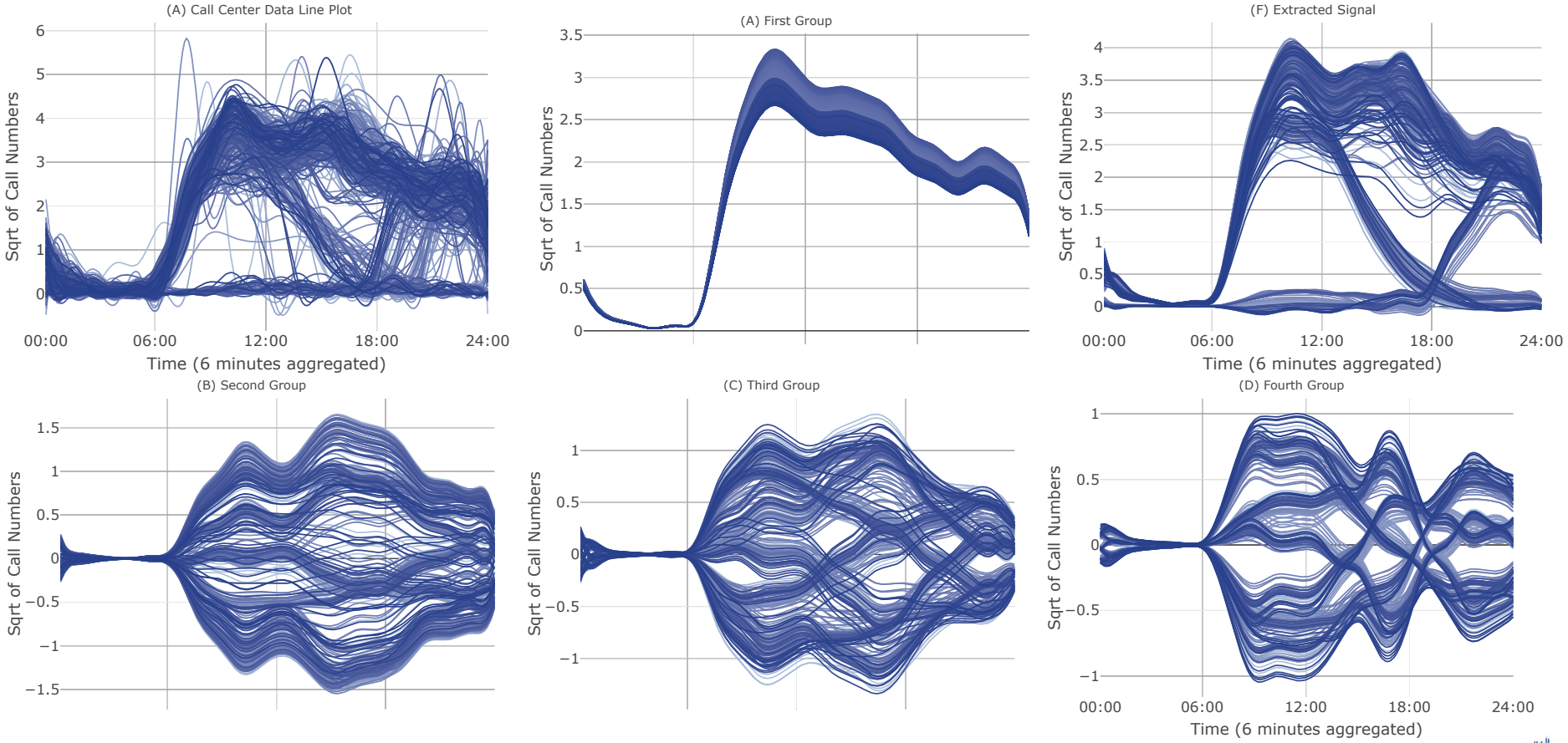








FSSA **Reconstruction**: Results for the Call Center Data



Multivariate SSA (MSSA) in a Nutshell

The multivariate time series of length N represented by $Y = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \dots, \overrightarrow{X_N}\}$ where $\overrightarrow{X_i} = \{X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(p)}\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory matrix with pK lagged vectors of Y as given by $\mathbf{T_H}$.

$$\mathbf{T_H} = [\mathbf{T}^{(1)} \quad \mathbf{T}^{(2)} \quad \mathbf{T}^{(3)} \quad . \quad . \quad \mathbf{T}^{(p)}],$$

where $$\mathbf{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & . & . & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & . & . & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_L^{(j)} & . & . & . & . & X_N^{(j)} \end{bmatrix}_{L \times K}$$

U=ssa(Y,L,kind='mssa')

Multivariate SSA (MSSA) in a Nutshell

The multivariate time series of length N represented by $Y = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \dots, \overrightarrow{X_N}\}$ where $\overrightarrow{X_i} = \{X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(p)}\}$

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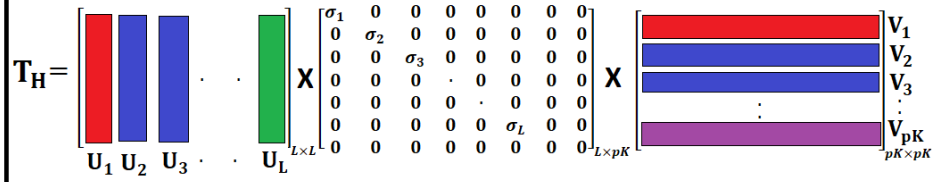
$$\mathrm{T_H} = [\mathrm{T}^{(1)} \quad \mathrm{T}^{(2)} \quad \mathrm{T}^{(3)} \quad \cdot \quad \cdot \quad \mathrm{T}^{(p)}],$$

where

$$\mathrm{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & \cdot & \cdot & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & \cdot & \cdot & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_L^{(j)} & \cdot & \cdot & \cdot & \cdot & X_N^{(j)} \end{bmatrix}_{L \times K}$$

2. Decompose the trajectory matrix using Singular Value Decomposition.



U=ssa(Y,L,kind='mssa')

U=ssa(Y,L,kind='mssa')

Multivariate SSA (MSSA) in a Nutshell

The multivariate time series of length N represented by $Y = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \dots, \overrightarrow{X_N}\}$ where $\overrightarrow{X_i} = \{X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(p)}\}$

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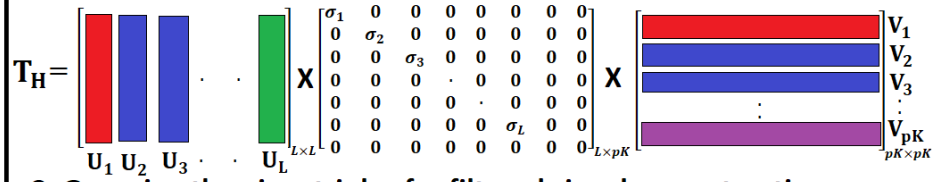
$$\mathrm{T_H} = [\mathrm{T}^{(1)} \quad \mathrm{T}^{(2)} \quad \mathrm{T}^{(3)} \quad \cdot \quad \cdot \quad \mathrm{T}^{(p)}],$$

where

$$\mathrm{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & \cdot & \cdot & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & \cdot & \cdot & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_L^{(j)} & \cdot & \cdot & \cdot & \cdot & X_N^{(j)} \end{bmatrix}_{L \times K}$$

2. Decompose the trajectory matrix using Singular Value Decomposition.



3. Grouping the eigentriples for filtered signal reconstruction

$$\mathrm{RT_{R_H}} = \sum_{i=1}^{s} \sigma_i \mathrm{U_i V_i^T}$$

U=ssa(Y,L,kind='mssa')

U=ssa(Y,L,kind='mssa')

reconstruct(U,groups)

Multivariate SSA (MSSA) in a Nutshell

The multivariate time series of length N represented by $Y = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \dots, \overrightarrow{X_N}\}$ where $\overrightarrow{X_i} = \{X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(p)}\}$

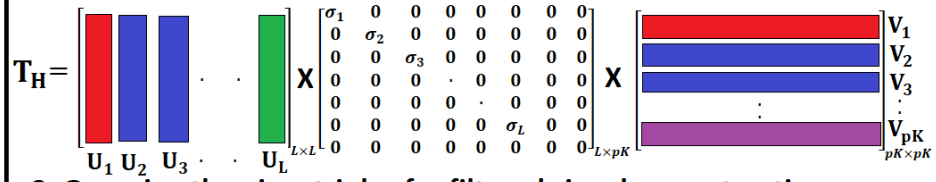
1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory matrix with pK lagged vectors of Y as given by $\mathrm{T_H}$.

$$\mathrm{T_H} = [\mathrm{T}^{(1)} \quad \mathrm{T}^{(2)} \quad \mathrm{T}^{(3)} \quad . \quad . \quad \mathrm{T}^{(p)}],$$

where

$$\mathrm{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & . & . & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & . & . & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_L^{(j)} & . & . & . & . & X_N^{(j)} \end{bmatrix}_{L\times K}$$

2. Decompose the trajectory matrix using Singular Value Decomposition.



3. Grouping the eigentriples for filtered signal reconstruction

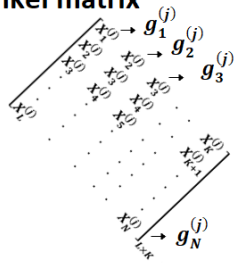
$$\mathrm{RT_{R_H}} = \sum_{i=1}^{s} \sigma_i \mathrm{U_i V_i^T}$$

4. Blockwise averaging the elements of the reconstruct trajectory matrix along antidiagonals gives a block Hankel matrix

$$Y_f = \{\overrightarrow{g_1}, \overrightarrow{g_2}, \overrightarrow{g_3}, \dots, \overrightarrow{g_N}\}$$

where $\overrightarrow{g_i} = \{g_i^{(1)}, g_i^{(2)}, \dots, g_i^{(p)}\}$

$\mathrm{RT}_{\mathrm{R_H}}^{(j)} =$



U=ssa(Y,L,kind='mssa')

U=ssa(Y,L,kind='mssa')

reconstruct(U,groups)

reconstruct(U,groups)

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Multivariate FSSA (MFSSA) in a Nutshell

The multivariate time series of length N represented by $Y = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \dots, \overrightarrow{X_N}\}$ where $\overrightarrow{X_i} = \{X_i^{(1)}, X_i^{(2)}, \dots, X_i^{(p)}\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory matrix with pK lagged vectors of Y as given by $\mathbf{T_H}$.

$$\mathbf{T_H} = [\mathbf{T}^{(1)} \quad \mathbf{T}^{(2)} \quad \mathbf{T}^{(3)} \quad . \quad . \quad \mathbf{T}^{(p)}],$$

where $$\mathbf{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & . & . & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & . & . & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_L^{(j)} & . & . & . & . & X_N^{(j)} \end{bmatrix}_{L \times K}$$

U=ssa(Y,L,kind='mssa')

Multivariate FSSA (MFSSA) in a Nutshell

The multivariate functional time series of length N represented by $Y = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \dots, \overrightarrow{X_N}\}$ where $\overrightarrow{X_i} = \{$, $, \dots, X_i^{(p)}\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory operator with K lagged vectors of Y as given by $\mathbf{T_V}$.

$$\mathbf{T_H} = [\mathbf{T}^{(1)} \quad \cancel{\mathbf{T}^{(2)} \quad \mathbf{T}^{(3)}} \quad . \quad . \quad \mathbf{T}^{(p)}],$$

$$\mathbf{T_V} = \begin{bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \\ . \\ . \\ . \\ \mathbf{T}^{(p)} \end{bmatrix} \begin{matrix} \text{curves} \\ \text{surfaces} \\ . \\ . \\ . \\ . \\ . \end{matrix} \quad \text{where} \quad \mathbf{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & . & . & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & . & . & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_L^{(j)} & . & . & . & . & X_N^{(j)} \end{bmatrix}_{L \times K}$$

U=fssa(Y,L,type='mfssa')

Multivariate FSSA (MFSSA) in a Nutshell

The multivariate functional time series of length N represented by $Y = \{\vec{X_1}, \vec{X_2}, \vec{X_3}, \dots, \vec{X_N}\}$ where $\vec{X_i} = \{$, , $\dots, X_i^{(p)}\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory operator with K lagged vectors of Y as given by $\mathbf{T_V}$.

$$\mathbf{T_H} = [\mathbf{T}^{(1)} \quad \cancel{\mathbf{T}^{(2)} \quad \mathbf{T}^{(3)}} \quad . \quad . \quad \mathbf{T}^{(p)}],$$

$$\mathbf{T_V} = \begin{bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \\ . \\ . \\ \mathbf{T}^{(p)} \end{bmatrix} \begin{matrix} \text{curves} \\ \text{surfaces} \\ . \\ . \\ . \\ . \end{matrix} \quad \text{where} \quad \mathbf{T}^{(1)} = \begin{bmatrix} \text{ } & \text{} & \text{} & . & . & X_K^{(1)} \\ \text{} & \text{} & X_4^{(1)} & . & . & X_{K+1}^{(1)} \\ \text{} & X_4^{(1)} & X_5^{(1)} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_L^{(1)} & . & . & . & . & X_N^{(1)} \end{bmatrix}_{L \times K}$$

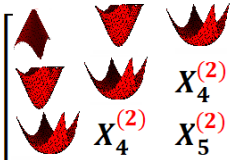
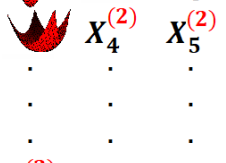
U=fssa(Y,L,type='mfssa')

Multivariate FSSA (MFSSA) in a Nutshell

The multivariate functional time series of length N represented by $Y = \{\vec{X_1}, \vec{X_2}, \vec{X_3}, \dots, \vec{X_N}\}$ where $\vec{X_i} = \{$ $, \dots, X_i^{(p)}\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory operator with K lagged vectors of Y as given by $\mathbf{T_V}$.

$$\mathbf{T_H} = [\mathbf{T}^{(1)} \quad \cancel{\mathbf{T}^{(2)} \quad \mathbf{T}^{(3)}} \quad . \quad . \quad \mathbf{T}^{(p)}],$$

$$\mathbf{T_V} = \begin{bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \\ . \\ . \\ \mathbf{T}^{(p)} \end{bmatrix} \begin{matrix} \text{curves} \\ \text{surfaces} \\ . \\ . \\ . \\ . \end{matrix} \quad \text{where} \quad \mathbf{T}^{(2)} = \begin{bmatrix} \text{  } & X_K^{(2)} \\ \text{} & X_{K+1}^{(2)} \\ \text{  } & \cdot \\ \cdots & \cdots \\ X_L^{(2)} \cdots X_N^{(2)} & \end{bmatrix}_{L \times K}$$

U=fssa(Y,L,type='mfssa')

Multivariate FSSA (MFSSA) in a Nutshell

The multivariate functional time series of length N represented by $Y = \{\vec{X_1}, \vec{X_2}, \vec{X_3}, \dots, \vec{X_N}\}$ where $\vec{X_i} = \{$ $, \dots, X_i^{(p)}\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. We form the block Hankel trajectory operator with K lagged vectors of Y as given by $\mathbf{T_V}$.

$$\mathbf{T_H} = [\mathbf{T}^{(1)} \quad \cancel{\mathbf{T}^{(2)} \quad \mathbf{T}^{(3)}} \quad . \quad . \quad \mathbf{T}^{(p)}],$$

$$\mathbf{T_V} = \begin{bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \\ . \\ . \\ . \\ \mathbf{T}^{(p)} \end{bmatrix} \begin{matrix} \text{curves} \\ \text{surfaces} \\ . \\ . \\ . \\ . \\ . \end{matrix} \quad \text{where} \quad \mathbf{T}^{(j)} = \begin{bmatrix} X_1^{(j)} & X_2^{(j)} & X_3^{(j)} & . & . & X_K^{(j)} \\ X_2^{(j)} & X_3^{(j)} & X_4^{(j)} & . & . & X_{K+1}^{(j)} \\ X_3^{(j)} & X_4^{(j)} & X_5^{(j)} & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ X_L^{(j)} & . & . & . & . & X_N^{(j)} \end{bmatrix}_{L \times K}$$

2. Decompose the trajectory matrix using Singular Value Decomposition.

$$\mathbf{T_H} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 & . & . & \mathbf{U}_L \end{bmatrix}_{L \times L} \mathbf{X} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{L \times pK} \mathbf{X} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_{pK} \end{bmatrix}_{pK \times pK}$$

U=fssa(Y,L,type='mfssa')

U=ssa(Y,L,kind='mssa')

Multivariate FSSA (MFSSA) in a Nutshell

The multivariate functional time series of length N represented by $Y = \{\vec{X_1}, \vec{X_2}, \vec{X_3}, \dots, \vec{X_N}\}$ where $\vec{X_i} = \{$ $, \dots, X_i^{(p)}\}$

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2. Decompose the trajectory operator using fSVD

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$$RT_{R_H} = \sum_{i=1}^{s} \sigma_i U_i V_i^T$$

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$$Y_f = \{\overrightarrow{g_1}, \overrightarrow{g_2}, \overrightarrow{g_3}, \dots, \overrightarrow{g_N}\}$$

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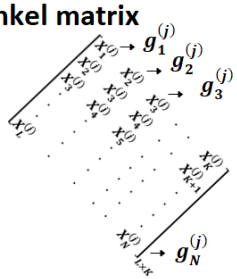

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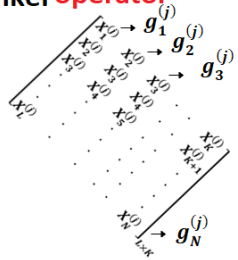
$RT_{R_V}^{(j)} =$ 

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(Trinka et al., 2022)

FSSA **Recurrent Forecasting** Algorithm

- The main idea behind the SSA recurrent forecasting algorithm is to express the next observation in the TS $\mathbf{g}_{N+M}$ as a weighted linear combination of the previous $L-1$ observations where the scalars a_j perform the weighting.

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  Under some regularity conditions we have the following:

Algorithm A - *SSA Recurrent Forecasting:*

$$g_i = \begin{cases} \tilde{y}_i & i = 1, \ldots, N, \\ \sum_{j=1}^{L-1} a_j g_{i+j-L} & i = N+1, \ldots, N+M. \end{cases}$$

For brevity, we call this method the *SSA R-Forecasting algorithm*.

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Under some regularity conditions we have the following:

Algorithm A- *FSSA Recurrent Forecasting (Practical implementation):*

$$\mathbf{c}_{g_i} = \begin{cases} \mathbf{c}_{\tilde{y}_i}, & i = 1, \ldots, N, \\ \sum_{j=1}^{L-1} \mathbf{A}_j \mathbf{c}_{g_{i+j-L}}, & i = N+1, \ldots, N+M. \end{cases}$$

where

$$\mathbf{A}_j = \mathbf{D}\left(\sum_{l=0}^{\infty}\left(\mathbf{D}^\top\mathbf{D}\right)^l\right)\mathbf{E}_j^\top\mathbf{G}^{\frac{1}{2}}, \qquad \text{for } j = 1, \ldots, L-1.$$

FSSA **Vector Forecasting** Algorithm

Algorithm B - *SSA Vector Forecasting:*

1. Define the L-lagged vectors

$$\mathbf{w}_j = \begin{cases} \mathbf{x}_j^{\mathfrak{s}} & j = 1, \ldots, K \\ \mathbf{Q}\mathbf{w}_{j-1} & j = K+1, \ldots, K+M, \end{cases}$$

   where $\mathbf{Q} : \mathbb{R}^L \to \mathbb{R}^L$

2. Form the matrix $\mathbf{W} \in \mathbb{R}^{L \times (K+M)}$ whose range is linearly spanned by the set $\{\mathbf{w}_i\}_{i=1}^{K+M}$.
3. Hankelize the matrix $\mathbf{W}$ in order to extract the TS $\mathbf{g}_{N+M}$.
4. The scalars $g_{N+1}, \cdots, g_{N+M}$ form the M terms of the SSA vector forecast.

We refer to this method as the *SSA V-Forecasting algorithm*.

FSSA **Vector Forecasting** Algorithm

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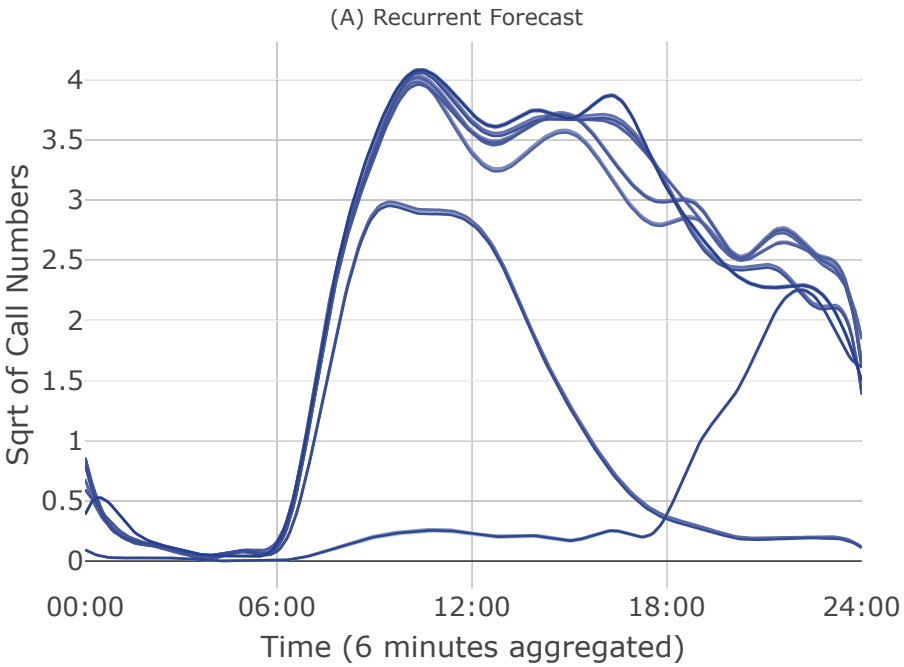
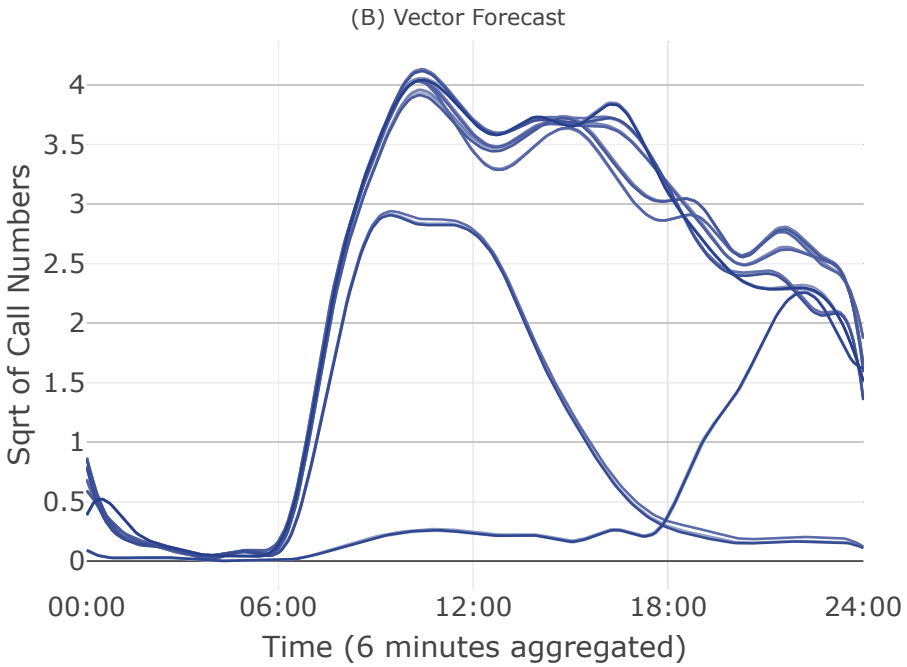
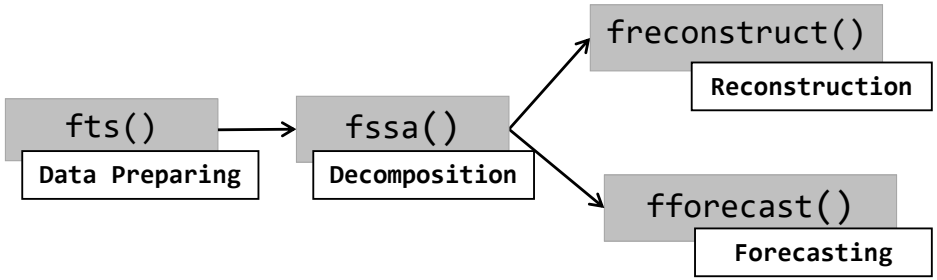
2. Form the operator $\boldsymbol{\mathcal{W}} \in \mathbb{H}^{L \times (K+M)}$ whose range is linearly spanned by the set $\{\boldsymbol{w}_i\}_{i=1}^{K+M}$.
3. Hankelize the operator $\boldsymbol{\mathcal{W}}$ in order to extract the FTS $\mathbf{g}_{N+M}$.
4. The functions $g_{N+1}, \cdots, g_{N+M}$ form the M terms of the FSSA vector forecast.

We refer to this method as the *FSSA V-Forecasting algorithm*.

Forecasting of the Call Center Data







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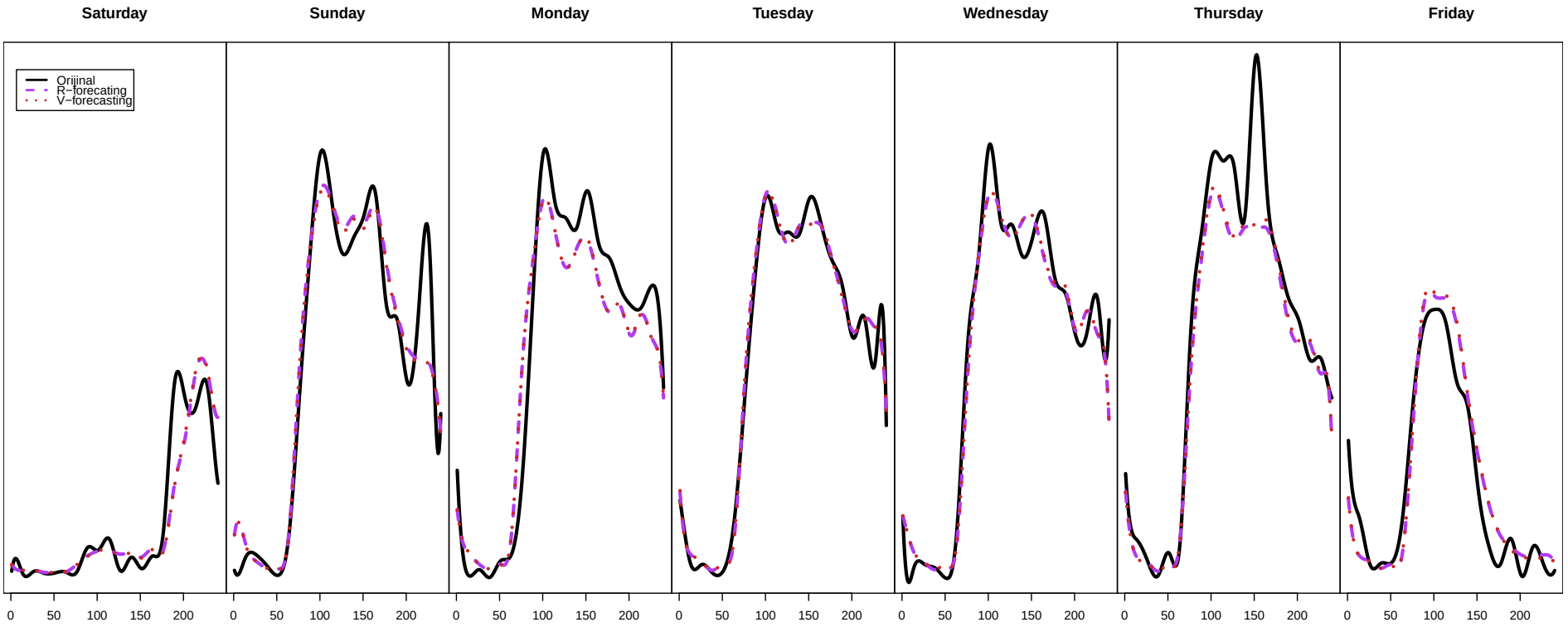
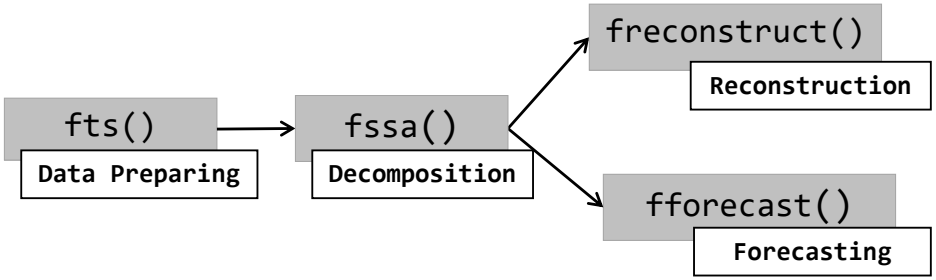




FSSA Applied to NDVI Data

- Define the NDVI functional time series (FTS) $\mathbf{y}_N = [y_1(s), ..., y_N(s)]$ where $y_j \in \mathbb{H} := \mathcal{L}^2([0,1])$, $\langle y_i, y_j \rangle = \int_0^1 y_i(s)\, y_j(s)\, ds$, $i, j = 1, ..., N$, and $N = 448$.

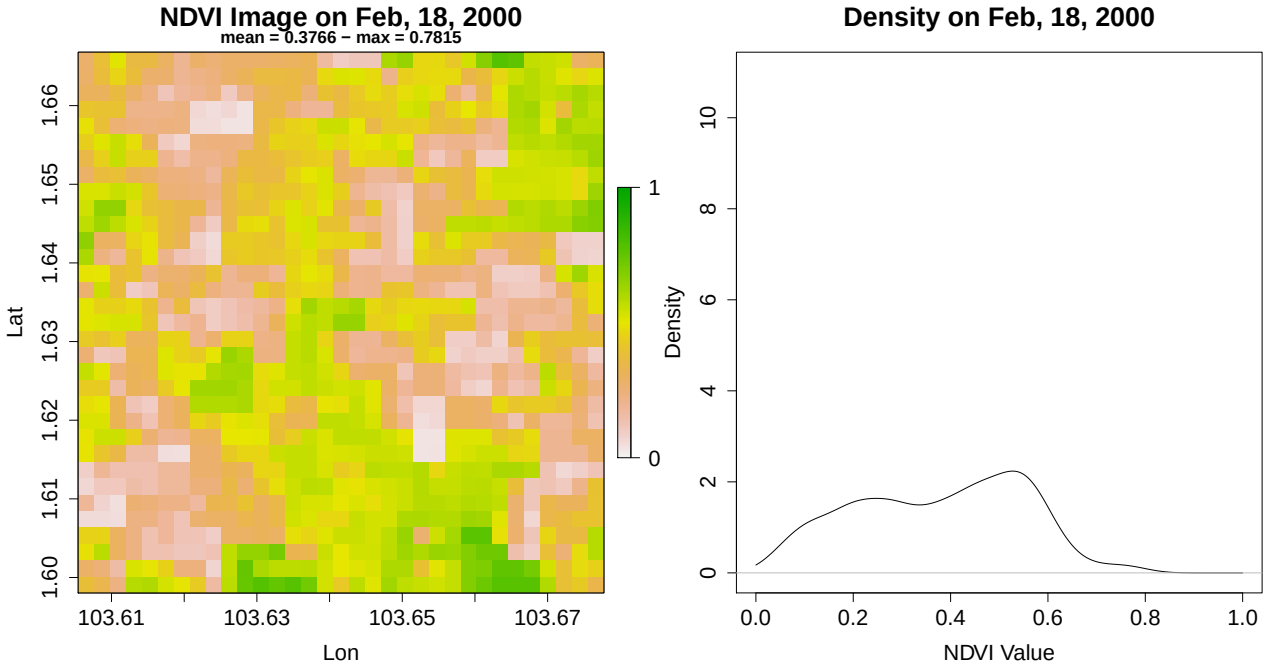
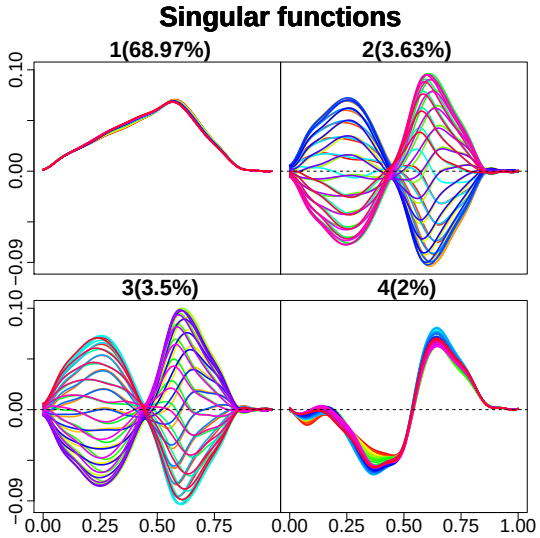


Figure: Density Animation

FSSA Applied to NDVI Data

- A reasonable grouping for reconstruction would be the first component by itself, the second with the third, and the fourth by itself with all other components capturing noise behavior.



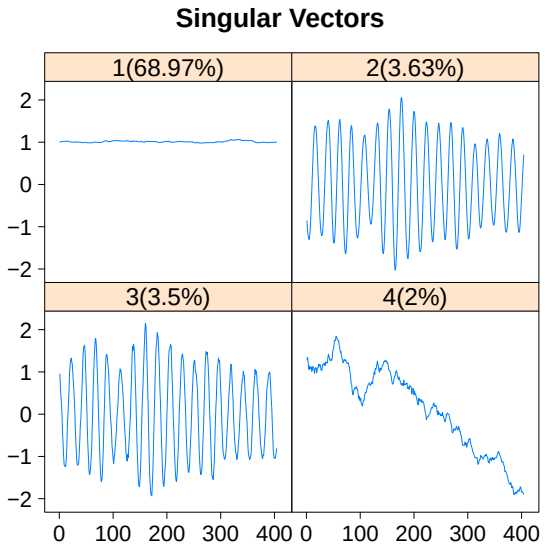


Figure: FSSA – Singular Functions and Singular Vectors

- We can use the singular vectors and singular functions plots to see what type of behavior each component captures.

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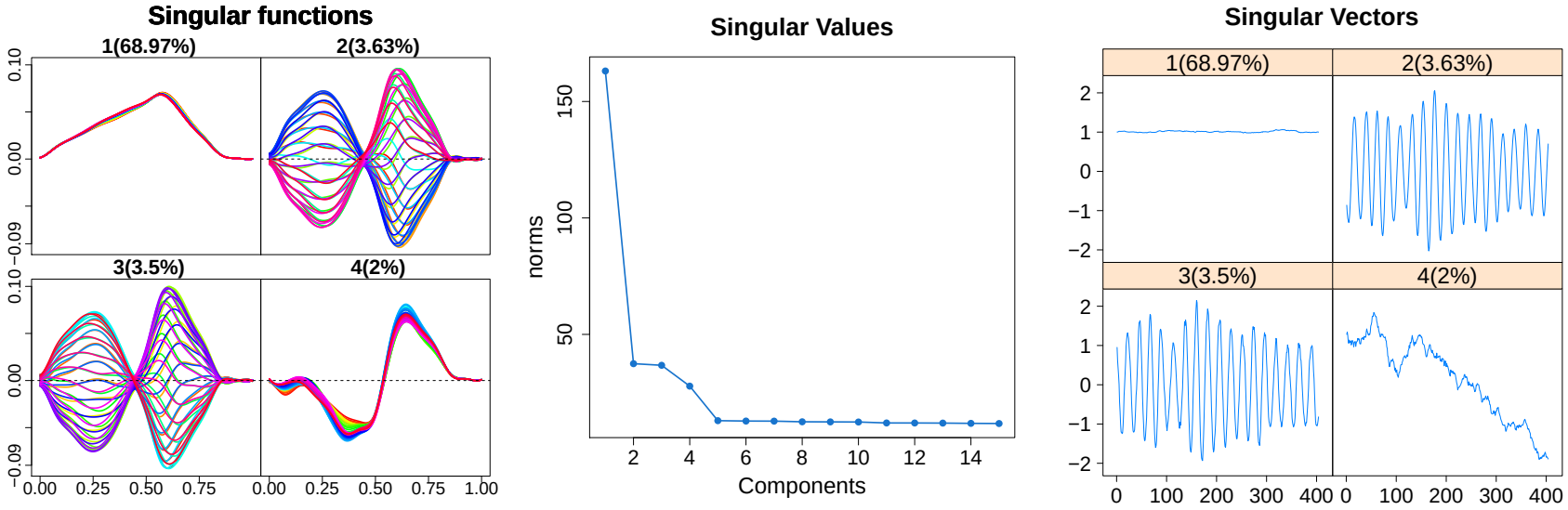

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FSSA Reconstruction

- We reconstruct the following operators based on our grouping

$$\mathcal{X}_1 = \sqrt{\lambda_1} u_1 \otimes v_1,$$

$$\mathcal{X}_2 = \sqrt{\lambda_2} u_2 \otimes v_2 + \sqrt{\lambda_3} u_3 \otimes v_3,$$

$$\mathcal{X}_3 = \sqrt{\lambda_4} u_4 \otimes v_4.$$

- As stated earlier, the FTS that corresponds with $\mathcal{X}_1$ captures mean behavior, the FTS that corresponds with $\mathcal{X}_2$ captures yearly seasonal behavior, and the FTS that corresponds with $\mathcal{X}_3$ is the interesting component that captures trend.

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FSSA Applied to NDVI Data cont.

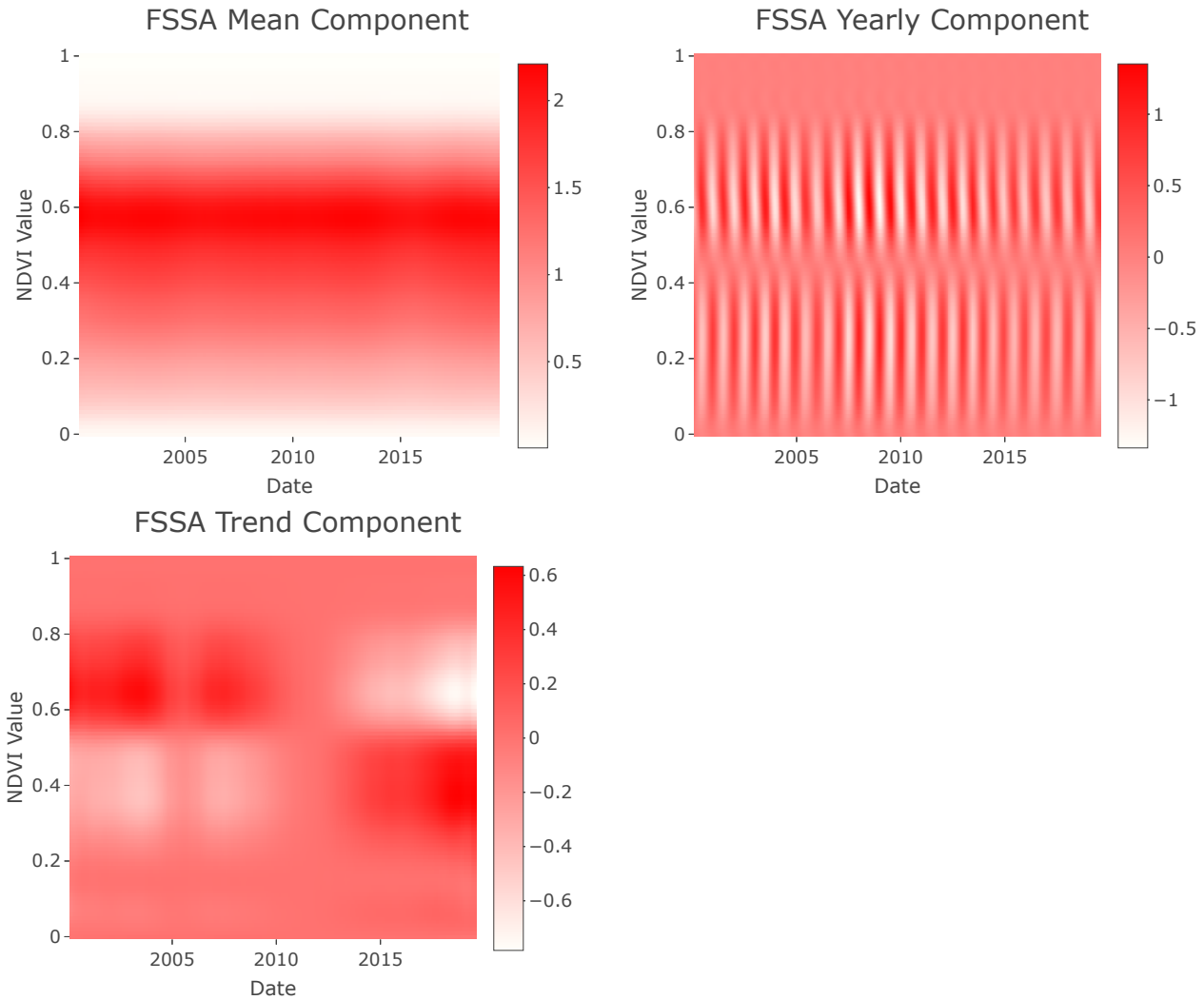

Figure: FSSA Reconstructions

FSSA Applied to NDVI Data cont.

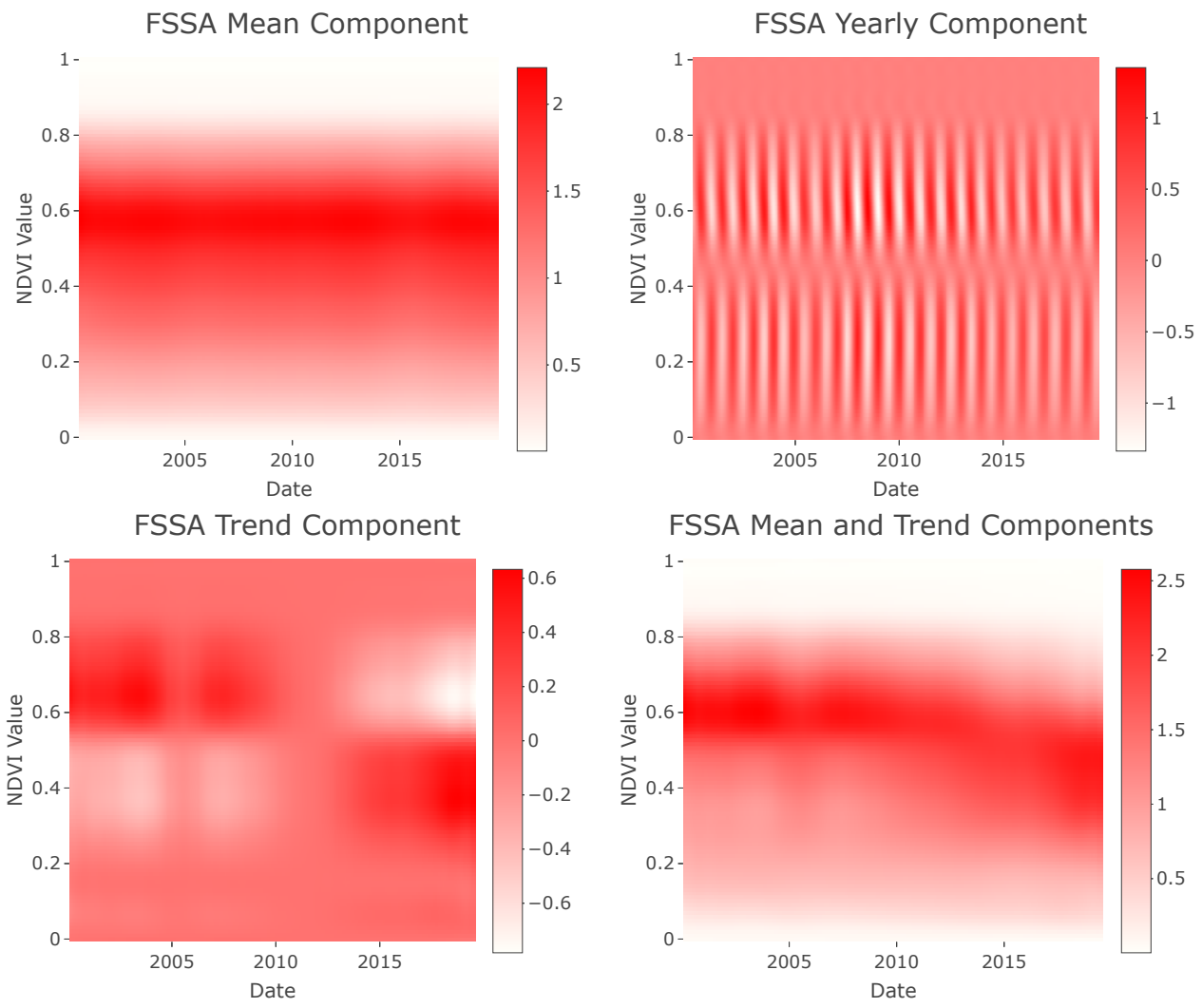

Figure: FSSA Reconstructions

Discussion

- It was determined in (Prasetyo et. al., 2015), the region specified in the images appears to be a hotspot for controlled burning for human use between 2000-2015.

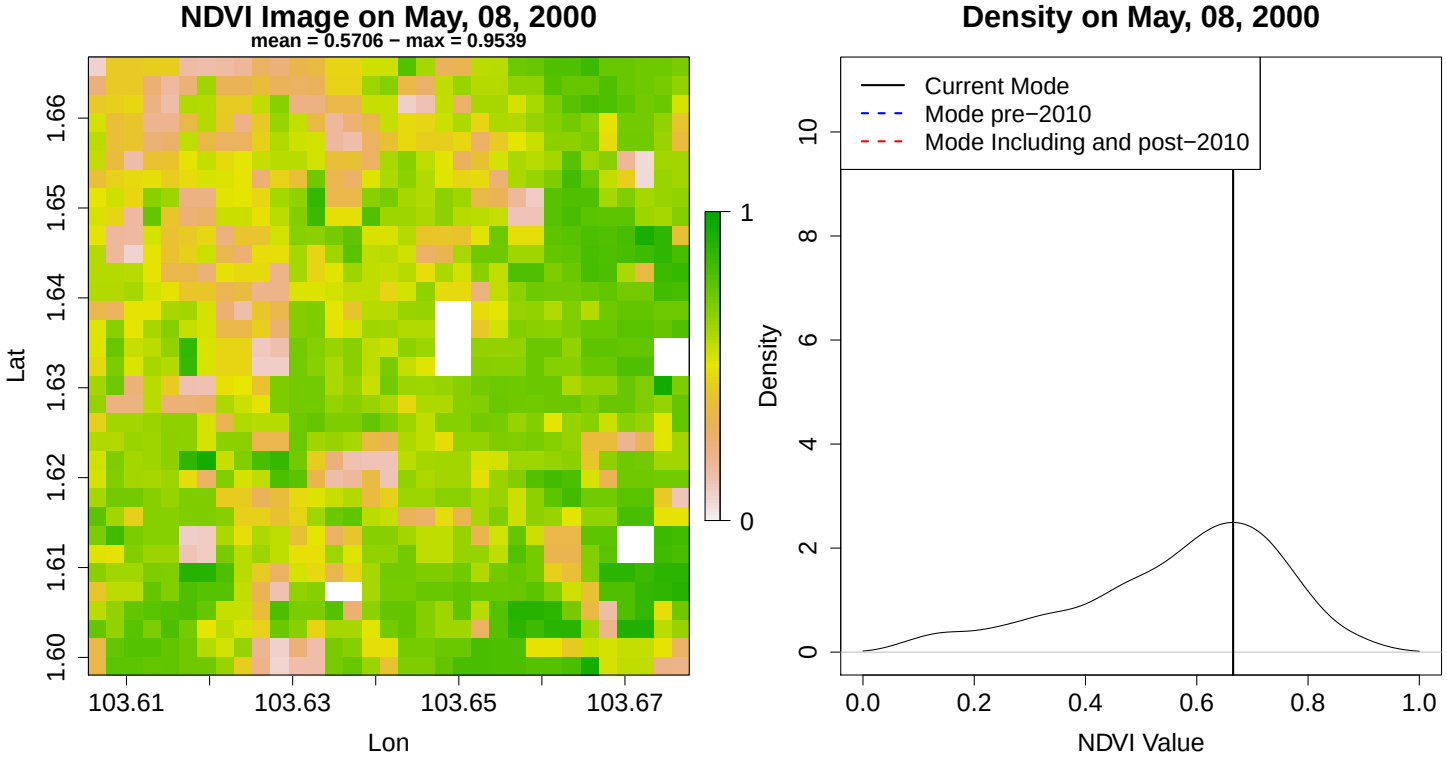

Figure: Southeast Jambi 8-16-2002

Figure: Southeast Jambi 9-10-2019

Conclusion

- We use the basics of Hilbert space in the FDA framework to introduce the concept of FSSA.
- The main contribution of this work is to present a rigorous methodology and computationally efficient algorithm (FSSA) to decompose, reconstruct and forecast the FTS, and MFTS.
- We have also developed an efficient and user-friendly R package (available on CRAN) including shiny web applications.
  - FSSA shiny [click for demonstration]
  - MFSSA shiny [click for demonstration]

Future (Ongoing) Work

- Hilbert Singular Spectrum Analysis (HSSA: e.g. Image) [Ongoing].

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Thank you!

- Colaborators:
  1. Hossein Haghbin, Assistant Professor, Persian Gulf University
  2. Jordan Trinka, AI Engineer, Booz Allen Hamilton
  3. Rahim Mahmoudvand, Associate Professor, Bu-Ali Sina University
  4. S. Morteza Najibi, Research Fellow, Lund University

Questions?

References

Haghbin, H., Morteza Najibi, S., Mahmoudvand, R., Trinka, J., and Maadooliat, M. (2021). Functional singular spectrum analysis. *Stat*, page e330. e330 STAT-20-0240.R1.

Haghbin, H., Trinka, J., and Maadooliat, M. (2024). Rfssa: An R package for functional singular spectrum analysis. *R Journal*, 16(2):Accepted.

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Trinka, J., Haghbin, H., and Maadooliat, M. (2022). Multivariate functional singular spectrum analysis: A nonparametric approach for analyzing multivariate functional time series. In Bekker, A., Ferreira, J., Arashi, M., and Chen, D., editors, *Innovations in multivariate statistical modelling: Navigating theoretical and multidisciplinary domains*, Emerging Topics in Statistics and Biostatistics. Springer Cham.

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FSSA Embedding

- Let $\mathbf{y}_N = [y_1, \cdots, y_N]$ be the FTS (of NDVI densities), choose $L < \frac{N}{2}$ to be the lag parameter (moving window), and define $K = N - L + 1$. The trajectory operator, $\mathcal{X} : \mathbb{R}^K \to \mathbb{H}^L$, is defined as

$$\mathcal{X}\boldsymbol{a} = \sum_{j=1}^{K} a_j \boldsymbol{x}_j, \quad \boldsymbol{a} \in \mathbb{R}^K \text{ and } \boldsymbol{x}_j = \left(y_j, y_{j+1}, \ldots, y_{j+L-1}\right). \tag{1}$$

- Evaluating $\mathcal{X}\boldsymbol{a}$ at a given $s \in [0, 1]$ is same as the matrix product $\mathbf{X}(s)\boldsymbol{a}$, where $\mathbf{X}(s)$ is an $L \times K$ Hankel matrix given by

$$\mathbf{X}(s) = \begin{bmatrix} y_1(s) & y_2(s) & \cdots & y_K(s) \\ \vdots & \vdots & \ddots & \vdots \\ y_L(s) & y_{L+1}(s) & \cdots & y_N(s) \end{bmatrix}. \tag{2}$$

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FSSA Embedding cont.

Theorem (1)

The operator $\mathcal{X} : \mathbb{R}^K \to \mathbb{H}^L$ *is linear, bounded, and has the adjoint,* $\mathcal{X}^* : \mathbb{H}^L \to \mathbb{R}^K$ *given by*

$$\mathcal{X}^*\mathbf{z} = \begin{bmatrix} \sum_{i=1}^{L} \langle y_i, z_i \rangle \\ \sum_{i=1}^{L} \langle y_{i+1}, z_i \rangle \\ \vdots \\ \sum_{i=1}^{L} \langle y_{i+K-1}, z_i \rangle \end{bmatrix} \tag{3}$$

for all $\mathbf{z} = [z_1, ..., z_L]^\top \in \mathbb{H}^L$.

FSSA Decomposition

- We form the operator $\mathcal{S} := \mathcal{X}\mathcal{X}^*$ such that $\mathcal{S} : \mathbb{H}^L \to \mathbb{H}^L$.

Theorem (2)

The operator $\mathcal{S}$ is linear, self-adjoint, positive, bounded, continuous, and compact.

- By Theorem 2 and the Hilbert-Schmidt theorem (Reed et. al., 1972), we know there exists an orthonormal basis $\{u_i\}_{i=1}^r$ of $\mathbb{H}^L$ with a corresponding family of scalars $\{\lambda_i\}_{i=1}^r$ such that

$$\mathcal{S}u_i = \lambda_i u_i. \tag{4}$$

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FSSA Decomposition and Grouping

Theorem (3)

Let $\{u_i\}_{i=1}^r$ *and* $\{\lambda_i\}_{i=1}^r$ *be the eigenelements of* $\mathcal{S}$ *and define*

$$\mathcal{S}^\dagger := \mathcal{X}^*\mathcal{X}. \tag{5}$$

Then the FSVD of $\mathcal{X}$ *is given as* $\mathcal{X} = \sum_{i=1}^r \sqrt{\lambda_i} v_i \otimes u_i$ *where* $v_i = \frac{\mathcal{X}^* u_i}{\sqrt{\lambda_i}}$ *is an eigenvector of* $\mathcal{S}^\dagger$.

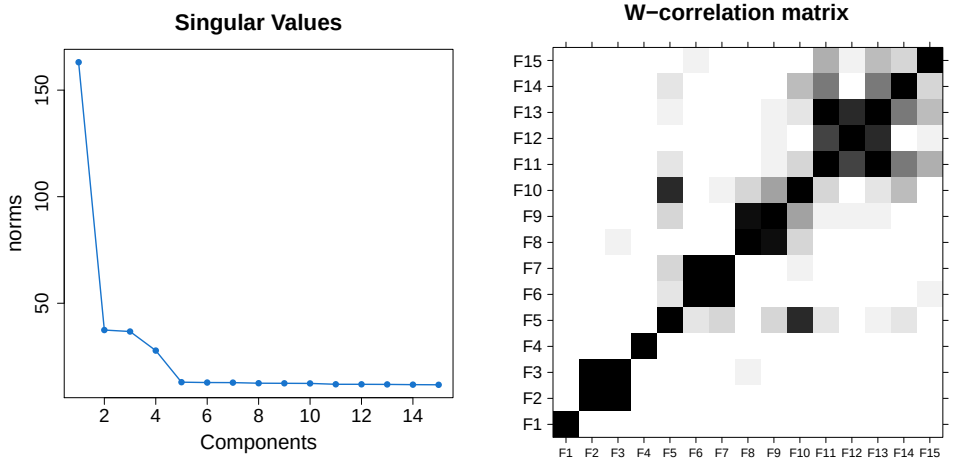

Figure: FSSA Decomposition Plots

FSSA in a Nutshell

The time series of length N represented by $Y = \{X_1, X_2, X_3 \ldots \ldots X_N\}$

1. Select an appropriate window length (L) following the constraint $2 < L < N/2$. Here N is the total number of data points and $K = N\text{-}L\text{+}1$.
   Form the trajectory matrix with K lagged vectors of Y as given by **T**.

Trajectory matrix

$$\mathbf{T} = \begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_K \\ X_2 & X_3 & X_4 & \cdots & X_{K+1} \\ X_3 & X_4 & X_5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_L & \cdots & \cdots & \cdots & X_{K+L-1} \end{bmatrix}_{L \times K}$$

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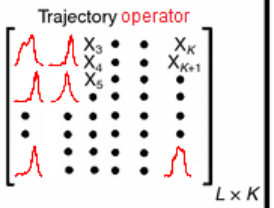
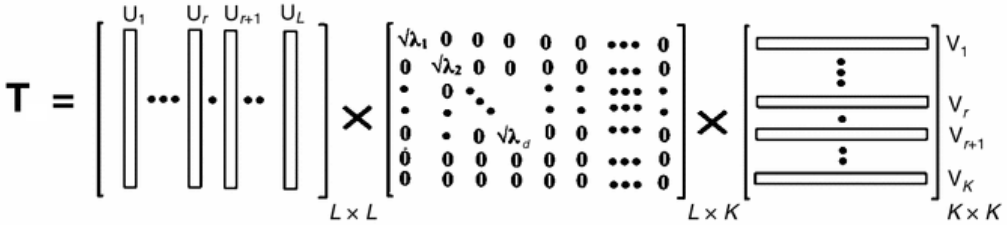


2. Decompose the trajectory matrix by applying Singular Value Decomposition (SVD)



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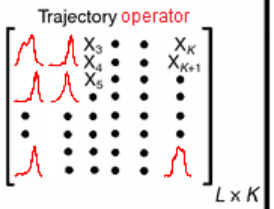
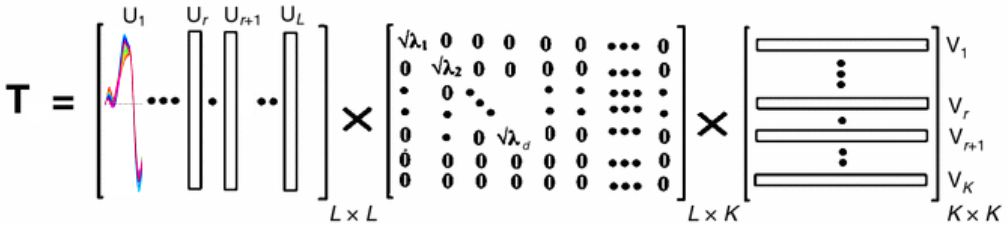


2. Decompose the trajectory operator by functional Singular Value Decomposition (FSVD)



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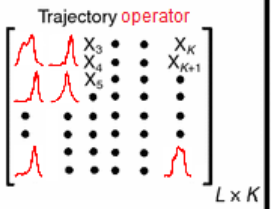

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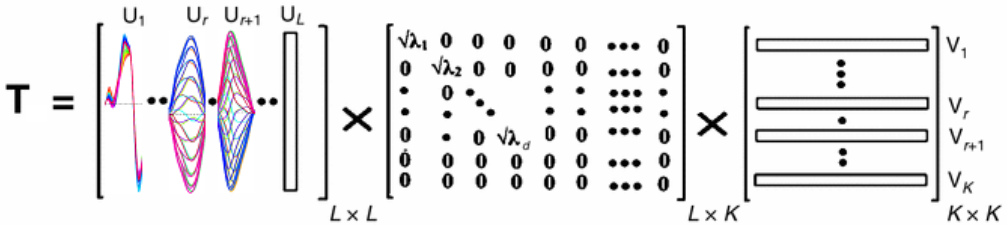

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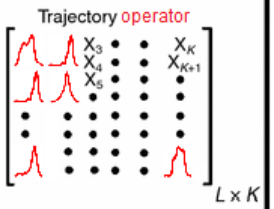

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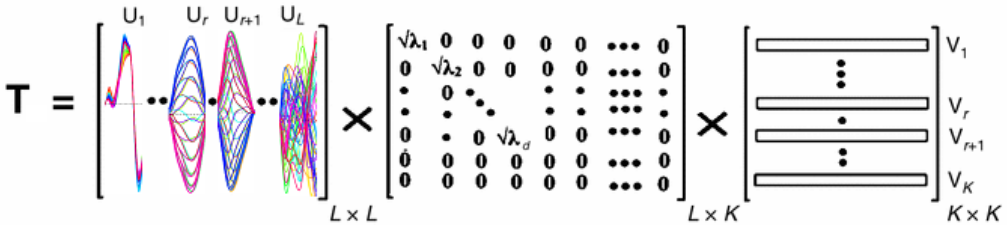

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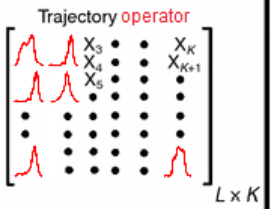

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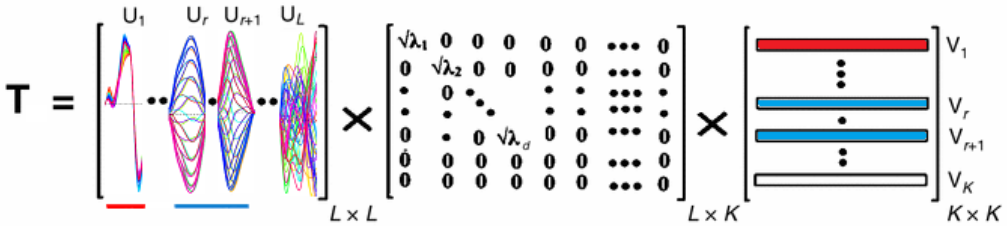
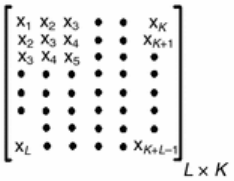


3. Grouping the eigentriples for filtered signal reconstruction



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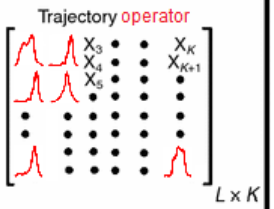

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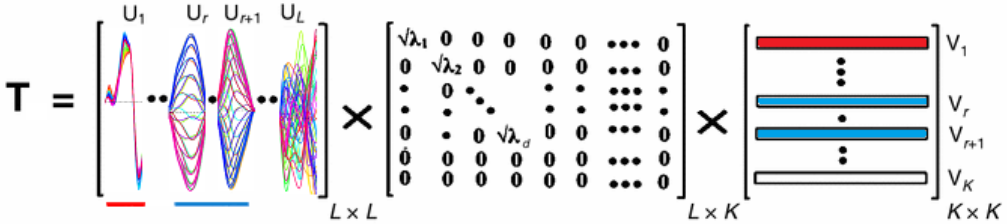
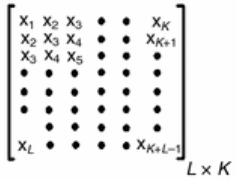


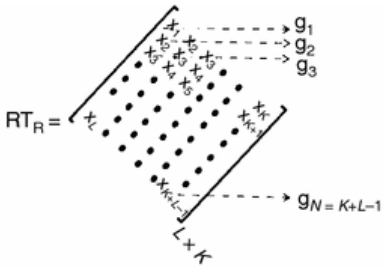
3. Grouping the eigentriples for filtered signal reconstruction

$$\mathbf{RT_R} = \sum_{i=1}^{s} \mathbf{U}_i \sqrt{\lambda_i} \mathbf{V}_i^{\mathbf{T}} =$$



4. Averaging the elements of reconstructed trajectory matrix along anti diagonals to get filtered data series represented by

$$Y_f = \{g_1, g_2, \ldots g_K, \ldots g_N\}$$



Theorem 1

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Back to slides

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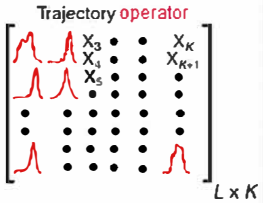

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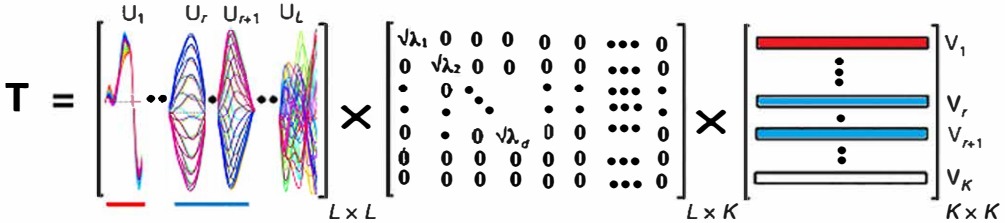

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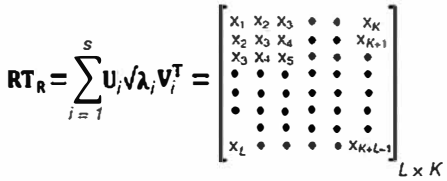

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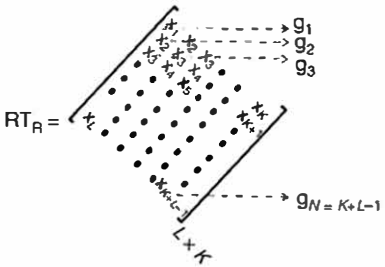

Theorem 1

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Back to slides