14. Given the joint density function

$$f(x,y) = 6x, \quad 0 < x < y < 1,$$

find the $E[Y \mid X]$ that is the regression line resulting from regressing Y on X.

Solution:

$E[Y|X] = \int_{-\infty}^{\infty} y f(y|x)\, dy$. $f(y|x) = f(x,y)/f_X(x)$.
$f_X(x) = \int_x^1 f(x,y)\, dy = \int_x^1 6x\, dy = 6xy|_x^1 = 6x - 6x^2 = 6x(1-x)$, for $0 < x < 1$.
$f(y|x) = \frac{6x}{6x(1-x)} = \frac{1}{1-x}$ for $0 < x < 1$.
This means

$$E[Y|X] = \int_x^1 y \cdot \frac{1}{1-x}\, dy = \left.\frac{y^2}{2(1-x)}\right|_x^1 = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2} = \frac{1}{2} + \frac{x}{2}.$$

17. If $f(x,y) = e^{-(x+y)}$, $x > 0$, and $y > 0$, find $\mathbb{P}\left(X + 3 > Y \,\middle|\, X > \frac{1}{3}\right)$.

Solution:

$$\mathbb{P}\left(X + 3 > Y \,\middle|\, X > \frac{1}{3}\right) = \frac{\mathbb{P}\left(X + 3 > Y,\, X > \frac{1}{3}\right)}{\mathbb{P}\left(X > \frac{1}{3}\right)}$$

$$\begin{aligned}
\mathbb{P}\left(X + 3 > Y,\, X > \frac{1}{3}\right) &= \int_{\frac{1}{3}}^{\infty} \int_0^{x+3} e^{-(x+y)}\, dy\, dx \\
&= \int_{\frac{1}{3}}^{\infty} \left. -e^{-(x+y)} \right|_0^{x+3} dx \\
&= \int_{\frac{1}{3}}^{\infty} -e^{-(2x+3)} + e^{-x}\, dx \\
&= \left. \frac{e^{-(2x+3)}}{2} - e^{-x} \right|_{\frac{1}{3}}^{\infty} \\
&= -\frac{e^{-11/3}}{2} + e^{-1/3}
\end{aligned}$$

$f_X(x) = \int_0^{\infty} e^{-(x+y)}\, dy = e^{-x}$

$$\begin{aligned}
\mathbb{P}\left(X > \frac{1}{3}\right) &= \int_{\frac{1}{3}}^{\infty} e^{-x}\,dx \\
&= -e^{-x}\Big|_{\frac{1}{3}}^{\infty} \\
&= e^{-1/3}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(X+3>Y \,\middle|\, X>\frac{1}{3}\right) &= \frac{\mathbb{P}\left(X+3>Y,\, X>\frac{1}{3}\right)}{\mathbb{P}\left(X>\frac{1}{3}\right)} \\
&= \frac{-\frac{e^{-11/3}}{2}+e^{-1/3}}{e^{-1/3}} \\
&= 1-\frac{1}{2}e^{-10/3} = 0.9822
\end{aligned}$$

26. Given the joint density function $f_{X,Y}(x,y) = x+y$, $0 \le x \le 1, 0 \le y \le 1$,

(a) Show that $f_{X,Y}(x,y) \ge 0$ for all x and y and that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$.

(b) Find the cumulative distribution function.

(c) Find the marginal means of X and Y.

(d) Find the marginal variances of X and Y.

Solution:

(a) By inspection $x+y$ is greater than or equal to zero in the ranges for x and y given. For property 2,

$$\begin{aligned}
\int_0^1\int_0^1 (x+y)\,dx\,dy &= \int_0^1 \left[\frac{x^2}{2}+xy\right]\Big|_0^1 dy \\
&= \int_0^1 \left(\frac{1}{2}+y\right) dy \\
&= \left[\frac{1}{2}y+\frac{y^2}{2}\right]\Big|_0^1 \\
&= \frac{1}{2}+\frac{1}{2} \overset{\checkmark}{=} 1
\end{aligned}$$

(b) The cumulative distribution function must be defined in the five regions: $x, y < 0$; $0 \le x < 1$ and $0 \le y < 1$; $x \ge 1$ and $0 \le y < 1$; $y \ge 1$ and $0 \le x < 1$; and $x, y \ge 1$.

For the region $0 \le x < 1$ and $0 \le y < 1$,

$$F_{X,Y}(x,y) = \int_0^x\int_0^y (x+y)\,dy\,dx = \frac{x^2y}{2}+\frac{xy^2}{2}.$$

For the region $x \geq 1$ and $0 \leq y < 1$,

$$F_{X,Y}(x, y) = \int_0^y \int_0^1 (x + y)\, dx\, dy = \frac{y}{2} + \frac{y^2}{2}.$$

For the region $y \geq 1$ and $0 \leq x < 1$,

$$F_{X,Y}(x, y) = \int_0^x \int_0^1 (x + y)\, dy\, dx = \frac{x}{2} + \frac{x^2}{2}.$$

$$F_{X,Y}(x, y) = \begin{cases} 0 & x, y < 0 \\ \frac{x^2 y}{2} + \frac{x y^2}{2} & 0 \leq x < 1, 0 \leq y < 1 \\ \frac{y}{2} + \frac{y^2}{2} & x \geq 1, 0 \leq y < 1 \\ \frac{x}{2} + \frac{x^2}{2} & y \geq 1, 0 \leq x < 1 \\ 1 & x, y \geq 1 \end{cases}$$

(c) To find the marginal means and variances, the marginal densities of x and y must be calculated.

$$f_X(x) = \int_0^1 (x + y)\, dy \; = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2} \text{ for } 0 \leq x \leq 1$$

Similarly, $f_Y(y) = y + \frac{1}{2}$ for $0 \leq y \leq 1$.
$E[X] = \int_0^1 x\left(x + \frac{1}{2}\right)\, dx = \frac{x^3}{3} + \frac{x^2}{4}\Big|_0^1 = \frac{7}{12}$
$E[Y] = \int_0^1 y\left(y + \frac{1}{2}\right)\, dy = \frac{y^3}{3} + \frac{y^2}{4}\Big|_0^1 = \frac{7}{12}$

(d) To calculate the variance, $E[X^2]$ and $E[Y^2]$ must be found:
$E[X^2] = \int_0^1 x^2\left(x + \frac{1}{2}\right)\, dx = \frac{x^4}{4} + \frac{x^3}{6}\Big|_0^1 = \frac{5}{12}$
$E[Y^2] = \int_0^1 y^2\left(y + \frac{1}{2}\right)\, dy = \frac{y^4}{4} + \frac{y^3}{6}\Big|_0^1 = \frac{5}{12}$
$Var[X] = E[X^2] - (E[X^2]) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{60-49}{144} = \frac{11}{144}$
$Var[Y] = E[Y^2] - (E[Y^2]) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{60-49}{144} = \frac{11}{144}$

31. Let X and Y have the joint density function

$$f_{X,Y}(x, y) = \begin{cases} Ky & -2 \leq x \leq 2,\ 1 \leq y \leq x^2 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find K so that $f_{X,Y}(x, y)$ is a valid **pdf**.

(b) Find the marginal densities of X and Y.

(c) Find $\mathbb{P}\left(Y > \frac{3}{2} \,\middle|\, X < \frac{1}{2}\right)$.

(a) Note that $-2 \leq x \leq 2$ and $1 \leq y \leq x^2$ implies that $1 \leq x^2$, which means that $(x \leq -1) \cup (x \geq 1)$ is implied in the ranges of the variables.

$$
\begin{aligned}
1 &\overset{\text{set}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{-2}^{-1} \int_{1}^{x^2} K y\,dy\,dx + \int_{1}^{2} \int_{1}^{x^2} K y\,dy\,dx \\
&= K\left[\int_{-2}^{-1} \frac{x^4-1}{2}\,dx + \int_{1}^{2} \frac{x^4-1}{2}\,dx\right] \\
&= K\left[\left(\frac{x^5}{10} - \frac{x}{2}\right)\Big|_{-2}^{-1} + \left(\frac{x^5}{10} - \frac{x}{2}\right)\Big|_{1}^{2}\right] \\
&= K\left[\left(\frac{-1}{10} + \frac{1}{2}\right) - \left(\frac{-32}{10} + 1\right) + \left(\frac{32}{10} - 1\right) - \left(\frac{1}{10} - \frac{1}{2}\right)\right] \\
&= K\left[\frac{4}{10} + \frac{22}{10} + \frac{22}{10} + \frac{4}{10}\right] \\
1 &= K\left[\frac{26}{5}\right] \\
\implies K &= \frac{5}{26}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_X(x) &= \int_y f_{X,Y}(x,y)\,dy \\
&= \int_1^{x^2} \frac{5}{26} y\,dy \\
&= \frac{5}{52} y^2 \Big|_1^{x^2} \\
&= \frac{5}{52}(x^4 - 1) \\
\implies f_X(x) &= \begin{cases} \frac{5}{52}(x^4-1), & (-2 \leq x \leq -1) \cup (1 \leq x \leq 2) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \int_x f_{X,Y}(x,y)\,dx \\
&= \int_{-2}^{-\sqrt{y}} \frac{5}{26} y\,dx + \int_{\sqrt{y}}^{2} \frac{5}{26} y\,dx \\
&= \frac{5}{26} xy \Big|_{-2}^{-\sqrt{y}} + \frac{5}{26} xy \Big|_{\sqrt{y}}^{2} \\
&= \frac{5}{26} \left[\left(-y^{3/2} + 2y \right) + \left(2y - y^{3/2} \right) \right] \\
&= \frac{5}{26} \left[4y - 2y^{3/2} \right] \\
\implies f_Y(y) &= \begin{cases} \frac{5}{13}(2y - y^{3/2}), & 1 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathbb{P}\left(Y > \frac{3}{2} \,\middle|\, X < \frac{1}{2} \right) &= \frac{\mathbb{P}\left(Y > \frac{3}{2}, X < \frac{1}{2}\right)}{\mathbb{P}\left(X < \frac{1}{2}\right)} \\
&= \frac{\int_{-2}^{-\sqrt{3/2}} \int_{3/2}^{x^2} \frac{5}{26} y\,dy\,dx}{\int_{-2}^{-1} \frac{5}{52}(x^4 - 1)\,dx} \\
&= \frac{\int_{-2}^{-\sqrt{3/2}} \frac{y^2}{2}\Big|_{3/2}^{x^2}\,dx}{\frac{1}{2}(\frac{x^5}{5} - x)\Big|_{-2}^{-1}} \\
&= \frac{\int_{-2}^{-\sqrt{3/2}} x^4 - \frac{9}{4}\,dx}{(\frac{-1}{5} + 1) - (\frac{-32}{5} + 2)} \\
&= \frac{\frac{x^5}{5} - \frac{9}{4}x\Big|_{-2}^{-\sqrt{3/2}}}{\frac{4}{5} + \frac{22}{5}} \\
&= \frac{\left(-\frac{9}{4}\frac{\sqrt{3/2}}{5} + \frac{9}{4}\sqrt{\frac{3}{2}} \right) - (\frac{-32}{5} + \frac{9}{2})}{\frac{26}{5}} \\
&= \frac{\frac{36}{20}\sqrt{\frac{3}{2}} + \frac{19}{10}}{\frac{26}{5}} \\
&= \frac{9\sqrt{\frac{3}{2}} + \frac{19}{2}}{26} \\
&= \frac{9\sqrt{6} + 19}{52} = 0.7893
\end{aligned}
$$

32. An engineer has designed a new diesel motor that is used in a prototype earth mover.

The prototype's diesel consumption in gallons per mile C follows the equation $C = 3 + 2X + \frac{3}{2}Y$, where X is a speed coefficient and Y is the quality diesel coefficient. Suppose the joint density for X and Y is $f_{X,Y}(x,y) = ky$, $0 \le x \le 2$, $0 \le y \le x$.

(a) Find k so that $f_{X,Y}(x,y)$ is a valid density function.

(b) Are X and Y independent?

(c) Find the mean diesel consumption for the prototype.

Solution:

(a)

$$
\begin{aligned}
1 &\overset{\text{set}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\
&= \int_0^2 \int_0^x ky\,dy\,dx \\
&= \int_0^2 \left.\frac{ky^2}{2}\right|_0^x dx \\
&= \int_0^2 \frac{kx^2}{2}\,dx \\
&= \left.\frac{kx^3}{6}\right|_0^2 \\
1 &= \frac{8k}{6} \\
\implies k &= \frac{3}{4}
\end{aligned}
$$

(b) For X and Y to be independent, $f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$.

$$
\begin{aligned}
f_X(x) &= \int_0^x \frac{3}{4}y\,dy = \left.\frac{3}{8}y^2\right|_0^x = \frac{3}{8}x^2 \text{ for } 0 \le x \le 2 \\
f_Y(y) &= \int_y^2 \frac{3}{4}y\,dx = \left.\frac{3}{4}xy\right|_y^2 = \frac{3}{4}(2y - y^2) \text{ for } 0 \le y \le 2
\end{aligned}
$$

$f_X(x) \cdot f_Y(y) = \frac{9}{32}x^2(2y - y^2) \neq \frac{3}{4}y = f_{X,Y}(x,y)$, so X and Y are not independent.

(c)

$$
\begin{aligned}
E[C] &= E\left[3 + 2X + \frac{3}{2}Y\right] = 3 + 2E[X] + \frac{3}{2}E[Y] \\
E[X] &= \int_0^2 x \cdot f_X(x)\,dx = \int_0^2 x \cdot \frac{3}{8}x^2\,dx = \left.\frac{3x^4}{32}\right|_0^2 = \frac{3}{2} \\
E[Y] &= \int_0^2 y \cdot \frac{3}{4}(2y - y^2)\,dy = \left.\frac{3}{4}\left(\frac{2y^3}{3} - \frac{y^4}{4}\right)\right|_0^2 = \frac{3}{4}\left(\frac{16}{3} - 4\right) = 1 \\
E[C] &= 3 + 2E[X] + \frac{3}{2}E[Y] \\
&= 3 + 2 \cdot \frac{3}{2} + \frac{3}{2} \cdot 1 = 7.5
\end{aligned}
$$

The average gas consumption is 7.5 gallons/mile.