MSSC 6010 | **Exam I (Fall 20 25)** | **Section 101**
Time Limit: 70 minutes

Print Your Name: Solution

Notes:

1 - DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO.

2 - GIVE ALL THE NECESSARY DETAILS TO GET FULL CREDITS.

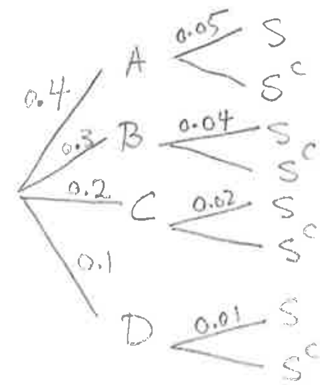
3 - NO ELECTRONIC DEVICES OTHER THAN A CALCULATOR MAY BE USED.

20 pts

1. Four different transportation companies (A, B, C, and D) deliver the products of an agricultural firm. Company A transports 40% of the products; company B, 30%; company C, 20%; and, finally, company D, 10%. During transportation, 5%, 4%, 2%, and 1% of the products spoil with companies A, B, C, and D, respectively. If one product is randomly selected,

   a) Obtain the probability that it is spoiled.
   b) If the chosen product is spoiled, derive the probability that company A has transported it.
   c) From a truck that belongs to company A and it delivers the products of this agricultural firm, we randomly selected five products, what is the probability that at most one of the products is spoiled?

$$P(S) = P(S\cap A) + \cdots + P(S\cap D)$$
$$= P(S|A)P(A) + \cdots + P(S|D)P(D)$$
$$= 0.4(0.05) + 0.3(0.04) + 0.2(0.02) + 0.1(0.01)$$
$$= 0.037 \qquad \Rightarrow \boxed{P(S) = 0.037} \text{ (a)}$$



$$P(A|S) = \frac{P(S|A)P(A)}{P(S)} = \frac{0.05 \times 0.4}{0.037}$$
$$\Rightarrow \boxed{P(A|S) = 0.54} \text{ (b)}$$

$X \sim Bin(n, \pi)$ where $n = 5$ & $\pi = P(S|A) = 0.05$

$$P(X \leq 1) = P(X=0) + P(X=1)$$
$$= \frac{5!}{0!\,5!}(0.05)^0(0.95)^5 + \frac{5!}{1!\,4!}(0.05)^1(0.95)^4$$
$$\Rightarrow \boxed{P(X \leq 1) = 0.9774} \text{ (c)}$$

PLEASE DO NOT WRITE IN THE FOLLOWING SPACE.

1	2	3	4	5	*Total*

20 pts

2. Let X be a random variable with probability density function

$$f(x) = ke^{-\frac{x}{\theta}}, \quad x \geq 0, \qquad \theta > 0.$$

a) Find k so that the given function is a valid pdf.
b) Find the moment generating function of X.
c) Find the cumulative distribution function $F(\cdot)$.
d) Let u to be an observation from $Unif(0,1)$, how can we obtain a realization from X using u.

(a) $\int_{-\infty}^{\infty} f(x|\theta)\,dx = 1 \Rightarrow \int_0^{\infty} ke^{-x/\theta}\,dx = -k\theta e^{-x/\theta}\Big|_0^{\infty} = 0-(-k\theta) = 1$

$\Rightarrow k = \frac{1}{\theta} \quad \Rightarrow f(x) = \frac{1}{\theta} e^{-x/\theta} \qquad x \geqslant 0$

(b) $M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \frac{1}{\theta} e^{-x/\theta}\,dx = \frac{1}{\theta}\int_0^{\infty} e^{(t-\frac{1}{\theta})x}\,dx$

$= \frac{1}{\theta}(t-\frac{1}{\theta})^{-1} e^{(t-\frac{1}{\theta})x}\Big|_0^{\infty} = 0 - \frac{1}{\theta}\left(\frac{t\theta-1}{\theta}\right)^{-1}$

$\Rightarrow M_X(t) = (1-\theta t)^{-1} \qquad$ for $|t| < \frac{1}{\theta}$

(c) $F(x) = \int_0^{x} \frac{1}{\theta} e^{-\frac{t}{\theta}}\,dt = -e^{-\frac{t}{\theta}}\Big|_0^{x} = 1-e^{-\frac{x}{\theta}}$

$\Rightarrow F(x) = 1-e^{-x/\theta} \qquad$ for $\quad x \geqslant 0$

(d) Let $\quad u = F(x) \Rightarrow \quad u = 1-e^{-x/\theta} \Rightarrow e^{-x/\theta} = 1-u$

$\Rightarrow -\frac{x}{\theta} = \ln(1-u) \Rightarrow \boxed{x = -\theta \ln(1-u)}$

x belongs to $F(\cdot)$

20 pts

3. Suppose that the average arrival rate at a local fast food drive through window is three cars per minute ($\lambda = 3$). Find

a) The probability that at most three cars arrive in 90 seconds.
b) The probability that less than one minute elapses before the first car arrives.
c) The probability that more than two minutes elapses before the third car arrives.
d) What is the average waiting time before the third car arrives?

Note: Give the details on the distributions you used in each part to get the full credit.

(a) $X \sim \text{Poisson}(\lambda t)$ where $\lambda = 3$ $t = 1.5$ (90 seconds in min)

$\Rightarrow X \sim \text{Poisson}(4.5)$

$$P(X \leq 3) = \sum_{x=0}^{3} \frac{(\lambda t)^x e^{-\lambda t}}{x!} = \sum_{x=0}^{3} \frac{4.5^x e^{-4.5}}{x!} \Rightarrow \boxed{P(X \leq 3) = 0.342}$$

(b) $Y \sim \text{Exp}(\lambda = 3)$ where $f(y) = \lambda e^{-\lambda y}$ $y \geq 0$

$$P(Y<1) = \int_0^1 \lambda e^{-\lambda y} dy = 1 - e^{-\lambda} = 1 - e^{-3} \Rightarrow \boxed{P(Y<1) = 0.95}$$

(c) $Z = Y_1 + Y_2 + Y_3 \Rightarrow Z \sim \text{Gamma}(\alpha = 3, \lambda = 3)$ $f(z) = \frac{1}{\Gamma(3)} \lambda^3 z^2 e^{-\lambda z}$

$$P(Z>2) = 1 - P(Z<2) = 1 - \int_0^2 \frac{27}{2} z^2 e^{-3z} dz$$

$$\Rightarrow \boxed{P(Z>2) = 1 - 25e^{-6} = 1 - 0.9380} \Rightarrow \boxed{P(Z>2) = 0.062}$$

(d) $W \sim \text{Gamma}(\alpha = 3, \lambda = 3)$

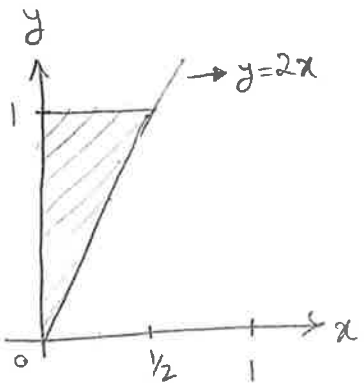
$$\boxed{E(W) = \frac{\alpha}{\lambda} = \frac{3}{3} = \text{a min}}$$

20 pts

4. If $f(x, y) = K(y - 2x)$ is a joint density function over $0 < x < 1, 0 < y < 1$, and $y > 2x$.

a) Find K so that the given function is a valid pdf.
b) Find the marginal density of X *and the conditional* density of. $X|Y$.
c) Are X and Y independent? Justify.

(a) $$\iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1 \int_0^{y/2} k(y-2x)\,dx\,dy = 1 \Rightarrow$$



$$= k \int_0^1 (yx - x^2)\Big|_{x=0}^{x=y/2} dy$$

$$= k \int_0^1 \left(\frac{y^2}{2} - \frac{y^2}{4}\right) dy = k \int_0^1 \frac{y^2}{4}\,dy$$

$$= k\,\frac{y^3}{12}\Big|_{y=0}^{y=1} = k\left(\frac{1}{12}\right)$$

$$\Rightarrow \boxed{k = 12}$$

(b) $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{y/2} 12(y-2x)\,dx = 12\left(yx - x^2\right)\Big|_{x=0}^{x=y/2} \Rightarrow \boxed{f_Y(y) = 3y^2 \quad 0<y<1}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{2x}^{1} 12(y-2x)\,dy = 12\left(\frac{y^2}{2} - 2xy\right)\Big|_{y=2x}^{y=1} = 12\left[\left(\tfrac{1}{2} - 2x\right) - \left(2x^2 - 4x^2\right)\right]$$

$$\Rightarrow \boxed{f_X(x) = 12(2x^2 - 2x + \tfrac{1}{2}) \qquad 0<x<\tfrac{1}{2}}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{12(y-2x)}{3y^2} \Rightarrow \boxed{f_{X|Y}(y) = 4(y-2x)y^{-2}, \quad 0<x<y/2}$$

(c) $f_{X,Y}(x,y) \neq f_X(x) f_Y(y) \Rightarrow X$ & Y are not independent.

20 pts

5. A particular unfair coin is constructed so that the probability of obtaining a tail is 2/3. The unfair coin is flipped twice. Define two random variables: Z = the number of heads in the second flip and W = the number of heads in two flips.

   a) Construct a table showing the joint probability distribution of both random variables Z and W including the marginal probabilities.
   b) Find the covariance between Z and W. Are they independent?
   c) Suppose the covariance between Z and W were 0. Would this imply that Z and W are independent?

$P(T)=2/3$, $P(H)=1/3$, $\Omega=\{HH, HT, TH, TT\} \Rightarrow \begin{cases} Z = \text{\# of heads in 2}^{nd} \text{ flip} = \{0,1\} \\ W = \text{\# of heads in 2 flips} = \{0,1,2\} \end{cases}$

(a)

Z \ W	0	1	2	$P_Z(z)$
0	$(2/3)^2$	$(2/3)(1/3)$	0	6/9
1	0	$(1/3)(2/3)$	$(1/3)^2$	3/9
$P_W(w)$	4/9	4/9	1/9	Total = 1

$\Rightarrow P_W(w) = \begin{cases} 4/9 & w=0,1 \\ 1/9 & w=2 \\ 0 & o/w \end{cases}$ & $P_Z(z) = \begin{cases} 2/3 & z=0 \\ 1/3 & z=1 \\ 0 & o/w \end{cases}$

(b) $Cov(Z,W) = E(ZW) - E(Z)E(W)$

$\begin{cases} E(ZW) = 0\times0\times 4/9 + 0\times1\times 2/9 + 0\times2\times0 + 1\times0\times0 + 1\times1\times 2/9 + 1\times2\times 1/9 \Rightarrow E(ZW) = 4/9 \\ E(Z) = 0\times 2/3 + 1\times 1/3 = 1/3 \Rightarrow E(Z) = 1/3 \\ E(W) = 0\times 4/9 + 1\times 4/9 + 2\times 1/9 = 6/9 \Rightarrow E(W) = 2/3 \end{cases}$

$\Rightarrow Cov(Z,W) = 4/9 - 1/3 \times 2/3 \Rightarrow Cov(Z,W) = 2/9$

$\Rightarrow Cov(Z,W) \neq 0 \Rightarrow$ $\boxed{\text{Z \& W are NOT indep.}}$

(c) No, it is not an "if and only if":

(X,Y) are Indep $\Rightarrow Cov(X,Y)=0$ But $Cov(X,Y)=0 \not\Rightarrow$ Indep.