42. The Laplace distribution, also known as a double exponential, has a **pdf** given by

$$f(x) = \frac{\lambda}{2} \cdot \mathrm{e}^{-\lambda\,|x-\mu|}\,, \text{ where } -\infty < x < \infty,\ -\infty < \mu < \infty,\ \lambda > 0.$$

(a) Find the theoretical mean and variance of a Laplace distribution. (Hint: Integrals of absolute values should be done as a positive and negative part, in this case, with limits from $-\infty$ to μ and from μ to ∞.)

(b) Let X_1 and X_2 be independent exponential random variables, each with parameter λ. The distribution of $Y = X_1 - X_2$ is a Laplace distribution with a mean of zero and a standard deviation of $\sqrt{2}/\lambda$. Set the seed equal to 3, and generate 25,000 X_1 values from an $Exp(\lambda = \frac{1}{2})$ and 25,000 X_2 values from another $Exp(\lambda = \frac{1}{2})$ distribution. Use these values to create the simulated distribution of $Y = X_1 - X_2$.

   (i) Superimpose a Laplace distribution over a density histogram of the Y values. (Hint: The R function `curve()` can be used to superimpose the Laplace distribution over the density histogram.)

   (ii) Is the mean of Y within 0.02 of the theoretical mean?

   (iii) Is the variance of Y within 2% of the theoretical variance?

Solution:

(a)

$$f(x) = \begin{cases} \frac{\lambda}{2} e^{\lambda x - \lambda\mu} & x \le \mu \\ \frac{\lambda}{2} e^{-\lambda x + \lambda\mu} & x > \mu \end{cases}$$

So, $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\mu} x f(x) + \int_{\mu}^{\infty} x f(x)\, dx$.

Let $\int_{-\infty}^{\mu} x f(x)\, dx = A$ and $\int_{\mu}^{\infty} x f(x)\, dx = B$, so $E[X] = A + B$.

$$\begin{aligned} A &= \int_{-\infty}^{\mu} x f(x)\, dx \\ &= \int_{-\infty}^{\mu} x \frac{\lambda}{2} e^{\lambda x - \lambda\mu}\, dx \\ &= \frac{1}{2} e^{-\lambda\mu} \int_{-\infty}^{\mu} x \lambda e^{\lambda x}\, dx \end{aligned}$$

Let $u = x$, $du = dx$, $dv = \lambda e^{\lambda x}$, $v = e^{\lambda x}$

$$\begin{aligned}
&= \frac{1}{2}e^{-\lambda\mu}\left[xe^{\lambda x}|_{-\infty}^{\mu} - \int_{-\infty}^{\mu} e^{\lambda x}\,dx\right] \\
&= \frac{1}{2}e^{-\lambda\mu}\left[\mu e^{\lambda\mu} - \frac{e^{\lambda x}}{\lambda}\bigg|_{-\infty}^{\mu}\right] \\
&= \frac{1}{2}e^{-\lambda\mu}\left[\mu e^{\lambda\mu} - \frac{e^{\lambda\mu}}{\lambda}\right] \\
&= \frac{\mu}{2} - \frac{1}{2\lambda}
\end{aligned}$$

$$\begin{aligned}
B &= \int_{\mu}^{\infty} xf(x)\,dx \\
&= \int_{\mu}^{\infty} x\frac{\lambda}{2}e^{-\lambda x+\lambda\mu}\,dx \\
&= \frac{1}{2}e^{\lambda\mu}\int_{\mu}^{\infty} x\lambda e^{-\lambda x}\,dx
\end{aligned}$$

Let $u = x$, $du = dx$, $dv = \lambda e^{-\lambda x}$, $v = -e^{\lambda x}$

$$\begin{aligned}
&= \frac{1}{2}e^{\lambda\mu}\left[-xe^{-\lambda x}|_{\mu}^{\infty} + \int_{\mu}^{\infty} e^{-\lambda x}\,dx\right] \\
&= \frac{1}{2}e^{\lambda\mu}\left[\mu e^{-\lambda\mu} - \frac{e^{-\lambda x}}{\lambda}\bigg|_{\mu}^{\infty}\right] \\
&= \frac{1}{2}e^{\lambda\mu}\left[\mu e^{-\lambda\mu} + \frac{e^{-\lambda\mu}}{\lambda}\right] \\
&= \frac{\mu}{2} + \frac{1}{2\lambda}
\end{aligned}$$

$E[X] = A + B = \frac{\mu}{2} - \frac{1}{2\lambda} + \frac{\mu}{2} + \frac{1}{2\lambda} = \mu.$

To find the variance, first compute $E[X^2] = \underbrace{\int_{-\infty}^{\mu} x^2 f(x)}_{C} + \underbrace{\int_{\mu}^{\infty} x^2 f(x)\,dx}_{D}$:

$$\begin{aligned}
C &= \int_{-\infty}^{\mu} x^2 f(x)\,dx \\
&= \int_{-\infty}^{\mu} x^2\frac{\lambda}{2}e^{\lambda x-\lambda\mu}\,dx \\
&= \frac{1}{2}e^{-\lambda\mu}\int_{-\infty}^{\mu} x^2\lambda e^{\lambda x}\,dx
\end{aligned}$$

Let $u = x^2$, $du = 2x\,dx$, $dv = \lambda e^{\lambda x}$, $v = e^{\lambda x}$

$$
\begin{aligned}
&= \frac{1}{2}e^{-\lambda\mu}\left[x^2 e^{\lambda x}|_{-\infty}^{\mu} - \int_{-\infty}^{\mu} 2xe^{\lambda x}\,dx\right] \\
&= \frac{1}{2}e^{-\lambda\mu}\left[\mu^2 e^{\lambda\mu} - 2\int_{-\infty}^{\mu} xe^{\lambda x}\,dx\right]
\end{aligned}
$$

Recall from the calculations of A that $\int_{-\infty}^{\mu} x\lambda e^{\lambda x}\,dx = \mu e^{\lambda\mu} - \frac{e^{\lambda\mu}}{\lambda}$

$$
\begin{aligned}
&= \frac{1}{2}e^{-\lambda\mu}\left[\mu^2 e^{\lambda\mu} - \frac{2}{\lambda}\left(\mu e^{\lambda\mu} - \frac{e^{\lambda\mu}}{\lambda}\right)\right] \\
&= \frac{\mu^2}{2} - \frac{\mu}{\lambda} + \frac{1}{\lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
D &= \int_{\mu}^{\infty} x^2 f(x)\,dx \\
&= \int_{\mu}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda x + \lambda\mu}\,dx \\
&= \frac{1}{2}e^{\lambda\mu}\int_{\mu}^{\infty} x^2\lambda e^{-\lambda x}\,dx
\end{aligned}
$$

Let $u = x^2$, $du = 2x\,dx$, $dv = \lambda e^{-\lambda x}$, $v = -e^{-\lambda x}$

$$
\begin{aligned}
&= \frac{1}{2}e^{\lambda\mu}\left[-x^2 e^{-\lambda x}|_{\mu}^{\infty} + \int_{\mu}^{\infty} 2xe^{-\lambda x}\,dx\right] \\
&= \frac{1}{2}e^{\lambda\mu}\left[\mu^2 e^{-\lambda\mu} + 2\int_{\mu}^{\infty} xe^{-\lambda x}\,dx\right]
\end{aligned}
$$

Recall from the calculations of B that $\int_{\mu}^{\infty} x\lambda e^{-\lambda x}\,dx = \mu e^{-\lambda\mu} + \frac{e^{-\lambda\mu}}{\lambda}$

$$
\begin{aligned}
&= \frac{1}{2}e^{\lambda\mu}\left[\mu^2 e^{-\lambda\mu} + \frac{2}{\lambda}\left(\mu e^{-\lambda\mu} + \frac{e^{-\lambda\mu}}{\lambda}\right)\right] \\
&= \frac{\mu^2}{2} + \frac{\mu}{\lambda} + \frac{1}{\lambda^2}
\end{aligned}
$$

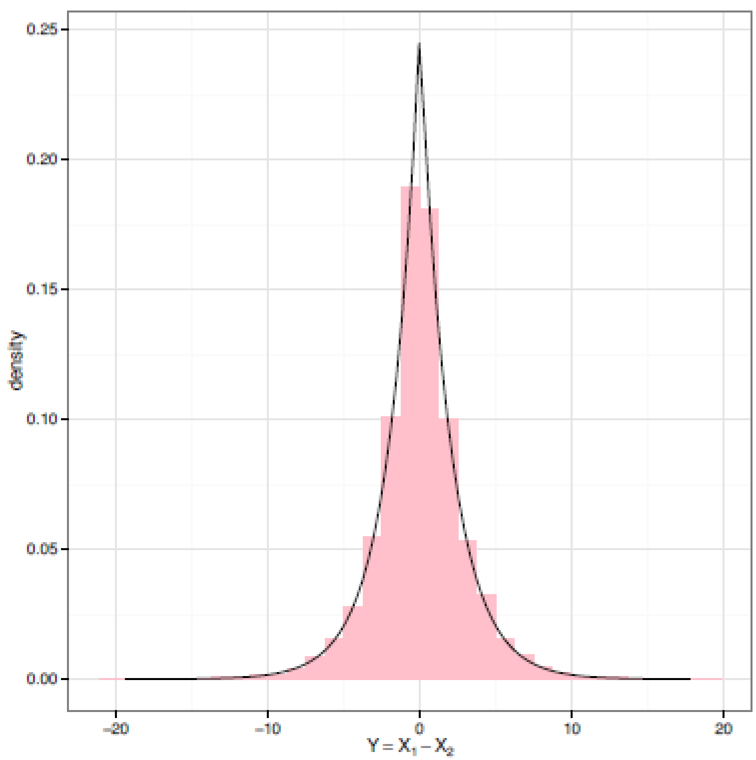
$E[X^2] = C + D = \left(\frac{\mu^2}{2} - \frac{\mu}{\lambda} + \frac{1}{\lambda^2}\right) + \left(\frac{\mu^2}{2} + \frac{\mu}{\lambda} + \frac{1}{\lambda^2}\right) = \mu^2 + \frac{2}{\lambda^2}$.
$Var[X] = E[X^2] - (E[X])^2 = \mu^2 + \frac{2}{\lambda^2} - \mu^2 = \frac{2}{\lambda^2}$

(b)

```
> set.seed(3)
> X1 <- rexp(25000, 1/2)
> X2 <- rexp(25000, 1/2)
> Y <- X1 - X2
```

(i)

```
> laplace <- function(x, lambda = 1, mu = 0){
+ lambda/2*exp(-lambda*abs(x - mu))
+ }
> DFL <- data.frame(x = Y)
> ggplot(data = DFL, aes(x = x)) +
+ geom_histogram(aes(y = ..density..), fill = "pink") +
+ theme_bw() +
+ stat_function(fun = laplace, arg = list(lambda = 1/2, mu = 0)) +
+ labs(x = expression(Y == X[1] - X[2]))
```



(ii) The mean of Y is 0.016 which is within 0.02 of the theoretical mean of 0.

```
> mean(Y)

[1] 0.01597546
```

(iii) The variance of Y is 8.0702 which is within 2% of the theoretical variance of 8.

```
> VY <- var(Y)
> VY

[1] 8.070168

> abs(VY - 8)/8*100

[1] 0.8770954
```

24. Give a general expression to calculate the quantiles of a Weibull random variable.

Solution:

The j^{th} quantile $(0 \leq j \leq 1)$ is the value x_j such that $\int_{-\infty}^{x_j} f(x)\,dx = j$.
For a Weibull,

$$f(x) = \begin{cases} \alpha\beta^{-\alpha}x^{\alpha-1}e^{-(x/\beta)^{\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{aligned} F(x_j) &= \int_0^{x_j} f(x)\,dx \\ &= 1 - e^{-(x_j/\beta)^{\alpha}} \quad x \geq 0 \end{aligned}$$

To find the quantile,

$$\begin{aligned} 1 - e^{-(x_j/\beta)^{\alpha}} &\overset{\text{set}}{=} j \\ e^{-(x_j/\beta)^{\alpha}} &= 1 - j \\ -(x_j/\beta)^{\alpha} &= \ln(1-j) \\ x_j &= \beta\big(-\ln(1-j)\big)^{1/\alpha} \end{aligned}$$

29. Let X be a random variable with probability density function

$$f(x) = 3\left(\frac{1}{x}\right)^4, \quad x \geq 1.$$

(a) Find the cumulative density function.

(b) Fix the seed at 98 (`set.seed(98)`), and generate a random sample of size $n = 100{,}000$ from X's distribution. Compute the mean, variance, and coefficient of skewness for the random sample.

(c) Obtain the theoretical mean, variance, and coefficient of skewness of X.

(d) How close are the estimates in (b) to the theoretical values in (c)?

Solution:

(a)

$$\begin{aligned}F_X(x) &= \int_1^x 3\left(\frac{1}{t}\right)^4 dt \\ &= -t^{-3}\Big|_1^x \\ &= -x^{-3}+1\end{aligned}$$

(b) To do the generation, find the relationship between a uniform and X.

$$\begin{aligned}F_X(x) = -x^{-3}+1 &\stackrel{\text{set}}{=} u \\ -x^{-3} &= u-1 \\ x^{-3} &= 1-u \\ x &= (1-u)^{-1/3} \\ x &= 1/(1-u)^{1/3}\end{aligned}$$

```
> set.seed(98)
> n <- 100000
> u <- runif(n, 0, 1)
> x <- 1/(1 -u)^(1/3)
> ans <- c(mean(x), var(x), mean((x - mean(x))^3)/sd(x)^3)
> names(ans) <- c("Mean", "Variance", "Skewness")
> ans

     Mean   Variance   Skewness
1.5001303  0.7145073 10.6328404
```

(c) Mean:

$$E[X] = \int_1^\infty x \cdot 3\left(\frac{1}{x}\right)^4 dx = \int_1^\infty 3x^{-3}\,dx = \frac{-3x^{-2}}{2}\Big|_1^\infty = \frac{3}{2}$$

Variance:

$$E[X^2] = \int_1^\infty x^2 \cdot 3\left(\frac{1}{x}\right)^4 dx = \int_1^\infty 3x^{-2}\,dx = -3x^{-1}\Big|_1^\infty = 3$$

$$Var[X] = E[X^2] - (E[X])^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

Coefficient of skewness:

$$\gamma_1 = \frac{E\left[(X-\mu)^3\right]}{\sigma^3}$$

$$\begin{aligned}
E\left[(X-\mu)^3\right] &= \int_1^{\infty} (x-1.5)^3 \cdot 3\left(\frac{1}{x}\right)^4 dx \\
&= \int_1^{\infty} 3x^{-4}(x^3 - 4.5x^2 + 6.75x - 3.375)\, dx \\
&= \int_1^{\infty} (3x^{-1} - 13.5x^{-2} + 20.25x^{-3} - 10.125x^{-4})\, dx \\
&= 3\ln(x) + 13.5x^{-1} - 10.125x^{-2} + \left.\frac{10.125x^{-3}}{3}\right|_1^{\infty} = \infty
\end{aligned}$$

$\mu_X = \frac{3}{2}$, $\sigma_X^2 = \frac{3}{4}$, and $\gamma_1 = \infty$.

```
> c(abs(ans[1] - 3/2)/(3/2)*100,
+ abs(ans[2] - 3/4)/(3/4)*100)

       Mean    Variance
0.008686381 4.732364625
```

The estimated mean and variance from (b) are both within 3% of their theoretical values. The skewness from (b) is not close to ∞.

32. Let X be a random variable with probability density function

$$f(x) = (\theta+1)(1-x)^{\theta}, \quad 0 \le x \le 1, \theta > 0.$$

(a) Verify that the area under $f(x)$ is 1.

(b) Find the cumulative density function.

(c) What is $\mathbb{P}(X \le .25 | \theta = 2)$?

(d) Fix the seed at 80 (`set.seed(80)`), and generate 100,000 realizations of X with $\theta = 2$. What are the mean and variance of the random sample?

(e) Calculate the theoretical mean and variance of X when $\theta = 2$.

(f) How close are the estimates in (d) to the theoretical values in (e)?

Solution:

(a) $\int_0^1 (\theta+1)(1-x)^\theta \, dx = -(1-x)^{\theta+1}\big|_0^1 = 0-(-1) = 1$

(b) $F_X(x) = \int_0^x (\theta+1)(1-t)^\theta \, dt = -(1-t)^{\theta+1}\big|_0^x = -(1-x)^{\theta+1}+1$

(c) $\mathbb{P}(X \le 0.25|\theta=2) = F(.25|\theta=2) = 1-(1-0.25)^{2+1} = 0.5781$

```
> 1 - (1 - 0.25)^(2 + 1)

[1] 0.578125

> # or
> f <- function(x) {
+     (2 + 1) * (1 - x)^2
+ }
> integrate(f, 0, 0.25)$value

[1] 0.578125
```

(d) To complete the simulation, set $u = F_X(x)$ and solve for x.

$$\begin{aligned} u &\overset{\text{set}}{=} F_X(x) \\ u &= -(1-x)^{\theta+1}+1 \\ (1-u)^{\frac{1}{\theta+1}} &= 1-x \\ x &= 1-(1-u)^{\frac{1}{\theta+1}} \end{aligned}$$

```
> n <- 100000
> u <- runif(n, 0, 1)
> x <- 1 - (1 - u)^(1/(2 + 1))
> ans <- c(mean(x), var(x))
> names(ans) <- c("Mean", "Variance")
> ans

      Mean   Variance
0.24993630 0.03733454
```

(e) Calculated for $\theta = 2$. Mean:

$$
\begin{aligned}
E[X|\theta=2] &= \int_0^1 x\cdot(2+1)(1-x)^2\,dx \\
&= \int_0^1 3x(1-2x+x^2)\,dx \\
&= \int_0^1 3x-6x^2+3x^3\,dx \\
&= \frac{3x^2}{2}-2x^3+\frac{3x^4}{4}\Big|_0^1 \\
&= \frac{3}{2}-2+\frac{3}{4}-0=\frac{1}{4}
\end{aligned}
$$

Variance:

$$
\begin{aligned}
E[X^2|\theta=2] &= \int_0^1 x^2\cdot(2+1)(1-x)^2\,dx \\
&= \int_0^1 3x^2(1-2x+x^2)\,dx \\
&= \int_0^1 3x^2-6x^3+3x^4\,dx \\
&= x^3-\frac{3x^4}{2}+\frac{3x^5}{5}\Big|_0^1 \\
&= 1-\frac{3}{2}+\frac{3}{5}-0=\frac{1}{10}
\end{aligned}
$$

$Var[X|\theta=2]=E[X^2|\theta=2]-(E[X|\theta=2])^2=\frac{1}{10}-\left(\frac{1}{4}\right)^2=\frac{3}{80}$

(f) The simulated mean (0.2499) and the simulated variance (0.0373) are both within 1% of their theoretical values.

```
> c(abs(ans[1] -1/4)/(1/4)*100, abs(ans[2] - 3/80)/(3/80)*100)

      Mean   Variance
0.02548176 0.44122917
```

34. A copper wire manufacturer produces conductor cables. These cables are of practical use if their resistance lies between 0.10 and 0.13 ohms per meter. The resistance of the cables follows a normal distribution, where 50% of the cables have resistance under 0.11 ohms and 10% have resistance over 0.13 ohms.

(a) Determine the mean and the standard deviation for cable resistance.

(b) Find the probability that a randomly chosen cable can be used.

(c) Find the probability that at least 3 out of 5 randomly chosen cables can be used.

Solution:

(a) Let X = resistance in ohms for a conductor cable. $X \sim N(\mu, \sigma)$. It is given that $\mathbb{P}(X < 0.11) = 0.5$ and $\mathbb{P}(X > 0.13) = 0.1$.

$$
\begin{aligned}
\mathbb{P}(X < 0.11) &= 0.5 \\
\mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{.11-\mu}{\sigma}\right) &= 0.5 \\
\mathbb{P}\left(Z < \frac{.11-\mu}{\sigma}\right) &= 0.5 \\
\implies Z_{0.5} &= \frac{.11-\mu}{\sigma} \\
0 &= \frac{.11-\mu}{\sigma} \\
\mu &= 0.11
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(X > 0.13) &= 0.1 \\
\mathbb{P}\left(\frac{X-0.11}{\sigma} > \frac{0.13-0.11}{\sigma}\right) &= 0.1 \\
\mathbb{P}\left(Z > \frac{0.13-0.11}{\sigma}\right) &= 0.1 \\
\implies Z_{0.9} &= \frac{0.13-0.11}{\sigma} \\
1.2816\sigma &= 0.02 \\
\sigma &= 0.0156
\end{aligned}
$$

$\therefore X \sim N(0.11, 0.0156)$

```
> mu <- 0.11
> sigma <- 0.02/qnorm(0.90)
> c(mu, sigma)

[1] 0.11000000 0.01560608
```

(b) $\mathbb{P}(0.10 < X < 0.13) = 0.6392$

```
> Area <- pnorm(0.13, mu, sigma) - pnorm(0.10, mu, sigma)
> Area

[1] 0.6391658
```

(c) Let Y = number of usable conductor cables. $Y \sim Bin(n = 5, \pi = 0.6392)$.
$\mathbb{P}(Y \geq 3) = 1 - P(Y \leq 2) = 0.7478$

```
> 1 - pbinom(2, 5, Area)

[1] 0.7477729
```