39. Let the random variable X be the sum of the numbers on two fair dice. Find an upper bound on $\mathbb{P}(|X-7| \geq 4)$ using Chebyshev's Inequality as well as the exact probability for $\mathbb{P}(|X-7| \geq 4)$.

Solution:

The probability density is

$$\begin{array}{c|ccccccccccc} x & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline p(x) & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} \end{array},$$

which implies $\mu_X = \sum_x x \cdot p(x) = 7$ and $\sigma_x^2 = \sum_x (x-\mu_x)^2 \cdot p(x) = 5.8\overline{3}$.
The bound given by Chebyshev's Inequality says

$$\mathbb{P}(|X-\mu_x| \geq k) \leq \frac{\sigma_x^2}{k^2}$$

$$\mathbb{P}(|X-7| \geq 4) \leq \frac{5.8\overline{3}}{4^2}$$

$$\implies \mathbb{P}(|X-7| \geq 4) \leq 0.3645833$$

```
> dicerolls <- expand.grid(1:6, 1:6)
> SDR <- apply(dicerolls, 1, sum)
> PDF <- fractions(table(SDR)/36)
> PDF

SDR
   2    3    4    5    6    7    8    9   10   11   12
1/36 1/18 1/12  1/9 5/36  1/6 5/36  1/9 1/12 1/18 1/36
```

```
> MX <- sum(2:12 * PDF)
> VX <- sum((2:12 - MX)^2 * PDF)
> SX <- sqrt(VX)
> c(MX, VX, SX)

[1] 7.000000 5.833333 2.415229

> UL <- VX/4^2
> UL

[1] 35/96
```

The exact probability is $\mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=11) + \mathbb{P}(X=12) = 1/6$.

48. Consider the random variable X, which takes the values 1, 2, 3, and 4 with probabilities 0.2, 0.3, 0.1, and 0.4, respectively. Calculate $E[X]$, $1/E[X]$, $E[1/X]$, $E[X^2]$, and $E[X]^2$, and check empirically that $E[X]^2 \neq E[X^2]$ and $E[1/X] \neq 1/E[X]$.

Solution:

```
> x <- 1:4
> px <- c(0.2, 0.3, 0.1, 0.4)
> EX <- sum(x*px)
> ErX <- sum(1/x*px)
> EX2 <- sum(x^2*px)
> ans <- c(EX, 1/EX, ErX, EX2, EX^2)
> names(ans) <- c("E(X)", "1/E(X)", "E(1/X)", "E(X^2)", "E(X)^2")
> ans

     E(X)    1/E(X)    E(1/X)    E(X^2)    E(X)^2
2.7000000 0.3703704 0.4833333 8.7000000 7.2900000

> EX2 == EX^2   # is E(X^2) = E(X)^2

[1] FALSE

> ErX == 1/EX   # is E(1/X) = 1/E(X)

[1] FALSE
```

Clearly, $E[X^2] = 8.7 \neq 7.29 = (E[X])^2$ and $E[1/X] = 0.4833 \neq 0.3704 = 1/E[X]$.

50. Find the values of k such that the following functions are probability density functions:
(a) $f(x) = kx^4/5$, $0 < x < 1$.

Solution:

In each case, k must be found so that the integral of $f(x)$ is equal to 1.
(a) $f(x) = kx^4/5$

$$\begin{aligned}\int_0^1 \frac{kx^4}{5}\,dx &\overset{\text{set}}{=} 1\\ \left.\frac{kx^5}{25}\right|_0^1 &= 1\\ \frac{k}{25} &= 1\\ k &= 25\end{aligned}$$

So, $f(x) = 5x^4$ and $F(x) = \int_0^x 5y^4\,dy = x^5$.

52. Consider an experiment where two dice are rolled. Let the random variable X equal the sum of the two dice and the random variable Y be the difference of the two dice.

(a) Find the mean of X.

(b) Find the variance of X.

(c) Find the skewness of X.

(d) Find the mean of Y.

(e) Find the variance of Y.

(f) Find the skewness of Y.

Solution:

(a)

```
> SS <- expand.grid(roll1 = 1:6, roll2 = 1:6)
> X <- apply(SS, 1, sum)
> x <- sort(unique(X))
> px <- xtabs(~X)/36
> EX <- sum(x*px)
> EX

[1] 7
```

(b)

```
> VX <- sum((x - EX)^2*px)
> VX

[1] 5.833333
```

(c)

```
> SX <- sum((x - EX)^3*px)/VX^(3/2)
> SX

[1] 0
```

(d)

```
> Y <- apply(SS, 1, diff)
> y <- sort(unique(Y))
> py <- xtabs(~Y)/36
> EY <- sum(y*py)
> EY

[1] 0
```

(e)

```
> VY <- sum((y - EY)^2*py)
> VY

[1] 5.833333
```

(f)

```
> SY <- sum((y - EY)^3*py)/VY^(3/2)
> SY

[1] 0
```

58. Consider the probability density function

$$f(x) = \frac{1}{36} x e^{-x/6}, \quad x > 0.$$

Derive the moment generating function, and calculate the mean and the variance.

Solution:

The moment generating function is calculated as $E[e^{tx}]$.

$$\begin{aligned} M_x(t) = E[e^{tx}] &= \int_0^\infty e^{tx} \cdot \frac{1}{36} x e^{-x/6} \, dx \\ &= \int_0^\infty \frac{1}{36} x e^{x\left(t-\frac{1}{6}\right)} \, dx \end{aligned}$$

Let $u = x$, $du = dx$, $dv = e^{x\left(t-\frac{1}{6}\right)} dx$, and $v = \frac{e^{x\left(t-\frac{1}{6}\right)}}{\left(t-\frac{1}{6}\right)}$

$$= \frac{1}{36} \left[x \frac{e^{x\left(t-\frac{1}{6}\right)}}{\left(t-\frac{1}{6}\right)} \Bigg|_0^\infty - \int_0^\infty \frac{e^{x\left(t-\frac{1}{6}\right)}}{\left(t-\frac{1}{6}\right)} \, dx \right]$$

If $t < \frac{1}{6}$, the moment generating function will exist, otherwise, it is infinite.

$$\begin{aligned} &= \frac{1}{36} \left[0 - \frac{e^{x\left(t-\frac{1}{6}\right)}}{\left(t-\frac{1}{6}\right)^2} \Bigg|_0^\infty \right] \\ &= \frac{1}{36} \cdot \frac{1}{\left(t-\frac{1}{6}\right)^2} \\ \implies M_x(t) &= (6t-1)^{-2} \text{ for } t < \frac{1}{6} \end{aligned}$$

$$\begin{aligned} E[X] &= \frac{d}{dt}(6t-1)^{-2}\Big|_{t=0} = -2(6t-1)^{-3} \cdot 6\big|_{t=0} = 12 \\ E[X^2] &= \frac{d^2}{dt^2}(6t-1)^{-2}\Big|_{t=0} = -2 \cdot -3(6t-1)^{-4} \cdot 6^2\big|_{t=0} = 216 \\ Var[X] &= E[X^2] - (E[X])^2 = 216 - 12^2 = 72 \end{aligned}$$

60. Prove that if a and b are real-valued constants, then

(1) $M_{X+a}(t) = E\left[e^{(X+a)t}\right] = e^{at} \cdot M_X(t)$.

(2) $M_{bX}(t) = E\left(e^{bXt}\right) = M_X(bt)$.

(3) $M_{\frac{X+a}{b}}(t) = E\left[e^{\left(\frac{X+a}{b}\right)t}\right] = e^{\frac{a}{b}t} \cdot M_X\left(\frac{t}{b}\right)$.

Solution:

It must be shown that if a and b are real-valued constants, then

(1) $M_{X+a}(t) = E\left[e^{(X+a)t}\right] = e^{at} \cdot M_X(t)$.

(2) $M_{bX}(t) = E\left(e^{bXt}\right) = M_X(bt)$.

(3) $M_{\frac{X+a}{b}}(t) = E\left[e^{\left(\frac{X+a}{b}\right)t}\right] = e^{\frac{a}{b}t} \cdot M_X\left(\frac{t}{b}\right)$.

(1)

$$\begin{aligned}
M_{X+a}(t) &= E\left[e^{(X+a)t}\right] \\
&= E\left[e^{Xt+at}\right] \\
&= E\left[e^{at} \cdot e^{tX}\right] \\
&= e^{at} \cdot E\left[e^{tX}\right] = e^{at} \cdot M_X(t)
\end{aligned}$$

(2)

$$\begin{aligned}
M_{bX}(t) &= E\left[e^{bXt}\right] \\
&= E\left[e^{Xbt}\right] = M_X(bt)
\end{aligned}$$

(3)

$$\begin{aligned}
M_{\frac{X+a}{b}}(t) &= E\left[e^{\left(\frac{X+a}{b}\right)t}\right] \\
&= E\left[e^{X \cdot \frac{t}{b}} \cdot e^{\frac{a}{b} \cdot t}\right] \\
&= e^{\frac{a}{b}t} \cdot M_X\left(\frac{t}{b}\right)
\end{aligned}$$