11. Traffic volume is an important factor for determining the most cost-effective method to surface a road. Suppose that the average number of vehicles passing a certain point on a road is 2 every 30 seconds.

(a) Find the probability that more than 3 cars will pass the point in 30 seconds.

(b) What is the probability that more than 10 cars pass the point in 3 minutes?

Solution:

(a) Let X = number of cars passing a certain point on the road in 30 seconds. $X \sim Pois(\lambda = 2)$. $\mathbb{P}(X > 3) = 1 - \mathbb{P}(X <= 3) = 0.1429$

```
> 1 - ppois(3, 2)

[1] 0.1428765
```

(b) Let Y = number of cars passing a certain point on the road in 3 minutes. $Y \sim Pois(\lambda = 12)$. $\mathbb{P}(Y > 10) = 1 - \mathbb{P}(X <= 10) = 0.6528$.

```
> 1 - ppois(10, 12)

[1] 0.6527706
```

17. Derive the mean and variance for the discrete uniform distribution.
(Hints: $\sum_{i=1}^n x_i = \dfrac{n(n+1)}{2}; \sum_{i=1}^n x_i^2 = \dfrac{n(n+1)(2n+1)}{6}$, when $x_i = 1, 2, \ldots, n$.)

Solution:

X is a discrete uniform, which means it takes on values in $\{1, 2, 3, \ldots, n\}$ with probability $\frac{1}{n}$ each.
$E[X] = \sum_{i=1}^n x_i \cdot \mathbb{P}(X = x_i) = \sum_{i=1}^n i \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^n i = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}$
$E[X^2] = \sum_{i=1}^n x_i^2 \cdot \mathbb{P}(X = x_i) = \sum_{i=1}^n i^2 \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^n i^2 = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$

$$
\begin{aligned}
Var[X] = E[X^2] - (E[X])^2 &= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{2n^2 + 3n + 1}{6} - \frac{n^2 + 2n + 1}{4} \\
&= \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} \\
&= \frac{n^2 - 1}{12}
\end{aligned}
$$

18. Suppose the percentage of drinks sold from a vending machine are 80% and 20% for soft drinks and bottled water, respectively.

(a) What is the probability that on a randomly selected day, the first soft drink is the fourth drink sold?

(b) Find the probability that exactly 1 out of 10 drinks sold is a soft drink.

(c) Find the probability that the fifth soft drink is the seventh drink sold.

(d) Verify empirically that $\mathbb{P}\big(Bin(n,\pi) \le r-1\big) = 1 - \mathbb{P}\big(NB(r,\pi) \le (n-r)\big)$, with $n = 10$, $\pi = 0.8$, and $r = 4$.

Solution:

Let X = number of waters (failures) purchased before the first soft drink is purchased. Then, $X \sim Geo(0.80)$.

(a) $\mathbb{P}(X = 3) = 0.0064$ or use $X \sim NB(1, 0.80)$.

```
> dgeom(3, 0.80)

[1] 0.0064

> dnbinom(3, 1, 0.80)

[1] 0.0064
```

(b) Let X = number of soft drinks sold. Then, $X \sim Bin(10, 0.80)$ and $\mathbb{P}(X = 1) = 0$.

```
> dbinom(1, 10, 0.80)

[1] 4.096e-06
```

(c) Let X = number of water purchased before the fifth soft drink is purchased. Then, $X \sim NB(5, 0.80)$ and $\mathbb{P}(X = 2) = 0.1966$.

```
> dnbinom(2, 5, 0.80)

[1] 0.196608
```

(d)

```
> A <- pbinom(3, 10, 0.80)
> B <- 1 - pnbinom(6, 4, 0.8)
> c(A, B)

[1] 0.0008643584 0.0008643584
```

38. Consider the function $g(x) = (x-a)^2$, where a is a constant and $E\left[(X-a)^2\right]$ is finite. Find a so that $E\left[(X-a)^2\right]$ is minimized.

Solution:

$$h(a) = E[(x-a)^2] = E[X^2] - 2aE[X] + a^2.$$

Then $h'(a) = -2E[X] + 2a \overset{\text{set}}{=} 0 \implies a = E[X]$. Since, $h''(a) = 2 > 0$, $a = E[X]$ minimizes $h(a)$.

40. If $X \sim Bin(n, \pi)$, use the binomial expansion to find the mean and variance of X. To find the variance, use the second factorial moment $E[X(X-1)]$ and note that $\frac{x}{x!} = \frac{1}{(x-1)!}$ when $x > 0$.

Solution:

$X \sim Bin(n, \pi) \implies \mathbb{P}(X = x) = \binom{n}{x}\pi^x(1-\pi)^{n-x}$.
Mean:

$$\begin{aligned} E[X] &= \sum_{x=0}^{n} x \cdot \binom{n}{x}\pi^x(1-\pi)^{n-x} \\ &= \sum_{x=0}^{n} \frac{x \cdot n!}{x!(n-x)!}\pi^x(1-\pi)^{n-x} \end{aligned}$$

Note that the first term in the sum $(x = 0)$ is zero

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}\pi^x(1-\pi)^{n-x}$$

Let $k = x - 1$.

$$\begin{aligned} &= \sum_{k=0}^{n-1} \frac{n!}{k!(n-k-1)!}\pi^{k+1}(1-\pi)^{n-k-1} \\ &= n\pi \underbrace{\sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!}\pi^k(1-\pi)^{n-1-k}}_{\text{Sum of all } Bin(n-1,\pi) \text{ probabilities } = 1} \end{aligned}$$

$$\therefore E[X] = n\pi$$

Finding $E[X(X-1)]$:

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{x!(n-x)!}\pi^x(1-\pi)^{n-x}$$

Note that the first two terms in the sum $(x = 0, 1)$ are zero

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}\pi^x(1-\pi)^{n-x}$$

Let $k = x - 2$, so $x = k + 2$.

$$= \sum_{k}^{n-2} \frac{n!}{k!(n-2-k)!}\pi^{k+2}(1-\pi)^{n-2-k}$$

$$= \pi^2 n(n-1) \underbrace{\sum_{k}^{n-2} \frac{(n-2)!}{k!(n-2-k)!}\pi^k(1-\pi)^{n-2-k}}_{\text{Sum of all } Bin(n-2,\pi) \text{ probabilities } = 1}$$

$$\therefore E[X(X-1)] = \pi^2 n(n-1)$$

The second factorial moment is $E[X(X-1)] = E[X^2] - E[X]$, so

$$\begin{aligned} Var[X] &= E[X(X-1)] + E[X] - (E[X])^2 \\ &= \pi^2 n(n-1) + n\pi - (n\pi)^2 \\ &= n^2\pi^2 - n\pi^2 + n\pi - n^2\pi^2 \\ &= n\pi(1-\pi) \end{aligned}$$