

MSSC6010-Fall2025

Homework 2 Solution

23. Assume that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(A \cap C) = 0.2$, $\mathbb{P}(C) = 0.4$, $\mathbb{P}(B) = 0.4$, $\mathbb{P}(A \cap B \cap C) = 0.1$, $\mathbb{P}(B \cap C) = 0.2$, and $\mathbb{P}(A \cap B) = 0.2$. Calculate the following probabilities:

- (a) $\mathbb{P}(A \cup B \cup C)$
- (b) $\mathbb{P}(A^c \cap (B \cup C))$
- (c) $\mathbb{P}((B \cap C)^c \cup (A \cap B)^c)$
- (d) $\mathbb{P}(A) - \mathbb{P}(A \cap C)$

Solution:

$$\begin{aligned}
 \text{(a)} \quad & \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C) \\
 & = 0.5 + 0.4 + 0.4 - 0.2 - 0.2 - 0.2 + 0.1 = 0.8 \\
 \text{(b)} \quad & \mathbb{P}(A^c \cap (B \cup C)) = \mathbb{P}(B \cup C) - \mathbb{P}((B \cup C) \cap A) \\
 & = \{\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)\} - \mathbb{P}(AB \cup AC) \\
 & = \{0.4 + 0.4 - 0.2\} - \{0.2 + 0.2 - 0.1\} = 0.3 \\
 \text{(c)} \quad & \mathbb{P}((B \cap C)^c \cup (A \cap B)^c) = 1 - \mathbb{P}((B \cap C) \cap (A \cap B)) = 1 - \mathbb{P}(A \cap B \cap C) \\
 & = 1 - 0.1 = 0.9 \\
 \text{(d)} \quad & \mathbb{P}(A) - \mathbb{P}(A \cap C) = 0.5 - 0.2 = 0.3
 \end{aligned}$$

25. Verify that $\mathbb{P}(F|E)$ satisfies the three axioms of probability.

Solution:

It must be shown that $\mathbb{P}(F|E) = \frac{\mathbb{P}(F \cap E)}{\mathbb{P}(E)}$ satisfies

- (1) $0 \leq \mathbb{P}(F|E) \leq 1$
- (2) $\mathbb{P}(\Omega|E) = 1$
- (3) $\mathbb{P}(\cup_{i=1}^{\infty} F_i|E) = \sum_{i=1}^n \mathbb{P}(F_i|E)$

(1) The left side is obvious since all probabilities must be greater than or equal to zero. Since $F \cap E \subset E$, it follows that $\mathbb{P}(F \cap E) \leq \mathbb{P}(E)$ which is less than or equal to one.

$$(2) \quad \mathbb{P}(\Omega|E) = \frac{\mathbb{P}(\Omega \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E)}{\mathbb{P}(E)} = 1$$

(3)

$$\begin{aligned}
 \mathbb{P}(\cup_{i=1}^{\infty} F_i|E) &= \frac{\mathbb{P}(\cup_{i=1}^{\infty} (F_i \cap E))}{\mathbb{P}(E)} \\
 &= \sum_{i=1}^{\infty} \frac{\mathbb{P}(F_i \cap E)}{\mathbb{P}(E)} \\
 &= \sum_{i=1}^n \mathbb{P}(F_i|E)
 \end{aligned}$$

33. A salesman in a department store receives household appliances from three suppliers: I, II, and III. From previous experience, the salesman knows that 2%, 1%, and 3% of the appliances from suppliers I, II, and III, respectively, are defective. The salesman sells 35% of the appliances from supplier I, 25% from supplier II, and 40% from supplier III. If an appliance randomly selected is defective, find the probability that it comes from supplier III.

Solution:

Let I , II , and III be events associated with suppliers and D be the event associated with a defective appliance.

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|I) \cdot \mathbb{P}(I) + \mathbb{P}(D|II) \cdot \mathbb{P}(II) + \mathbb{P}(D|III) \cdot \mathbb{P}(III) \\ &= 0.02 \times 0.35 + 0.01 \times 0.25 + 0.03 \times 0.40 = 0.0215\end{aligned}$$

$$\mathbb{P}(III|D) = \frac{\mathbb{P}(D|III) \cdot \mathbb{P}(III)}{\mathbb{P}(D)} = \frac{0.03 \times 0.4}{0.0215} = 0.5581$$

37. John and Peter play a game with a coin such that $\mathbb{P}(\text{head}) = p$. The game consists of tossing a coin twice. John wins if the same result is obtained in the two tosses, and Peter wins if the two results are different.

- (a) At what value of p is neither of them favored by the game?
- (b) If p is different from your answer in (a), who is favored?

Solution:

$$\mathbb{P}(\text{John wins}) = p^2 + (1-p)^2 \text{ and } \mathbb{P}(\text{Peter wins}) = 1 - \mathbb{P}(\text{John wins}).$$

- (a) When $\mathbb{P}(\text{John wins}) = \mathbb{P}(\text{Peter wins}) = 1/2$, the game is fair.

$$\begin{aligned}\mathbb{P}(\text{John wins}) &= p^2 + (1-p)^2 = \frac{1}{2} \\ 2p^2 + 2(1-p)^2 &= 1 \\ 4p^2 - 4p + 1 &= 0 \\ (2p-1)^2 &= 0 \\ \implies p &= \frac{1}{2}\end{aligned}$$

If $p = 1/2$ both of them have the same probability of winning the game.

- (b) Since $(2p-1)^2 > 0$ for all $p \neq 1/2$, John wins for any different answer than that in (a).

41. Two independently wealthy philatelists, Alvin and Bob, are interested in buying rare stamps at a private auction. For each stamp up for auction, given that the previous bid did not win, Alvin or Bob wins on their i^{th} bid with probability p . Assume that Alvin always makes the first bid.

(a) Find the probability that Alvin wins the first auction.

(Hint: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ if $|r| < 1$.)

Solution:

(a) Let $A = \text{Alvin buys a stamp}$. Let $S_i = \text{stamp is won on the } i^{\text{th}} \text{ bid}$. Note that $(1-p) < 1$, so that the geometric series simplification holds.

$$\begin{aligned}\mathbb{P}(S_i) &= (1-p)^{i-1} \cdot p \\ \mathbb{P}(A) &= \mathbb{P}(S_1 \cup S_3 \cup S_5 \cup \dots) \\ &= p + (1-p)^2 \cdot p + (1-p)^4 \cdot p + \dots \\ &= p [1 + (1-p)^2 + (1-p)^4 + (1-p)^6 + \dots] \\ &= p \cdot \frac{1}{1 - (1-p)^2} \\ &= \frac{p}{1 - (1 - 2p + p^2)} \\ &= \frac{1}{2 - p}\end{aligned}$$

44. Consider tossing three fair coins. The eight possible outcomes are

$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$.

Define X as the random variable “number of heads showing when three coins are tossed.” Obtain the mean and the variance of X . Simulate tossing three fair coins 10,000 times. Compute the simulated mean and variance of X . Are the simulated values within 2% of the theoretical answers?

Solution:

```
> library(MASS)
> SS <- expand.grid(toss1 = 0:1, toss2 = 0:1, toss3 = 0:1)
> SCT <- apply(SS, 1, sum)
> PDF <- fractions(table(SCT)/8)
> PDF

SCT
 0   1   2   3 
1/8 3/8 3/8 1/8
```

```

> x <- 0:3
> MX <- sum(x * PDF)
> VX <- sum((x - MX)^2 * PDF)
> c(MX, VX)
[1] 1.50 0.75

> set.seed(13)
> sims <- 10000
> flips <- 3
> heads <- numeric(sims)
> for (i in 1:sims) {
+   heads[i] <- sum(sample(0:1, size = flips, replace = TRUE))
+ }
> SPDF <- xtabs(~heads)/sims
> SPDF

heads
 0      1      2      3
0.1289 0.3746 0.3683 0.1282
> SMX <- sum(x * SPDF)
> SVX <- sum((x - SMX)^2 * SPDF)
> c(SMX, SVX)

[1] 1.4958000 0.7641824

> abs((MX - SMX)/MX) * 100
[1] 7/25

> abs((VX - SVX)/VX) * 100
[1] 34743/18373

```

The mean of X is $3/2$ and the variance of X is $3/4$. The simulated mean of X is 1.4958 and the simulated variance of X is 0.7642. Both simulated values are within 2% of the theoretical answers.