Chapter 6

Monte Carlo Integration and Variance Reduction

6.1 *Compute a Monte Carlo estimate of*

$$\int_0^{\pi/3} sin\, t\, dt$$

and compare your estimate with the exact value of the integral.

The simple Monte Carlo estimator is

$$(b-a)\int_a^b g(x)dx = \frac{\pi}{3}\left\{\frac{1}{m}\sum_{i=1}^{m} sin(u)\right\},$$

where u is generated from Uniform$(0, \pi/3)$.

```
m <- 10000
x <- runif(m, 0, pi/3)
theta.hat <- pi/3 * mean(sin(x))
print(theta.hat)

## [1] 0.5030077
```

The exact value of the integral is 0.5. Repeating the estimation 1000 times gives an estimate of the standard error:

```
y <- replicate(1000, expr = {
    x <- runif(m, 0, pi/3)
    theta.hat <- pi/3 * mean(sin(x)) } )
    mean(y)
```

```
## [1] 0.4999822

    sd(y)

## [1] 0.002743902
```

6.2 *Compute a Monte Carlo estimate of the standard normal cdf, by generating from the Uniform(0,x) distribution. Compare your estimates with the normal cdf function* `pnorm`*. Compute an estimate of the variance of your Monte Carlo estimate of* $\Phi(2)$*, and a 95% confidence interval for* $\Phi(2)$*.*

```
    x <- seq(.1, 2.5, length = 10)
    m <- 10000
    cdf <- numeric(length(x))
    for (i in 1:length(x)) {
        u <- runif(m, 0, x[i])
        g <- x[i] * exp(-(u^2) / 2)
        cdf[i] <- mean(g) / sqrt(2 * pi) + 0.5
    }

    Phi <- pnorm(x)
    print(round(rbind(x, cdf, Phi), 3))

##     [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8]  [,9] [,10]
## x   0.10 0.367 0.633 0.900 1.167 1.433 1.700 1.967 2.233 2.500
## cdf 0.54 0.643 0.737 0.816 0.879 0.924 0.954 0.974 0.988 0.991
## Phi 0.54 0.643 0.737 0.816 0.878 0.924 0.955 0.975 0.987 0.994
```

To estimate the variance of the MC estimate of $\Phi(2)$, replicate the experiment. Then apply the CLT to construct an approximate 95% confidence interval for $\Phi(2)$.

```
    est <- replicate(1000, expr = {
        u <- runif(m, 0, 2)
        g <- 2 * exp(-(u^2) / 2)
        mean(g) / sqrt(2 * pi) + 0.5
        } )

    pnorm(2)

## [1] 0.9772499

    c(mean(est), sd(est))
```

```
## [1] 0.977209570 0.002317498

    mean(est) + qnorm(c(.025,.975)) * sd(est)

## [1] 0.9726674 0.9817518
```

6.3 *Compute a Monte Carlo estimate $\hat{\theta}$ of*

$$\theta = \int_0^{0.5} e^{-x}\, dx$$

by sampling from Uniform(0, 0.5), and estimate the variance of $\hat{\theta}$. Find another Monte Carlo estimator θ^* by sampling from the exponential distribution. Which of the variances (of $\hat{\theta}$ and $\hat{\theta}^*$) is smaller, and why?

[The exact value of the integral is $\theta = 1 - e^{-.5} \doteq 0.3934693$.]

The simple Monte Carlo estimator is

$$\hat{\theta} = (b-a)\int_a^b g(x)dx = \frac{1}{2}\left\{\frac{1}{m}\sum_{i=1}^{m} e^{-u}\right\},$$

where u is generated from Uniform$(0, \frac{1}{2})$.

```
    m <- 10000
    u <- runif(m, 0, .5)
    theta <- .5 * mean(exp(-u))
    theta

## [1] 0.3924437

    est <- replicate(1000, expr = {
        u <- runif(m, 0, .5)
        theta <- .5 * mean(exp(-u))
    } )

    mean(est)

## [1] 0.393469

    c(var(est), sd(est))

## [1] 3.140757e-07 5.604246e-04
```

Let

$$\hat{\theta}^* = \frac{1}{m}\sum_{i=1}^{m} I(v < 0.5),$$

where v is generated from standard exponential distribution.

```
m <- 10000
v <- rexp(m, 1)
theta <- mean(v <= .5)
theta
```

```
## [1] 0.3938
```

```
est1 <- replicate(1000, expr = {
    v <- rexp(m, 1)
    theta <- mean(v <= .5)
} )

mean(est1)
```

```
## [1] 0.3935925
```

```
c(var(est1), sd(est1))
```

```
## [1] 2.391485e-05 4.890281e-03
```

```
var(est) / var(est1)
```

```
## [1] 0.01313308
```

The simulation suggests that $Var(\hat{\theta}) < Var(\hat{\theta}^*)$. In this example we can compute the exact variance of the estimators for comparison.

$$Var(\hat{\theta}^*) = \frac{\theta(1-\theta)}{m} = (1-e^{-1/2})(e^{-1/2})/m \doteq 2.386512e-05.$$

The variance of $g(U)$ is

$$\begin{aligned}
Var(e^{-U}) &= \int_0^{1/2} 2e^{-2u}du - \left[\int_0^{1/2} 2e^{-u}du\right]^2 \\
&= 1 - e^{-1} - 4(1-e^{-1/2})^2 \\
&= -e^{-1} - 1 - 4(1 - 2e^{-1/2} + e^{-1}) \\
&= 1 - e^{-1} - 4 + 8e^{-1/2} - 4e^{-1} \\
&= 8e^{-1/2} - 5e^{-1} - 3.
\end{aligned}$$

The variance of $\hat{\theta}$ is

$$\frac{Var(\frac{1}{2}g(U))}{m} \doteq \frac{0.01284807}{4m} \doteq 3.212018e-07.$$

Then

$$\frac{Var(\hat{\theta})}{Var(\hat{\theta}^*)} = \frac{0.01284807/4}{(1-e^{-1/2})(e^{-1/2})} \doteq 0.01345905.$$

6.6 *Consider the antithetic variate approach to estimating*

$$\theta = \int_0^1 e^x dx.$$

Compute $Cov(e^U, e^{1-U})$ and $Var(e^U + e^{1-U})$, where $U \sim$ Uniform(0,1). What is the percent reduction in variance of $\hat{\theta}$ that can be achieved using antithetic variates (compared with simple MC)?

$$\begin{aligned}
Cov(e^U, e^{1-U}) &= E[e^U e^{1-U}] - E[e^U]E[e^{(1-U}] \\
&= e - (e-1)^2 \doteq -0.2342106; \\
Var(e^U) &= E[e^{2U}] - (E[e^U])^2 = \frac{1}{2}(e^2-1) - (e-1)^2 \doteq 0.2420356; \\
Cor(e^U, e^{1-U}) &= \frac{Cov(e^U, e^{1-U})}{\sqrt{Var(e^U)}\sqrt{Var(e^{1-U})}} = \frac{e-(e-1)^2}{\frac{1}{2}(e^2-1)-(e-1)^2}.
\end{aligned}$$

(The variances of e^U and e^{1-U} are equal because U and $1-U$ are identically distributed.)

Suppose $\hat{\theta}_1$ is the simple MC estimator and $\hat{\theta}_2$ is the antithetic estimator. Then if U and V are iid Uniform (0,1) variables, we have

$$Var(\frac{1}{2}(e^U + e^V)) = \frac{1}{4}2Var(e^U) = \frac{1}{2} \cdot \frac{1}{2}(e^2 - 1 - (e-1)^2) \doteq 0.1210178.$$

If antithetic variables are used,

$$\begin{aligned}
Var(\frac{1}{2}(e^U + e^{1-U})) &= \frac{1}{4}(2Var(e^U) + 2Cov(e^U, e^{1-U})) \\
&= \frac{1}{2}\left(\frac{1}{2}(e^2-1) - (e-1)^2) + e - (e-1)^2\right) \\
&\doteq 0.003912497.
\end{aligned}$$

The reduction in variance is

$$\frac{Var(\hat{\theta}_1) - Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)} = \frac{0.1210178 - 0.003912497}{0.1210178} = 0.96767,$$

or 96.767%.

6.7 *Refer to Exercise 6.6. Use a Monte Carlo simulation to estimate θ by the antithetic variate approach and by the simple Monte Carlo method. Compute an empirical estimate of the percent reduction in variance using the antithetic variate.*

```
    m <- 10000
    mc <- replicate(1000, expr = {
        mean(exp(runif(m)))})
    anti <- replicate(1000, expr = {
        u <- runif(m/2)
        v <- 1-u
        mean((exp(u) + exp(v))/2) })
    v1 <- var(mc)
    v2 <- var(anti)
    c(mean(mc), mean(anti))

## [1] 1.718507 1.718364

    c(v1, v2)

## [1] 2.548294e-05 7.865248e-07

    (v1 - v2) / v1

## [1] 0.9691352
```

In this simulation the reduction in variance printed on the last line above is close to the theoretical value 0.96767 from Exercise 6.6.

6.9 *The Rayleigh density is*

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}, \qquad x \geq 0, \sigma > 0.$$

Implement a function to generate samples from a Rayleigh(σ) distribution, using antithetic variables. What is the percent reduction in variance of $\frac{X+X'}{2}$ compared with $\frac{X_1+X_2}{2}$ for independent X_1, X_2?

Here $F(x) = 1 - \exp(-x^2/(2\sigma^2), x > 0$ and

$$u = 1 - e^{-x^2/(2\sigma^2)} \Rightarrow F^{-1}(u) = \sigma(-2\log(1-u))^{1/2}.$$

```
Ray1 <- function(n, sigma) {
    u <- runif(n)
    return(sigma * sqrt(-2 * log(u)))
    }
Ray2 <- function(n, sigma) {
    u <- runif(n / 2)
    x1 <- sigma * sqrt(-2 * log(u))
    x2 <- sigma * sqrt(-2 * log(1-u))
    return(c(x1, x2))
    }

m <- 10000
sigma <- 2
r1 <- replicate(1000, mean(Ray1(2, sigma)))
r2 <- replicate(1000, mean(Ray2(2, sigma)))
var(r1)
```

```
## [1] 0.8416862
```

```
var(r2)
```

```
## [1] 0.05339187
```

The approximate percent reduction in variance is

```
100*(var(r1) - var(r2)) / var(r1)
```

```
## [1] 93.65656
```

6.12 *Let $\hat{\theta}_f^{IS}$ be an importance sampling estimator of $\theta = \int g(x)dx$, where the importance function f is a density. Prove that if $g(x)/f(x)$ is bounded, then the variance of the importance sampling estimator $\hat{\theta}_f^{IS}$ is finite.*

Suppose that f is a density, $\theta = \int g(x)dx < \infty$, and $\left|\frac{g(x)}{f(x)}\right| \leq M < \infty$. Let $\hat{\theta} = \hat{\theta}_f^{IS}$. Then

$$\begin{aligned} Var\hat{\theta} = E[\hat{\theta}^2] - (E[\hat{\theta}])^2 &= E\left[\frac{1}{m}\sum_{i=1}^{m}\left(\frac{g(X_i)}{f(X_i)}\right)^2 f(X_i)\right] - \theta^2 \\ &= \int \frac{g(x)^2}{f(x)}\,dx - \theta^2 = \int \frac{g(x)}{f(x)} g(x)\,dx - \theta^2 \\ &\leq M \int g(x)dx - \theta^2 = M\theta - \theta^2 < \infty. \end{aligned}$$