MATH 4720 / MSCS 5720

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Chapter 7

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INFERENCE ABOUT POPULATION STANDARD DEVIATION (σ)

- $H_0: \sigma = \sigma_0$ **(pre-assigned value)**
- $H_a: \sigma > \sigma_0$

  or $\sigma < \sigma_0$

  or $\sigma \neq \sigma_0$

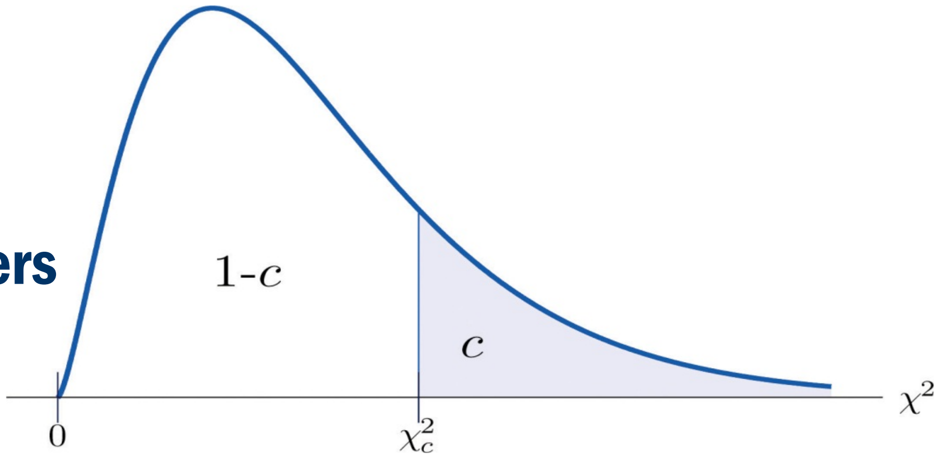
- **Assumption: Data is drawn from a normal population.**

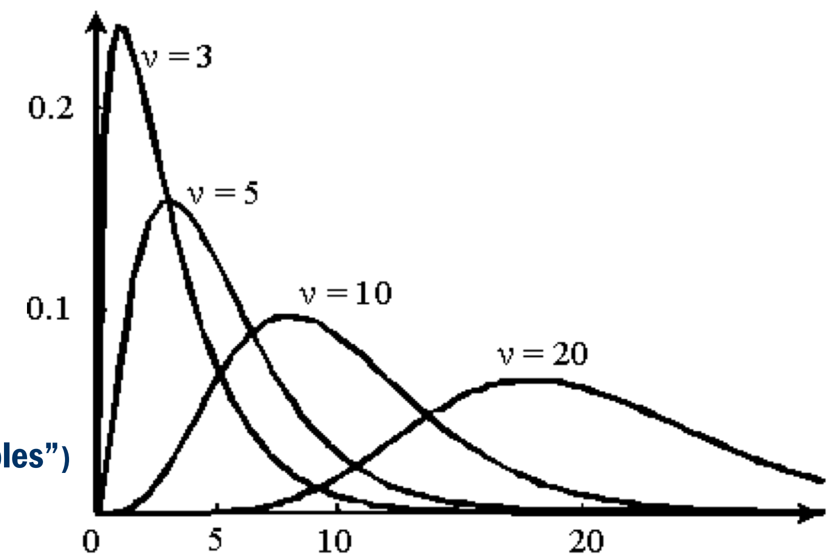
- **T.S.** $\chi^2 = \dfrac{(n-1)s^2}{{\sigma_0}^2}$

- **Decision Rule: (**$\mathrm{df} = n - 1$**)**
  - $H_a: \sigma > \sigma_0$: **Reject** H_0 **if** $\chi^2 > \chi^2_{\alpha}$
  - $H_a: \sigma < \sigma_0$: **Reject** H_0 **if** $\chi^2 < \chi^2_{1-\alpha}$
  - $H_a: \sigma \neq \sigma_0$: **Reject** H_0 **if** $\chi^2 > \chi^2_{\alpha/2}$ **or** $\chi^2 < \chi^2_{1-\alpha/2}$

CHI-SQUARED (χ^2) DISTRIBUTION

- **Right skewed distribution**
- **Defined over positive numbers**
- **Parameter: df=ν**
- **How to write:**
  - $\chi^2(\nu)$
- **[Chi-Squared Calculator]**
- χ^2-table ("D2L > Useful Links > Z, T and χ^2 Tables")
  - $P(\chi^2 \geq c_\alpha)$
- **Ti-84: χ^2cdf(lower, upper, df)**





Source: www.jrigol.com

EXAMPLE

- **Hypothesis test on σ is usually done for quality control purposes.**

Book Example 7.1

- **A machine fills 500-gram coffee container. The machine was designed to fill the average weight of 506.6 grams and standard deviation of 4 grams.**

$$\text{weight} \sim N(506.6,\ 4^2)$$

- **Why the mean is 506.6?**
  - **Ans.** $P(Y < 500) = 0.05$. **Only 5% of the containers contain less than 500 grams of coffee.**

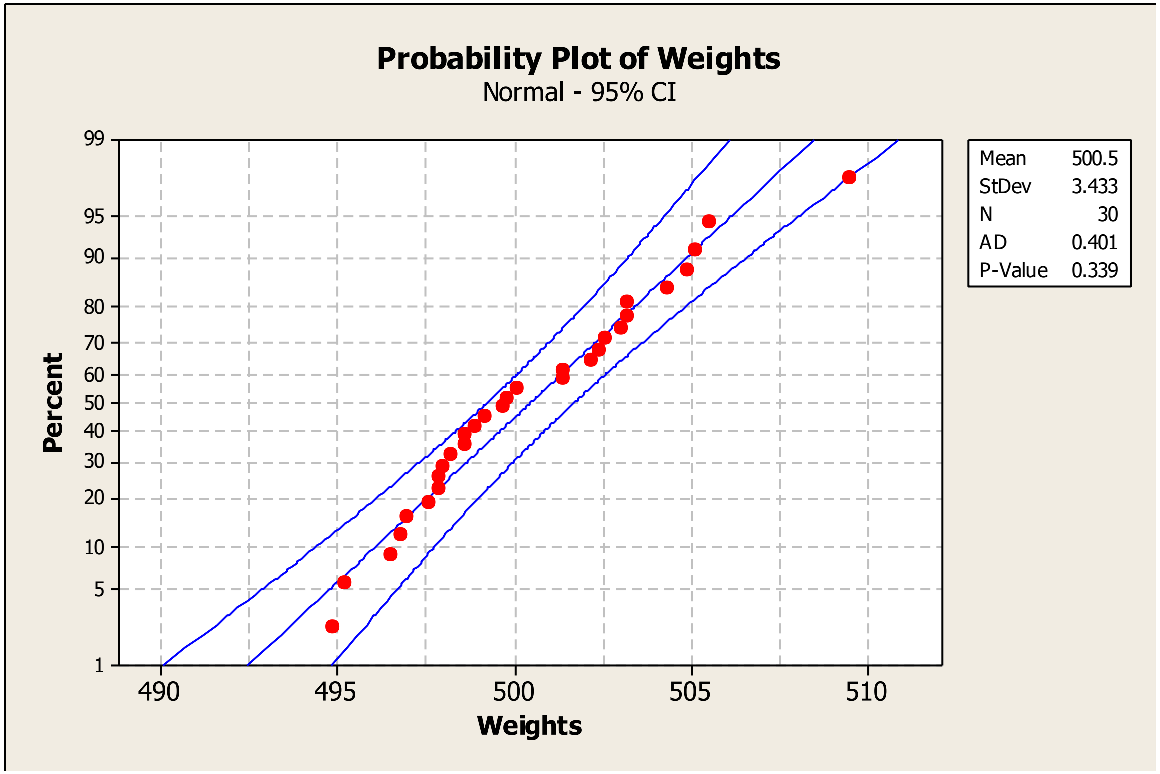
- **To maintained the quality control,**

  Variance (equivalently st. dev.) should not be very high.

EXAMPLE CONT'D

- **Sample of** 30 **containers is taken, and the weights of the coffee are recorded.**
- $\bar{y} = 500.453, \quad s = 4.433$
- **The process will be considered out of control if** $\sigma > 4.0$.
- **Is there a sufficient evidence to conclude that the filling process is out of control?** $\alpha = 0.05$.
- $H_0: \sigma = 4.0 \quad vs. \quad H_a: \sigma > 4.0$
- **Assumption: Check for normality?**

CHECK FOR NORMALITY



- **Normal probability plot confirms that the data is normally distributed.**

EXAMPLE CONT'D

- $T.S.\quad \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(30-1)4.433^2}{4.0^2} = 35.62$
- $df = n - 1 = 29,\ \alpha = 0.05,\ \chi_\alpha^2 = 42.56$
- **Reject** H_0 **in favor of** H_a **if** $\chi^2 > \chi_\alpha^2 = 42.56$
- **Conclusion: Is** $\chi^2 > 42.56?$ **No.**
- **Fail to reject** H_0**. We do not have sufficient evidence to conclude that the process is out of control.**
- **Minitab**
  Stat → Basic Statistics → 1 Variance

CONFIDENCE INTERVAL

- $100(1-\alpha)\%$ **confidence interval of** σ
- **Formula:**

For Variance: $$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

For St. Dev.: $$\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}$$

- **Back to Example 7.1: Estimate** σ **using 95% confidence interval**
- $\alpha = 0.05,\ \ df = 29,\ \chi^2_{\alpha/2} = 45.72,\ \ \chi^2_{1-\alpha/2} = 16.05$
- **95% C.I. of** σ: $$\sqrt{\frac{29*4.433^2}{45.72}} < \sigma < \sqrt{\frac{29*4.433^2}{16.05}}$$

$$3.53 < \sigma < 5.96$$

- **[Chi-Squared Calculator]**

BACK TO POOL T-TEST ASSUMPTIONS:

- Note that, whenever we use t-statistics, there are assumptions. Also note that two samples are drawn from two populations. μ_1 is the mean of population 1, and σ_1 is its standard deviation. μ_2 is the mean of population 2, and σ_2 its standard deviation

- **Assumption 1:** $\sigma_1 = \sigma_2$

- **Assumption 2:** $n_1 \geq 30, n_2 \geq 30$. If not, we assume that both samples are drawn from normal populations.

INFERENCE ABOUT POPULATION STANDARD DEVIATION (σ)

- $H_0: \sigma_1 = \sigma_2$
- $H_a: \sigma_1 > \sigma_2$

  or $\sigma_1 \neq \sigma_2$

- **Assumption: Data is drawn from a normal population.**

- **T.S.** $F = \dfrac{\max\left(s_1^2, s_2^2\right)}{\min\left(s_1^2, s_2^2\right)}$ **This F follows F-distribution if H_0 is true.**

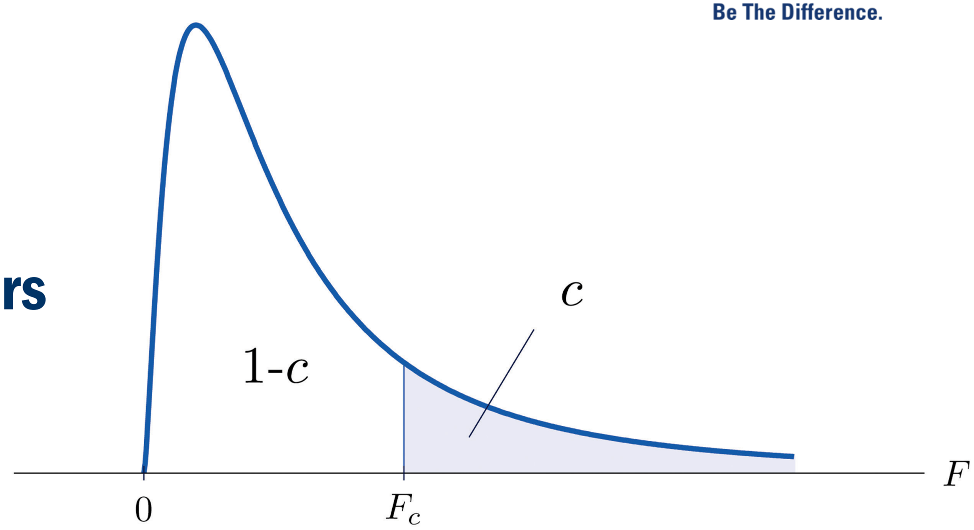
- **Decision Rule: ($\mathrm{df}_1 = n_{\mathrm{numerator}} - 1, \mathrm{df}_2 = n_{\mathrm{denominator}} - 1$)**
  - $H_a: \sigma_1 > \sigma_2$: **Reject** H_0 **if** $F \geq F_{\alpha}(\mathrm{df}_1, \mathrm{df}_2)$
  - $H_a: \sigma_1 \neq \sigma_2$: **Reject** H_0 **if** $F \geq F_{\alpha/2}(\mathrm{df}_1, \mathrm{df}_2)$ **or** $F \leq F_{1-\alpha/2}(\mathrm{df}_1, \mathrm{df}_2)$

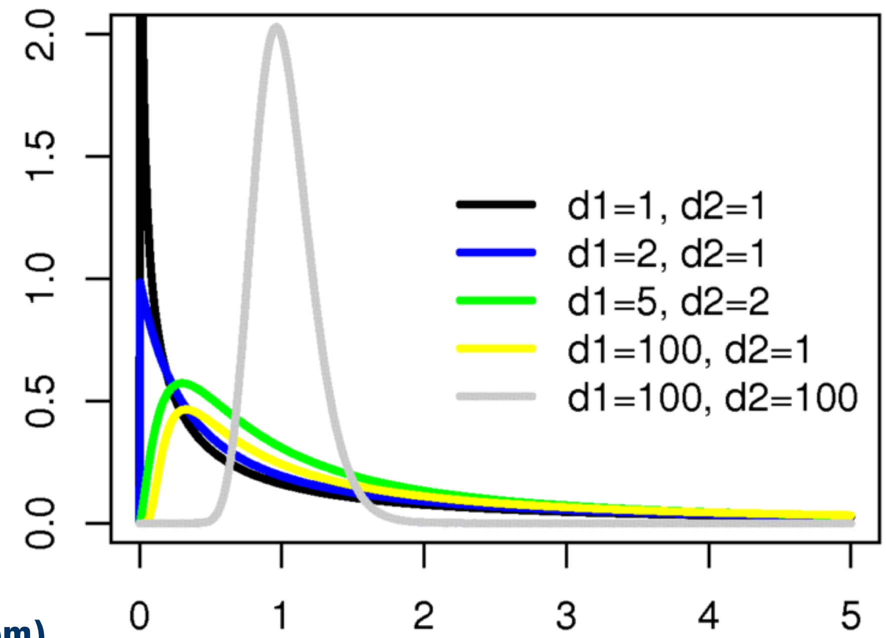
- **Minitab**

  Stat → Basic Statistics → 2 Variances

F DISTRIBUTION

- **Right skewed distribution**
- **Defined over positive numbers**
- **Parameters: df$_1$=ν_1, df$_2$=ν_2**
- **How to write:**
  - $F(\nu_1, \nu_2)$
- $F(\nu_1, \nu_2) = \dfrac{\frac{\chi^2(\nu_1)}{\nu_1}}{\frac{\chi^2(\nu_2)}{\nu_2}}$
- **[F Calculator]**
- **Ti-84: Fcdf(lower, upper, dfNumer, dfDenom)**





Source: Wikipedia

BACK TO THE WEIGHT LOSS EXAMPLE

- A study was conducted to see he effectiveness of a weight loss program. Two different groups of 10 subjects each were selected. The control group did not participate in the program. The data on weight loss was collected at the end of six months.

- **Control** | **Experimental**
- $n_1 = 10$ | $n_2 = 10$
- $\bar{y}_1 = 2.1\ lb$ | $\bar{y}_2 = 4.2\ lb$
- $s_1 = 0.5\ lb$ | $s_2 = 0.7\ lb$

- Is there a sufficient evidence at $\alpha = 0.05$ to conclude that the program is effective? **(We used pooled t-test)**
- Assumptions:
  - $\boldsymbol{\sigma_1 = \sigma_2}$
  - The weight loss for both groups are normally distributed.

BACK TO THE WEIGHT LOSS EXAMPLE

	Control	Experimental
•	$n_1 = 10$ $\bar{y}_1 = 2.1\ lb$ $s_1 = 0.5\ lb$	$n_2 = 10$ $\bar{y}_2 = 4.2\ lb$ $s_2 = 0.7\ lb$

- $H_0: \sigma_1 = \sigma_2$
- $H_a: \sigma_1 \neq \sigma_2$

- **T.S.** $F = \frac{\max(s_1^2, s_2^2)}{\min(s_1^2, s_2^2)} = \frac{0.7^2}{0.5^2} = 1.96$
- **Reject H_0 if**
  - $F \geq F_{0.05/2}(df_1 = n_2 - 1, df_2 = n_1 - 1) = 4.03$ **or**
  - $F \leq F_{1-0.05/2}(df_1 = n_2 - 1, df_2 = n_1 - 1) = 0.25$

- **Conclusion: Is $F \geq 4.03$ or $F \leq 0.25$?**
  No, since $F = 1.96$. We fail to reject H_0.
  We don't have any evidence against the assumption $\sigma_1 = \sigma_2$.

TEST FOR COMPARING VARIANCES FOR MORE THAN TWO POPULATIONS

- $H_0: \sigma_1 = \sigma_2 = \cdots = \sigma_t$
- H_a: Population variancies are not all equal

- **F test can be extended to more than two populations:**
  - **Hartley's F_{max} test:**
    - F_{max} test is sensitive to departures from normality.
    - E.g. Sampling from non-normal but equal variances population:
      - F_{max} will reject H_0 and declare the variances to be unequal.

- **Brown-Forsythe-Levene (BFL) test:**
  - BFL replace the j^{th} observation from sample i, y_{ij}, with z_{ij}, where
  - $z_{ij} = |y_{ij} - \tilde{y}_i|$, where $\tilde{y}_i$ is the sample median of the i^{th} sample.

VARIANCES FOR MORE THAN TWO POPULATIONS (CONT'D)

- **Brown-Forsyth-Levene (BFL) test:**

- $H_0: \sigma_1 = \sigma_2 = \cdots = \sigma_t$
  H_a: Population variancies are not all equal

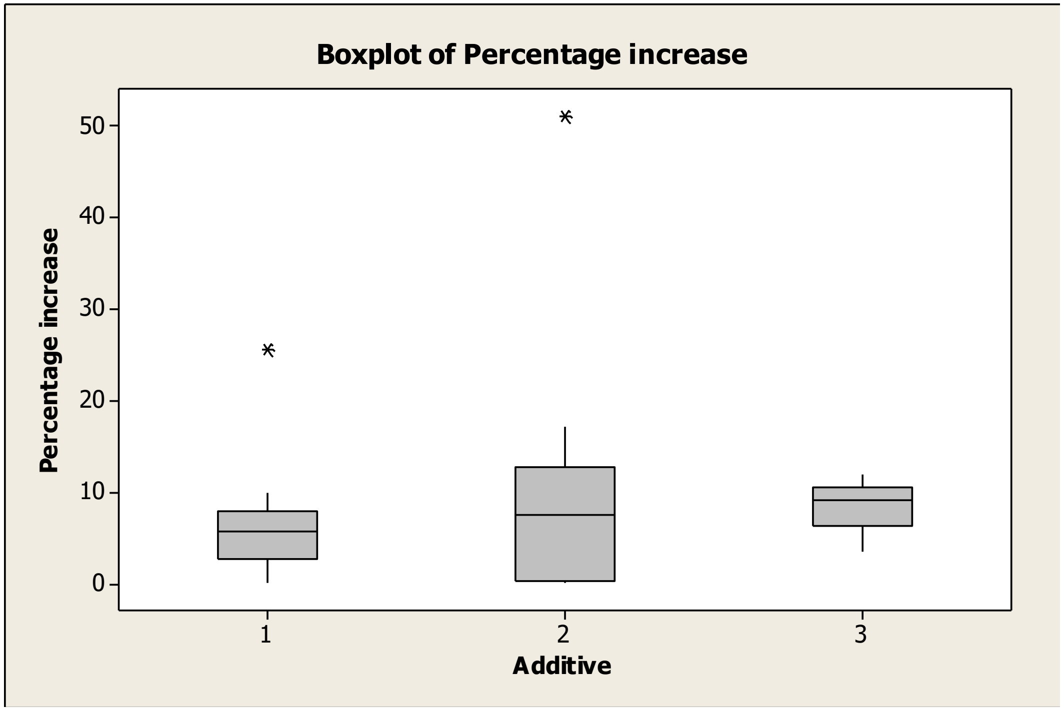
- **T.S.** $L = \dfrac{\sum_{i=1}^{t} n_i(\bar{z}_{i.} - \bar{z}_{..})^2/(t-1)}{\sum_{i=1}^{t}\sum_{j=1}^{n_i}(z_{ij} - \bar{z}_{i.})^2/(N-t)}$

- **Decision rule (**$\mathrm{df}_1 = t-1, \mathrm{df}_2 = N-t$**)**
  - **Reject** H_0 **if** $L \geq F_{\alpha,\mathrm{df}_1,\mathrm{df}_2}$

- **Here** $N = \sum_{i=1}^{t} n_i$

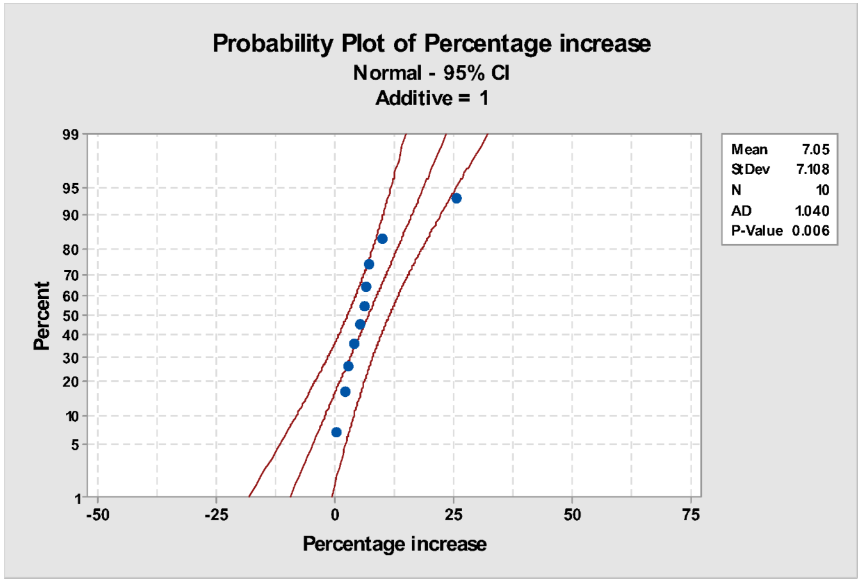
BOOK EXAPMLE 7.9:

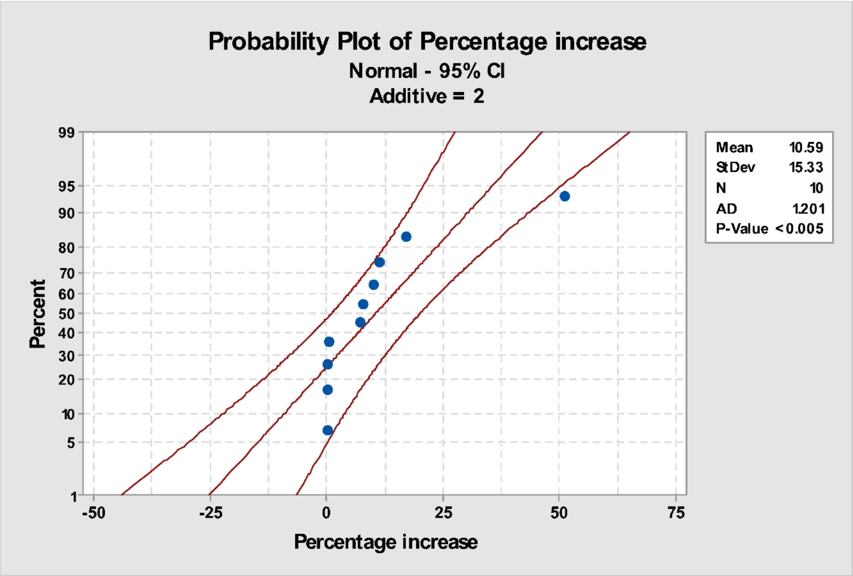
- Three different additives that are marketed for increasing the miles per gallon (mpg) for automobiles
- The percentage increase in mpg was recorded for a 250-mile test drive for each additive for 10 randomly assigned cars.



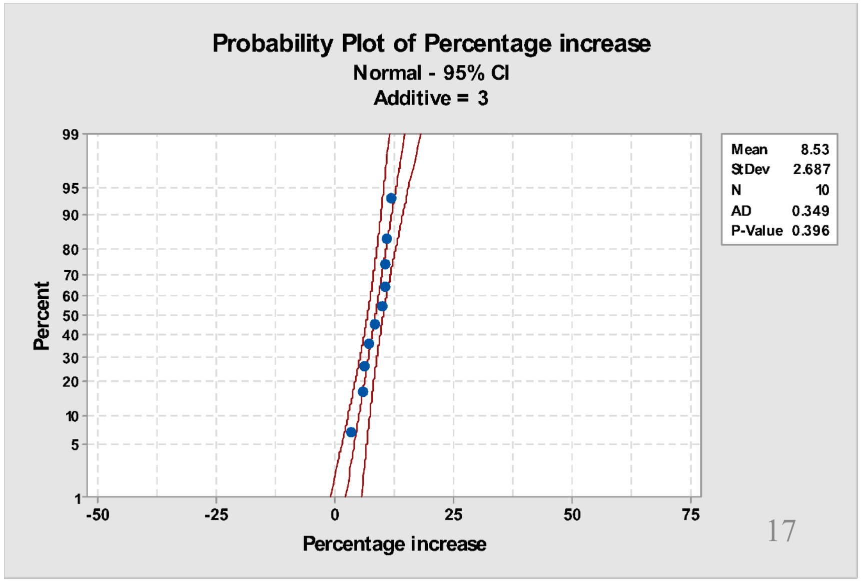
- Is there a difference between the three additive with respect to their variability?
- [Minitab]
  Stat ⟶ ANOVA ⟶ Test for Equal Variances

NORMAL PROBABILITY PLOTS





- **Additive 1 and 2 do not appear to be normal**



- **Avoid Hartley's F_{max} test**
- **Use Brown-Forsythe-Levene (BFL) test**

BOOK EXAPMLE 7.9 (CONT'D)

- **Brown-Forsyth-Levene (BFL) test:**

- $H_0: \sigma_1 = \sigma_2 = \sigma_3$
- H_a: Population variancies are not all equal

The value of BFL's test statistics, in an alternative form, is given by

$$L = \frac{(T_2 - T_1)/(t-1)}{T_1/(N-t)} = \frac{(1{,}978.4 - 1{,}742.6)/(3-1)}{1{,}742.6/(30-3)} = 1.827$$

The rejection region for the BFL test is to reject H_0 if $L \geq F_{\alpha,t-1,N-t} = F_{.05,2,27} = 3.35$. Because $L = 1.827$, we fail to reject H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2$ and conclude that there is insufficient evidence of a difference in the population variances of the percentage increase in mpg for the three additives.

- p. value $> \alpha$

- **Fail to reject H_0**

- **insufficient evidence of a difference in the population variances of the percentage increase in mpg for the three additives.**