# MATH 4720 / MSCS 5720

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## Chapter 6 (Part C)

Department of Mathematics, Statistics and Computer Science

# CHAPTER 6 (PART C)

- **Comparing Two Population Means**
  - Independent Samples
  - Dependent Samples
- **Two sample t-test (Independent Samples)**
  - Pooled t-test
  - Unequal variance t-test
- **Paired t-test (Dependent Samples)**
- **Power Analysis**
  - Independent Samples
  - Dependent Samples
- **Non-parametric Tests**
  - Sign test (from Chapter 5, test for median $M$)
  - Wilcoxon Rank-Sum (or Mann–Whitney) Test (two independent samples)
  - Wilcoxon Signed-Rank Test (dependent samples)

# NONPARAMETRIC INFERENCE

- In both one-sample and two-sample t-tests, we assumed that either the sample size is $\geq 30$ or the samples are drawn from normal populations.

- What if $n < 30$, and the distribution is non-normal?

- In such cases, we usually use non-parametric tests.

- No assumptions on the distribution means no parameters.

# MOTIVATION

- **Example: Suppose the weights of cereal boxes is not normally distributed.**

- **Median weight of cereal boxes supposed to be $16.37$ oz.**

- **Take a sample of $5$ boxes: $16.01$, $15.98$, $16.23$, $15.5$, $16.2$**

- **What is the probability that all of these boxes have weight less than $16.37$ oz.?**
  - **Ans.** $\frac{1}{2^5} = 0.0315.$

- **This is the $p - value$.**

- **Note that to answer this, we did not need distributional assumption.**

# EXAMPLE (CONT'D)

- **Now if the sample is:** 16.01, 15.98, 16.23, 15.5, 16.47
- **Here one** 4 **out of five values are less than** 16.37 **oz.**
- $H_0: median = 16.37$
- $H_a: median < 16.37$
- $p - value$
- $= P\begin{pmatrix}\text{four or more values are less than } 16.37 \\ \text{if } H_0 \text{ is true}\end{pmatrix}$
- $= P(Y \geq 4)$, $Y \sim Binomial(n = 5, \pi = 0.5)$
  $= 0.1875$
- [Binomial Calculator]

# NON-PARAMETRIC ONE SAMPLE INFERENCE (SECTION 5.9)

- **Sign Test**

**Data:** $y_1, y_2, \ldots, y_n$

- $H_0: median = m_0$
- $H_a: median > m_0$

  or $median < m_0$

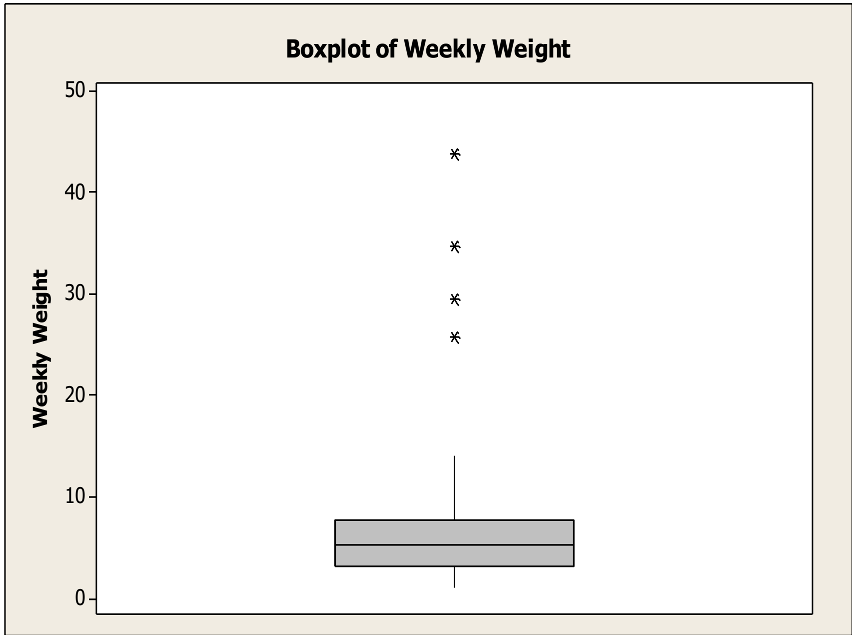
  or $median \neq m_0$

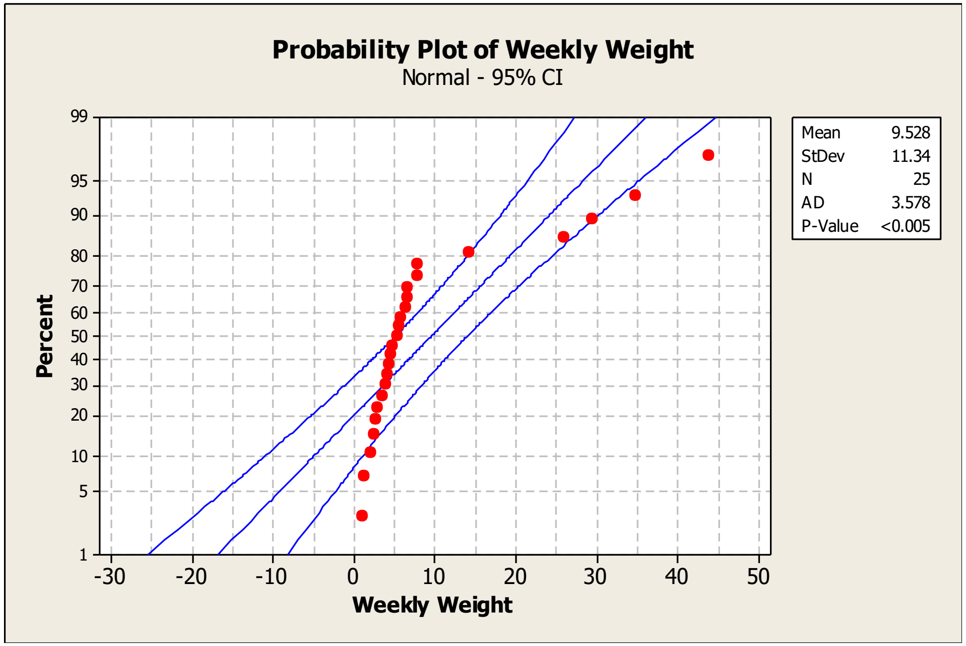
- **Test Statistics** $B = \#\ of\ data\ values > m_0$

- **Decision Rule:**
  - $H_a: median > m_0$: **Reject** $H_0$ **in favor of** $H_a$ **if** $B \geq n - B_\alpha$
  - $H_a: median < m_0$: **Reject** $H_0$ **in favor of** $H_a$ **if** $B \leq B_\alpha$
  - $H_a: median \neq m_0$: **Reject** $H_0$ **in favor of** $H_a$ **if** $B \leq B_{\alpha/2}$ **or** $B \geq n - B_{\alpha/2}$

# BOOK EXAMPLE 5.20

- A landfill company wants to determine if the average weekly amount of household recyclable wastes material is more than 5 lbs. The data is collected from 25 households.




- Here, the word "average" should not be interpreted literally.
- [Minitab]

# EXAMPLE CONT'D

- $n = 25 < 30$, **and the data is not normally distributed.**
- $H_0\!: median = 5$ vs. $H_a\!: median > 5$
- **Stat $\longrightarrow$ Nonparametrics $\longrightarrow$ 1- Sample Sign**

**Sign Test for Median: Weekly Weight**

```
Sign test of median =  5.000 versus > 5.000

                N  Below  Equal  Above       P  Median
Weekly Weight  25     12      0     13  0.5000   5.300
```

- **T.S. B = # of data values greater** $5$ **lbs =** $13$
- **p-value is** $0.5000$, **we fail to reject** $H_0$ **in favor of** $H_a$.
- **Thus we cannot conclude that the median household recyclable waste is grater than** $5$ **pounds per week.**

# EXAMPLE CONT'D

- **Minitab Output**

- **95% Confidence Interval for the Median**

**Sign CI: Weekly Weight**

Sign confidence interval for median

| | N | Median | Achieved Confidence | Confidence Interval Lower | Confidence Interval Upper | Position |
|---|---|---|---|---|---|---|
| Weekly Weight | 25 | 5.300 | 0.8922 | 4.200 | 6.700 | 9 |
| | | | 0.9500 | 3.959 | 6.700 | NLI |
| | | | 0.9567 | 3.900 | 6.700 | 8 |

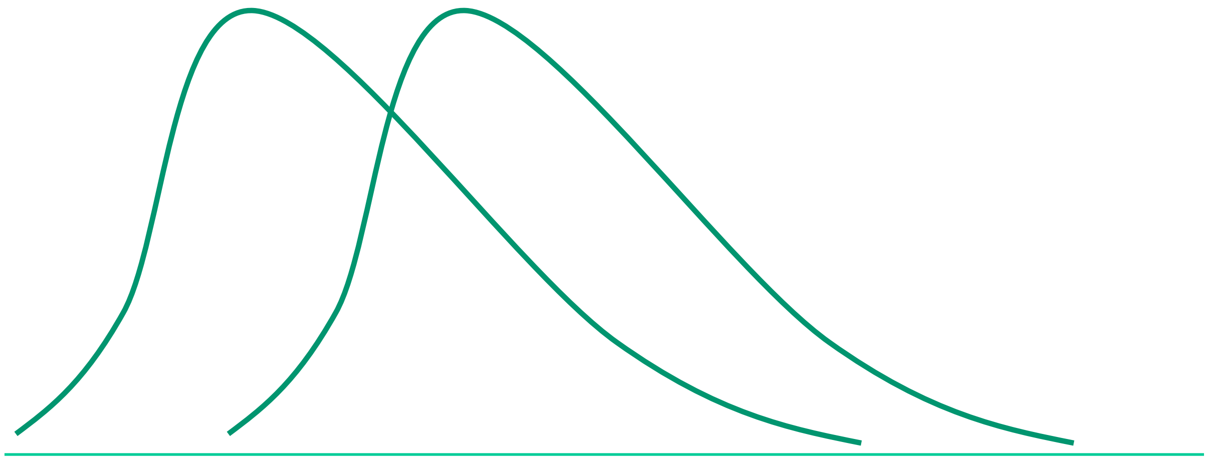
# NON-PARAMETRIC TWO SAMPLES TEST

- **Non-parametric Two Independent Samples Test**

| **Group 1** | **Group 2** |
|---|---|
| $y_{11}, y_{12}, \ldots, y_{1n_1}$ | $y_{21}, y_{22}, \ldots, y_{2n_2}$ |

- $n_1 < 30$ **and/or** $n_2 < 30$

- **Data is generated from non-normal distributions.**



# WILCOXON RANK-SUM TEST (MANN–WHITNEY U TEST)

- $H_0$: Two distributions are identical (i. e. $med_1 = med_2$)
- $H_a$: dist. of 1 is shifted to the right of dist. 2 ($med_1 > med_2$)
- **or** dist. of 1 is shifted to the left of dist. 2 ($med_1 < med_2$)
- **or** Two distributions are not identical ($med_1 \neq med_2$)

- **T.S. Combine both the samples. Rank all values of the combined sample from lowest to the highest.**

$$T = \text{sum of the ranks in sample 1}$$

- **Decision Rule:(We will use computer output)**
  - $H_a: med_1 > med_2$: **Reject** $H_0$ **if** $T > T_U$
  - $H_a: med_1 < med_2$: **Reject** $H_0$ **if** $T < T_L$
  - $H_a: med_1 \neq med_2$: **Reject** $H_0$ **if** $T > T_{U^*}$ or $T < T_{L^*}$

## A SIMPLE EXAMPLE

- $H_0: med_1 = med_2$

$$H_a: med_1 > med_2$$

| | Group 1 | Group 2 |
|---|---|---|
| | 32, 33, 55, 60, 61 | 23, 25, 56, 33, 21 |
| • **Rank** | **4 5.5 7 9 10** | **2 3 8 5.5 1** |

- **T.S.** $T =$ sum of the ranks of sample 1 $= 35.5$

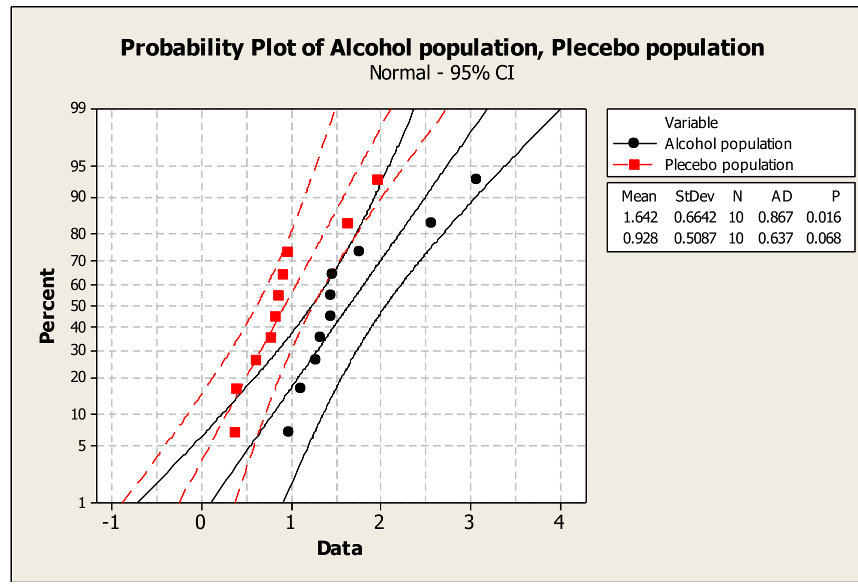
- **If** $T > T_U$**, we would say that dist. of group 1 is to the right of dist. of group 2**

- **Note that** $T_U$ **is determined in such a way that probability of false conclusion is** $\alpha = 0.05$.

# BOOK EXAMPLE 6.5

- An investigator is interested to study the effect of alcohol on reaction time. The following data is collected on the reaction time to an instruction.

| Group 1: Alcohol | Group 2: Placebo |
|---|---|
| 10 subjects | 10 subjects |


- [Minitab]

# EXAMPLE 6.5: (CONT'D)

- $n_1 = 10,\ n_2 = 10,$ **and the distributions are non-normal**
- $H_0\!: med_1 = med_2$ vs. $H_a\!: med_1 > med_2$
- **Minitab Output**

**Stat → Nonparametrics → Mann-Whitney**

**Mann-Whitney Test and CI: Alcohol, Placebo**

```
          N  Median
Alcohol  10   1.445
Placebo  10   0.845


Point estimate for ETA1-ETA2 is 0.610
95.5 Percent CI for ETA1-ETA2 is (0.250,1.080)
W = 140.0
Test of ETA1 = ETA2 vs ETA1 > ETA2 is significant at 0.0046
```

- **Conclusion: Since** $p - value$ **of** $0.0046$ **is small, we reject** $H_0$ **in favor of** $H_a$**. Thus, we conclude that the reaction time for the Alcohol population is significantly higher than that for the Placebo population.**

# NON-PARAMETRIC TWO DEPENDENT SAMPLE TEST

| Subject | $y_1$ | $y_2$ | $d = y_1 - y_2$ |
|---|---|---|---|
| 1 | $y_{11}$ | $y_{21}$ | $d_1$ |
| 2 | $y_{12}$ | $y_{22}$ | $d_2$ |
| . | . | . | . |
| . | . | . | . |
| n | $y_{1n}$ | $y_{2n}$ | $d_n$ |

- $n < 30$ and the **differences** are **not normally** distributed
- In such case, a **nonparametric** method must be used.

# WILCOXON SIGNED-RANK TEST

- $H_0 : med_d = 0$ **(median of the difference = 0)**
- $H_a : med_d > 0$

  **or** $med_d < 0$

  **or** $med_d \neq 0$

- **T.S. Rank the absolute values of the differences.**
  - $T_-$ **= sum of the ranks of negative differences**
  - $T_+$ **= sum of the ranks of positive differences**
  - $T$ **= smaller of** $T_+$ **and** $T_-$

- **Decision Rule: (We will use computer output)**
  - $H_a : med_d > 0$: **Reject** $H_0$ **if** $T_- < T_U^-$
  - $H_a : med_d < 0$: **Reject** $H_0$ **if** $T_+ < T_L^+$
  - $H_a : med_d \neq 0$: **Reject** $H_0$ **if** $\mathrm{T} < T^*$

# A SIMPLE EXAMPLE

| Subject | $y_1$ | $y_2$ | $d = y_1 - y_2$ | Rank of $\lvert y_1 - y_2 \rvert$ |
|---|---|---|---|---|
| 1 | 30 | 24 | 6 | 5 |
| 2 | 20 | 22 | -2 | 1.5 |
| 3 | 32 | 30 | 2 | 1.5 |
| 4 | 41 | 37 | 4 | 4 |
| 5 | 27 | 30 | -3 | 3 |

- **Test Statistics:**
- $T_- = 4.5$
- $T_+ = 10.5$
- $T = \text{smaller}(10.5, 4.5) = 4.5$

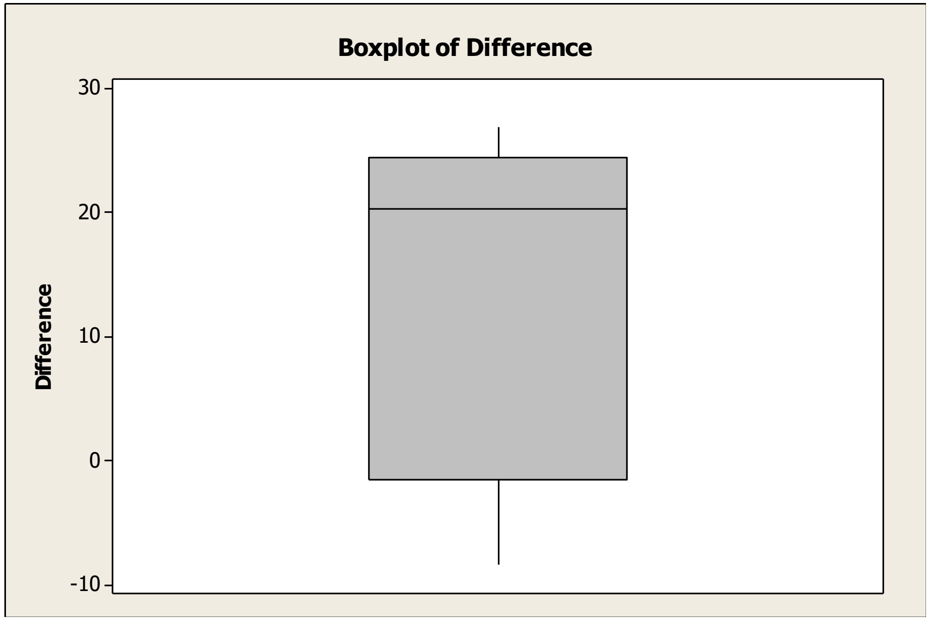
# BOOK EXAMPLE 6.9

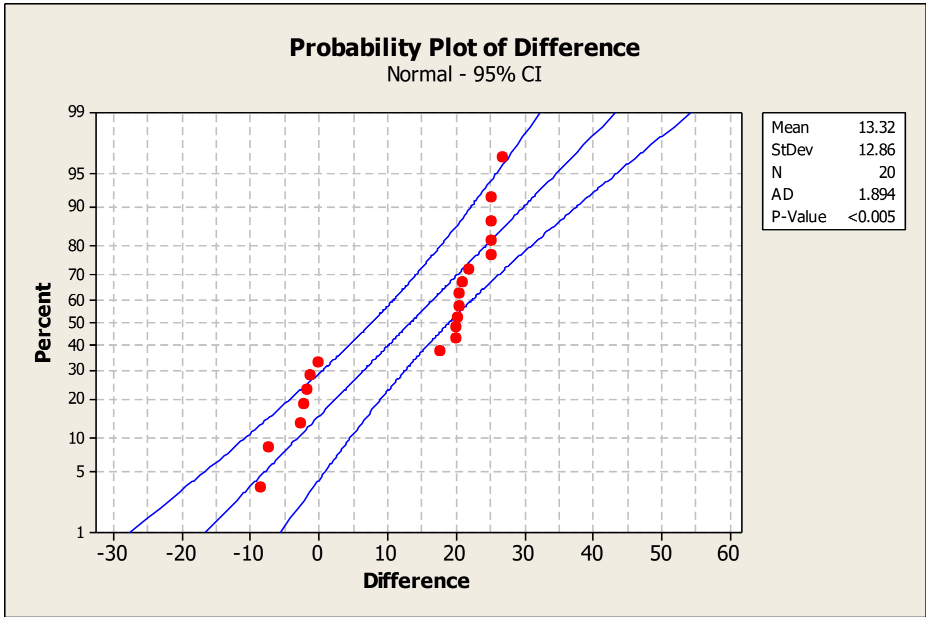
- **Does Brand A fertilizer produce more grass than Brand B?**

| Field | Brand A | Brand B | Difference | Field | Brand A | Brand B | Difference |
|---|---|---|---|---|---|---|---|
| 1 | 211.4 | 186.3 | 25.1 | 11 | 208.9 | 183.6 | 25.3 |
| 2 | 204.4 | 205.7 | −1.3 | 12 | 208.7 | 188.7 | 20.0 |
| 3 | 202.0 | 184.4 | 17.6 | 13 | 213.8 | 188.6 | 25.2 |
| 4 | 201.9 | 203.6 | −1.7 | 14 | 201.6 | 204.2 | −2.6 |
| 5 | 202.4 | 180.4 | 22.0 | 15 | 201.8 | 181.6 | 20.1 |
| 6 | 202.0 | 202.0 | 0 | 16 | 200.3 | 208.7 | −8.4 |
| 7 | 202.4 | 181.5 | 20.9 | 17 | 201.8 | 181.5 | 20.3 |
| 8 | 207.1 | 186.7 | 20.4 | 18 | 201.5 | 208.7 | −7.2 |
| 9 | 203.6 | 205.7 | −2.1 | 19 | 212.1 | 186.8 | 25.3 |
| 10 | 216.0 | 189.1 | 26.9 | 20 | 203.4 | 182.9 | 20.5 |

- **Minitab**

# BOOK EXAMPLE 6.9 (CONT'D)




- $H_0\colon med_d = 0 \text{ vs } H_a\colon med_d > 0$

# BOOK EXAMPLE 6.9 (CONT'D)

- **Minitab Output**
  - **Stat → Nonparametrics → 1-Sample Wilcoxon**

**Wilcoxon Signed Rank Test: Difference**

Test of median = 0.000000 versus median > 0.000000

| | N | N for Test | Wilcoxon Statistic | P | Estimated Median |
|---|---|---|---|---|---|
| Difference | 20 | 19 | 169.0 | 0.002 | 11.80 |

- **Conclusion: Since the $p - value = 0.002$ is small, we reject $H_0$ in favor of $H_a$. Thus conclude that Brand A fertilizer produce more grass than Brand B.**
- **95% Confidence interval**

**Wilcoxon Signed Rank CI: Difference**

| | N | Estimated Median | Achieved Confidence | Confidence Interval Lower | Confidence Interval Upper |
|---|---|---|---|---|---|
| Difference | 20 | 11.8 | 95.0 | 8.7 | 21.4 |