MATH 4720 / MSSC 5720

Instructor: Mehdi Maadooliat

Chapter 5 (Part C)

Department of Mathematical and Statistical Sciences

CHAPTER 5 (PART C)

- Confidence Interval (CI)
- CI for μ, when σ is known
- Choosing Sample Size for Estimating μ
- A Statistical Test for μ
- Type I and Type II Errors
- Power of a Test
- Choosing Sample Size for Testing μ
- Level of Significance
- P-value
- Inference about μ, when σ is unknown

LEVEL OF SIGNIFICANCE

- **The approach to hypothesis testing we discussed so far is called classical approach. Note that it is based on a decision rule for a given α.**

- **A question arises: What is an appropriate α?**

- **Many statisticians object to this approach. They say that an appropriate approach should be that we should indicate how much evidence the data provide in favor of H_0.**

- **What to do?**

- $p - value$

$$= P(\text{oberving the evidence more extreme than the evidence by the data if } H_0 \text{ is true})$$

P-VALUE APPROACH

- **Example:**
  An experiment was done on 35 petri dishes which were first cultured with E.Coli bacteria, and then a solution of the antibacterial soap was added. After 24-hour incubation period, bacterial counts were recorded on each of 35 dishes.

- **Sample Information:** $\bar{y} = 31.2, \quad s = 8.4$

- $H_0: \mu = 33$

- $H_a: \mu < 33$

- **T.S.** $z = \frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{31.2 - 33}{8.4/\sqrt{35}} = -1.27$

- [Applet: P-value]

P-VALUE APPROACH CONT'D

- $H_0: \mu = 33$ **vs** $H_a: \mu < 33$. **T.S.** $z = -1.27$
- $p - value$

  $= P(\text{oberving the evidence more extreme than the evidence by the data} \mid H_0 \text{ is true})$

  $= P(Z \leq -1.27) = 0.1020$
- **Since this evidence for H_0 is not small enough, we do note have sufficient evidence to reject H_0.**

- **Suppose** $\bar{y} = 30.2$ **instead of** 31.2

  $$z = \frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{30.2 - 33}{8.4/\sqrt{35}} = -1.972$$
- $p - value = P(Z \leq -1.972) = 0.0244$
- **Since this evidence for H_0 is very small, we should reject H_0.**

ANOTHER INTERPRETATION OF P-VALUE:

- $p - value$
- $\quad =$ smallest level of α at which H_0 can be rejected.

- **In the previous example:**

- **When** $\bar{y} = 30.2, \quad p - value = 0.0244.$ **This is the smallest level at which** H_0 **can be rejected.**

- **If** $\alpha = 0.05, \quad p - value < \alpha$ **, so Reject** H_0
- **If** $\alpha = 0.01, \quad p - value > \alpha$ **, so Fail to Reject** H_0

- **In other word:**
  - **You Reject at any** $\alpha > 0.0244$
  - **You Fail to Reject at any** $\alpha < 0.0244$

GENERAL APPROACH

- **If** $\mathrm{p-value} < \alpha$, **Reject** H_0 **in favor of** H_a**.**

- **Computing p-value**

  - $H_0: \mu = \mu_0$
  - **Compute** $z = \frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$

  - $H_a: \mu > \mu_0$ $\quad \mathrm{p-value} = \mathrm{P}(Z > \mathrm{computed\ z})$
  - $H_a: \mu < \mu_0$ $\quad \mathrm{p-value} = \mathrm{P}(Z < \mathrm{computed\ z})$
  - $H_a: \mu \neq \mu_0$ $\quad \mathrm{p-value} = 2 * \mathrm{P}(Z > |\mathrm{computed\ z}|)$

- **Applet: P-value**

THREE EXAMPLES

- **Let** $\alpha = 0.05$ **in the following examples.**

1. $H_0: \mu = 5.0 \;\; vs. \; H_a: \mu > 5.0$

   $n = 16, \;\; \bar{y} = 6.5, \;\;\; \sigma = 2.0$

   Given that sample is drawn from normal population.

   - $z = \frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{6.5 - 5.0}{2.0/\sqrt{16}} = 3.0$
   - $\text{p}-\text{value} = P(Z > 3.0)$

     $= normcdf(3.0, \infty, 0{,}1)$

     $= 0.0013$

   - **Conclusion: Since this** $\text{p}-\text{value} < \alpha = 0.05$**,** **we reject** H_0 **in favor of** H_a**.**

THREE EXAMPLES CONT'D

2. $H_0: \mu = 150 \;\; vs. \;\; H_a: \mu < 150$

   $\text{n} = 50, \quad \bar{y} = 147.4, \quad \sigma = 10$

   – $z = \frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{147.4 - 150}{10/\sqrt{50}} = -1.84$

   – $\text{p}-\text{value} = P(Z < -1.84)$
   $= normcdf(-\infty, -1.84, 0, 1)$
   $= \; 0.0329$

   – **Conclusion: Since this** $\text{p}-\text{value} < \alpha = 0.05$**,**
   we reject H_0 **in favor of** H_a**.**

THREE EXAMPLES CONT'D

3. $H_0: \mu = 35 \quad vs. \quad H_a: \mu \neq 35$

   $n = 25, \quad \bar{y} = 40, \quad \sigma = 15$

   Given that sample is drawn from normal population.

   – $z = \frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{40-35}{15/\sqrt{25}} = 1.67$

   – $\text{p}-\text{value} = 2 * P(Z > 1.67)$
   $= 2 * normcdf(1.67, \infty, 0, 1)$
   $= 0.095$

   – **Conclusion: Since** $\text{p}-\text{value} \nless 0.05$.
   We don't have sufficient evidence to reject H_0 in favor of H_a.

INFERENCE ABOUT $\boldsymbol{\mu}$ WHEN $\boldsymbol{\sigma}$ IS UNKNOWN

- **Hypothesis Testing**
- $H_0: \mu = \mu_0$ **vs** $H_a: \mu > \mu_0$
  or $\mu < \mu_0$
  or $\mu \neq \mu_0$

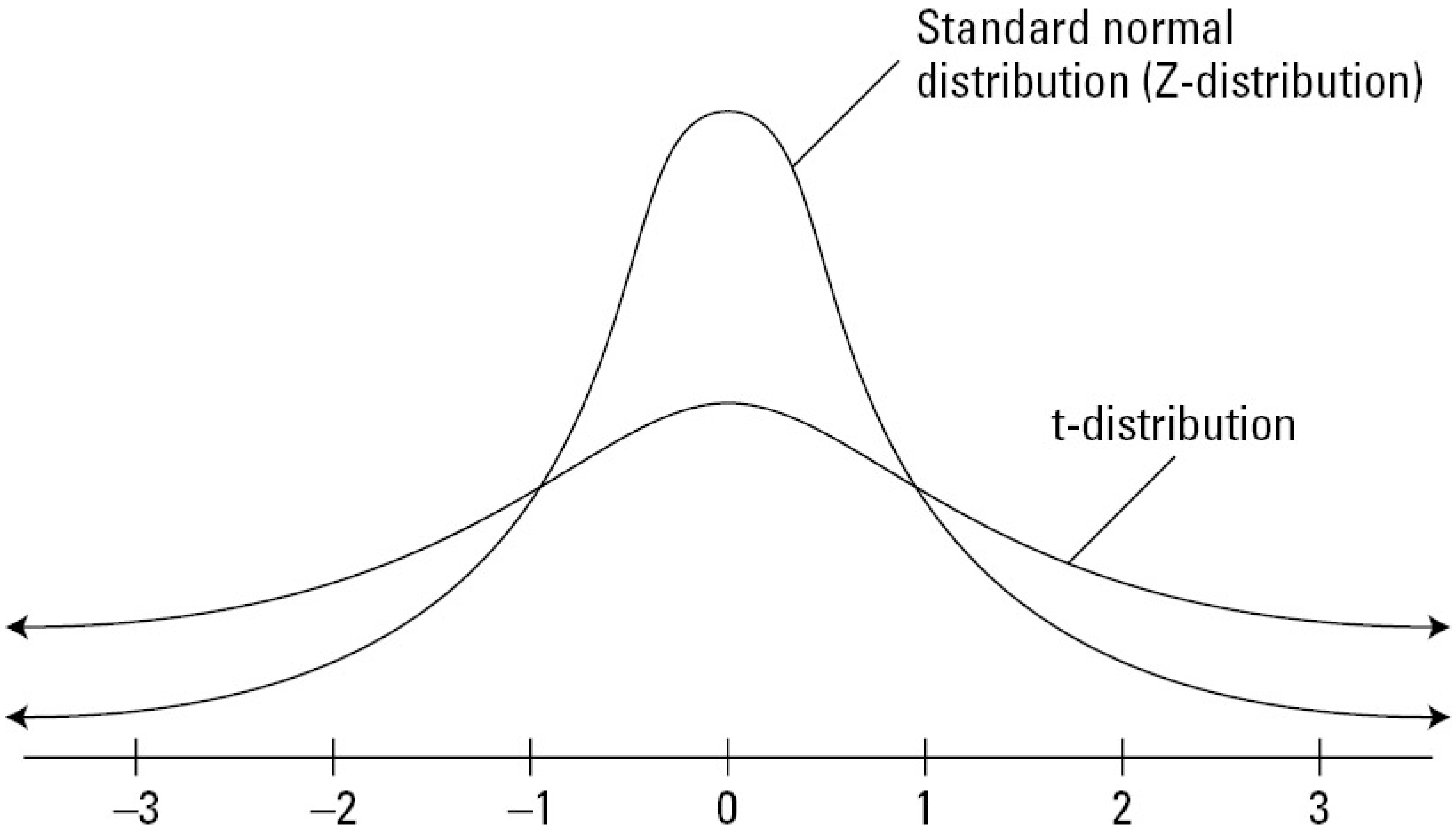
- **Sample:** $\{y_1, y_2, \dots, y_n\}$
- $\bar{y}$ – sample mean, s – sample st.dev.

- **T.S.** $\hat{z} = \frac{\bar{y}-\mu_0}{\frac{s}{\sqrt{n}}}$ (σ replaced by sample st.dev.)

- **The distribution of $\hat{z}$ is no longer N(0, 1).**

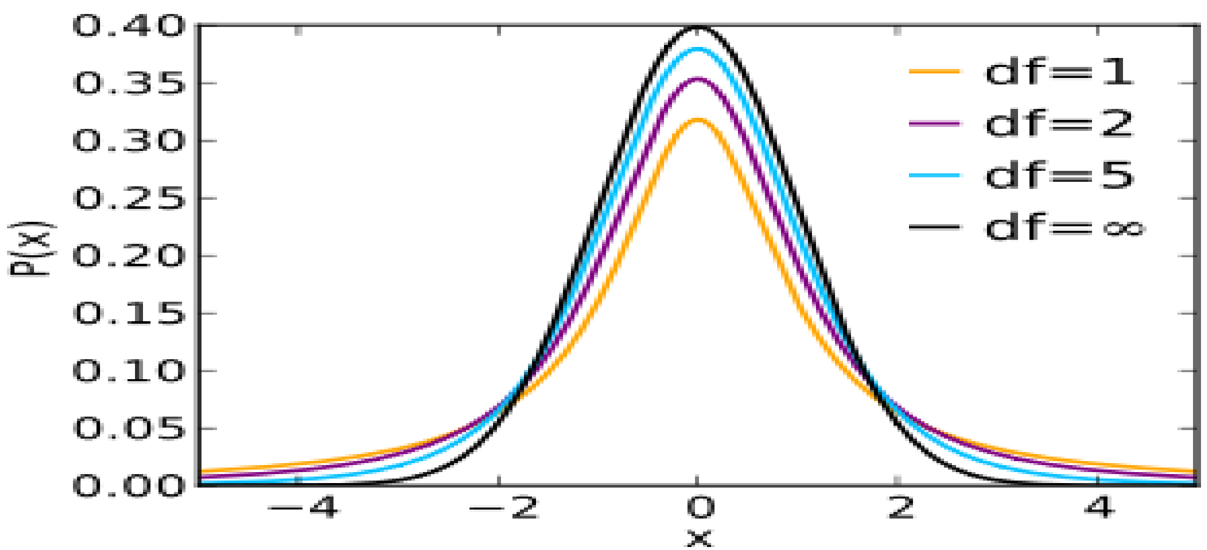
STUDENT T-DISTRIBUTION

- $$t = \frac{\bar{y} - \mu}{\frac{s}{\sqrt{n}}}$$
- **The distribution of t is called student-t distribution.**



REMARKS ON T-DISTRIBUTION

1. t-dist. has similar shape as $N(0,1)$, but is flatter than $N(0,1)$.
2. The t-distribution is symmetric around 0 as $N(0,1)$ is.
3. It has a range from $-\infty$ to ∞ as the range of $N(0,1)$.
4. Unlike $N(0,1)$, the t-distribution depends on the **degrees of freedom** df.
5. As the $\mathrm{df} \to \infty$, t-distribution approaches to $N(0,1)$.



- Applet: t-distribution vs $N(0,1)$

INFERENCE ABOUT μ WHEN σ IS UNKNOWN

- **Hypothesis Testing**
- $H_0: \mu = \mu_0$ **vs** $H_a: \mu > \mu_0$
  or $\mu < \mu_0$
  or $\mu \neq \mu_0$

- **Decision Rule** $\mathrm{df} = n - 1$
  - $H_a: \mu > \mu_0$: **Reject** H_0 **in favor of** H_a **if** $t > t_\alpha$
  - $H_a: \mu < \mu_0$: **Reject** H_0 **in favor of** H_a **if** $t < -t_\alpha$
  - $H_a: \mu \neq \mu_0$: **Reject** H_0 **in favor of** H_a **if** $|t| > t_{\alpha/2}$

- **Here** t_A **is a notation for value of** t **so that the area to the right is** A**.**
  - **t-table ("D2L > Useful Links > Z, T and Chi^2 Tables")**
    - $P(t \geq t_\alpha)$, **where** t **is a t-distribution with** $\mathrm{df} = n - 1$.
  - **t calculator (df)**
  - **Calculator**

EXAMPLE

- Consider a population of hypertension group whose average systolic blood pressure (SBP) is 150. You want to determine whether a new treatment is effective in reducing SBP. A clinical trial was conducted on 25 patients of this population. After 6 months of treatment, SBP was recorded on each subject.
- $\bar{y} = 147.2, \quad s = 5.5$
- Is there a sufficient evidence at $\alpha = 0.05$ that the new treatment is effective?
- Assume that the distribution of SBP is normal.

EXAMPLE CONT'D

- $H_0: \mu = 150 \;\; vs. \;\; H_a: \mu < 150$
- **TS:** $t = \frac{\bar{y} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{147.2 - 150}{\frac{5.5}{\sqrt{25}}} = -2.55$
- **Decision Rule: Reject H_0 in favor of H_a if** $t < -t_\alpha$
- $\mathrm{df} = n - 1 = 25 - 1 = 24, \;\; \alpha = 0.05$
- $t_\alpha = \mathrm{invT}(0.95, 24) = 1.711$
- **Reject H_0 in favor of H_a if** $t < -1.711$**.**
- **Conclusion: Is $t < -1.711$? Yes, since $t = -2.55$. Thus we reject H_0, and we have sufficient evidence to conclude that the new treatment is effective.**

P–VALUE APPROACH

- $H_0: \mu = 150 \;\; vs. \;\; H_a: \mu < 150$, **TS:** $t = -2.55$

- $\text{p}-\text{value} = P(t < -2.55)$
  $= \text{tcdf}(-\infty, -2.55, 24) =$ **0.0088**

- **Since** $\text{p}-\text{value} < \alpha = 0.05$**, we reject** H_0 **in favor of** H_a**. We have sufficient evidence to conclude that the new treatment is effective.**

- $\text{P}-\text{value}$ **Formula:**
  - $H_a: \mu > \mu_0$, $\text{p}-\text{value} = P(t > \text{computed t})$
  - $H_a: \mu < \mu_0$, $\text{p}-\text{value} = P(t < \text{computed t})$
  - $H_a: \mu \neq \mu_0$: $\text{p}-\text{value} = 2 * P(t > |\text{computed t}|)$

ESTIMATION OF μ USING A CONFIDENCE INTERVAL

- $100(1-\alpha)\%$ **Confidence Interval of** μ **is**

$$\bar{y} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

- **Assumption: Either**
  - $n \geq 30$
  - **the sample is drawn from a normal population.**

BOOK EXERCISE 5.39

- A discount tire manufacturer claim that its tires can be driven at least $35{,}000$ miles on the average. A consumer testing agency suspects that this claim is false. A study was conducted on 15 cars in which the testing tires were mounted.

$$n = 15,\ \bar{y} = 31.47,\ \ s = 5.04$$ **(in thousand miles)**

(a) Estimate the mean miles driven by the tires using 99% confidence interval.

(b) Is there a sufficient evidence ($\alpha = 0.01$) that the manufacturer's claim is false.

FIRST LET'S DO (B)

- $n = 15, \; \bar{y} = 31.47, \;\; s = 5.04$
- **(b)** $H_0 : \mu = 35 \quad vs. \;\; H_a : \mu < 35$
- **TS :** $t = \frac{\bar{y} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{31.47 - 35}{\frac{5.04}{\sqrt{15}}} = -2.71$
- **Decision Rule:** $\alpha = 0.01. \;\; df = 15 - 1 = 14, \; t_\alpha = 2.624$
- **Reject** H_0 **in favor of** H_a **if** $t < -t_\alpha = -2.624$
- **Conclusion: Is** $t < -2.624?$ **Yes. Reject** H_0 **in favor of** H_a**.**
- **We have sufficient evidence to conclude that the miles driven by the tires is less than** $35{,}000$ **miles on the average.**

NOW LET'S DO (A)

- $n = 15,\ \bar{y} = 31.47,\ \ s = 5.04$

(a) 99% confidence interval of μ.

- **Formula:** $\bar{y} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$
- **Assumption: sample is drawn from a normal population**

- $\alpha = 0.01,\ \alpha/2 = 0.005,\ \ \mathrm{df}\ =\ 14,\ t_{\alpha/2} = 2.977$

- 99% **CI of μ:** $31.47 \pm 2.977 * \frac{5.04}{\sqrt{15}}$

  31.47 ± 3.90

- $27.59 < \mu < 35.3$

SUMMARY

Case 1: One Variable

	Numerical Variable, σ known	**Numerical Variable,** σ unknown
Parameter of Interest:	Mean, μ	Mean, μ
Confidence Interval Formula:	$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$	$\bar{x} \pm t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$ with df = $n - 1$
Name of Hypothesis Test, H_0	One Sample Z Test, $H_0 : \mu = \mu_0$	One Sample T Test, $H_0 : \mu = \mu_0$
Test Statistic Formula:	$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$	$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$ with df = $n - 1$
***p*-value:**	$H_a : \mu \neq \mu_0$, p-value = $2P(Z \geq \lvert z \rvert)$ $H_a : \mu > \mu_0$, p-value = $P(Z \geq z)$ $H_a : \mu < \mu_0$, p-value = $P(Z \leq z)$	$H_a : \mu \neq \mu_0$, p-value = $2P(T \geq \lvert t \rvert)$ $H_a : \mu > \mu_0$, p-value = $P(T \geq t)$ $H_a : \mu < \mu_0$, p-value = $P(T \leq t)$