MATH 4720 / MSSC 5720

Instructor: Mehdi Maadooliat

Chapter 6 (Part C)

Department of Mathematical and Statistical Sciences

CHAPTER 6 (PART C)

- **Comparing Two Population Means**
  - Independent Samples
  - Dependent Samples
- **Two sample t-test (Independent Samples)**
  - Pooled t-test
  - Unequal variance t-test
- **Paired t-test (Dependent Samples)**
- **Power Analysis**
  - Independent Samples
  - Dependent Samples
- **Non-parametric Tests**
  - Sign test (from Chapter 5, test for median M)
  - Wilcoxon Rank-Sum (or Mann–Whitney) Test (two independent samples)
  - Wilcoxon Signed-Rank Test (dependent samples)

NONPARAMETRIC INFERENCE

- In both one-sample and two-sample t-tests, we assumed that either the sample size is ≥ 30 or the samples are drawn from normal populations.

- What if $n < 30$, and the distribution is non-normal?

- In such cases, we usually use non-parametric tests.

- No assumptions on the distribution means no parameters.

MOTIVATION

- **Example:** Suppose the weights of cereal boxes is not normally distributed.

- Median weight of cereal boxes supposed to be 16.37 oz.

- Take a sample of 5 boxes: 16.01, 15.98, 16.23, 15.5, 16.2

- What is the probability that all of these boxes have weight less than 16.37 oz.?
  - Ans. $\frac{1}{2^5} = 0.0315.$

- This is the $p - value$.

- Note that to answer this, we did not need distributional assumption.

EXAMPLE (CONT'D)

- **Now if the sample is:** $16.01, 15.98, 16.23, 15.5, 16.47$
- **Here one** 4 **out of five values are less than** 16.37 **oz.**
- $H_0: median = 16.37$
- $H_a: median < 16.37$
- $p - value$
- $= P\begin{pmatrix}\text{four or more values are less than } 16.37 \\ \text{if } H_0 \text{ is true}\end{pmatrix}$
- $= P(Y \geq 4)$, $Y \sim Binomial(n = 5, \pi = 0.5)$
  $= 0.1875$
- [Binomial Calculator]

NON-PARAMETRIC ONE SAMPLE INFERENCE (SECTION 5.9)

- **Sign Test**

Data: $y_1, y_2, \dots, y_n$

- $H_0: median = m_0$
- $H_a: median > m_0$

  or $median < m_0$

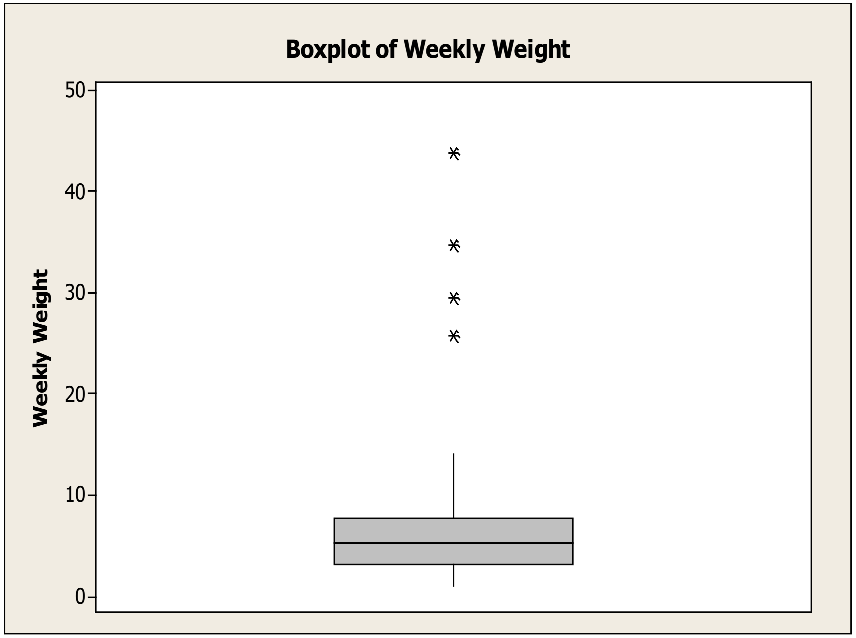
  or $median \neq m_0$

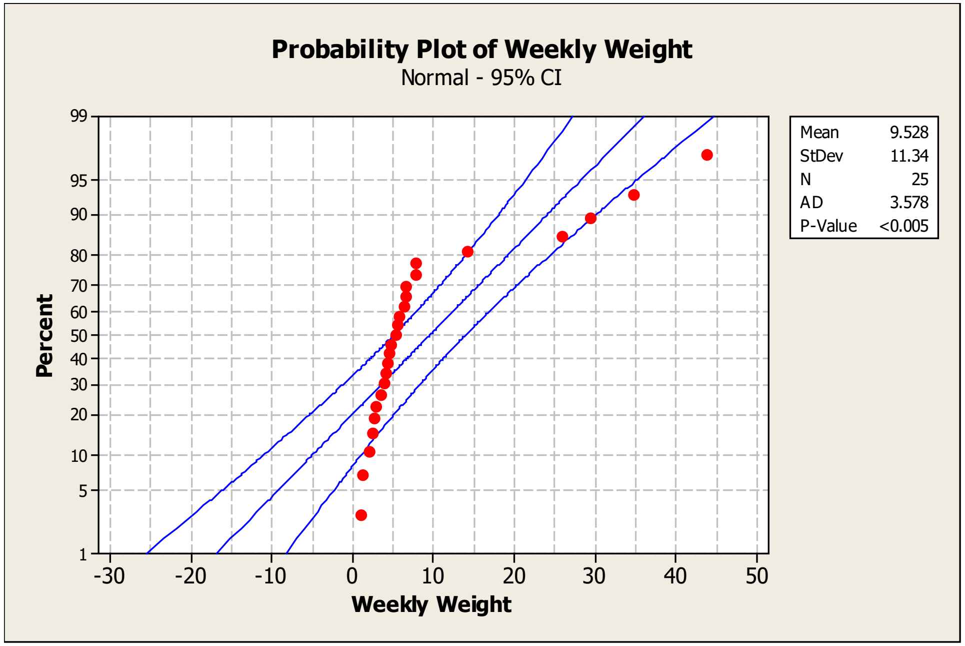
- **Test Statistics** $B = \#\, of\ data\ values > m_0$

- **Decision Rule:**
  - $H_a: median > m_0$: **Reject** H_0 **in favor of** H_a **if** $B \geq n - B_\alpha$
  - $H_a: median < m_0$: **Reject** H_0 **in favor of** H_a **if** $B \leq B_\alpha$
  - $H_a: median \neq m_0$: **Reject** H_0 **in favor of** H_a **if** $B \leq B_{\alpha/2}$ **or** $B \geq n - B_{\alpha/2}$

BOOK EXAMPLE 5.22

- A landfill company wants to determine if the average weekly amount of household recyclable wastes material is more than 5 lbs. The data is collected from 25 households.





- Here, the word "average" should not be interpreted literally.

EXAMPLE CONT'D

- $n = 25 < 30$, **and the data is not normally distributed.**
- $H_0: median = 5$ vs. $H_a: median > 5$

In R: binom.test(sum(exmp5.20$WeeklyWt > 5), length(exmp5.20$WeeklyWt), p=0.5, alternative="greater")

```
Exact binomial test

data:  sum(exmp5.20$WeeklyWt > 5) and length(exmp5.20$WeeklyWt)
number of successes = 13, number of trials = 25, p-value = 0.5
alternative hypothesis: true probability of success is greater than 0.5
```

- **T.S. B = # of data values greater 5 lbs = 13**
- **p-value is 0.5000, we fail to reject H_0 in favor of H_a.**
- **Thus we cannot conclude that the median household recyclable waste is grater than 5 pounds per week.**

EXAMPLE CONT'D (IN R)

- **95% Confidence Interval for the Median**

```
> library("BSDA")
Error in library("BSDA") : there is no package called 'BSDA'
> install.packages("BSDA")

> library("BSDA")
> SIGN.test(exmp5.20$WeeklyWt)

        One-sample Sign-Test

data:  exmp5.20$WeeklyWt
s = 25, p-value = 5.96e-08
alternative hypothesis: true median is not equal to 0
95 percent confidence interval:
 3.931247 6.700000
sample estimates:
median of x
        5.3

Achieved and Interpolated Confidence Intervals:

                    Conf.Level  L.E.pt  U.E.pt
Lower Achieved CI       0.8922 4.2000     6.7
Interpolated CI         0.9500 3.9312     6.7
Upper Achieved CI       0.9567 3.9000     6.7
```

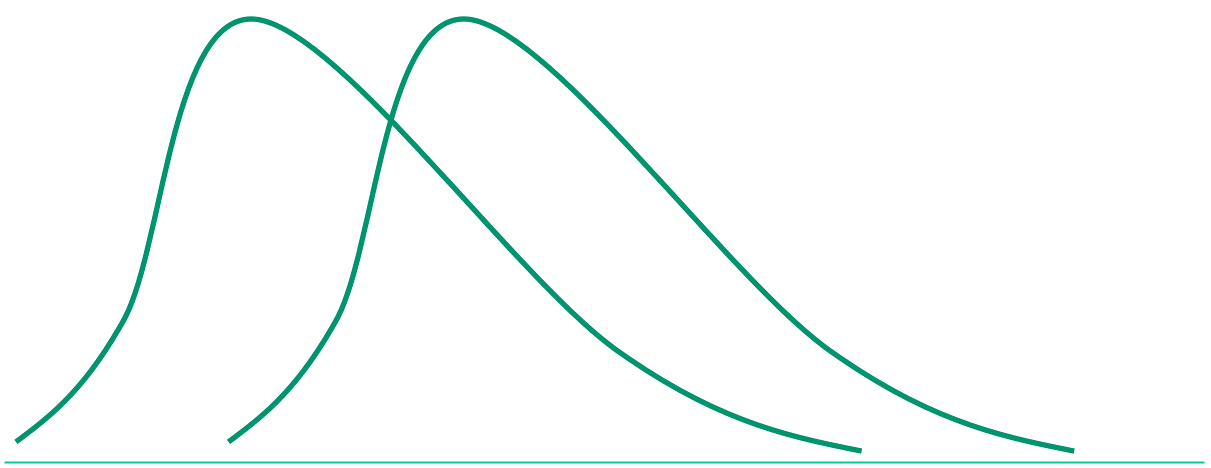
NON-PARAMETRIC TWO SAMPLES TEST

- **Non-parametric Two Independent Samples Test**

Group 1	Group 2
$y_{11}, y_{12}, \dots, y_{1n_1}$	$y_{21}, y_{22}, \dots, y_{2n_2}$

- $n_1 < 30$ **and/or** $n_2 < 30$

- **Data is generated from non-normal distributions.**



WILCOXON RANK-SUM TEST (MANN–WHITNEY U TEST)

- H_0: Two distributions are identical (i.e. $med_1 = med_2$)
- H_a: dist. of 1 is shifted to the right of dist. 2 ($med_1 > med_2$)
- **or** dist. of 1 is shifted to the left of dist. 2 ($med_1 < med_2$)
- **or** Two distributions are not identical ($med_1 \neq med_2$)

- **T.S. Combine both the samples. Rank all values of the combined sample from lowest to the highest.**

$$T = \text{sum of the ranks in sample 1}$$

- **Decision Rule:(We will use computer output)**
  - $H_a: med_1 > med_2$: **Reject** H_0 **if** $T > T_U$
  - $H_a: med_1 < med_2$: **Reject** H_0 **if** $T < T_L$
  - $H_a: med_1 \neq med_2$: **Reject** H_0 **if** $T > T_{U^*}$ or $T < T_{L^*}$

A SIMPLE EXAMPLE

- $H_0: med_1 = med_2$

$$H_a: med_1 > med_2$$

	Group 1	Group 2
	$32, 33, 55, 60, 61$	$23, 25, 56, 33, 21$
• **Rank**	**4 5.5 7 9 10**	**2 3 8 5.5 1**

- **T.S.** $T =$ sum of the ranks of sample 1 $= 35.5$

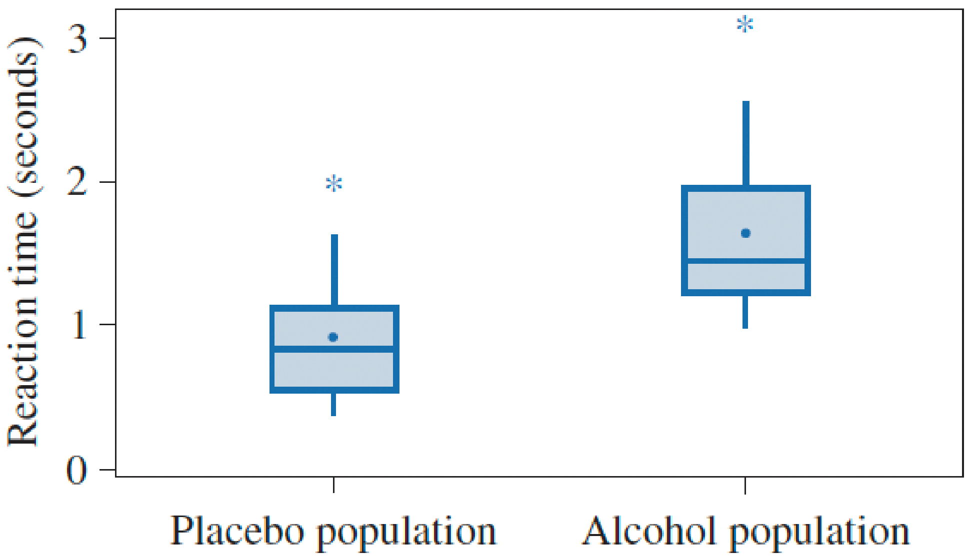
- **If $T > T_U$, we would say that dist. of group 1 is to the right of dist. of group 2**

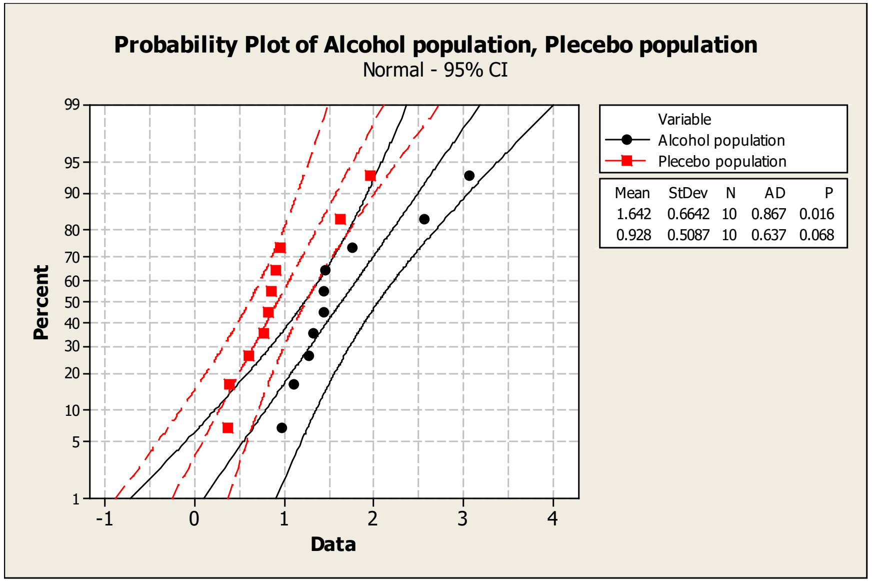
- **Note that T_U is determined in such a way that probability of false conclusion is $\alpha = 0.05$.**

BOOK EXAMPLE 6.5

- **An investigator is interested to study the effect of alcohol on reaction time. The following data is collected on the reaction time to an instruction.**

Group 1: Placebo	Group 2: Alcohol
10 subjects	10 subjects





» **From Minitab**

EXAMPLE 6.5: (CONT'D)

- $n_1 = 10,\ n_2 = 10,$ **and the distributions are non-normal**
- $H_0\!: med_1 = med_2$ vs. $H_a\!: med_1 < med_2$
- **In R:**
  - wilcox.test(exmp6.5$Placebo, exmp6.5$Alcohol, alternative = "less")

```
    Wilcoxon rank sum exact test


data:  exmp6.5$Placebo and exmp6.5$Alcohol
W = 15, p-value = 0.003421
alternative hypothesis: true location shift is less than 0
```

- **Conclusion: Since $p - value$ of 0.0034 is small, we reject H_0 in favor of H_a. Thus, we conclude that the reaction time for the Alcohol population is statistically significantly higher than that for the Placebo population.**

EXAMPLE 6.5: (CONT'D)

- **Confidence Interval**
- **In R:**
  - wilcox.test(exmp6.5$Placebo, exmp6.5$Alcohol, conf.int = T)

```
 Wilcoxon rank sum exact test

data:  exmp6.5$Placebo and exmp6.5$Alcohol
W = 15, p-value = 0.006841
alternative hypothesis: true location shift is not equal to 0
95 percent confidence interval:
 -1.08 -0.25
sample estimates:
difference in location
             -0.61
```

NON-PARAMETRIC TWO DEPENDENT SAMPLE TEST

Subject	y_1	y_2	$d = y_1 - y_2$
1	y_{11}	y_{21}	d_1
2	y_{12}	y_{22}	d_2
.	.	.	.
.	.	.	.
n	y_{1n}	y_{2n}	d_n

- $n < 30$ **and the differences are not normally distributed**
- **In such case, a nonparametric method must be used.**

WILCOXON SIGNED-RANK TEST

- $H_0: med_d = 0$ **(median of the difference = 0)**
- $H_a: med_d > 0$

  or $med_d < 0$

  or $med_d \neq 0$

- **T.S. Rank the absolute values of the differences.**
  - T_- **= sum of the ranks of negative differences**
  - T_+ **= sum of the ranks of positive differences**
  - T **= smaller of** T_+ **and** T_-

- **Decision Rule: (We will use computer output)**
  - $H_a: med_d > 0$: **Reject** H_0 **if** $T_- < T_U^-$
  - $H_a: med_d < 0$: **Reject** H_0 **if** $T_+ < T_L^+$
  - $H_a: med_d \neq 0$: **Reject** H_0 **if** $\mathrm{T} < T^*$

A SIMPLE EXAMPLE

Subject	y_1	y_2	$d = y_1 - y_2$	Rank of $\lvert y_1 - y_2 \rvert$
1	30	24	6	5
2	20	22	-2	1.5
3	32	30	2	1.5
4	41	37	4	4
5	27	30	-3	3

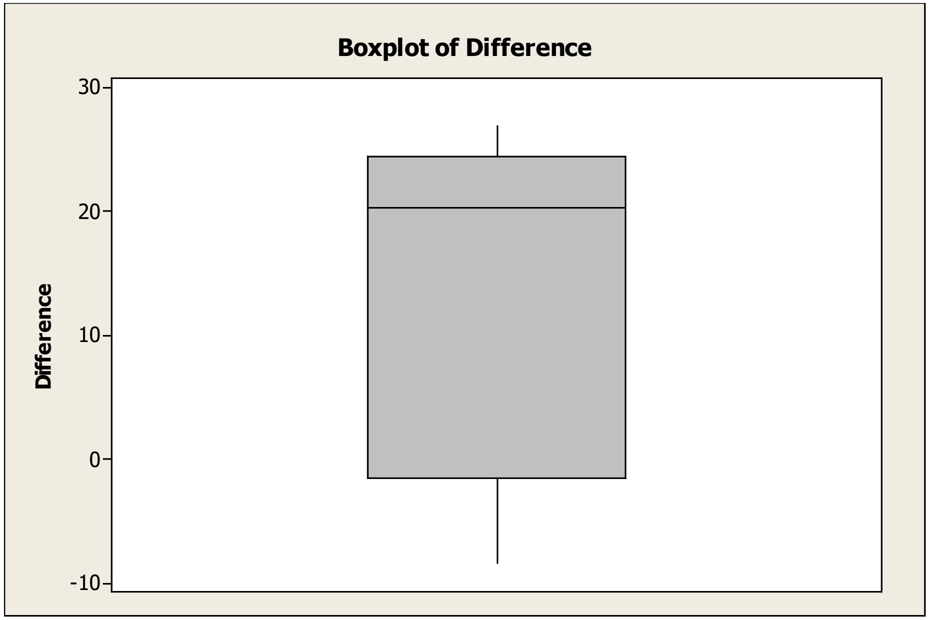
- **Test Statistics:**
- $T_{-} = 4.5$
- $T_{+} = 10.5$
- $T = \text{smaller}(10.5, 4.5) = 4.5$

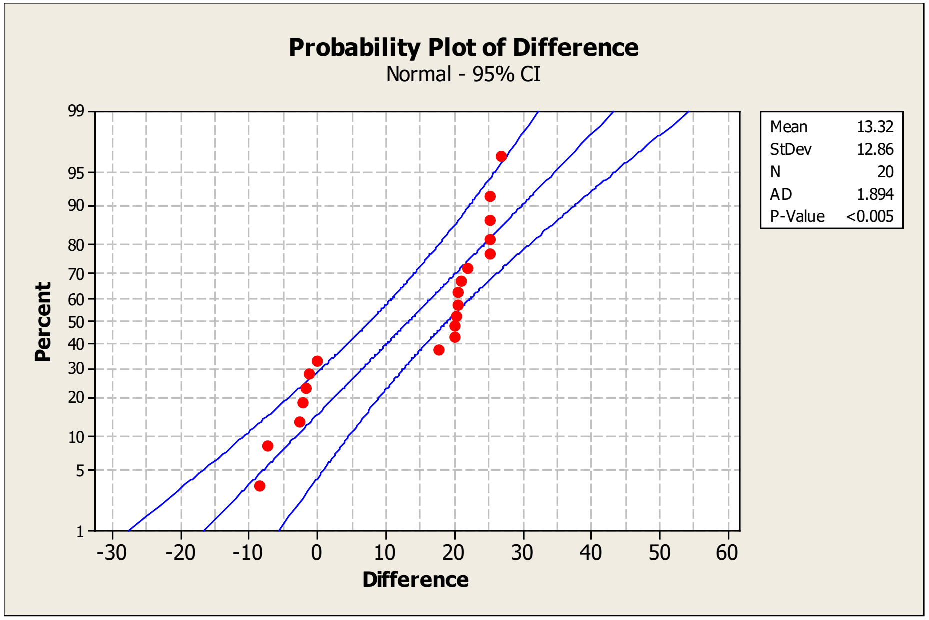
BOOK EXAMPLE 6.9

- **Does Brand A fertilizer produce more grass than Brand B?**

Field	Brand A	Brand B	Difference	Field	Brand A	Brand B	Difference
1	211.4	186.3	25.1	11	208.9	183.6	25.3
2	204.4	205.7	−1.3	12	208.7	188.7	20.0
3	202.0	184.4	17.6	13	213.8	188.6	25.2
4	201.9	203.6	−1.7	14	201.6	204.2	−2.6
5	202.4	180.4	22.0	15	201.8	181.6	20.1
6	202.0	202.0	0	16	200.3	208.7	−8.4
7	202.4	181.5	20.9	17	201.8	181.5	20.3
8	207.1	186.7	20.4	18	201.5	208.7	−7.2
9	203.6	205.7	−2.1	19	212.1	186.8	25.3
10	216.0	189.1	26.9	20	203.4	182.9	20.5

BOOK EXAMPLE 6.9 (CONT'D)





- $H_0: med_d = 0$ vs $H_a: med_d > 0$

BOOK EXAMPLE 6.9 (CONT'D)

- **In R:**
  - wilcox.test(exmp6.9$BrandA, exmp6.9$BrandB, paired = T, alternative = "greater") **or**
  - wilcox.test(exmp6.9$diff, alternative = "greater")

```
Wilcoxon signed rank test with continuity correction

data:  exmp6.9$diff
V = 169, p-value = 0.001548
alternative hypothesis: true location is greater than 0
```

- **Conclusion: Since the $p - value = 0.002$ is small, we reject H_0 in favor of H_a. Thus conclude that Brand A fertilizer produce more grass than Brand B.**
- **95% Confidence interval**
  - wilcox.test(exmp6.9$diff, conf.int = T)

```
Wilcoxon signed rank test with continuity correction

alternative hypothesis: true location is not equal to 0
95 percent confidence interval:
 8.70002 22.64996
```