MATH 4720 / MSCS 5720

Instructor: Mehdi Maadooliat

Chapter 4 (Part A)

Department of Mathematical and Statistical Sciences

CHAPTER 4 (PART A)

- Probability
- Random process, Event, Sample space
- Set theory review
- Conditional Probability and Independence
- Bayes' Theorem
- Law of total probability
- Random Variables
  - Discrete
    - Binomial
    - Poisson
  - Continuous
    - Normal
- Normal Approximation to $Binomial(n, \pi)$
- Sampling Distribution

PROBABILITY

Common misconceptions about probability

- Paulos (1988) tells the story of a weather forecaster on American TV who reported that there was *a 50% chance of rain on Saturday*, and *a 50% chance of rain on Sunday*, from which he concluded that there was *a 100% chance of rain on the weekend.*
- *New Scientist* reported a story about an inspector in the Food and Drug Administration who visited a restaurant in Salt Lake City famous for its quiches made from four fresh eggs. The inspector told the owner that according to FDA research *every fourth egg has salmonella bacteria*, so the restaurant should only *use three eggs* in a quiche.

(Source: http://www.edge.org/3rd_culture/gigerenzer03/gigerenzer_print.html)

Paulos JA (1988) *Innumeracy: Mathematical illiteracy and its consequences*. Vintage Books, New York

RANDOMNESS AND PROBABILITY

- We call a **phenomenon random** if individual outcomes are uncertain but there is nonetheless a regular distribution of outcomes in a large number of repetitions

- The **probability** of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions. If a random experiment is repeated *n* times then,

$$\frac{\text{number of times A occurs}}{\text{number of trials, } n} \rightarrow P(\text{A}), \text{ the probability that event A occurs}$$

- Example: [Tossing coins]

PROBABILITY DEFINITIONS

- **Random process:** a process whose outcome can not be predicted with certainty
- **Sample space (S):** the collection of all possible outcomes to a random process
- **Event (A,B):** a collection of possible outcomes
  - **Simple:** one outcome
  - **Compound:** more than one outcome
- **Probability:** a number between 0 and 1 (inclusive) indicating the likelihood that the event will occur

SET THEORY REVIEW

- **Union** ($A \cup B$) – all elements in A or B
- **Intersection** ($A \cap B$) - all elements in A **and** B
- **Complement** ($\bar{A}$) – all elements **not** in the set
- **Mutually exclusive** – two events are mutually exclusive if they have no outcomes in common

- [Venn Diagram]

- The proportion of times that an event A occurs converges to the probability of A, $P(A)$, as the number of repetitions becomes large.

PROBABILITY WITH EQUALLY LIKELY OUTCOMES

- If a sample space is composed of k **equally likely** outcomes with m outcomes contained in A, then $P(A) = m/k$.

- Two way classification for the salary data:

	Education	Engineering	Total
Female	856	232	1088
Male	220	924	1144
Total	1076	1156	2232

- **Randomly** select a student from the survey group, what is the probability that
  - the student is female?
  - the student is an engineer?
  - the student is a female engineer?
  - the student is a female or an engineer?

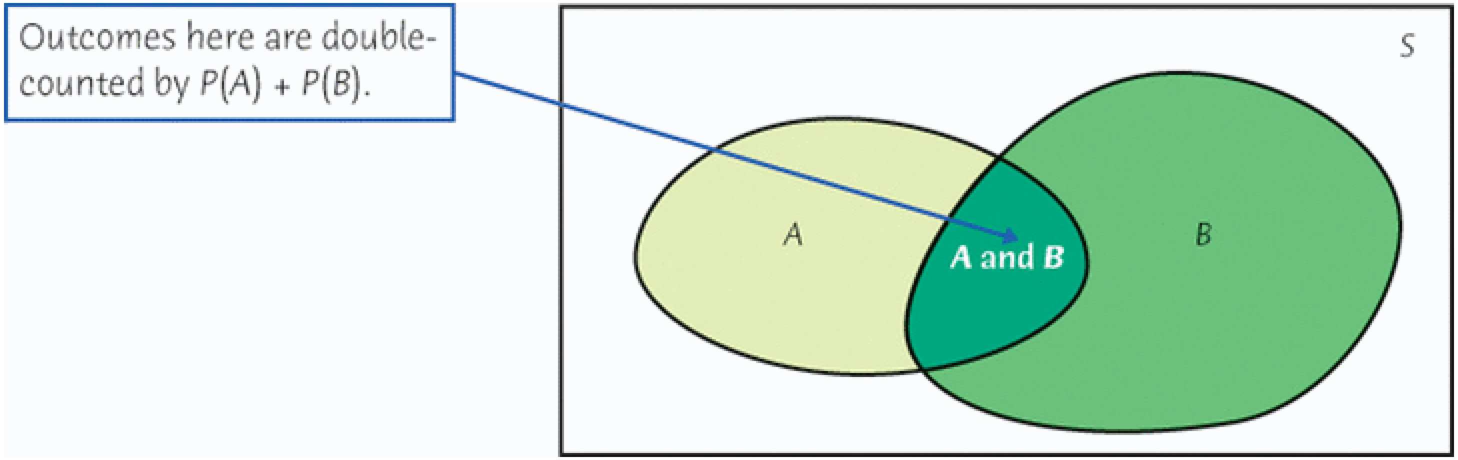
PROPERTIES OF PROBABILITY

- **Axioms:**
  - $P(A) \geq 0$
  - $P(S) = 1$
  - If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$
- **Properties that follow:**
  - $0 \leq P(A) \leq 1$
  - $P(\varnothing) = 0$
  - $P(\overline{A}) = 1 - P(A)$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

ADDITION RULE IN GENERAL

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



A RAINY WEEKEND?

- Recall the TV weather forecaster who announced that if the probability of rain on Saturday is 50% and the probability of rain on Sunday is 50%, then the probability of rain over the weekend is 100%.

- If the probability that it rains on both Saturday and Sunday is 35%, what is the probability of rain over the weekend?

Events: $A =$ it rains on Saturday, $B =$ it rains on Sunday.

BACK TO THE SALARY EXAMPLE

- **Consider the following two way classification table between major and gender for the salary data:**

	Education	Engineering	Total
Female	856	232	1088
Male	220	924	1144
Total	1076	1156	2232

- **If we randomly select a student from this group, what is the probability that**
  - **the student is not a male educator?**
  - **the student is a female or an engineer?**

CONDITIONAL PROBABILITY AND INDEPENDENCE

- The **conditional probability** of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) > 0$.

- A and B are **independent** if $P(A|B) = P(A)$.
- If the events are independent, knowing B occurs does not change the probability that A occurs.
- If $P(A) > 0$ and $P(B) > 0$, then A independent of B means $P(A \cap B) = P(A)P(B)$.
- What are some examples of events that are independent?

BACK TO THE EXAMPLES

- **For the salary example, is the gender of the student selected independent of the student's major?**

	Education	Engineering	Total
Female	856	232	1088
Male	220	924	1144
Total	1076	1156	2232

APPLYING THE MULTIPLICATION RULE TO INDEPENDENT EVENTS: THE BAD EGG?

- ***New Scientist*** **reported a story about an inspector who visited a restaurant in Salt Lake City famous for its quiches made from four fresh eggs. The inspector told the owner that according to FDA research** ***every fourth egg has salmonella bacteria*****, so the restaurant should only** ***use three eggs*** **in a quiche.**

Now, what is the probability

a. use 1 egg and it is bad

b. use 2 eggs and at least one is bad

c. use 3 eggs and at least one is bad

d. use 4 eggs and at least one is bad

BAYES' THEOREM

- **In some situations, we know the conditional probability $P(B|A)$ but are much more interested in $P(A|B)$.**

- **For example, diagnostic tests provide**

$$P(\text{Test} + |\text{disease})$$

but we are interested in

$$P(\text{disease}|\text{Test} +)$$

		True health status	
		Disease	No disease
Test result	Positive	✓	**False positive**
	Negative	**False negative**	✓

(a)

- **Bayes' Theorem provides a method for finding $P(A|B)$ from $P(B|A)$**

BAYES' FORMULA

- If A and B are any events whose probabilities are not 0 or 1, then

- $P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|\bar{A})P(\bar{A})}$

- How do we get this?

POLYGRAPH TEST

- Some employers use lie detector tests to screen applicants. Lie detector tests are not completely reliable. Suppose that in a lie detector test, 64% of lies are identifies as lies and that 17% of true statements are also identified as lies. A company gives job applicants a polygraph test, asking, "Did you tell the truth on your job application? Suppose that 94% of the job applicants tell the truth during the polygraph test. What is the probability that a person who fails the test was actually telling the truth?

LAW OF TOTAL PROBABILITY AND BAYES RULE

- If $A_1, \ldots, A_n$ are mutually exclusive, each event has positive probability and one of the events must occur, then

$$P(B) = \sum_{j=1}^{n} P(B \mid A_j) P(A_j)$$

- If $P(B) > 0,$ the above result implies Bayes rule:

$$P(A_k \mid B) = \frac{P(B \mid A_k) P(A_k)}{\sum_{j=1}^{n} P(B \mid A_j) P(A_j)}$$

VOLTAGE REGULATOR EXAMPLE

- **In a batch of voltage regulators, 60% came from supplier 1, 30% from supplier 2 and 10% from supplier 3.**
- 95% **of regulators from supplier 1 work**
- 60% **of regulators from supplier 2 work**
- 50% **of regulators from supplier 3 work**

- **If a regulator randomly selected from the batch works, what is the probability it came from supplier 1?**

RANDOM VARIABLES

- **Random Variable: Y is a random variable if it assumes values randomly**

- **Examples:**
  - **Toss a coin 2 times. $Y = \#$ of heads**
  - **$Y = \#$ of accidents in Wisconsin Ave. per day**
  - **$Y =$ time(in minutes) until next accident in Wisconsin Ave.**

- **A discrete random variable takes on a finite or countable number of values.**

DISCRETE RANDOM VARIABLE

- **We will identify the distribution of a discrete random variable Y by its probability distribution function, $P(Y = y)$.**

- **The probability associated with every value of Y lies between 0 and 1.**

- **The sum of the probabilities for all values of Y is equal to 1.**

- **The probabilities for a discrete random variable are additive. Hence, the probability that $Y = 1$ or 2 is equal to $P(Y = 1) + P(Y = 2)$.**

BINOMIAL DISTRIBUTION

- **Binomial Experiment:**
  - The experiment consists of n identical trials.
  - Each trial can result in one of two outcomes (S or F)
  - The probability of success, $P(S)$, is a constant π for all trials
  - Trials are independent
  - X counts the number of successes observed in n trials

- The random variable X is said to have a **Binomial distribution** with parameters n and π.

- $X \sim Binomial(n, \pi)$

BINOMIAL DISTRIBUTION CONT…

- **Let** $Y \sim Binomial(n, \pi)$
- **Note that** Y **is discrete with possible values** $0, 1, \ldots, n$
- $P(Y = k) = \frac{n!}{k!(n-k)!}\pi^k(1-\pi)^{n-k}, \; k = 0,1, \ldots, n$
- **Example: Toss a coin two times.**
  - $Y = \#\ of\ heads$
  - $Y \sim Binomial(2, 1/2)$

ANOTHER EXAMPLE

- Suppose there are 10 multiple choice questions each with 4 multiple choices.
  - Q.1: A B C D
  - Q,2: A B C D
  - .
  - .
  - .
  - Q.10: A B C D

MULTIPLE CHOICE EXAMPLE

- **Suppose a student answer each question by random guessing.**

- $Y = \#\ of\ correct\ answers.$

- **What is the distribution of Y?**

  1. **What is the probability that the student gets all correct answers?**
  2. **What is the probability that the students gets at least 7 correct answers?**

EXAMPLE CONT.

- $Y \sim Binomial(10, ¼)$
- $P(Y = k) = \frac{n!}{k!(n-k)!} \pi^k (1 - \pi)^{n-k}$

1. $P(Y = 10) = \frac{10!}{10!(10-10)!} \left(\frac{1}{4}\right)^{10} \left(1 - \frac{1}{4}\right)^{10-10}$
$$= 9.5 \times 10^{-7}$$

2. $P(Y \geq 7) = P(Y = 7) + P(Y = 8)$
$$+P(Y = 9) + P(Y = 10)$$
$$= 0.0035$$

- [Binomial Calculator]
- **R Code:**

```
> dbinom(10, size = 10, prob = 0.25)
[1] 9.536743e-07
> 1-pbinom(6, size = 10, prob = 0.25)
[1] 0.003505707
```

EXAMPLE CONT.

- **Using Calculator:**
- **Binomialpdf**$(n, \pi, k) = P(Y = k)$
- **Binomialcdf**$(n, \pi, k) = P(Y \leq k)$
  - $P(Y = 10) = binomialpdf(10, 0.25, 10)$
    $= 9.5 \times 10^{-7}$
  - $P(Y \geq 7) = 1 - P(Y \leq 6)$
    $= 1 - binomialcdf(10, 0.25, 6)$
    $= 0.0035$
- **This implies that, if you only make a random guess, it is highly unlikely that you will get all 10 answers correct, or even 7 or more answers correct.**

EXAMPLE

- **Genetic theory suggest that 1 in 1000 adult population have a particular genetic disorder. What is the probability that in a sample of 25 adults at least one adult have the genetic disorder.**

  – $Y \sim Binomial(25, 0.001)$

  – $$\begin{aligned} P(Y \geq 1) &= 1 - P(Y = 0) \\ &= 1 - binomialpdf(25, 0.001, 0) \\ &= 1 - 0.975 = 0.025 \end{aligned}$$

- **This implies that it is NOT likely that in a sample of 25 adults, at least one has genetic disorder.**
- **Binomial Calculator**

NURSE EMPLOYMENT CASE

- **Contract requires 90% of records handled timely**
- **32 of 36 sample records handled timely, she was fired!**
- **If the proportion of all records handled timely is 0.9, what is the probability that 32 or fewer would be handled timely in a sample of 36?**

  - $Y \sim Binomial(36, 0.90)$
  - $P(Y \leq 32) = binomialcdf(36, 0.90, 32)$
  - $= 0.4915$

- **Binomial Calculator**

POISSON DISTRIBUTION

- **Let Y be the number of accidents at a particular intersection during a year.**
- **Y is discrete with possible values $0, 1, 2, \ldots ,$ (we cannot assign a maximum value).**
- **Binomial distribution does not make sense in this case.**
- **For such situation we use *Poisson* distribution.**

POISSON DISTRIBUTION

- Let Y be the number of occurrence of an event during a time period (or in a given region), then under certain condition, the distribution of Y can be described as Poisson.

- Notation: $Poisson(\mu)$.

- $\mu -$ average number of events during time period (or in a given region)

- $$P(Y = y) = \frac{\mu^y e^{-\mu}}{y!}, \quad y = 0,1,2, \ldots$$

- Poisson Calculator

EXAMPLE 1

- **Suppose the average number of accidents at a particular intersection is 20 per year. What is the probability that during next year more than 12 accidents would occur?**
  - $Y \sim Poisson(\mu = 20)$

$$P(Y > 12) = 1 - P(Y \leq 12)$$
$$= 1 - Poissoncdf(0, 12, 20)$$
$$= 1 - 0.0390 = 0.961$$

- **What is the probability that 60 or more accidents would occur during the next 5 years?**
  - $Y \sim Poisson(\mu = 20 * 5) = Poisson(\mu = 100)$

$$P(Y \geq 60) = 1 - P(Y \leq 59)$$
$$= 1 - Poissoncdf(0, 59, 100)$$
$$\approx 1.000$$

[Poisson Calculator]

EXAMPLE 2

- **It is generally observed that the average number of infected trees per acre in a forest is 0.1. It was observed that a 100 acre forest has 17 infected trees. What is the probability that 17 or more trees are infected in the forest if the average per acre is 0.1?**

  - $Y \sim Poisson(\mu = 0.1 * 100) = Poisson(\mu = 10)$

  - $P(Y \geq 17) = 1 - P(Y \leq 16)$
  $$= 1 - Poissoncdf(0, 16, 10)$$
  $$= 1 - 0.9730 = 0.0270$$

- **Poisson Calculator**

POISSON APPROXIMATION OF THE $\boldsymbol{Binomial(n,\pi)}$ DISTRIBUTION

- **Let $Y \sim Binomial(n,\pi)$.**
  If n is very large, and π is very small, then

$$Y \approx Poisson(\mu = n\pi)$$

- **Example: Suppose the prevalence of a disease is 1 in $10{,}000$ among adults. What is the probability that a city with $100{,}000$ adults has 15 or more adults carrying the disease?**

$$Y \approx Poisson\left(\mu = 100000 * \frac{1}{10000}\right) = Poisson(\mu = 10)$$

$$P(Y \geq 15) = 1 - Poissoncdf(0, 14, 10)$$

$$= 1 - 0.9165 = 0.0835$$

- **Poisson Calculator**