MATH 4720 / MSSC 5720

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Clarification on Binomial Distribution

Department of Mathematical and Statistical Sciences

THE BINOMIAL PROBABILITY DISTRIBUTION

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- Consider the following probability experiment. I give you a surprise four-question multiple-choice quiz.
- You have not studied the material, and therefore you decide to answer the four questions by randomly guessing.
- Here are some questions for you?

Answer Page to Quiz

Directions: Circle the best answer to each question.

1. a	b	c
2. a	b	c
3. a	b	c
4. a	b	c

1. How many of the four questions are you likely to have answered correctly?
2. How likely are you to have more than half of the answers correct?
3. What is the probability that you selected the correct answers to all four questions?
4. What is the probability that you selected wrong answers for all four questions?
5. If an entire class answers the quiz by guessing, what do you think the class "average" number of correct answers will be?

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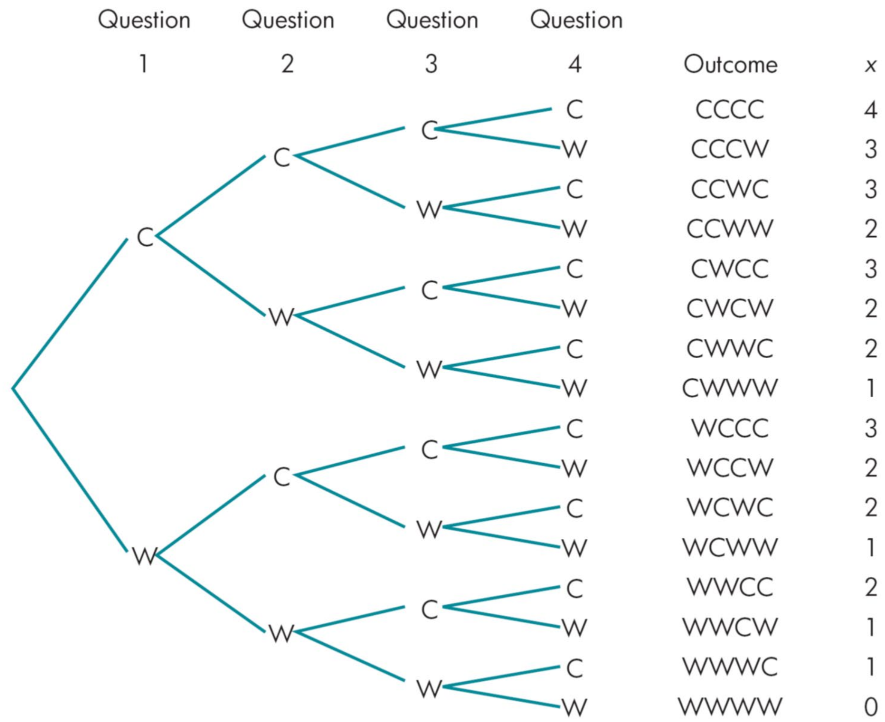
- To find the answers to these questions, let's start with a tree diagram



- Each of the four questions is answered with the correct answer (C) or with a wrong answer (W).
- x is the "number of correct answers" on one person's quiz when the quiz was taken by randomly guessing.

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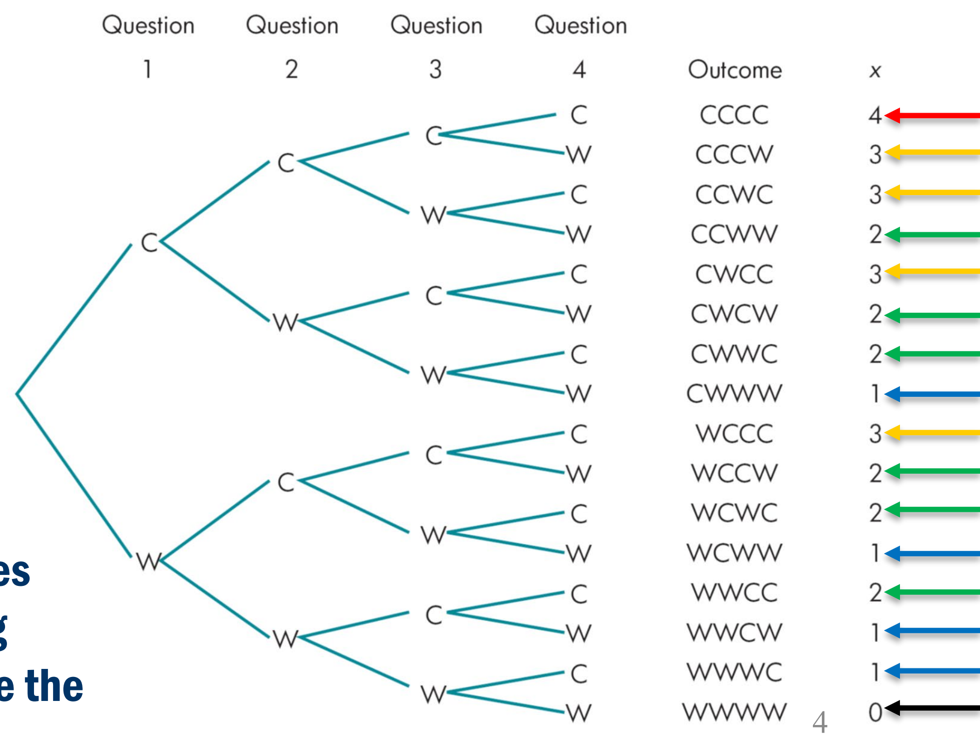
- **Notice that:**
  - The event $x = 4$, "four correct answers," is shown on the top branch.
  - The event $x = 0$, "zero correct answers," is shown on the bottom branch.
  - The event $x = 1$ occurs on four different branches.
  - The event $x = 2$ occurs on six branches.
  - The event $x = 3$ occurs on four branches.

  - Each individual question has only one correct answer.
  - The probability of selecting the correct answer to each question is $\frac{1}{3}$.
  - The probability that a wrong answer is selected is $\frac{2}{3}$.
  - The probability of each value of x can be found by calculating the probabilities of all the branches and then combining the probabilities for branches that have the same x values.



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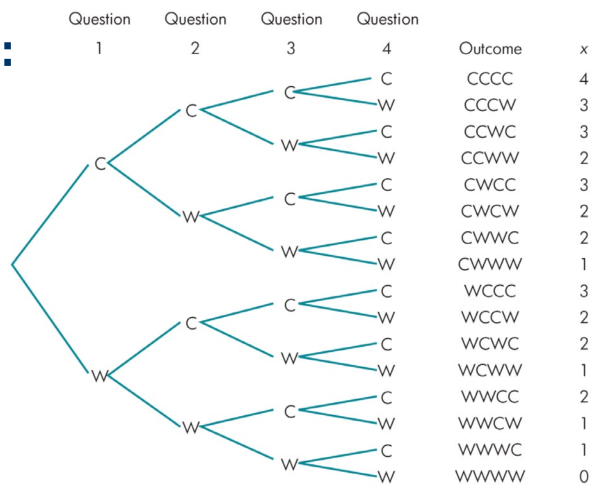
- $P(x = 0)$ **is the probability that the correct answers are given for zero questions.**
  - $P(x=0) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \left(\frac{2}{3}\right)^4 = \frac{16}{81} = \mathbf{0.198}$
  - **Note: Answering each individual question is a separate and independent event, thereby we can use:**
  - $P\ (A \text{ and } B) = P(A) \cdot P(B)$



- $P(x = 4)$ **is the probability that correct answers are given for all four questions.**
  - $P(x=4) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \left(\frac{1}{3}\right)^4 = \frac{1}{81} = \mathbf{0.012}$

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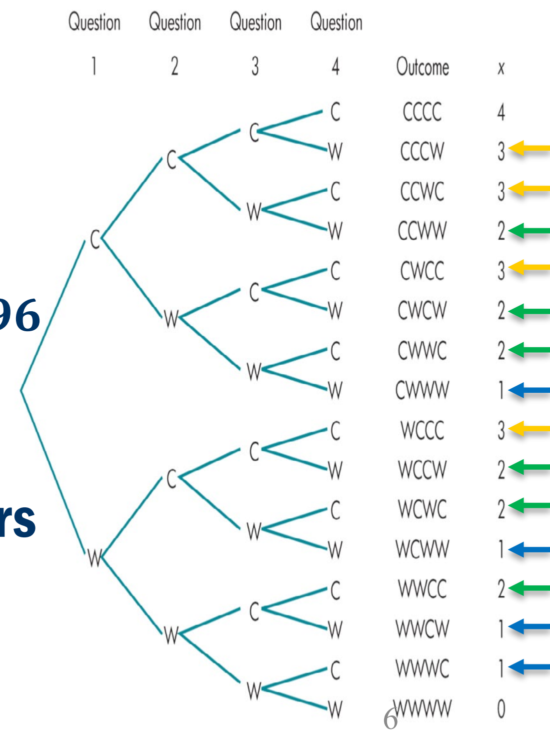
- $P\ (x = 1)$ **is the probability that the correct answer is given for exactly one question and wrong answers are given for the other three (there are four branches: CWWW, WCWW, WWCW, WWWC—and each has the same probability):**
  - $P(x = 1) = 4 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = 4 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^3 = \mathbf{0.395}$

- $P\ (x = 2)$ **is the probability that correct answers are given for exactly two questions and wrong answers are given for the other two (there are six branches) :**
  - $P(x = 2) = 6 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = 6 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^2 = \mathbf{0.296}$

- $P\ (x = 3)$ **is the probability that correct answers are given for exactly three questions and wrong answers are given for the other one(there are four branches) :**
  - $P(x = 3) = 4 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = 4 \times \left(\frac{1}{3}\right)^3 \times \frac{2}{3} = \mathbf{0.099}$



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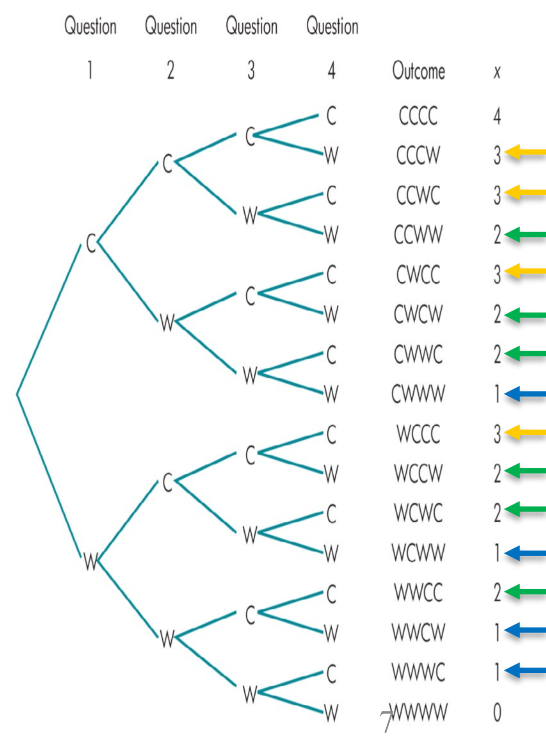
- $P(x=0) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \left(\frac{2}{3}\right)^4 = \frac{16}{81} = \mathbf{0.198}$
- $P(x=1) = 4 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = 4 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^3 = \mathbf{0.395}$
- $P(x=2) = 6 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = 6 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^2 = \mathbf{0.296}$
- $P(x=3) = 4 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = 4 \times \left(\frac{1}{3}\right)^3 \times \frac{2}{3} = \mathbf{0.099}$
- $P(x=4) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \left(\frac{1}{3}\right)^4 = \frac{1}{81} = \mathbf{0.012}$
- **In general:**
- $\boxed{P(x=k) = \frac{4!}{k!(4-k)!} \times \left(\frac{1}{3}\right)^k \times \left(\frac{2}{3}\right)^{4-k}, \quad \textbf{for } k = 0,1,2,3,4}$
- **Probability distribution:**

x	$P(x)$
0	0.198
1	0.395
2	0.296
3	0.099
4	0.012
	1.000

Probability Distribution for the Four-Question Quiz



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- **Now we can answer those five questions.**

x	$P(x)$
0	0.198
1	0.395
2	0.296
3	0.099
4	0.012
	1.000

Probability Distribution for the Four-Question Quiz

1. **How many of the four questions are you likely to have answered correctly?**

➢ **The most likely occurrence would be to get one answer correct; it has a probability of 0.395.**

2. **How likely are you to have more than half of the answers correct?**

➢ **Having more than half correct is represented by $x = 3$ or 4; their total probability is $0.099 + 0.012 = 0.111$. (You will pass this quiz with 11% chance by random guess.)**

3. **What is the probability that you selected the correct answers to all four questions?**

➢ $P(\text{all four correct}) = P(x = 4) = 0.012$. **(All correct occurs only 1% of the time.)**

4. **What is the probability that you selected wrong answers for all four questions?**

➢ $P(\text{all four wrong}) = P(x = 0) = 0.198$. **(That's almost 20% of the time.)**

5. **If an entire class answers the quiz by guessing, what do you think the class "average" number of correct answers will be?**

➢ **The class average is expected to be $\frac{1}{3}$ of 4, or 1.33 correct answers.**

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- **Many experiments are composed of repeated trials whose outcomes can be classified into one of two categories: success or failure.**
  - Examples of such experiments are coin tosses, right/wrong quiz answers, and other, more practical experiments such as determining whether a product did or did not do its prescribed job and whether a candidate gets elected or not.

- **There are experiments in which the trials have many outcomes that, under the right conditions, may fit this general description of being classified in one of two categories.**
  - For example, when we roll a single die, we usually consider six possible outcomes.
  - However, if we are interested only in knowing whether a "one" shows or not, there are really only two outcomes: the "one" shows or "something else" shows.

- **The experiments just described are called binomial probability experiments.**

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- **Binomial probability experiment:** An experiment that is made up of repeated trials that possess the following properties:
  1. There are n repeated identical independent trials.
  2. Each trial has two possible outcomes (**success** or **failure**).
  3. $P(\text{success}) = \pi$, $P(\text{failure}) = 1 - \pi$, and $\pi + (1 - \pi) = 1$.
  4. The **binomial random variable** X is the count of the number of successful trials that occur; X may take on any integer value from zero to n.

- **Binomial probability function** For a binomial experiment, let π represent the probability of a "success" and $1 - \pi$ represent the probability of a "failure" on a single trial. Then $P(X = k)$, the probability that there will be exactly k successes in n trials, is

- $P(X = k) = \binom{n}{k} \pi^k (1 - \pi)^{n-k}, \quad \text{for } k = 0, 1, 2, \cdots, n$

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- $P(X = k) = \frac{n!}{k!(n-k)!}\pi^k(1-\pi)^{n-k}, \quad \text{for } k = 0, 1, 2, \cdots, n$

- **When you look at the probability function, you notice that it is the product of three basic factors:**
  1. **The number of ways that exactly k successes can occur in n trials,**
  $$\frac{n!}{k!\,(n-k)!}$$
  2. **The probability of exactly k successes, π^k**
  3. **The probability of failure on the remaining $(n-k)$ trials, $(1-\pi)^{n-k}$**