MATH 4720 / MSSC 5720

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Chapter 6 (Part B)

Department of Mathematical and Statistical Sciences

CHAPTER 6 (PART B)

- **Comparing Two Population Means**
  - Independent Samples
  - Dependent Samples
- **Two sample t-test (Independent Samples)**
  - Pooled t-test
  - Unequal variance t-test
- **Paired t-test (Dependent Samples)**
- **Power Analysis**
  - Independent Samples
  - Dependent Samples
- **Non-parametric Tests**
  - Sign test (from Chapter 5, test for median M)
  - Wilcoxon Rank-Sum (or Mann–Whitney) Test (two independent samples)
  - Wilcoxon Signed-Rank Test (dependent samples)

COMPARING TWO DEPENDENT POPULATION MEANS (REMINDER)

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- **Dependent Samples**

- Here, there is only one group of subjects, but two different measurements are taken from this group.

Subject	Before	After
1	y_{11}	y_{21}
2	y_{12}	y_{22}
.	.	.
.	.	.
n	y_{1n}	y_{2n}

- Since the subjects are same for before and after measurements, the two samples are dependent.

COMPARING TWO DEPENDENT POPULATION MEANS (CONT'D)

Subject	Before (y_1)	After (y_2)	Difference $d = y_1 - y_2$
1	y_{11}	y_{21}	d_1
2	y_{12}	y_{22}	d_2
.	.	.	.
.	.	.	.
n	y_{1n}	y_{2n}	d_n

- μ_1 = **Mean Before,** μ_2 = **Mean After**
- $\mu_d = \mu_1 - \mu_2$
- $H_0: \mu_1 = \mu_2 \equiv \mu_d = 0$
- $H_a: \mu_1 > \mu_2 \equiv \mu_d > 0$

  or $\mu_1 < \mu_2 \equiv \mu_d < 0$

  or $\mu_1 \neq \mu_2 \equiv \mu_d \neq 0$

COMPARING TWO DEPENDENT POPULATION MEANS (CONT'D)

- **Assumption:** $n \geq 30$ **or the differences are normally distributed.**

- **T.S.** $$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

- **Decision Rule:** $(df = n - 1)$
  - $H_a: \mu_d > 0$: **Reject** H_0 **in favor of** H_a **if** $t > t_\alpha$
  - $H_a: \mu_d < 0$: **Reject** H_0 **in favor of** H_a **if** $t < -t_\alpha$
  - $H_a: \mu_d \neq 0$: **Reject** H_0 **in favor of** H_a **if** $|t| > t_{\alpha/2}$

- **Note that this method is same as one sample t-test for the sample of differences. We call it paired t-test.**

COMPARING TWO DEPENDENT POPULATION MEANS (CONT'D)

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- **p-values** can be calculated in the similar manner.

- Confidence Interval for the difference $\mu_d = \mu_1 - \mu_2$

$$\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n}}$$

- **Example**: Consider a drug that can be used to reduce blood pressure for the hypertensive individuals.

- Objective: Is this drug effective?

- Sample of 10 hypertensive individuals use the drug for four weeks.

EXAMPLE CONT'D

Subject	Before y_1	After y_2	Difference $d = y_1 - y_2$
1	143	124	19
2	153	129	24
3	142	131	11
4	139	145	-6
5	172	152	20
6	176	150	26
7	155	125	30
8	149	142	7
9	140	145	-5
10	169	160	9

- $\bar{d} = 13.5, \; s_d = 12.48$
- **Does the data provide sufficient evidence that the drug is effective in reducing the blood pressure?**

EXAMPLE CONT'D

- μ_1 = **Mean Before,** μ_2 = **Mean After**
- $\mu_d = \mu_1 - \mu_2$

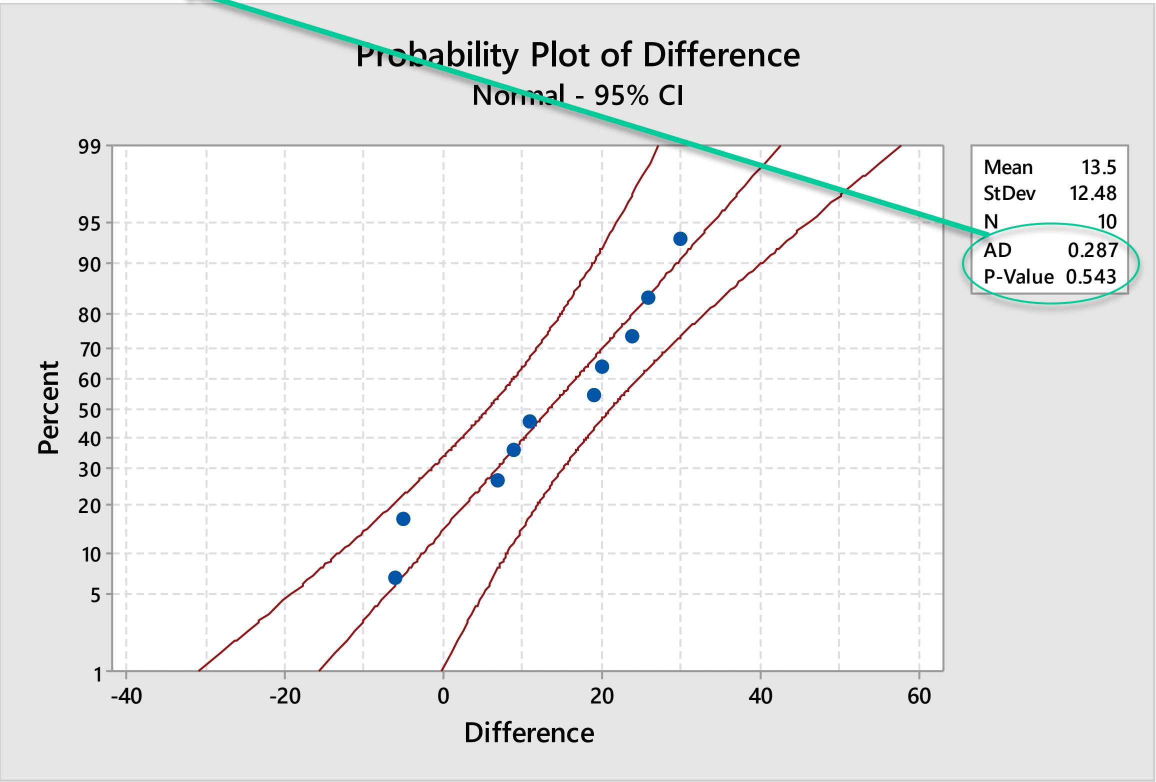
- $H_0: \mu_1 = \mu_2 \equiv \mu_d = 0$
- $H_a: \mu_1 > \mu_2 \equiv \mu_d > 0$

- **Assumption: Differences are normally distributed. This can be tested by normal probability plot of the differences.**
  - **In R:** shapiro.test(d)

- **Note that here the distribution of y_1 and y_2 is not important.**

NORMAL PROBABILITY PLOT

- H_0: **Data is normally distributed**
  - $P - Value < \alpha$, **therefore Fail to reject** H_0



- **TS.** $$t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{13.5}{\frac{12.48}{\sqrt{10}}} = 3.42$$

EXAMPLE CONT'D

- **Decision Rule:** $\alpha = 0.05, \;\; df = n - 1 = 9$
- **Reject** H_0 **in favor of** H_a **if** $t > t_\alpha = 1.833$
- **Conclusion: Is** $t > 1.833$**? Yes, since** $t = 3.42$**. We reject** H_0 **in favor of** H_a**, and conclude that the drug is effective in reducing blood pressure.**

- **Estimate the difference in the mean Blood Pressures using a 95% CI.**
- **Formula:** $\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n}}$
- $df = 9, \frac{\alpha}{2} = 0.025\,,\; t_{\alpha/2} = 2.262$
- **95% CI:** $13.5 \pm 2.262 \frac{12.48}{\sqrt{10}}, \quad i.e\; 4.57 < \mu_d < 22.42$

BOOK EXAMPLE: 6.8

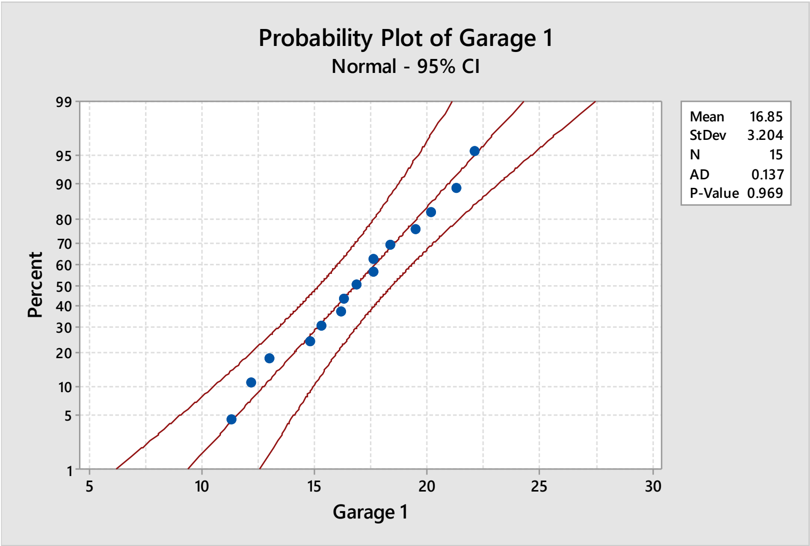
- **15 cars involved in accidents were taken to two garages (Garage I and Garage II).**

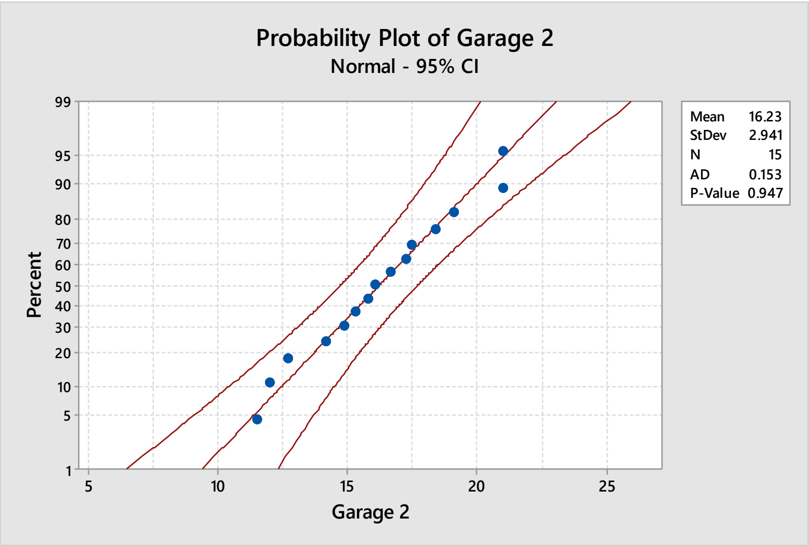
TABLE 6.14
Repair estimates (in hundreds of dollars)

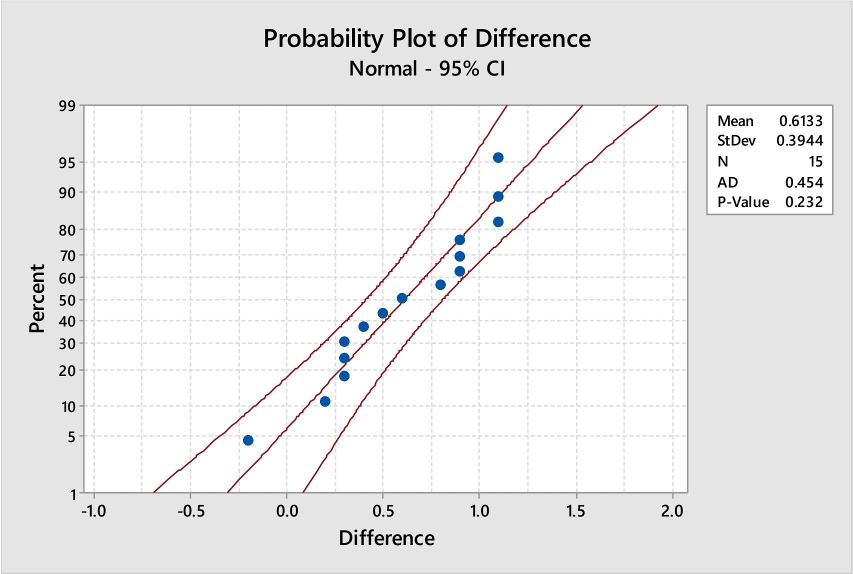
Car	Garage I	Garage II
1	17.6	17.3
2	20.2	19.1
3	19.5	18.4
4	11.3	11.5
5	13.0	12.7
6	16.3	15.8
7	15.3	14.9
8	16.2	15.3
9	12.2	12.0
10	14.8	14.2
11	21.3	21.0
12	22.1	21.0
13	16.9	16.1
14	17.6	16.7
15	18.4	17.5
Totals:	$\bar{y}_1 = 16.85$	$\bar{y}_2 = 16.23$
	$s_1 = 3.20$	$s_1 = 2.94$

- **Given $\alpha = 0.05,$ is there a significant difference between Garage I and Garage II?**

NORMAL PROBABILITY PLOTS







EXAMPLE 6.8 (CONT'D)

- **Is it fine to do two independent sample t-test?**
- **NO**
- **WRONG ANALYSIS:**

- **Assume** $\sigma_1 \neq \sigma_2$

Two-Sample T-Test and CI: Garage I, Garage II

```
Two-sample T for Garage I vs Garage II

             N   Mean  StDev  SE Mean
Garage I    15  16.85   3.20     0.83
Garage II   15  16.23   2.94     0.76


Difference = mu (Garage I) - mu (Garage II)
Estimate for difference:  0.61
95% CI for difference:  (-1.69, 2.92)
T-Test of difference = 0 (vs not =): T-Value = 0.55  P-Value = 0.589  DF = 27
```

  - **In R:** t.test(exmp6.8$Garage1,exmp6.8$Garage2)

- **Assume** $\sigma_1 = \sigma_2$

```
Difference = mu (Garage I) - mu (Garage II)
Estimate for difference:  0.61
95% CI for difference:  (-1.69, 2.91)
T-Test of difference = 0 (vs not =): T-Value = 0.55  P-Value = 0.589  DF = 28
Both use Pooled StDev = 3.0754
```

  - **In R:** t.test(exmp6.8$Garage1,exmp6.8$Garage2, var.equal = T)

EXAMPLE 6.8 (CONT'D)

- **Correct test is the paired t-test. Let μ_1 = Garage I, μ_2 = Garage II**
- $\mu_d = \mu_1 - \mu_2$
- $H_0: \mu_1 = \mu_2 \equiv \mu_d = 0$
- $H_a: \mu_1 \neq \mu_2 \equiv \mu_d \neq 0$
- **TS.** $t = \dfrac{\bar{d}}{s_d/\sqrt{n}} = \dfrac{0.613}{\frac{0.394}{\sqrt{15}}}$

  $t = 6.02$

	Garage 1	Garage 2	Difference
1	17.6	17.3	0.3
2	20.2	19.1	1.1
3	19.5	18.4	1.1
4	11.3	11.5	-0.2
5	13.0	12.7	0.3
6	16.3	15.8	0.5
7	15.3	14.9	0.4
8	16.2	15.3	0.9
9	12.2	12.0	0.2
10	14.8	14.2	0.6
11	21.3	21.0	0.3
12	22.1	21.0	1.1
13	16.9	16.1	0.8
14	17.6	16.7	0.9
15	18.4	17.5	0.9

One-Sample T: Difference

In R:
```
t.test(exmp6.8$diff)
t.test(exmp6.8$Garage1, exmp6.8$Garage2, paired = T)
```

Variable	N	Mean	StDev	SE Mean	95% CI	T	P
Difference	15	0.613	0.394	0.102	(0.395, 0.832)	6.02	0.000

- **Decision Rule ($df = 14$):**
- **Reject H_0 if $|t| > t_{\frac{\alpha}{2}}$**
- $6.02 > 2.145$, **so we reject H_0.**
- **There is a significant difference between garage estimates**

POWER ANALYSIS

Independent Samples:

- $H_0: \mu_1 = \mu_2$
- $H_a: \mu_1 > \mu_2$
- or $\mu_1 < \mu_2$ $\Big\}$ One-tail test
- or $\mu_1 \neq \mu_2 \rightarrow$ Two-tail test

- The samples sizes $n_1 = n_2 = n$ needed to correctly discover a difference in testing of hypothesis with the power $P(\Delta_a)$ when the difference in the means is $\geq \Delta_a$ is given by

- $n_1 = n_2 = \frac{2\sigma^2(z_\alpha + z_\beta)^2}{\Delta^2}$ - One-tail test

- $n_1 = n_2 = \frac{2\sigma^2(z_{\alpha/2} + z_\beta)^2}{\Delta^2}$ - Two-tail test

- Note that $\beta = 1 - Power$

POWER ANALYSIS CONT'D

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Dependent Samples:

- $H_0: \mu_d = 0$
- $H_a: \mu_d > 0$ — One-tail test
- or $\mu_d < 0$ — One-tail test
- or $\mu_d \neq 0$ → Two-tail test

- The samples sizes n needed to correctly discover a difference in testing of hypothesis with the power $P(\Delta_a)$ when the difference in the means is $\geq \Delta_a$ is given by

- $n = \frac{\sigma_d^2 (z_\alpha + z_\beta)^2}{\Delta^2}$ - One-tail test

- $n = \frac{\sigma_d^2 (z_{\alpha/2} + z_\beta)^2}{\Delta^2}$ - Two-tail test

- Note that $\beta = 1 - Power$

BOOK EXAMPLE 6.10

- **In building construction, the set-up time needed for concrete to reach solid state is an important factor. An additive is developed to speed up this set-up time. An experiment is to be designed to test if the additive does work.**

- **Without Additive** | **With additive**

  n_1 | n_2

- **How many sample runs $n_1 = n_2$ need to be performed to correctly discover with 90% power that "With Additive" reduces the set-up time by testing hypothesis at $\alpha = 0.05$ when the true average reduction is 1.5 hours or more?**

- **It is known from previous experience that $\sigma = 2.4\ hours$.**

- $H_0: \mu_1 = \mu_2 \quad vs. \quad H_a: \mu_1 > \mu_2$.

BOOK EXAMPLE 6.10 CONT'D

- **Two independent sample:**
  - Two-sample t-test
- **Formula:** $n_1 = n_2 = \frac{2\sigma^2 (z_\alpha + z_\beta)^2}{\Delta^2}$

- $\alpha = 0.05, \beta = 1 - Power = 1 - 0.90 = 0.10,$
- $\Delta = 1.5, \sigma = 2.4$

- $z_\alpha = 1.645,\ z_\beta = 1.28$

- $n_1 = n_2 = \frac{2(2.4^2)(1.645+1.28)^2}{1.5^2} = 43.8 \approx 44.$

SUMMARY

Case 2: Two Numerical Variables – Population Standard Deviations unknown

	Independent Samples $\sigma_1 = \sigma_2$	Independent Samples $\sigma_1 \neq \sigma_2$	Paired Samples
Parameters of Interest:	Mean Difference, $\mu_1 - \mu_2$	Mean Difference, $\mu_1 - \mu_2$	Mean Difference, $\mu_1 - \mu_2$
Confidence Interval Formula:	$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ $df = n_1 + n_2 - 2$, where $s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$	$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ $df = \frac{(n_1-1)(n_2-1)}{(1-c)^2(n_1-1)+c^2(n_2-1)}$, where $c = \frac{\frac{s_1^2}{n_1}}{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$	$\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n}}$ with df = $n - 1$ where the subscript "d" denotes the calculation is performed on the differences "Sample1" – "Sample2"
Name of Hypothesis Test, H_0	Two-sample T Test, $H_0: \mu_1 = \mu_2$ or $H_0: \mu_1 - \mu_2 = 0$	Two-sample T Test, $H_0: \mu_1 = \mu_2$ or $H_0: \mu_1 - \mu_2 = 0$	Paired T Test, $H_0: \mu_1 = \mu_2$ or $H_0: \mu_1 - \mu_2 = 0$
Test Statistic Formula:	$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$	$t' = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}}$	$t = \frac{\bar{d}}{s_d/\sqrt{n}}$ with df = $n - 1$
P-value:	$H_a: \mu_1 \neq \mu_2$, p-value = $2P(T \geq \lvert t \rvert)$ $H_a: \mu_1 > \mu_2$, p-value = $P(T \geq t)$ $H_a: \mu_1 < \mu_2$, p-value = $P(T \leq t)$		$H_a: \mu_1 \neq \mu_2$, p-value = $2P(T \geq \lvert t \rvert)$ $H_a: \mu_1 > \mu_2$, p-value = $P(T \geq t)$ $H_a: \mu_1 < \mu_2$, p-value = $P(T \leq t)$

WHAT IF ASSUMPTIONS ARE VIOLATED?

- **In both one-sample and two-sample t-tests, we assumed that either the sample size is ≥ 30 or the samples are drawn from normal populations.**

- **What if $n < 30$, and the distribution is non-normal?**

- **In such cases, we usually use non-parametric tests.**

- **No assumptions on the distribution means no parameters.**