MATH 4720 / MSSC 5720

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Chapter 8 (Part B)

Department of Mathematical and Statistical Sciences

BACK TO ANOVA: WHAT IF EQUALITY OF THE VARIANCES FAIL?

- **The assumption that the sample are generated from normal distribution is not very important as long as the total sample size is large.**

- **Note that conceptually the test statistic** $F = \dfrac{SS_B/df_B}{SS_E/df_E}$ **still makes sense.**

- **The major problem is with the assumption** $\sigma_1 = \sigma_2 = \cdots = \sigma_t$**. If this cannot be assumed, F- test must not be used.**

- **If** $H_0:\ \sigma_1 = \sigma_2 = \cdots = \sigma_t$ **is rejected, then one approach is to transform the data if the variances** σ^2 **is a function of the mean** μ**.**

EQUALITY OF VARIANCES FAIL CONT'D

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Be The Difference.

- **Transforming the data:**

Treatment Levels

1	2	3	. . .	t
y_{11}	y_{21}	y_{31}	. . .	y_{t1}
y_{12}	y_{22}	y_{32}	. . .	y_{t2}
.	.	.		.
.	.	.		.
y_{1n_1}	y_{2n_2}	y_{3n_3}	. . .	y_{tn_t}
$N(\mu_1, \sigma_1^2)$	$N(\mu_2, \sigma_2^2)$	$N(\mu_3, \sigma_3^2)$	. . .	$N(\mu_t, \sigma_t^2)$

- **If** $\sigma^2 \propto \mu$, **then use** $Y_T = \sqrt{Y}$ **or** $\sqrt{Y + 0.375}$
- **If** $\sigma^2 \propto \mu^2$, **then use** $Y_T = \ln(Y)$ **or** $\ln(Y + 1)$
- **If** $\sigma^2 \propto \mu(1-\mu)$, **then use** $Y_T = \sin^{-1}\sqrt{Y}$

BOOK EXAMPLE 8.4

- **Biologists believe that Mississippi river causes the oxygen level to be depleted near the Gulf of Mexico. To test this hypothesis water samples are taken at different distances from the mouth of Mississippi river, and the amounts of dissolve oxygen (in ppm) are recorded**

Distance

Oxygen Content

1 KM	5 KM	10 KM	20 KM
1	4	20	37
5	8	26	30
.	.	.	.
.	.	.	.
2	3	24	33
$\bar{y}_{1.} = 2.2$	$\bar{y}_{2.} = 4.6$	$\bar{y}_{3.} = 21.2$	$\bar{y}_{4.} = 31.4$
$s_1 = 1.476$	$s_2 = 2.119$	$s_3 = 4.7333$	$s_4 = 5.522$

- **R**

EXAMPLE 8.4 CONT'D

- H_0: $\sigma_1 = \sigma_2 = \cdots = \sigma_t$
  - **In R:** levene.test(unlist(exmp8.4), rep(1:4,each=10))

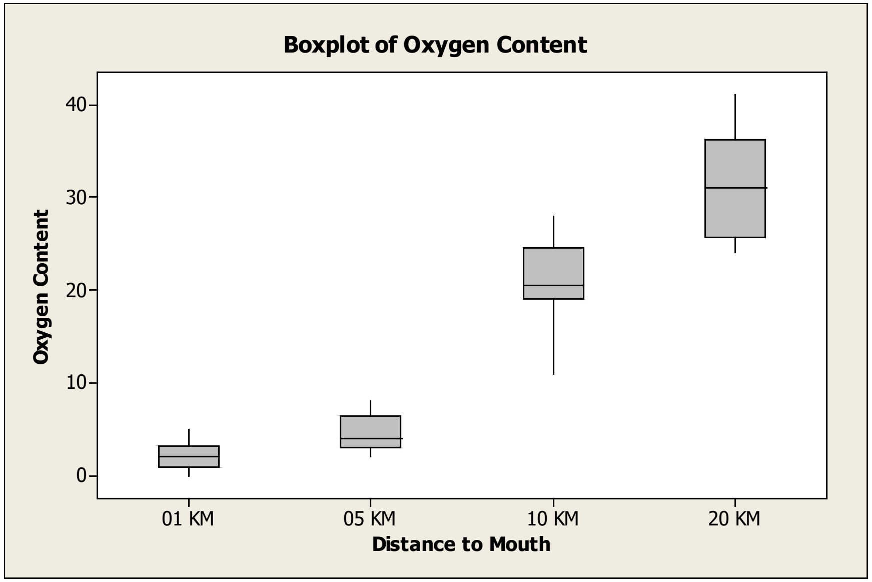
Test for Equal Variances: Oxygen Content versus Distance to Mouth

```
95% Bonferroni confidence intervals for standard deviations

Distance
to Mouth   N    Lower    StDev    Upper
   01 KM  10  0.92361  1.47573   3.2636
   05 KM  10  1.32602  2.11870   4.6855
   10 KM  10  2.96214  4.73286  10.4668
   20 KM  10  3.45583  5.52167  12.2113


Bartlett's Test (Normal Distribution)
Test statistic = 17.17, p-value = 0.001


Levene's Test (Any Continuous Distribution)
Test statistic = 3.70, p-value = 0.020
```



- **Levene's test** $p - value$ **is** 0.02.
- **Reject the** H_0 **: Equality of variances**
- **Let's calculate** $\frac{s_i^2}{\bar{y}_{i.}}$ **for** $i = 1,2,3,4$

EXAMPLE 8.4 CONT'D

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$\bar{y}_{1.} = 2.2$	$\bar{y}_{2.} = 4.6$	$\bar{y}_{3.} = 21.2$	$\bar{y}_{4.} = 31.4$
$s_1 = 1.476$	$s_2 = 2.119$	$s_3 = 4.7333$	$s_4 = 5.522$

- $\frac{s_1^2}{\bar{y}_{1.}} = 0.99$ $\frac{s_2^2}{\bar{y}_{2.}} = 0.97$ $\frac{s_3^2}{\bar{y}_{3.}} = 1.06$ $\frac{s_4^2}{\bar{y}_{4.}} = 0.97$
- **So, nearly,** $\text{Variance} \propto Mean$**.**
- **We use the transformation** $Y_T = \sqrt{Y + 0.375}$
- **Now, the ANOVA on this transformed data can be performed.**

Distance

Transformed Data	1 KM	5 KM	10 KM	20 KM
	1.173	2.092	4.514	6.114
	2.318	2.894	5.136	5.511
	.	.	.	.
	.	.	.	.
	1.541	1.837	4.937	5.777
	$\bar{y}_{1.} = 1.54$	$\bar{y}_{2.} = 2.19$	$\bar{y}_{3.} = 4.62$	$\bar{y}_{4.} = 5.62$
	$s_1^2 = 0.24$	$s_2^2 = 0.22$	$s_3^2 = 0.29$	$s_4^2 = 0.24$

A COMPREHENSIVE MODELING APPROACH
(USEFUL FOR MODEL ASSESSMENT AND EXTENSIONS)

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Be The Difference.

- **To generalized the ANOVA, it is easier to think of one-factor ANOVA in the following way:**

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}, \qquad j = 1,2,\ldots,n_i, \qquad i = 1,2,..,t$$

- **Here,**
  - τ_i **is the effect due to** i^{th} **treatment,**
  - μ **is the overall effect irrespective of the treatment,**
  - ϵ_{ij}**s are the random errors**

- **Assumption:**
  - **The random errors** $\epsilon_{ij}s$ **are independent and normally distributed**
  - $Var(\epsilon_{ij}) = \sigma^2$ **(a constant value)**

HOW ABOUT THE HYPOTHESIS TEST?

- **To test the effect of the treatment, we can test**

- $H_0: \tau_i = 0,$ **for all** $i = 1,2, \dots, t$
- $H_a: \tau_i \neq 0,$ **for some** i

- **Test Statistics and decision rule are same as before**

- **TS:** $F = \frac{SS_B/df_B}{SS_E/df_E}$

- **Decision Rule: Reject** H_0 **in favor of** H_a **if**
  - $F > F_\alpha\,(df_B, df_E)$

CHECKING THE ASSUMPTIONS

- To check the assumption, we first estimate the errors ϵ_{ij} by r_{ij} (called residuals)

$$r_{ij} = y_{ij} - \hat{\mu} - \hat{\tau}_i$$

- To test, normal distribution of errors ϵ_{ij}, we look at the normal probability plot of r_{ij}.

- To test that the Var$(\epsilon_{ij}) =$ constant, we look at the scatter plot of the residuals r_{ij} and the predicted values $\hat{y}_{ij}$, where

$$\hat{y}_{ij} = \hat{\mu} + \hat{\tau}_i$$

GOING BACK TO EXAMPLE 8.4

- **Biologists believe that Mississippi river causes the oxygen level to be depleted near the Gulf of Mexico. To test this hypothesis water samples are taken at different distances from the mouth of Mississippi river, and the amounts of dissolve oxygen (in ppm) are recorded**

Distance

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- **R**

WRONG ANOVA

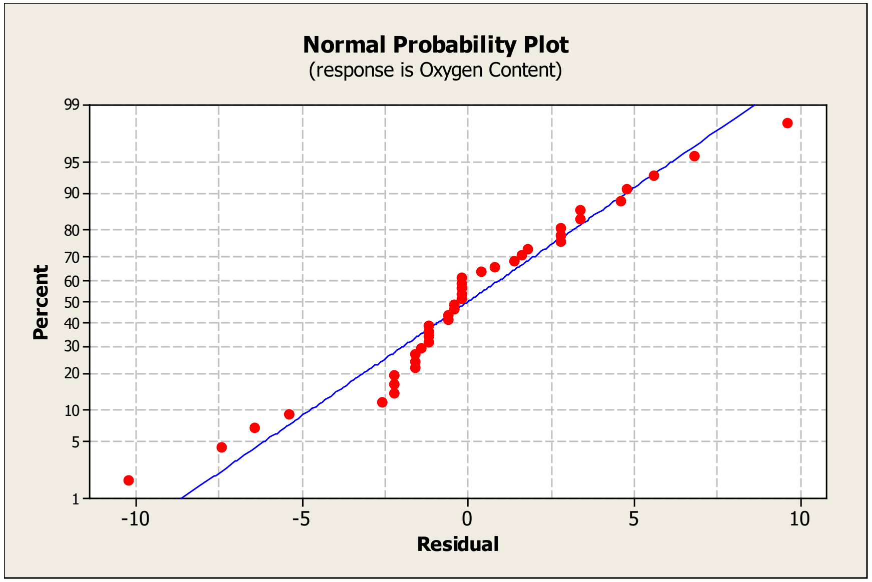
- **In R:**
  - model1 <- aov(unlist(exmp8.4)~ factor(rep(1:4,each=10)))
  - summary(model1)

One-way ANOVA: Oxygen Content versus Distance to Mouth

Source	DF	SS	MS	F	P
Distance to Mouth	3	5793.1	1931.0	129.70	0.000
Error	36	536.0	14.9		
Total	39	6329.1			

S = 3.859   R-Sq = 91.53%   R-Sq(adj) = 90.83%

  - plot(model1)

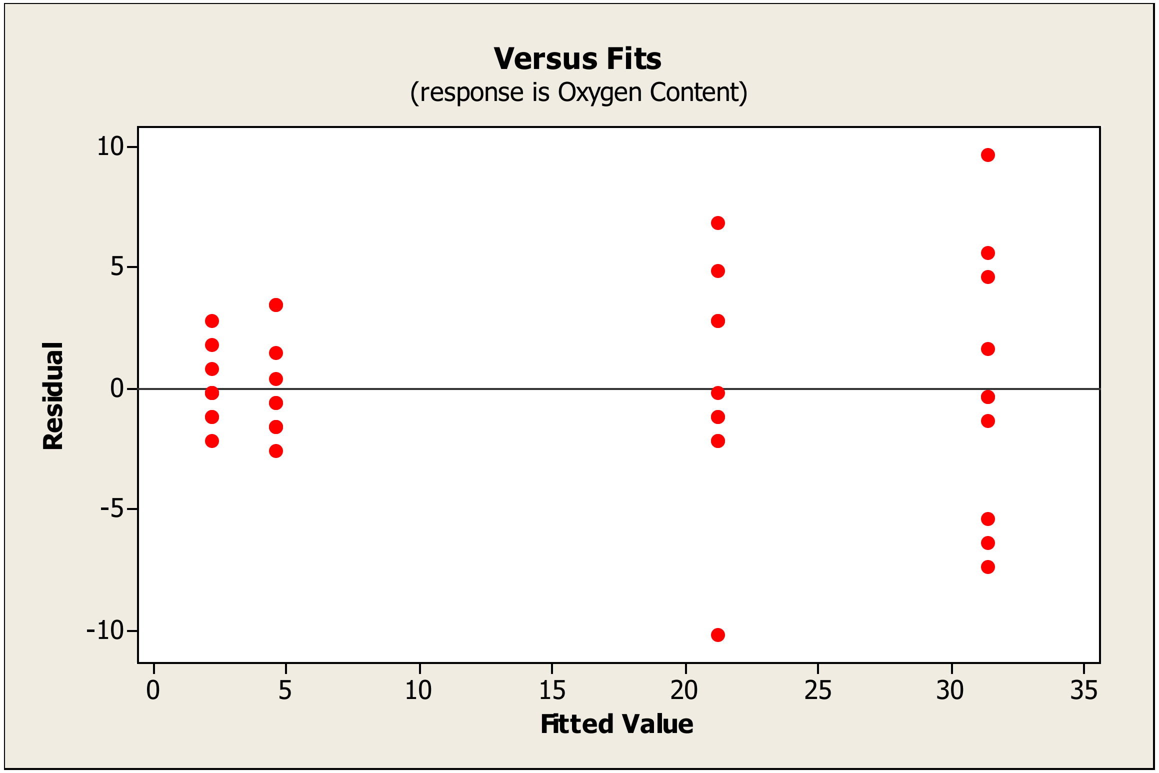


- **Although the normal probability plot is not very closed to straight line**

- **But we have relatively large total sample size $n = 40$**

WRONG ANALYSIS CONT'D

- **Fitted values versus Residuals:**
  - **Scatterplot of $\hat{y}_{ij}$ versus r_{ij}**



- **Due to cone shape, we can conclude that $Var(\epsilon_i)$ is not constant.**
- **We can further say that this variance is a function of the mean of y_i.**

CORRECT ANALYSIS
BASED ON TRANSFORMED DATA

- $Var(y_i) = Var(\epsilon_i) \propto E(y_i)$ **the mean of y_i.**
- $Y_T = \sqrt{Y + 0.375}$

> model2 <- aov(unlist(sqrt(exmp8.4+0.375))~ factor(rep(1:4,each=10)))
> summary(model2)

One-way ANOVA: Y_t versus Distance to Mouth

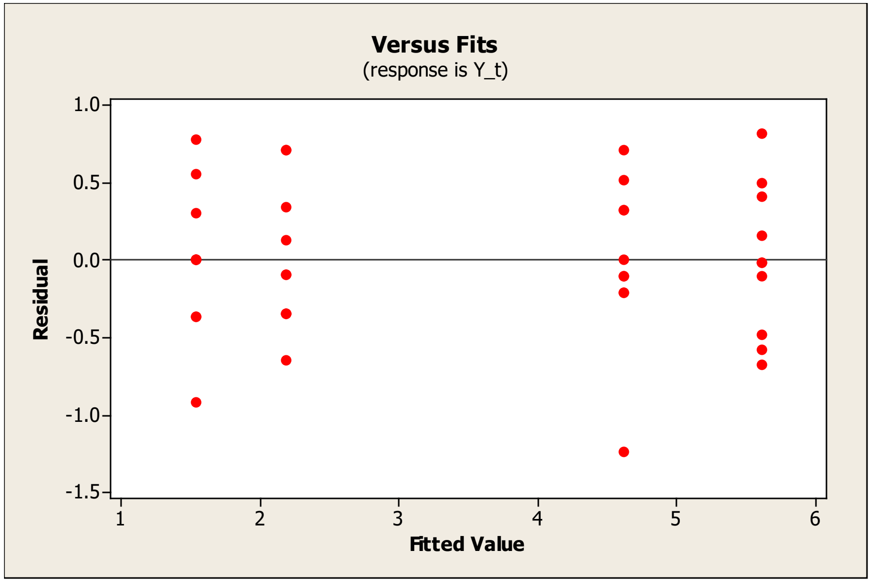
Source	DF	SS	MS	F	P
Distance to Mouth	3	113.095	37.698	153.30	0.000
Error	36	8.853	0.246		
Total	39	121.948			

S = 0.4959   R-Sq = 92.74%   R-Sq(adj) = 92.14%

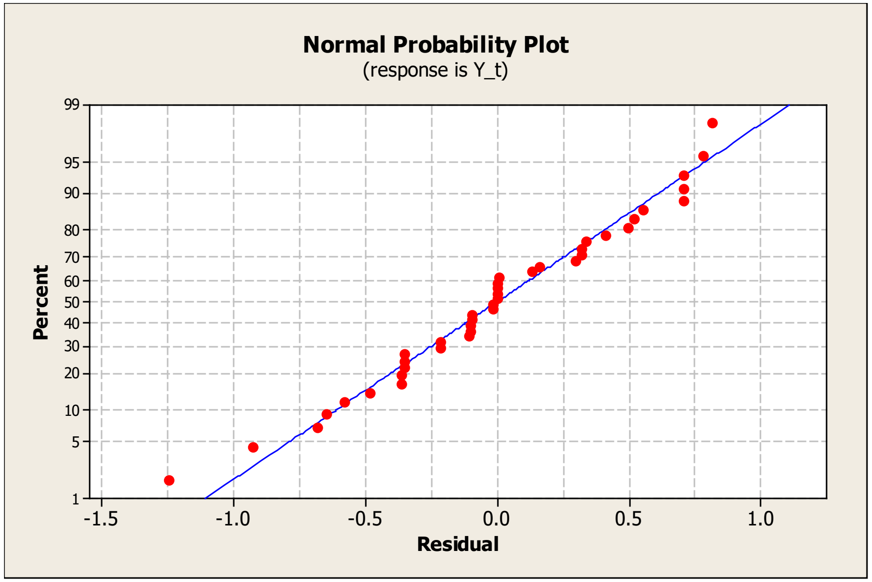
- **For the transformed variable ANOVA we need to check the assumptions**
  - **Normal distributions of the errors**
  - $Var(\epsilon_i)$**= constant are satisfied.**

CHECKING THE ASSUMPTIONS BASED ON TRANSFORMED DATA

- plot(model2)
- $Var(\epsilon_i)$= **constant**

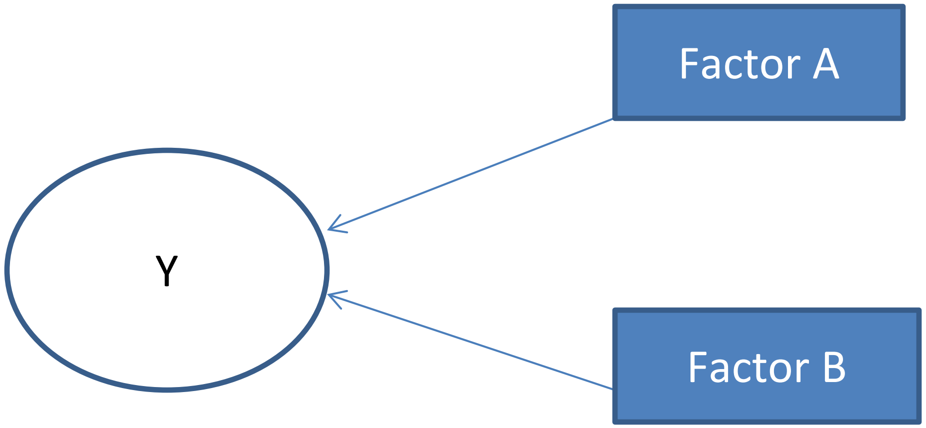


- **Normal distributions of the errors**



TWO-FACTOR ANALYSIS OF VARIANCE

- **Two Way ANOVA**
- **In R**
  - aov(Y ~ Factor.A * Factor.B)



Levels

- **Factor A: Low High**
- **Factor B Low Medium High**

A COMPREHENSIVE MODELING APPROACH

- **The observation Y is affected by two factors A and B**

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

- **Here**
  - α_i - **effect of i^{th} level of factor A**
  - β_j - **effect of j^{th} level of factor B**
  - γ_{ij} - **called the interaction effect of A and B**
  - ϵ_{ijk} - **random errors**

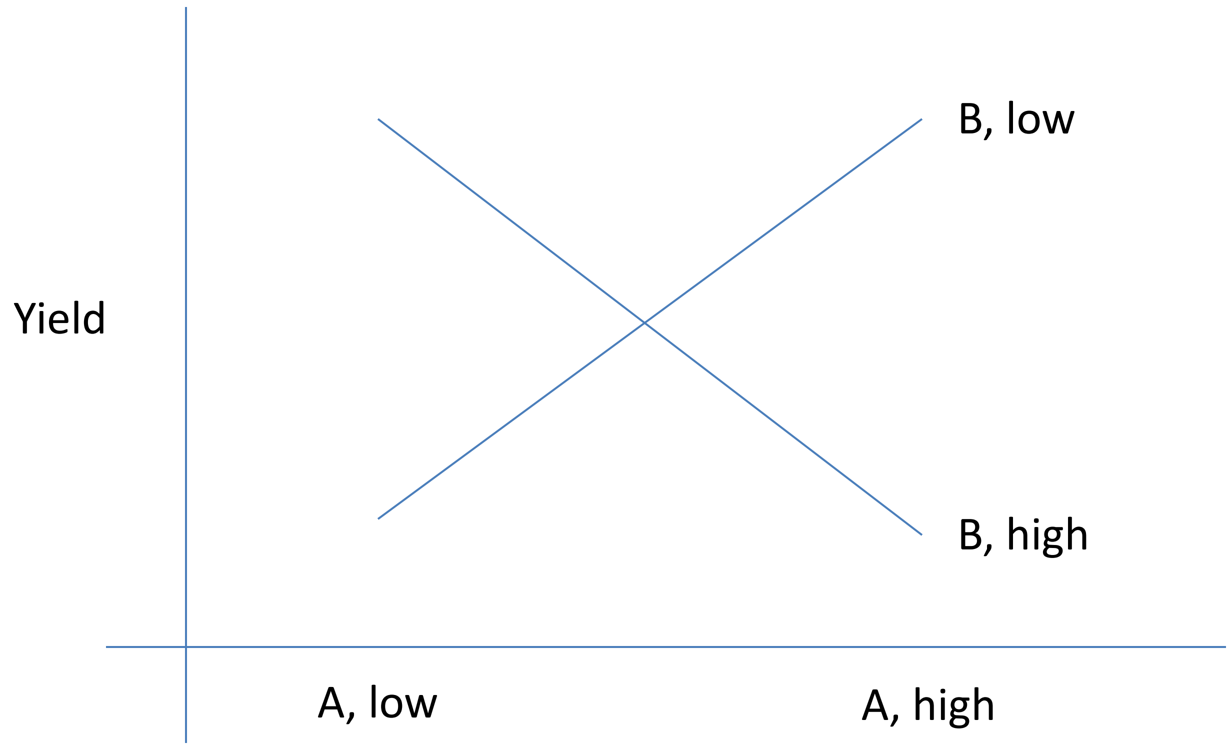
- **Assumptions:**

  (1) Errors are normally distributed

  (2) $Var(\epsilon_{ijk}) =$ **Constant.**

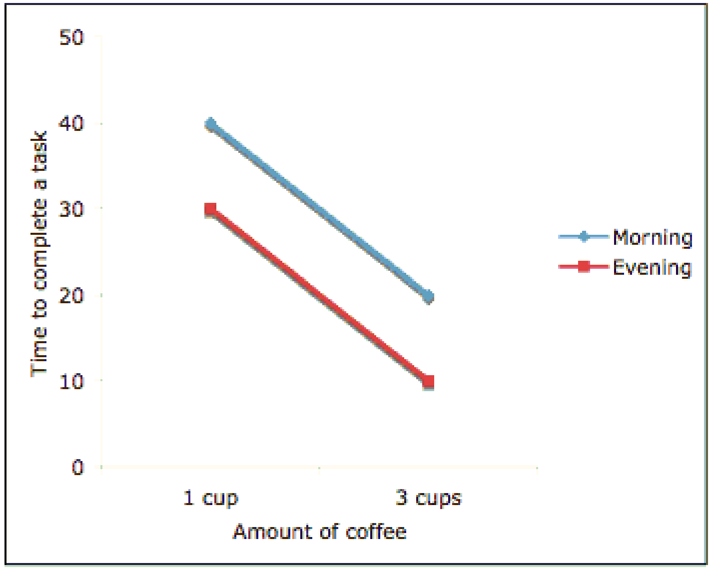
WHAT IS INTERACTION EFFECT?

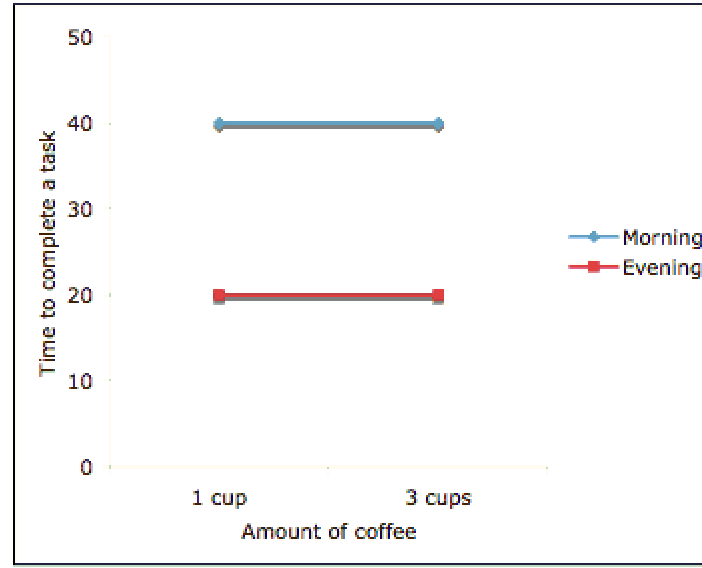
- **Meaning of Interaction Effect**



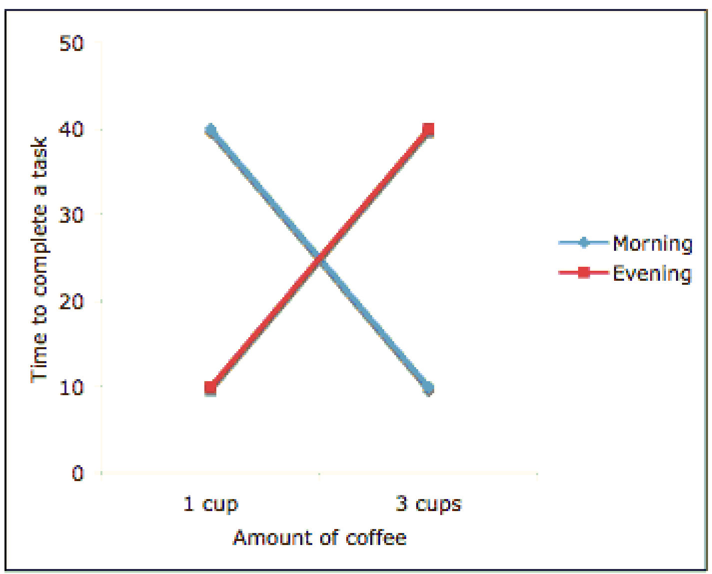
WHAT IS INTERACTION EFFECT? CONT'D

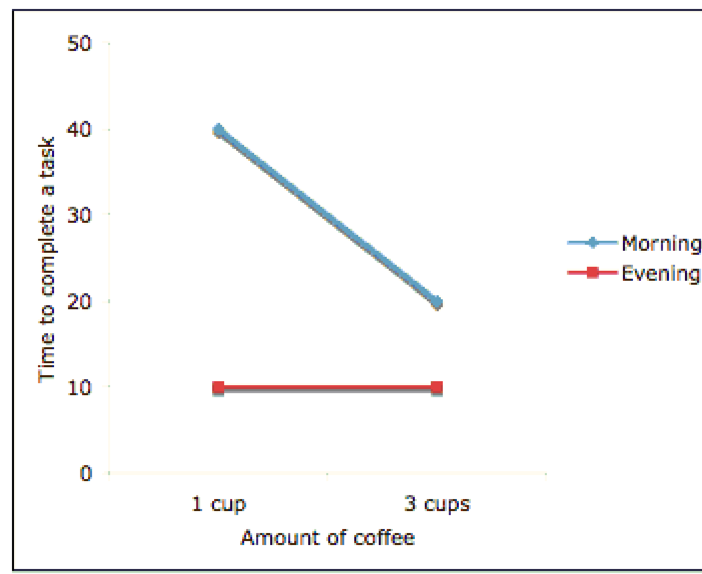
- **No Interaction**





- **Interaction**





AUSTRALIAN INSTITUTE OF SPORT EXAMPLE

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- **Response Variable: Weight**

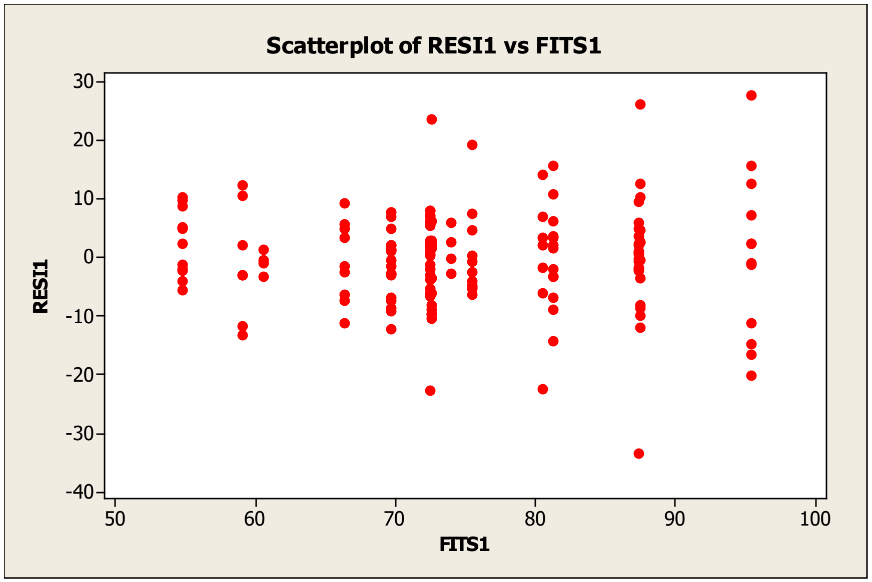
- **Factor A: Gender**
- **Factor B: Sport**

- **Two-way ANOVA: yield versus Factor A, Factor B**
  - **In R:**
    - summary(model3 <- aov(Wt ~ gender * sport, data=ais2))

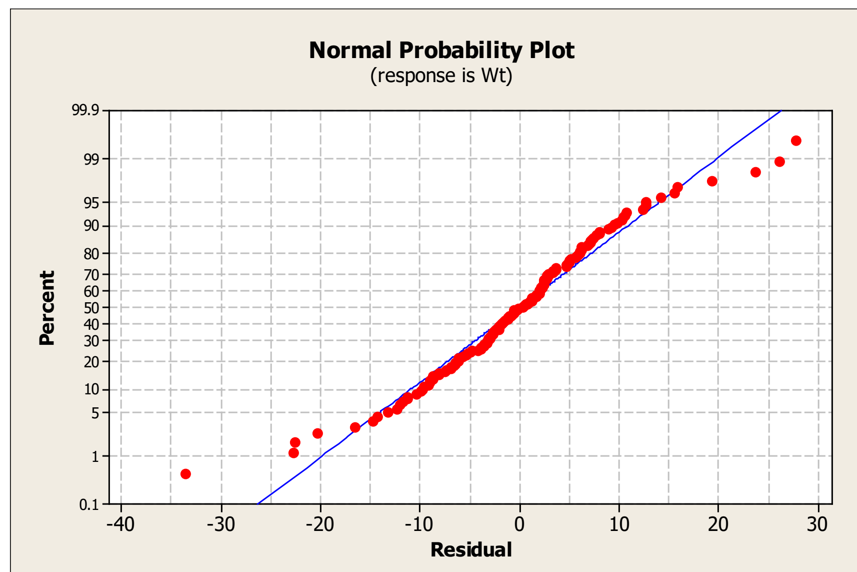
	DF	SS	MS	F value	Pr(>F)
Gender	1	7424	7424	95.845	<2e-16 ***
Sport	6	10975	1829	23.614	<2e-16 ***
Interaction	6	185	31	0.398	0.879
Error	144	11155	77		

CHECKING THE ASSUMPTIONS BASED ON TRANSFORMED DATA

- plot(model3)
- $Var(\epsilon_i)$= **constant**



- **Normal distributions of the errors**



ANOVA RESULTS BASED ON THE TRANSFORMED DATA

- **Assumptions of errors seems to be satisfied**

- $H_0: \gamma_{ij} = 0$ vs. $H_a: \gamma_{ij} \neq 0$
  - **TS.** $F = 0.398$, $\text{p-value} = 0.897$
  - **There is no significant interaction**

- $H_0: \alpha_i = 0$ vs. $H_a: \alpha_i \neq 0$
  - **TS.** $F = 95.845$, $\text{p-value} < 2 * 10^{-16}$
  - **Significant effect of Gender**

- $H_0: \beta_j = 0$ vs. $H_a: \beta_j \neq 0$
  - **TS.** $F = 23.614$, $\text{p-value} < 2 * 10^{-16}$
  - **Significant effect of Sport**

INTERACTION

- **In R:**
  - with(ais2, {interaction.plot(sport, gender, Wt, fixed = TRUE)})
  - with(ais2, {interaction.plot(gender, sport, Wt, fixed = TRUE)})