8 Optical Flow

In Visual Computing, people seem like to use

$$I_\spadesuit = \frac{\partial I}{\partial \spadesuit}, \qquad u = \frac{\partial x}{\partial t}, \qquad v = \frac{\partial y}{\partial t},$$

where the subindex means a derivative if and only if we are talking about I.

8.1 Applications

1 tracking 2 structure from motion 3 stabilization 4 compression 5 Mosaicing

8.2 Brightness constancy

Definition of Optical Flow "Apparent motion of brightness patterns". Ideally, the optical flow is the projection of the three-dimensional velocity vectors on the image.

Caution required 1 Uniform, rotating sphere $\mathcal{OF} = 0$ 2 No motion, but changing lighting $\mathcal{OF} \neq 0$

8.3 Mathematical formulation

$I(x, y, t)$ = brightness at (x, y) at time t.

Brightness constancy assumption:

$$I\left(\frac{\mathrm{d}x}{\mathrm{d}t}\delta t, y + \frac{\mathrm{d}y}{\mathrm{d}t}\delta t, t + \delta t\right) = I(x, y, t)$$

Optical flow constraint equation:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial I}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial I}{\partial t} = 0$$

8.4 The aperture problem

The motion of an edge seen through an aperture is (in some cases) inherently ambiguous. E.g. the edge is physically moving upwards, but the edge motion alone is consistent with many other possible motions, and in this case the edge e.g. appears to move diagonally.

8.5 Optical Flow meaning

Estimate of observed projected motion field. Not always well defined! Compare: 1 Motion Field (or Scene Flow), projection of 3-D motion field 2 Normal Flow: observed tangent motion 3 Optic Flow: Apparent motion of the brightness pattern: Apparent motion of the brightness pattern (hopefully equal to motion field) 4 Consider barber pole illusion

Planar motion Ideal motions of a plane, X, Y being the horizontal and vertical direction and Z normal to the image plane: 1 translation in X 2 translation in Z 3 rotation around Z 4 rotation around Y

8.6 Regularization: Horn & Schunck algorithm

The Horn-Schunck algorithm assumes smoothness in the flow over the whole image. thus, it tries to minimize distortions in flow and prefers solutions which show more smoothness. The flow is formulated as a global energy functional which is the sought to be minimized. This function is given for two-dimensional image streams as Eq. 5: The associated ELE are

$$\frac{\partial L}{\partial \dot{x}} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \frac{\partial \dot{x}}{\partial x}} - \frac{\partial}{\partial y}\frac{\partial L}{\partial \frac{\partial \dot{x}}{\partial y}} = 0,$$

$$\frac{\partial L}{\partial \dot{y}} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \frac{\partial \dot{y}}{\partial x}} - \frac{\partial}{\partial y}\frac{\partial L}{\partial \frac{\partial \dot{y}}{\partial y}} = 0.$$

this gives

$$\frac{\partial I}{\partial x}\left(\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \frac{\partial I}{\partial t}\right) - \alpha^2 \Delta\dot{x} = 0,$$

$$\frac{\partial I}{\partial y}\left(\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \frac{\partial I}{\partial t}\right) - \alpha^2 \Delta\dot{y} = 0.$$

with $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

Remarks 1 Coupled PDE solved using iterative methods and finite differences

$$\ddot{x} = \Delta\dot{x} - \lambda\left(\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \dot{I}\right)\frac{\partial I}{\partial x},$$

$$\ddot{y} = \Delta\dot{y} - \lambda\left(\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \dot{I}\right)\frac{\partial I}{\partial y}.$$

2 More than two frames allow a better estimation of $\dot{I}$. 3 Information spreads from corner-type patterns. 4 Errors at boundaries 5 Example of *regularisiation*: selection principle for the solution of illposed problems.

8.7 Lucas-Kanade: Integrate over a Patch

The Lucas-Kanade method assumes that the displacement of the image contents between two nearby instants (frames) is small and approximately constant within a neighborhood of the point p under consideration. thus the optical flow equation can be assumed to hold for all pixels within a window centered at p. Namely, the local image flow (velocity) vector $(\dot{x}, \dot{y})$ must satisfy

$$\frac{\partial I(q_k)}{\partial x}\dot{x} + \frac{\partial I(q_k)}{\partial y}\dot{y} = -\frac{\partial I(q_k)}{\partial t},$$

for $k = 1, \ldots, n$ and q_k the pixels inside the window. These equations can be written in matrix form

$$A\mathbf{v} = \mathbf{b} \tag{1}$$

where $\mathbf{x} = (x\ y)^T$, $\quad \mathbf{v} = (\dot{x}\ \dot{y})^T$ and

$$A_{ij} = \frac{\partial I(q_i)}{\partial x_j}, \qquad \mathbf{b}_i = -\frac{\partial I(q_i)}{\partial t}.$$

Eq. 1 is overdetermined, so do compromise solution by the least squares principle Eq. 6

8.8 Gradient-Based Estimation

Assume *brightness constancy*. Let $f_1(x)$ and $f_2(x)$ be 1D signals (images) at two time instants. Let $f_2 = f_1(x - \delta)$, where δ denotes translation.

$$\begin{aligned} &\rightsquigarrow f_1(x) - f_2(x) \\ &= \delta f_1'(x) + O\left(\delta^2\right) \end{aligned}$$

$$\rightsquigarrow \delta \approx \frac{f_1(x) - f_2(x)}{f_1'(x)} \tag{2}$$

Assume displaced image well approximated by first-order Taylor series

$$\begin{aligned} &I(\mathbf{x} + \mathbf{u}, t + 1) \\ &\approx I(\mathbf{x}, t) + \mathbf{u} \cdot \nabla I(\mathbf{x}, t) + I_t(\mathbf{x}, t) \end{aligned} \tag{3}$$

Insert Eq. 2 in Eq. 3 to get

$$\nabla I(\mathbf{x}, t) \cdot \mathbf{u} + I_t(\mathbf{x}, t) = 0.$$

This i s called the *gradient constraint equation.*

8.9 Pyramid/Coarse-to-fine

Limits of the (local) gradient method: 1 Fails when intensity structure within window is poor
2 Fails when displacement is large (typical operating range is motion of 1 pixel per iteration!). Linearization of brightness is suitable only for small displacements.
3 Brightness is no strictly constant in images. Actually less problematic than it appears, since we can pre-filter images to make them look similar.

8.10 Parametric motion models

Global miton models offer:
1 More constrained solutions than smoothness (Horn-Schunck)
2 Integration over a large area than a translation-only model can accomodate (Lucas-Kanade)

9 Questions

- $$I_{\text{comp}} = I_\alpha I_a + (1 - I_\alpha) I_b$$
- MAP, Maximum a posteriori detector.
- graph cuts
- Solve MRFs with graph cuts
- impulse respons $t(-x, -y)$
- Canny nonmaxima suppression
- Entropy Coding (Huffman code)
- Aperture problem: normal flow
- Lucas-Kanade: Iterative refinement/local gradient method
- Coarse-to-fine-estimation

A Big equations

$$\begin{aligned} \mathcal{F}[h](u, v) &= \frac{1}{2\ell}\int_{-\ell}^{\ell}\mathrm{d}x_1\ \exp(-i2\pi u x_1) \cdot \underbrace{\int_{-\infty}^{\infty}\mathrm{d}x_2\,\delta(x_2)\exp(-i2\pi v x_2)}_{=1} \\ &= \operatorname{sinc}(2\pi u \ell) \end{aligned} \tag{4}$$

$$\begin{aligned} E = \iint \mathrm{d}x\mathrm{d}y \Bigg[&\left(\frac{\partial I}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial I}{\partial t}\right)^2 \\ &+ \alpha^2(\|\nabla\dot{x}\| + \|\nabla\dot{y}\|)^2 \Bigg] \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbf{v} = &\begin{pmatrix} \sum_i w_i I_x(q_i)^2 & \sum_i w_i I_x(q_i) I_y(q_i) \\ \sum_i w_i I_x(q_i) I_y(q_i) & \sum_i w_i I_y(q_i)^2 \end{pmatrix}^{-1} \\ &\cdot \begin{pmatrix} -\sum_i w_i I_x(q_i) I_t(q_i) \\ -\sum_i w_i I_y(q_i) I_t(q_i) \end{pmatrix} \end{aligned} \tag{6}$$