Why is $X^T X$ Symmetric?

be $\underset{K \times K}{A} = \underset{K\times M}{X^T} \underset{M \times K}{X}$ $\qquad$ A_{ij} is the element of A in row i and column j

$$A_{ij} = (i\text{-th Row of } X^T) \text{ Times } (j\text{-th colum of } X)$$

$$\overset{\text{Def. of transposition}}{=} (i\text{-th Colum of } X)^T \text{ Times } (j\text{-th colum of } X)$$

with

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \dots X_{1j} \dots X_{1K} \\ \dots & & \\ X_{i1} & X_{i2} & X_{i3} \dots X_{ij} \dots X_{iK} \\ X_{M1} & X_{M2} & X_{M3} \dots X_{Mj} \dots X_{MK} \end{bmatrix}$$

$\rightarrow$ To prove that $A_{ij} = A_{ji}$

$$\begin{pmatrix} X_{1i} \\ X_{2i} \\ | \\ X_{Mi} \end{pmatrix}^T \begin{pmatrix} X_{1j} \\ X_{2j} \\ | \\ X_{Mj} \end{pmatrix}$$

$$(X_{1i}, X_{2i}, —, X_{Mi}) \begin{pmatrix} X_{1j} \\ X_{2j} \\ | \\ X_{Mj} \end{pmatrix}$$

$$= X_{1i} X_{1j} + X_{2i} X_{2j} + \dots + X_{Mi} X_{Mj} \quad *$$

What is A_{ji}? As above is $(j\text{-th colum of } X)^T$ Times $(i\text{-th colum of } X)$

$$(X_{1j}, X_{2j}, —, X_{Mj}) \begin{pmatrix} X_{1i} \\ X_{2i} \\ | \\ X_{Mi} \end{pmatrix}$$

$$= X_{1j} X_{1i} + X_{2j} X_{2i} + — + X_{Mj} X_{Mi} \quad **$$

$\Rightarrow$ $* = **$

$A_{ij} = A_{ji}$

$\Rightarrow A$ is Symmetric

$\Rightarrow X^T X$ is Symmetric