$$\mathrm{Cov}\left(\sum_i y_i,\ \sum_j (x_j-\bar{x})\, y_j\right) \overset{???}{=} \sum_j (x_j-\bar{x})\,\sigma^2 \longleftarrow \text{To PROVE}$$

$$= \sum_j (x_j-\bar{x})\,\mathrm{Cov}\left(\sum_i y_i,\ y_j\right)$$

"$x_j$ and $\bar{x}$" are constants here

$$= \underbrace{(x_1-\bar{x})\,\mathrm{Cov}\left(\sum_i y_i,\ y_1\right)}_{j=1} + \underbrace{(x_2-\bar{x})\,\mathrm{Cov}\left(\sum_i y_i,\ y_2\right)}_{j=2} + \underbrace{(x_3-\bar{x})\,\mathrm{Cov}\left(\sum_i y_i,\ y_3\right)}_{j=3} + \dots + \underbrace{(x_m-\bar{x})\,\mathrm{Cov}\left(\sum_i y_i,\ y_m\right)}_{j=m}$$

$j=1$ ↓

$$(x_1-\bar{x})\left\{\mathrm{Cov}(y_1,y_1) + \mathrm{Cov}(y_2,y_1) + \mathrm{Cov}(y_3,y_1) + \dots + \mathrm{Cov}(y_i,y_1) + \dots + \mathrm{Cov}(y_m,y_1)\right\}$$

$\underbrace{\Rightarrow \text{only } \mathrm{Cov}(y_1,y_1) \neq 0 \text{, others are } 0}$

$j=2$ ↓

$\vdots$

$\underbrace{\Rightarrow \text{only } \mathrm{Cov}(y_2,y_2) \neq 0}$

$j=3$ ↓

$$(x_3-\bar{x})\left\{\mathrm{Cov}(y_1,y_3) + \mathrm{Cov}(y_2,y_3) + \mathrm{Cov}(y_3,y_3) + \dots + \mathrm{Cov}(y_i,y_3) + \dots + \mathrm{Cov}(y_m,y_3)\right\}$$

$\underbrace{\Rightarrow \text{only } \mathrm{Cov}(y_3,y_3) \neq 0}$

$j=m$ ↓

$\vdots$

$\underbrace{\Rightarrow \text{only } \mathrm{Cov}(y_m,y_m) \neq 0}$

$$= (x_1-\bar{x})\,\mathrm{Cov}(y_1,y_1) + (x_2-\bar{x})\,\mathrm{Cov}(y_2,y_2) + \dots + (x_m-\bar{x})\,\mathrm{Cov}(y_m,y_m)$$

$$= \sum_{j=1}^{m} (x_j-\bar{x})\,\underbrace{\mathrm{Cov}(y_j,y_j)}_{=\sigma^2\ \forall j} = \sum_j (x_j-\bar{x})\,\sigma^2$$

→ PROVED