Financial econometrics

Chapter 3, Portfolio analysis & CAPM

Overview

Section 1, Introduction

Introduction

Markowitz portfolio analysis is a mathematical procedure to determine the optimum portfolios in which to invest Francis and Kim, 2013.

The objective of portfolio analysis is to find the set of **efficient portfolios**:

- Have the greatest expected return for a given level of risk, or, conversely,
- Offer the lowest risk for a given level of expected return.

$\rightarrow$ The collection of all the efficient portfolios comprises a curve in risk-return space called the **efficient frontier**.

Introduction

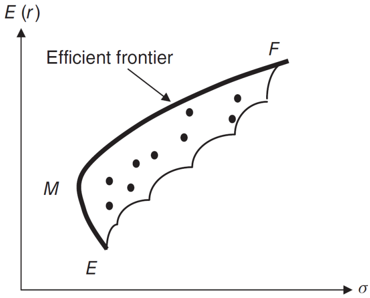


Figure: Opportunity Set and efficient frontier in (σ, μ) space.

The objective is to find the dark curve from E to F, the efficient frontier, from some **opportunity set** of potential investments (individual stocks, bonds, ...).

Introduction, general framework

Though the idea in Fig. 1 appears simple, yet there is a lot going on

- Preferences: $A \succ B$, what is this?
- Decision rules: how to tell whether $A \succ B$ or $B \succ A$?
- Utility functions: how much A is preferred over B, and higher utility for A means $A \succ B$?
- Estimation: $E(r_1), E(r_1), \sigma_1, \sigma_2, \sigma_{12}$ where do come from?

Introduction, decision rules I

Decision rules are used to establish a (partial) ordering in a set of *lotteries* $A, B, C, \ldots$. E.g. how to tell if $A \succ B$?

Example (Cash vs cash)

You are offered \$1000 now or \$1050 now

- What would you prefer?

Example (Cash vs future cash)

You are offered \$1000 now or \$1050 in a year

- What would you prefer?

Introduction, decision rules II

Example (Dice vs cash)

A dice is rolled, you take home whatever the outcome is:

- How much would you pay for playing this game?

Example (St. Petersburg Paradox)

A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake begins at 2 dollars and is doubled every time tails appears. The first time heads appears, the game ends and the player wins whatever is the current stake.

- How much would you pay for playing this game?

Introduction, decision rules III

Example (St. Petersburg Paradox, solution)

- You win 2^k if tail appears at the k-th toss
- The expected value of the game is

$$\begin{aligned} \mathbb{E}(X) &= 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + 16 \cdot \frac{1}{16} \dots \\ &= 1 + 1 + 1 + 1 + \dots \\ &= \infty \end{aligned}$$

  The expected value is an infinite amount of money: you should be willing to pay whatever (even astronomical) price the casino asks you to enter the game!
- ... by you wouldn't really pay any between \$500, \$5,000, \$50,000 ... there is a problem!

Introduction, decision rules IV

Example (St. Petersburg Paradox, solution (cont'd))

- Take a logarithmic utility function $U(w) = \log w$
- The initial utility, at your initial wealth w is $\log w$
- after k tails you utility is $\log\left(w + 2^k - c\right)$, with c the price for entering the game
- Fine, but k tails happen with a probability $1/2^k$: you don't know when the game will end, you don't know how much your wealth X will be: better to consider the *expected utility* of the game:

$$\mathbb{E}(U(X)) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \log\left(w + 2^k - c\right)$$

$\rightarrow$ The problem is that you are considering only the expected value!

Introduction, decision rules V

- Investors/gamblers implicitly base their evaluations on the fact that the outcome is random
- Implicitly now different outcomes have different probabilities
- Implicitly they know that the outcome is a random variable, with some variance
- Implicitly they balance expected value *and* variance
- And perhaps also higher moments...
- $\rightarrow$ We need some 'more sophisticated' rules that look at the variance of the outcome too.

Introduction, decision rules VI

Example (St. Petersburg Paradox, solution (cont'd))

- the change in utility that you get by playing the game by paying c is:

$$\Delta\,\mathbb{E}(U(X)) = \sum_{k=1}^{+\infty} \frac{1}{2^k}\left[\log\left(w + 2^k - c\right) - \log w\right] < +\infty$$

- This is a formula relating gambler's wealth and how much he should be willing to pay
- He should be willing to pay any c that gives positive change in expected utility
- A person with $w = 1,000,000$ should be willing to pay \$20.88, a person with \$1,000 should pay up to \$10.95, a person with \$2 should pay \$3.35 (and thus should borrow \$1.35)

Definition (Mean-Variance rule)

The M-V rule is employed when ranking various prospects and constructing efficient portfolios:

We say that prospect A dominates B by the M-V rule if the following two conditions hold:

$$\mathbb{E}(r_A) \geq \mathbb{E}(r_B) \qquad \text{and} \qquad \sigma_A^2 \leq \sigma_B^2$$

and there is at least one strict inequality.
Conversely, if the following holds,

$$\mathbb{E}(r_A) > \mathbb{E}(r_B) \qquad \text{and} \qquad \sigma_A^2 > \sigma_B^2$$

there is no dominance by the M-V rule.

Mean-Variance rule II

If there is no dominance by the M-V rule, we conclude that both the investments A and B are in the *efficient set*.

In the efficient set we cannot say which prospects should be selected:

- this is a personal choice, depending on investor's preferences
- we can however assert that with the M-V *partial* ordering no one should select a prospect in the inefficient set

Is this a good rule?

Mean-Variance rule III

Example

Prospect F yields \$5 or \$10 with equal probability, and prospect G yields \$10 or \$20 with equal probability.

- Would you prefer F or G?
- G has higher mean and higher variance than F: the M-V rule assigns both of them to the efficient set
- the M-V rule is unable to distinguish the two prospects, even if we clearily prefer G

The expected utility of an investment depends also on higher moments: with the M-V rule we ignore moments higher than two:

- The M-V rule is practical, is intuitive and makes sense economically, yet not necessarily the best as it imposes a simplified framework where higher moments are neglected.

Mean-Variance rule IV

It can be shown that the expected utility is indeed a function of portfolio's mean, variance and higher moments:

$$
\begin{aligned}
\mathbb{E}[U(W_t)] &\approx U(W_t) + \frac{1}{2!}U^{''}[\mathbb{E}(W_t)]\sigma_W^2 + \frac{1}{3!}U^{'''}[E(W_t)]\mu_3 + \dots \\
&\approx U(W_0(1+\mathbb{E}(r_p))) + \frac{1}{2}U^{''}[W_0(1+\mathbb{E}(r_p))]W_0\sigma_p^2
\end{aligned}
$$

with W_0 the initial wealth, W_t the wealth at the end of the period, and p a portfolios whose return is r_p, and variance σ_p^2.

Question: what $\partial\,\mathbb{E}[U(W_t)]/\partial\sigma_p^2$ represents?

Inputs

The inputs to the portfolio analysis of a set of n candidate assets are:

- n expected returns: $\{\mathbb{E}(r_i)\}$, $i = 1, \ldots, n$
- n variances of returns $\{\sigma_i^2\}$, $i = 1, \ldots, n$
- $\left(n^2 - n\right)/2$ covariance $\{\sigma_{ij}\}$, $i = 1, \ldots, n, \quad j = 1, \ldots, n$
- these all refer to a single period (usually a year). I.e, r_i is the (random) return that an investor will get in a year from now.

$\rightarrow$ Need to find the portfolio weights w_i that are in the above sense optimal, within the **budget constraint** or **balance sheet identity**:

$$\sum_{i=1}^{n} w_i = 1$$

Question: can weights be negative?

Inputs

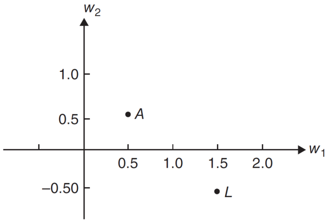


Figure: Two portfolios

- Portfolio A: 50%-50% invested in two assets
- Portfolio L: a leveraged portfolio with 150% of the original capital invested in asset 1 and 50% percent of the capital invested in asset 2. e.g., suppose that portfolio L had \$1,000 invested. Then, the portfolio manager purchased \$1,500 of assets 1, so $w = 1.5$. To cover the \$500 shortage, the portfolio manager issues \$500 worth of a assets 2,s o $w_2 = -0.5$. Note that $w_1 + w_2 = 1$.

Section 2, Graphical Portfolio Analysis

Two-asset analysis, isomeans

The portfolio expected return is

$$\mathbb{E}(r_p) = w_1\,\mathbb{E}(r_1) + w_2\,\mathbb{E}(r_2),$$

rewrite it as follows:

$$w_2 = \frac{\mathbb{E}(r_p)}{\mathbb{E}(r_2)} - \frac{\mathbb{E}(r_1)}{\mathbb{E}(r_2)} w_1$$

This is a linear equation:

- intercept: $\mathbb{E}(r_p)/\,\mathbb{E}(r_2)$
- slope: $-\,\mathbb{E}(r_1)/\,\mathbb{E}(r_2)$
- $\mathbb{E}(r_1), \mathbb{E}(r_2)$ are quantities estimated/provided by an analyst.

Isomeans

Three unknowns w_1, w_2 and portfolio's expected return $\mathbb{E}(r_p)$

- thus the slope is fixed at $\mathbb{E}(r_1)/\mathbb{E}(r_2)$
- a countably infinity number of lines exist that each consist of combinations of w_1 and w_2 that will yield a given expected return: **isomeans lines**.
- the higher the expected return, the higher the slope the higher the isomean
- all isomeans are parallel regardless of the expected return, with slope $-\mathbb{E}(r_1)/\mathbb{E}(r_2)$

Isomeans

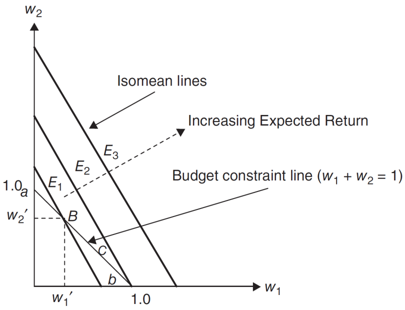


Figure: Isomenas and budget constraint.

- Investors all desire to reach the highest possible isomean line.
- However, they cannot invest infinite amounts because of the budget constraint, $w_1 + w_2 = 1$ ($w_2 = 1 - w_1$ line)

Isomeans

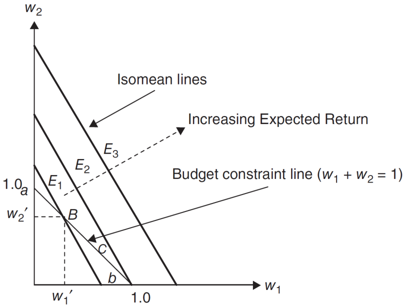


Figure: Isomenas and budget constraint.

- The isomean E_3 does not intersect the budget constraint line: the corresponding amount of expected portfolio return is not attainable.
- E_1 is attainable by investing w_1' in asset 1 and w_2' in asset 2
- The highest expected return is on E_2 attainable by investing everything in asset 1 $(w_1 = 1)$

Example (also in Matlab)

Example

Company	i	w_i	$E(r_i)$	σ_i^2	σ_{ij}
Excelon	1	w_1	2.07%	48.20	-2.65
Jorgenson	2	w_2	1.01%	34.25	-2.65

From $\mathbb{E}(r_p) = w_1\,\mathbb{E}(r_1) + w_2\,\mathbb{E}(r_2)$, we get

$$\mathbb{E}(r_p) = 2.07w_1 + 1.01w_2$$

since $w_1 = 1 - w_2$, the isomean is:

$$w_2 = \frac{\mathbb{E}(r_p)}{1.01} - \frac{2.07}{1.01}w_1 = 0.99\,\mathbb{E}(r_p) - 2.05w_1$$

The intercept depends on the amount $\mathbb{E}(r_p)$ the manager is seeking. The maximum attainable return is with $w_1 = 1$ and $w_0 = 0$, i.e. 2.07%.

Two-asset analysis, isovariance ellipses

The variance of returns from a two-security portfolio is

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

- this is an **ellipse**
- There are many combinations of the weights w_1 and w_2 that produce the same level of portfolio variance σ_p^2: **isovariances**.

Isovariance

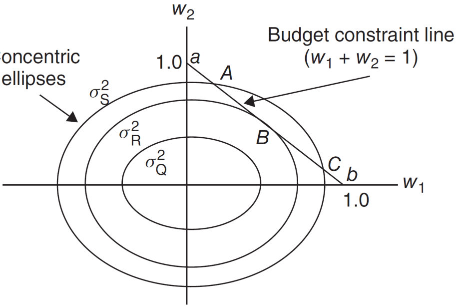


Figure: Isovariance

- The larger the ellipse the greater the variance it represents $\sigma^2_S > \sigma^2_R \sigma^2_Q$
- The ellipses do not intersect, as a point (w_1, w_2) can represent only one level of portfolio variance

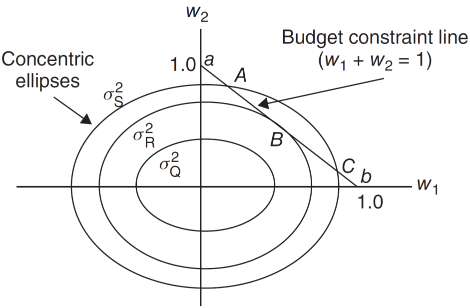


Figure: Isovariance

- Not all desired levels of variance are attainable
- A, C: same isovariance and feasible (know nothing about the corresponding expected portfolio return)
- B: **minimum variance portfolio** (MVP)

Example (also in Matlab)

Example

To identify the MVP corresponding to the previous example, one can compute

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

for different values of w_1, e.g. $0.01, 0.02, \ldots, 0.99, 1$, and identify the optimal weight $w_1^\star$ such that weights $(w_1^\star, 1 - w_1^\star)$ determine the MVP.

The $(\sigma, \mathbb{E}(p))$ space

Every point on the budget constraint line (identifies a certain portfolio:

- ! each point (w_1, w_2) can be mapped in the $(\sigma, \mathbb{E}(p))$ space ($\rightarrow$ see the Matlab example!)

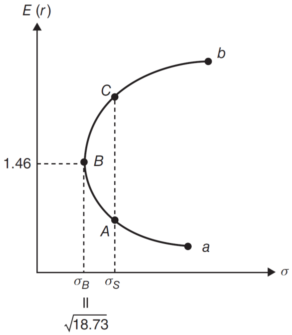


Figure: The $(\sigma, \mathbb{E}(p))$ space and investment opportunity set

The $(\sigma, \mathbb{E}(p))$ space

- The lower section of the budget line *BCb* in Fig. 5 is mapped into the upper part of the opportunity set in Fig. 7, *BCb*
- The upper section of the budget line *BAa* in Fig. 5 is mapped into the lower part of the opportunity set in Fig. 7, *BAa*
- Inefficient portfolios are denoted by *BAa*
- A, C in Fig. 5 are on the same ellipse: they have the same variance in Fig. 7
- C is on a higher isomean than A, desipite the same variance
- **Dominant assets** have:
  - The maximum return in their class of risk, or
  - The minimum risk in their level of returns

Legitimate portfolios

Markowitz tells us an efficient portfolio must meet three conditions:

1. It must have the maximum return in its risk class
2. It must have the minimum risk in its return class
3. It must not involve any negative weights

Negative weights:

- Financially speaking, denying negative weights means no leverage or short sales are permitted.
- More realistically, negative weights are possible and have a rational economic interpretation
- Only public investment funds that are constrained by regulations need adhere to the third condition.
- $\rightarrow$ not a problem for us, as much as $\sum w_i = 1$.

Section 3, Efficient Portfolio

Introduction

Here we follow an analytical approach.

Mathematical techniques yield the same answers as the graphical technique. However, the mathematical techniques are more general, more precise, and offer the tools for solving larger problems.

Risk and Returns for portfolios

r_p denotes the return from a portfolio consisting of n securities, the expected return can be expressed as

$$\mathbb{E}(r_p) = \sum_{i=1}^{n} w_i \, \mathbb{E}(r_i)$$

and its variance as,

$$\begin{aligned} \sigma_p^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\ &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} w_i w_j \sigma_{ij} \end{aligned}$$

that is a sum of n variances and $\left(n^2 - n\right)$ covariance terms.

Overall, the portfolio variance equals the sum of the entries in the following matrix:

$$\Sigma = \begin{pmatrix} w_1 w_1 \sigma_{11} & w_1 w_2 \sigma_{12} & \cdots & w_1 w_n \sigma_{1n} \\ w_2 w_1 \sigma_{21} & w_2 w_2 \sigma_{22} & \cdots & w_2 w_n \sigma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_n w_1 \sigma_{n1} & w_n w_2 \sigma_{n2} & \cdots & w_n w_n \sigma_{nn} \end{pmatrix} = \begin{pmatrix} w_1^2 \sigma_1^2 & w_1 w_2 \sigma_{12} & \cdots & w_1 w_n \sigma_{1n} \\ w_2 w_1 \sigma_{21} & w_2^2 \sigma_2^2 & \cdots & w_2 w_n \sigma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ w_n w_1 \sigma_{n1} & w_n w_2 \sigma_{n2} & \cdots & w_n^2 \sigma_n^2 \end{pmatrix} \tag{1}$$

with $w_i w_j \sigma_{ij} = w_j w_i \sigma_{ji}$ (symmetry).

Example

For a portfolio of 20 securities, there are 20 variances and 380 covariances!

Expected returns, variances, and covariances are parameters that the analyst must estimate

The opportunity set

The **opportunity set**, also known as the **feasible set**, represents all portfolios that could be formed from a group of n securities, plus the n securities.

The set of all possible portfolios formed from a group of n securities lies either **on or within** the boundary of the opportunity set.

The two-security case

Be the correlation between the two assets, denoted by ρ_{12} (recall that $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$).

$$
\begin{aligned}
\mathbb{E}(R_p) &= w_1\,\mathbb{E}(r_1) + w_2\,\mathbb{E}(r_2) \\
\sigma_p &= \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}} \\
&= \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}
\end{aligned}
$$

where, as for the budget constraint, $w_2 = 1 - w_1$.

The weights that minimize the portfolio variance are a standard calculus problem...

Minimizing the risk in the two-security case

We take the first derivative of the portfolio's variance with respect to the weight of asset 1, w_1 set that derivative equal to zero, and solve for w_1. The derivative is

$$\begin{aligned}\frac{\partial \sigma_p^2}{\partial w_1} &= \frac{\partial}{\partial w_1}\left[w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{12}\right] \\ &= 2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2 + 2(1-2w_1)\sigma_{12}\end{aligned}$$

Setting this to zero and solving for w_1, yields:

$$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \qquad w_2 = 1 - w_1 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

Example (also in Matlab)

Example

With respec to to the previous example where MVP was found with a trial and error approach, now we compute:

$$w_1 = \frac{34.25 - (-2.65)}{48.20 + 34.25 - 2(-2.65)} = 0.4205$$

$$w_2 = 1 - 0.4205 = 0.5795;$$

Minimizing the risk in the three-security case

Now, with three assets the opportunity set ‘widens’: possible portfolios lie either on or within the boundary of the opportunity set:

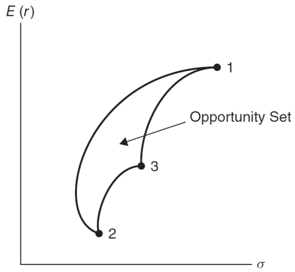


Figure: Opportunity set with 3 securities and no short sales allowed

If short sales are allowed, the opportunity set has open ends on the right-hand side (see later)

Minimizing the risk in the three-security case

In this case,

$$
\begin{aligned}
\mathbb{E}(R_p) &= w_1\,\mathbb{E}(r_1) + w_2\,\mathbb{E}(r_2) + w_3\,\mathbb{E}(r_3) \\
\sigma_p^2 &= w_1^2\sigma_{11} + w_2^2\sigma_{22} + w_3^2\sigma_{33} + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23} \\
&= (\sigma_{11} + \sigma_{33} - 2\sigma_{13})w_1^2 + (2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23})w_1w_2 + \ldots \\
&+ (\sigma_{22} + \sigma_{33} - 2\sigma_{23})w_2^2 + (-2\sigma_{33} + 2\sigma_{13})w_1 + \ldots \\
&+ (-2\sigma_{33} + 2\sigma_{23})w_2 + \sigma_{33}
\end{aligned}
$$

with $w_3 = 1 - w_2 - w_1$.

Minimizing the risk in the three-security case:

$$\frac{\partial \sigma_p^2}{\partial w_1} = 2(\sigma_{11} + \sigma_{33} - 2\sigma_{13})w_1 + (2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23})w_2 + \ldots$$
$$\ldots + (-2\sigma_{33} + 2\sigma_{13}) = 0$$
$$\frac{\partial \sigma_p^2}{\partial w_2} = (2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23})w_1 + (\sigma_{22} + \sigma_{33} - 2\sigma_{23})w_2 + \ldots$$
$$\ldots + (-2\sigma_{33} + 2\sigma_{23}) = 0$$

The above is a system of two equations in two unknowns w_1 and w_2: the simultaneous solution gives the weights for the MVP.

Example

There is an exhaustive Matlab example!

Markowitz diversification

Definition

Markowitz diversification involves combining securities with less-than-perfect positive correlation in order to reduce risk in the portfolio without sacrificing any of the portfolio's expected return.

In general, the lower the correlations (or, equivalently, covariances) of the assets in a portfolio, the less risky the resulting portfolio will be. This is true regardless of how risky the portfolio's assets are when analyzed in isolation.

Example

Example

Security 1 has $\mathbb{E}(r_1) = 0.05$, $\sigma_1 = 0.2$, Security 2 has $\mathbb{E}(r_1) = 0.15$, $\sigma_2 = 0.4$, $w_1 = 2/3$, $w_2 = 1/3$. The correlation between r_1 and r_2 is ρ_{12}.

$$\mathbb{E}(r_p) = \frac{2}{3}0.05 + \frac{1}{3}0.15 = 0.083 \quad \text{(or 8.3\%)}$$

$$\sigma_p = \sqrt{\left(\frac{1}{3}\right)^2 0.20^2 + \left(\frac{1}{3}\right)^2 0.4^2 + 2\frac{2}{3}\frac{1}{3}\rho_{12}(0.2)(0.4)}$$
$$= \sqrt{0.0356 + 0.0365\rho_{12}}$$

Example (continues)

Example

Although the expected return of this portfolio is fixed at 8.3% percent for the investment weights assumed for securities 1 and 2, the risk of the portfolio varies with the correlation coefficient ρ_{12}.

- if $\rho_{12} = 1$, $\sigma_p^2 = 26.7\%$
- if $\rho_{12} = 0$, $\sigma_p^2 = 18.9\%$
- if $\rho_{12} = -1$, $\sigma_p^2 = 0\%$
- I.e., the risk decreases as ρ_{12} moves from 1 to -1

Markowitz diversification

Following the previous example (same expected returns) make the weight vary with different levels for ρ_{12}:

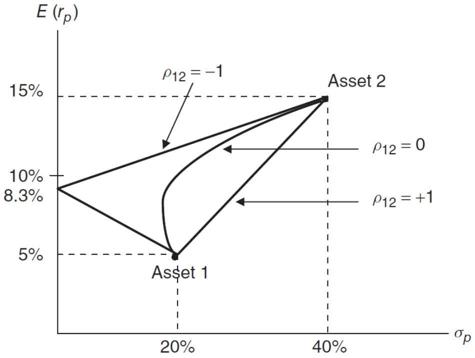


Figure: Diversification (without short sale)

Markowitz diversification

The figure shows that

- lowering the ρ_{12} reduces the risk when combining securities 1 and 2
- when $\rho_{12} = 1$ the straight line is an upper bound for the portfolio's risk (std. deviation)
- with $\rho_{12} = -1$ the risk is minimized
- with $-1 < \rho_{12} < 1$ the line is curved and concave

Markowitz diversification

- Observe that the standard deviation of the portfolio is less than or equal to the weighted average of two securities' standard deviations. In fact for $\rho_{12} \leq 1$

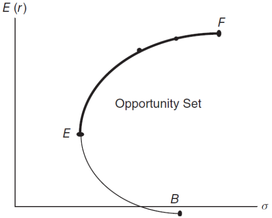
$$
\begin{aligned}
\sigma_p^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \\
&\leq w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 + w_2\sigma_2)^2
\end{aligned}
$$

  That is, $\sigma_p \leq w_1\sigma_1 + w_2\sigma_2$ and the equality holds only for $\rho_{12} = 1$.
- On the other hand, the expected portfolio returns **is** the weighted average of two securities' expected returns:

$\rightarrow$ Risk can be reduced by means of Markowitz diversification without decreasing return at all!

Efficient frontier without the risk-free asset

The upper left quadrant between points E and F is the efficient frontier of the opportunity set. It is comprised of efficient portfolios.



Definition

An efficient portfolio is the portfolio that has either (1) more return than any other portfolio in its risk class (that is, any other asset with the same variability of returns), or (2) less risk than any other security with the same return. These portfolios are sometimes said to be Markowitx efficient.

The n-security case

Consider how a portfolio on the efficient frontier can be identified.

What is desired are weights for the portfolio that minimize the portfolio variance σ_p^2, at a given level of the expected return $\mathbb{E}(r_p) \rightarrow$ The problem

involves finding the weights that minimize the portfolio variance, that is:

$$\text{Minimize } \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

... subject to two mathematical constraints.

Introduction

1 Require that some desired level of expected return is achieved

$$\sum_{i=1}^{n} w_i\,\mathbb{E}(r_i) = \mathbb{E}(r_p) \quad \rightarrow \quad \mathbb{E}(r_p) - \sum_{i=1}^{n} w_i\,\mathbb{E}(r_i) = 0$$

2 Impose that familiar requirement that the sum of the weights is 1:

$$\sum_{i=1}^{n} w_i = 1 \quad \rightarrow \quad 1 - \sum_{i=1}^{n} w_i = 0$$

(short sales are allowed)

These are called **Lagrangian constraints**

Introduction

The Lagrangian objective of our constrained risk-minimization problem is:

$$\text{Minimize } L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} + \lambda\left[\mathbb{E}(r_p) - \sum_{i=1}^{n}\mathbb{E}(r_i)w_i\right] + \gamma\left[1 - \sum_{i=1}^{n} w_i\right]$$

where λ and γ are called **Lagrangian multipliers**. The factor $1/2$ is there merely for convenience.

Introduction

The minimum-risk portfolio is found by setting the partial derivatives to zero, and solve the simultaneous system of $n+2$ equations

$$\begin{cases} \frac{\partial L}{\partial w_1} = w_1\sigma_{11} + w_2\sigma_{12} + \cdots + w_n\sigma_{1n} - \lambda\,\mathbb{E}(r_1) - \gamma = 0 \\ \ldots \\ \frac{\partial L}{\partial w_i} = w_i\sigma_{i1} + w_2\sigma_{i2} + \cdots + w_n\sigma_{in} - \lambda\,\mathbb{E}(r_i) - \gamma = 0 \\ \ldots \\ \frac{\partial L}{\partial w_n} = w_n\sigma_{n1} + w_2\sigma_{n2} + \cdots + w_n\sigma_{nn} - \lambda\,\mathbb{E}(r_n) - \gamma = 0 \\ \frac{\partial L}{\partial \lambda} = w_1\,\mathbb{E}(r_1) + w_2\,\mathbb{E}(r_2) + \cdots + w_n\,\mathbb{E}(r_n) - \mathbb{E}(r_p) = 0 \\ \frac{\partial L}{\partial \gamma} = w_1 + w_2 + \cdots + w_n - 1 = 0 \end{cases}$$

... this is a system of **linear** equations, very tedious to solve. We shall take a different route.

Solving the problem

The two constraints can be rewritten as

$$\sum_{i=1}^{n} w_i \, \mathbb{E}(r_i) = \mathbb{E}(r_p) \quad \rightarrow \quad \boldsymbol{w}^\top \boldsymbol{E} = \mathbb{E}(r_p)$$

$$\sum_{i=1}^{n} w_i = 1 \quad \rightarrow \quad \boldsymbol{w}^\top \mathbf{1} = 1$$

where $\boldsymbol{w}^\top = (w_1, w_2, \ldots, w_n)$, $\boldsymbol{E} = (\mathbb{E}(r_1), \mathbb{E}(r_2), \ldots, \mathbb{E}(r_n))^\top$, and $\mathbf{1}$ is a $(n \times 1)$ vector of ones.

Solving the problem

Thus,

$$\text{Minimize } \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad \rightarrow \quad \text{Minimize } \sigma_p^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$$

where Σ is the $(n \times n)$ covariance matrix in eq. (1).

Solving the problem

Overall, the problem in concise notation reads:

$$\begin{aligned} \text{Minimize } L &= \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \\ \text{subject to } \boldsymbol{w}^\top \boldsymbol{E} &= \mathbb{E}(r_p) \\ \boldsymbol{w}^\top \boldsymbol{1} &= 1 \end{aligned}$$

Or, in other words:

$$\begin{aligned} \text{Minimize } L &= \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \\ \text{subject to } \mathbb{E}(r_p) - \boldsymbol{w}^\top \boldsymbol{E} &= 0 \\ 1 - \boldsymbol{w}^\top \boldsymbol{1} &= 0 \end{aligned}$$

Therefore the Lagrangian objective reads:

$$\text{Minimize } L = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \lambda\Big[\mathbb{E}(r_p) - \boldsymbol{w}^\top \boldsymbol{E}\Big] + \gamma\Big(1 - \boldsymbol{w}^\top \boldsymbol{1}\Big)$$

Solving the problem

The corresponding relevant system of partial derivatives for finding the risk-minimizing weights is:

$$\frac{\partial L}{\partial \boldsymbol{w}} = \Sigma \boldsymbol{w} - \lambda \boldsymbol{E} - \gamma \mathbf{1} = \mathbf{0} \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = \mathbb{E}(r_p) - \boldsymbol{w}^\top \boldsymbol{E} = 0 \tag{3}$$

$$\frac{\partial L}{\partial \gamma} = 1 - \boldsymbol{w}^\top \mathbf{1} = 0 \tag{4}$$

... all the derivatives are straightforward, just keep in mind that in (2) $\partial\big(\boldsymbol{w}^\top \Sigma \boldsymbol{w}\big)/\boldsymbol{\partial w}^\top = \boldsymbol{w}^\top \Sigma^\top + \boldsymbol{w}^\top \Sigma$ and that Σ is symmetric

Solving the problem

The three solutions are respectively:

- From eq. (2)

$$\boldsymbol{w} = \lambda \Sigma^{-1} \boldsymbol{E} + \gamma \Sigma^{-1} \mathbf{1} \tag{5}$$

- Combining eq. (5) with eq. (3):

$$\begin{aligned} \mathbb{E}(r_p) = \boldsymbol{w}^\top \boldsymbol{E} = \boldsymbol{E}^\top \boldsymbol{w} &= \boldsymbol{E}^\top \left(\lambda \Sigma^{-1} \boldsymbol{E} + \gamma \Sigma^{-1} \mathbf{1}\right) \\ &= \lambda \left(\boldsymbol{E}^\top \Sigma^{-1} \boldsymbol{E}\right) + \gamma \left(\boldsymbol{E}^\top \Sigma^{-1} \mathbf{1}\right) \\ &= \lambda B + \gamma A^\top = \lambda B + \gamma A \end{aligned} \tag{6}$$

- Combining eq. (5) with eq. (4):

$$\begin{aligned} \mathbf{1} = \boldsymbol{w}^\top \mathbf{1} = \mathbf{1}^\top \boldsymbol{w} &= \mathbf{1}^\top \left(\lambda \Sigma^{-1} \boldsymbol{E} + \gamma \Sigma^{-1} \mathbf{1}\right) \\ &= \lambda \left(\mathbf{1}^\top \Sigma^{-1} \boldsymbol{E}\right) + \gamma \left(\mathbf{1}^\top \Sigma^{-1} \mathbf{1}\right) \\ &= \lambda A + \gamma C \end{aligned}$$

Solving the problem

Solving eq. (6) and eq. (4) w.r.t. λ and γ yields

$$\lambda = \frac{C\,\mathbb{E}(r_p) - A}{D} \tag{7}$$

$$\gamma = \frac{B - A\,\mathbb{E}(r_p)}{D} \tag{8}$$

where,

$$A = \mathbf{1}^\top \Sigma^{-1} \boldsymbol{E} \qquad B = \boldsymbol{E}^\top \Sigma^{-1} \boldsymbol{E}$$

$$C = \mathbf{1}^\top \Sigma^{-1} \mathbf{1} \qquad D = BC - A^2$$

The inverse of a positive-definite matrix is also positive definite, thus: $B > 0$, $C > 0$, and it can be shown that $BC > A^2$ so $D > 0$ as well.

Solving the problem

Now plug eq. (7) and (8) in (5):

$$\begin{aligned}
\boldsymbol{w}_p &= \lambda\Sigma^{-1}\boldsymbol{E} + \gamma\Sigma^{-1}\mathbf{1} \\
&= \left[\frac{C\,\mathbb{E}(r_p) - A}{D}\right]\Sigma^{-1}\boldsymbol{E} + \left[\frac{B - A\,\mathbb{E}(r_p)}{D}\right]\Sigma^{-1}\mathbf{1} \\
&= \frac{1}{D}\left[C\,\mathbb{E}(r_p)\Sigma^{-1}\boldsymbol{E} - A\Sigma^{-1}\boldsymbol{E} + B\Sigma^{-1}\mathbf{1} - A\,\mathbb{E}(r_p)\Sigma^{-1}\mathbf{1}\right] \\
&= \frac{1}{D}\left[B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\boldsymbol{E}\right] + \frac{1}{D}\left[C\Sigma^{-1}\boldsymbol{E} - A\Sigma^{-1}\mathbf{1}\right]\mathbb{E}(r_p) \\
&= \boldsymbol{g} + \boldsymbol{h}\,\mathbb{E}(r_p)
\end{aligned}$$

where,

$$\begin{aligned}
\boldsymbol{g} &= \frac{1}{D}\left(B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\boldsymbol{E}\right) \\
\boldsymbol{h} &= \frac{1}{D}\left(C\Sigma^{-1}\boldsymbol{E} - A\Sigma^{-1}\mathbf{1}\right)
\end{aligned}$$

Solving the problem

With the representation

$$\boldsymbol{w}_p = g + h\,\mathbb{E}(r_p) \tag{9}$$

we see that the optimal weights of the mean-variance efficient portfolios are a linear function of the given level of the expected return of the portfolio, because g and h are constants. Concluding remark:

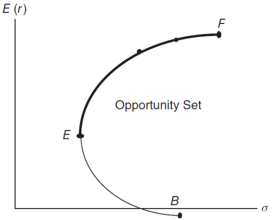
- Any efficient portfolio can be represented as eq.(9)
- Any portfolio that can be represented as eq.(9) is an efficient portfolio
- The variance of the efficient portfolio is

$$\sigma_p^2 = \boldsymbol{w}_p^\top \Sigma \boldsymbol{w}_p$$

The two-fund separation theorem

Theorem

All portfolios on the mean-variance efficient frontier can be formed as a linear combination of any two portfolios (or funds) on the efficient frontier



- Let portfolios p_1 and p_2 be on the efficient frontier EF
- Assume $\mathbb{E}(r_{p_1}) \neq \mathbb{E}(r_{p_2})$ and be q a portfolio obtained combining the two:

$$\mathbb{E}(r_q) = \alpha\,\mathbb{E}(r_{p_1}) + (1-\alpha)\,\mathbb{E}(r_{p_2})$$

The two-fund separation theorem

The investment weights of portfolio q are consequently

$$\boldsymbol{w}_q = \alpha \boldsymbol{w}_{p_1} + (1-\alpha)\boldsymbol{w}_{p_2}$$

Therefore,

$$\begin{aligned}
\boldsymbol{w}_q &= \alpha[\boldsymbol{g} + \boldsymbol{h}\,\mathbb{E}(r_{p_1})] + (1-\alpha)[\boldsymbol{g} + \boldsymbol{h}\,\mathbb{E}(r_{p_2})] \\
&= \boldsymbol{g} + \boldsymbol{h}[\alpha\,\mathbb{E}(r_{p_1}) + (1-\alpha)\,\mathbb{E}(r_{p_2})] \\
&= \boldsymbol{g} + \boldsymbol{h}\,\mathbb{E}(r_q)
\end{aligned}$$

Thus:

- q is an efficient portfolio.

! as a consequence, the entire efficient frontier can be generated by a linear combination of two efficient portfolios.

Efficient Frontier without the risk-free asset

- Point E is typically referred to as the global minimum variance portfolio (MVP) because no other portfolio exists that has lower variance.
- Point F is typically referred to as the maximum return portfolio because no other portfolio exists that has a higher level of expected return.

Equation for the efficient frontier I

By using (9), the variance of a portfolio can be expressed as

$$
\begin{aligned}
\sigma_p^2 &= \boldsymbol{w}_p \Sigma \boldsymbol{w}_p && (10)\\
&= [\boldsymbol{g} + \boldsymbol{h}\,\mathbb{E}(r_p)]^\top \Sigma [\boldsymbol{g} + \boldsymbol{h}\,\mathbb{E}(r_p)] && (11)
\end{aligned}
$$

recalling the definitions of $\boldsymbol{g}$ and $\boldsymbol{h}$ and rearranging the above

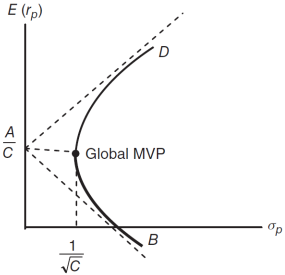
$$\sigma_p^2 = \frac{C}{D}\left[\mathbb{E}(r_p) - \frac{A}{C}\right]^2 + \frac{1}{C}$$

This corresponds to

- a parabola in the mean-variance space
- a hyperbola in the mean-standard deviation space, with vertex

$$[\sigma_p, \mathbb{E}(r_p)] = \left(\frac{1}{\sqrt{C}}, \frac{A}{C}\right)$$

Equation for the efficient frontier II



What are the weights of the MVP?
Weights are determined as for eq (5), that is:

$$\boldsymbol{w}_{MVP} = \lambda \Sigma^{-1} \boldsymbol{E} + \gamma \Sigma^{-1} \mathbf{1}$$

Equation for the efficient frontier III

Since $\mathbb{E}(r_{MVP}) = A/C$ and $D = BC - A^2$, for the MVP the Lagrangian multipliers become:

$$\lambda = \frac{C\,\mathbb{E}(r_{MVP}) - A}{D} = 0 \qquad \text{and} \qquad \gamma = \frac{B - A\,\mathbb{E}(r_{MVP})}{D} = \frac{1}{C}$$

Therefore:

$$\begin{aligned} \boldsymbol{w}_{MVP} &= \lambda \Sigma^{-1} \boldsymbol{E} + \gamma \Sigma^{-1} \mathbf{1} \\ &= 0 + \frac{1}{C} \Sigma^{-1} \mathbf{1} \\ &= \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}} \end{aligned}$$

Optimal portfolio

Theorem

The optimal portfolio for a risk-averse investor will be located on the efficient frontier.

This theorem can be proven intuitively by noting that the risk-averse investor's **convex** indifference curves represent increasing expected utility as one moves from one curve (utility isoquant) to another in a **northwesterly direction**.

Because (1) the investor seeks a portfolio on the **most northwestern** indifference curve possible, (2) the opportunity set is **concave** on its northwest boundary, and (3) the northwest boundary of the opportunity set is by definition the efficient frontier, the investor's optimal portfolio will be an efficient portfolio.

Optimal portfolio

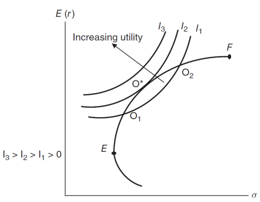


Figure: Optimal portfolio with different risk aversion.

- The optimal portfolio is the efficient portfolio $O^\star$, which is a tangency point between the efficient frontier and the highest feasible indifference curve.
- All investors (no matter what their risk aversion is) maximize their utility on the efficient frontier.
- All rational investors will chose a portfolio on the efficient frontier.

Introducing the risk-free asset

- Every candidate asset has positive variance, $\sigma^2 > 0$
- We assume the existence of the risk-free asset, $\sigma^2 = 0$
- Now we can borrow and lend at the risk-free rate r_f.

Introducing the risk-free asset

Consider what happens when a risk-free security is held in a long position in conjunction with a long position in a risky asset. Denote the proportions held in the risky asset and the risk-free asset as w_1 and $(1 - w_1)$ respectively. The expected return of this portfolio is defined by the unambiguously linear equation:

$$\mathbb{E}(r_p) = (1 - w_1)r_f + w_1\,\mathbb{E}(r_1) \tag{12}$$

The variance of this portfolio is:

$$\begin{aligned} \sigma_p^2 &= w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_f^2 + 2w_1(1 - w_1)\sigma_{1f} \\ &= w_1^2\sigma_1^2 + (1 - w_1)^2 0 + 2w_1(1 - w_1)0 \\ &= w_1^2\sigma_1^2 \end{aligned}$$

from which,

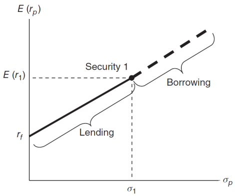
$$w_1 = \frac{\sigma_p}{\sigma_1} \tag{13}$$

Introducing the risk-free asset

Now substitute eq.(13) in eq.(12) and simplify, to obtain

$$\mathbb{E}(r_p) = r_f + \left(\frac{\mathbb{E}(r_1) - r_f}{\sigma_1}\right)\sigma_p \tag{14}$$

This is a linear relation, $\mathbb{E}(r_p)$ is linear in σ_p: any portfolio obtained by mixing the risk-free rate and a risky asset lies on a straight line.



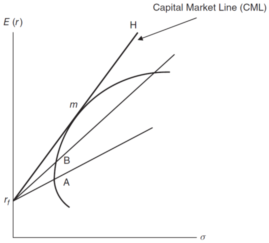
Note that this is the actual opportunity set.

Introducing the risk-free asset

- Assuming that $r_f < \mathbb{E}(r_1)$, if $0 < w_1 < 1$ then $0 < \sigma_p < \sigma_1$ and $r_f < \mathbb{E}(r_p) < \mathbb{E}(r_1)$.
- with $w_f = 1 - w_1 < 0$ the risk-free rate is being borrowed. Such a portfolio will lie anywhere anywhere on the dotted line.
- Freely combining the risk-free asset with any risky portfolio will result in a new portfolio somewhere on the straight line connecting the two.

Introducing the risk-free asset

By combining the risk-free asset with a portfolio of n assets, we immediately see that there is a preferred portfolio that generates a frontier that dominates any other:



- The combinations of r_f with the portfolio m along the line $r_f mH$ have the highest return at any level of risk.
- any ray with a highest slope is unfeasible as it would lie outside the opportunity set of the n assets.

Introducing the risk-free asset

The result is that the new efficient frontier is a straight line where all efficient portfolios are simply linear combinations of the risk-free asset and the tangency portfolio m.

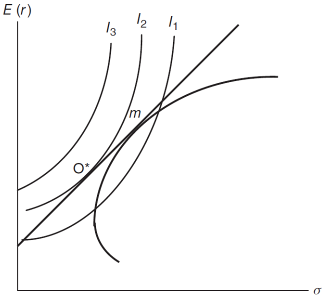
The corresponding straight-line equation is (analogous to eq.(14)):

$$\mathbb{E}(r_p) = r_f + \left(\frac{\mathbb{E}(r_m) - r_f}{\sigma_m} \right) \sigma_p$$

This new efficient frontier is called the **capital market line** (CML).

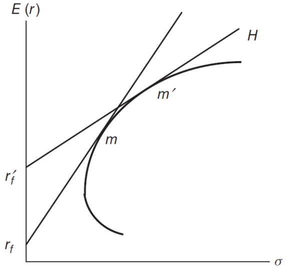
Introducing the risk-free asset

Given a linear efficient frontier and convex indifference curves for a risk averse investor, the efficient frontier theorem still holds and the investor optimally will choose a portfolio on this new linear efficient frontier.



Introducing the risk-free asset

The tangency portfolios, m change their locationas the level of the risk-free interest rate changes. If the risk free rate changes from r_f to r_f' the tangency portfolio moves from m to m'.



It it intuitive that the higher r_f, the more on the upper-right m is: so higher levels of utility can be achieved.

Efficient portfolio with a risk free asset

How do we find the weights $\boldsymbol{w}_p$ of an efficient portfolio with a given $\mathbb{E}(r_p)$ in this case?

Now we have n assets and the risk free rate, therefore $n+1$ candidates for constructing efficient portfolios. The optimal weights for the efficient

portfolio when the riskless asset exists are obtained from the following objective:

$$\text{Minimize } L = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \tag{15}$$

$$\text{Subject to } \boldsymbol{w}^\top \boldsymbol{E} + w_f r_f = \mathbb{E}(r_p) \tag{16}$$

where $w_f = 1 - \boldsymbol{w}^\top \mathbf{1}$ is the weight for the risk-free asset.

Efficient portfolio with a risk free asset

Note that the variance of the portfolio formed by the $n+1$ assets is the same as the variance of the portfolio formed by the n risky assets ony:

$$\sigma_p^2 = \boldsymbol{w}_p^\top \Sigma \boldsymbol{w}_p \tag{17}$$

The corresponding Lagrangian objective for eq.(15)-(16) is:

$$\text{Minimize } L = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \lambda\left[\mathbb{E}(r_p) - \boldsymbol{w}^\top \boldsymbol{E} - \left(1 - \boldsymbol{w}^\top \mathbf{1}\right) r_f\right]$$

Efficient portfolio with a risk free asset

By setting the derivatives to zero

$$\frac{\partial L}{\partial \boldsymbol{w}} = \Sigma \boldsymbol{w} - \lambda(\boldsymbol{E} - r_f \mathbf{1}) = \mathbf{0}$$
$$\frac{\partial L}{\partial \lambda} = \mathbb{E}(r_p) - \boldsymbol{w}^\top \boldsymbol{E} - \left(1 - \boldsymbol{w}^\top \mathbf{1}\right) r_f = 0$$

(Because Σ is a positive definite covariance matrix, the first-order conditions are necessary and sufficient for a global optimum.) From the above we have:

$$\boldsymbol{w} = \lambda \Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1}) \tag{18}$$
$$\mathbb{E}(r_p) = r_f + \boldsymbol{w}^\top(\boldsymbol{E} - r_f \mathbf{1}) \tag{19}$$

Substituting eq.(18) into eq.(19)

$$\mathbb{E}(r_p) = r_f + \lambda(\boldsymbol{E} - r_f \mathbf{1})^\top \Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1}) \tag{20}$$

Efficient portfolio with a risk free asset

From eq.(20), the lagrangian multiplier is

$$\lambda = \frac{\mathbb{E}(r_p) - r_f}{H} \tag{21}$$

where,

$$H = (\boldsymbol{E} - r_f \mathbf{1})^\top \Sigma^{-1} (\boldsymbol{E} - r_f \mathbf{1})$$

(Since Σ^{-1} is positive definite $H > 0$). Now plug eq.(21) into (18):

$$\boldsymbol{w}_p = \Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1}) \left[\frac{\mathbb{E}(r_p) - r_f}{H} \right] \tag{22}$$

and the optimal weight for the riskless asset is:

$$w_f = 1 - \boldsymbol{w}_p^\top \mathbf{1}$$

Efficient portfolio with a risk free asset

As for the n-security case, eq.(22) can be expressed as a linear function of the desired level of the expected portfolio return $\mathbb{E}(r_p)$:

$$\boldsymbol{w}_p = \boldsymbol{u} + \boldsymbol{v}\,\mathbb{E}(r_p)$$

where

$$\boldsymbol{u} = -\Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1})\frac{r_f}{H}, \qquad \boldsymbol{v} = +\Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1})\frac{1}{H}$$

are fixed.

$\rightarrow$ The two-fund separation theorem applies in this case too.

The tangency portfolio

- The tangency portfolio is the portfolio of risky assets on the efficient frontier at the point where the capital market line (CML) is tangent to the efficiency frontier.
- Linear combinations of the tangency portfolio and the risk-free asset compose the CML.

The variance of the portfolio is (plug eq. (22) in (17))

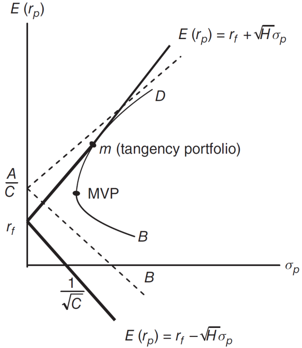
$$
\begin{aligned}
\sigma_p^2 &= \boldsymbol{w}_p^\top \Sigma \boldsymbol{w}_p \\
&= \left(\Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1})\left[\frac{\mathbb{E}(r_p) - r_f}{H}\right]\right)^\top \Sigma \left(\Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1})\left[\frac{\mathbb{E}(r_p) - r_f}{H}\right]\right) \\
&= (\mathbb{E} - r_f \mathbf{1})^\top \Sigma^{-\top} \Sigma \Sigma^{-1} (\mathbb{E} - r_f \mathbf{1}) \left[\frac{\mathbb{E}(r_p) - r_f}{H}\right]^2 \\
&= (\mathbb{E} - r_f \mathbf{1})^\top \Sigma^{-1} (\mathbb{E} - r_f \mathbf{1})^\top \left[\frac{\mathbb{E}(r_p) - r_f}{H}\right]^2 = H \left[\frac{\mathbb{E}(r_p) - r_f}{H}\right]^2 \\
&= (\mathbb{E}(r_p) - r_f)^2 / H
\end{aligned}
$$

The tangency portfolio

Thus,

$$\sigma_p = \begin{cases} +[\mathbb{E}(r_p) - r_f]/\sqrt{H} & \text{if } \mathbb{E}(r_p) \geq r_f \\ -[\mathbb{E}(r_p) - r_f]/\sqrt{H} & \text{if } \mathbb{E}(r_p) < r_f \end{cases}$$

These are two lines with intercept r_f and slopes $\pm\sqrt{H}$:



The tangency portfolio

- It can be shown that if $r_f > A/C = (\mathbf{1}^\top \Sigma^{-1} \boldsymbol{E})/(\mathbf{1}^\top \Sigma^{-1} \mathbf{1})$ a tangent portfolio does not exist on the upper-half of the frontier BD
- It can be shown that if $r_f < A/C$ a tangency portfolio on the upper-half of the frontier exists (but not in the lower-half)
- If $r_f = A/C$, the lines overlap the asymptotes of the efficient frontier composed only of risky assets: there is no tangency at all.

The tangency portfolio

What is portfolio m?

- it does *not* contain the riskless asset.
- as such, at the point m, $w_f = 0$.
- thus for the tangency portfolio m:

$$w_f = 1 - \boldsymbol{w}_m^\top \mathbf{1} = 0$$

Using eq.(22),

$$\begin{aligned} \boldsymbol{w}_m^\top \mathbf{1} &= (\boldsymbol{E} - r_f \mathbf{1})^\top \Sigma^{-1} \mathbf{1} \left[\frac{\mathbb{E}(r_p) - r_f}{H} \right] \\ &= (A - r_f C) \left[\frac{\mathbb{E}(r_p) - r_f}{H} \right] = 1 \end{aligned} \tag{23}$$

from which,

$$\frac{\mathbb{E}(r_p) - r_f}{H} = \frac{1}{A - r_f C}$$

So, by eq.(22)

$$\boldsymbol{w}_m = \frac{\Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1})}{A - r_f C} \tag{24}$$

The corresponding expected return and variance of the tangency portfolio m are:

$$\mathbb{E}(r_p) = \boldsymbol{E}^\top \boldsymbol{w}_m = \frac{\boldsymbol{E}^\top \Sigma^{-1}(\boldsymbol{E} - r_f \mathbf{1})}{A - r_f C} = \frac{B - r_f A}{A - r_f C}$$

$$\sigma_p^2 = \boldsymbol{w}_m^\top \Sigma \boldsymbol{w}_m = \frac{(\boldsymbol{E} - \mathbf{1} r_f)^\top \Sigma^{-1} \Sigma \Sigma^{-1} (\boldsymbol{E} - \mathbf{1} r_f)}{(A - r_f C)^2} = \frac{H}{(A - r_f C)^2}$$

The market portfolio, another perspective I

Claim

Let $\boldsymbol{w}_p$ be the weight vector of n risky assets for the efficient portfolio p. Then, the weights for the tangency portfolio can be obtained by normalizing $\boldsymbol{w}_p$. I.e.,

$$\boldsymbol{w}_m = \frac{\boldsymbol{w}_p}{\boldsymbol{w}_p^\top \mathbf{1}}$$

In fact, we can obtain eq.(24) again:

$$\boldsymbol{w}_m = \frac{\boldsymbol{w}_p}{\boldsymbol{w}_p^\top \mathbf{1}} = \frac{\Sigma^{-1}(\boldsymbol{E} - r_f\mathbf{1})\left[\frac{\mathbb{E}(r_p)-r_f}{H}\right]}{(\boldsymbol{E} - r_f\mathbf{1})^\top\Sigma^{-1}\mathbf{1}\left[\frac{\mathbb{E}(r_p)-r_f}{H}\right]}$$

$$= \frac{\Sigma^{-1}(\boldsymbol{E} - r_f\mathbf{1})}{(\boldsymbol{E} - r_f\mathbf{1})^\top\Sigma^{-1}\mathbf{1}} = \frac{\Sigma^{-1}(\boldsymbol{E} - r_f\mathbf{1})}{A - r_f C}$$

The market portfolio, another perspective II

This means that given an efficient portfolio on the new (linear) efficient frontier, the tangency portfolio m, can be obtained by rescaling the weights such that all the wealth is invested in the n risky assets and nothing in r_f.
$\rightarrow$ in fact,

$$\boldsymbol{w}_m = \frac{\boldsymbol{w}_p}{\boldsymbol{w}_p^\top \mathbf{1}} = \frac{\boldsymbol{w}_p}{\sum_{i=1}^{n} w_{p_i}}$$

Example

Be $\boldsymbol{w}_p = (0.2, 0.3, 0.4, w_f = 0.1)^\top$, then
$\boldsymbol{w}_m = (0.2, 0.3, 0.4, w_f = 0)^\top / 0.9$, that is
$\boldsymbol{w}_m = (0.222, 0.333, 0.444, w_f = 0)^\top$.

Why do we call the portfolio with weights $\boldsymbol{w}_m$ *market* portfolio? (See the slide around eq. (25)).

Section 4, CAPM

CAPM

After Markowitz developed the two-parameter portfolio analysis model, researchers began investigating the stock market implications that would occur if all investors used the Markowitz two-parameter model to make their investment decisions. The results is the **Capital Asset Pricing Model (CAPM)**.

Capital market theory is based on the assumptions underlying portfolio analysis, because the theory is essentially an accumulation of the logical implications of portfolio analysis.

Assumptions

The initial portfolio theory assumptions are:

1. Investors in capital assets (defined as all terminal-wealth-producing assets) are risk-averse one-period expected-utility-of-terminal-wealth maximizers. Equivalently, investors are **risk averse** and **maximize their expected utility of returns over a one-period** planning horizon.
2. Investors find it possible to make their portfolio decisions **solely on the basis of the mean and standard deviation of the terminal wealth** (or, equivalently, rates of return) associated with the alternative portfolios
3. The mean and standard deviation of terminal wealth (or, equivalently, rates of return) associated with these portfolios are **finite numbers that exist and can be estimated** or measured.

Assumptions

4. There are a collection of assumptions that underlie most economic theories. All capital assets are **infinitely divisible**, meaning that fractions of shares can be bought or sold. Investors are assumed to be **price takers** (instead of price setters). Finally, taxes and transactions costs are assumed to be **nonexistent**.

Investors who conform to the preceding assumptions 1-4 will prefer Markowitz efficient portfolios over inefficient portfolios.

Assumptions

Additional assumptions for the CAPM theory are:

5. There is a **single risk-free interest** rate at which all borrowing and lending takes place.
6. All assets, including human capital, are **marketable**.
7. Capital **markets are perfect**, meaning that (a) all **information** is freely and instantly available to everyone (transparency prevails), (b) no margin requirements exist[1], and (c) investors have **unlimited opportunities to borrow, lend, or sell assets short**.
8. Investors all have **homogeneous expectations** over the same one-period investment horizon, and they all have the same perceptions regarding each security's **ex-ante** expected return, variance, and covariance during the universal planning horizon. These uniform perceptions are sometimes called **idealized uncertainty**.

[1]this is the difference between the current value of the security offered for loan (called collateral) and the value of loan granted.

The market portfolio

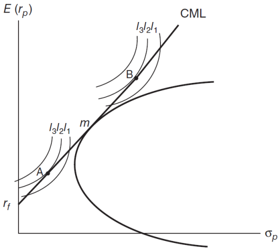
In this equilibrium all investors' optimal portfolio choices have been aggregated into one huge market portfolio, denoted m, and, supply equals demand for every asset. After aggregating all investors' optimal portfolio choices, the **market portfolio** must contain every marketable asset in the proportion w_i,

$$w_i = \frac{\text{total market value of asset } i}{\text{total market value of all assets in the market}} \tag{25}$$

- The market portfolio is a risky portfolio containing all risky securities **in the proportions in which they are supplied**.
- Accordingly, **the return on the market portfolio is the weighted average of the returns of all securities in the market**.
- It's an ideal construct: it **cannot be observed** (imperfect proxies are e.g. the S&P 500 index).

The separation theorem

- Assumptions 5, 6, 7 imply that all investors face the linear efficient frontier called the CML.
- Assumption 8 (homogeneous expectations) ensure that all investors will envision efficient frontiers identically.
- However, they have different utility functions and thus different preferences: they will select different portfolios on the CML.
- From the separation theorem: **all** investors are concerned with **just** m, the market portfolio, and r_f the risk free rate.



Implications

- **The separation theorem implies that all investors, whether timid or aggressive, should hold the same mix of risky securities in their optimum portfolio.** Then, after they select their optimal portfolio, they should use borrowing or lending at the risk-free interest rate r_f to attain their preferred risk class.
- This is the opposite of portfolio management where the portfolio manager should design a portfolio to match the client's personality.
- Timid and aggressive investors should both own m and differ only in the way they finance it.

Efficient frontier equation

Assumption 8, implies that all the investors face the very same frontier, the CML.

We obtained its equation in eq.(14), for any efficient portfolio p:

$$\mathbb{E}(r_p) = r_f + \left[\frac{\mathbb{E}(r_m) - r_f}{\sigma_m} \right] \sigma_p$$

$\rightarrow$ The CML is the dominant efficient frontier facing investors. **Each investor** will select their personal optimal portfolio from the CML by finding the tangency point between their indifference curves and the CML.

The CAPM

We have determined that:
the expected return of an efficient portfolio is a linear function of its standard deviation of return.

Utility theory suggests that investors should seek investments that have the maximum expected return in their risk class.

- Investor's happiness is derived from the expected utility $\mathbb{E}(U) = f(\sigma, \mathbb{E}(r))$
- Wealth-seeking and risk-adverse investors maximize their expected utility
- To do so, they (1) maximize the expected return for a given risk class (2) minimize the risk at any given rate of expected return

Systematic vs. idiosyncratic risks

- **Systematic risk** arises from market structure or dynamics which produce shocks or uncertainty faced by *all* the agents in the market.
- **Idiosyncratic risk** is the risk to which only *specific* agents or industries are vulnerable.
- The idiosyncratic risk can be reduced or eliminated through **diversification**; but since all market actors are vulnerable to systematic risk, it cannot be limited through diversification.

Systematic vs. idiosyncratic risks

Example

- **Systematic risk**: inflation, climate change, currencies fluctuations, wars, demographic risks, technology, politics.
  After all, the above factors although 'external to markets' constitute the environment in which markets operate, thus represent risk-factors for the markets, ideally not removable.
- **Idiosyncratic risk**: tax policy for certain goods/class of assets, customers' demand for some product, for a mining company an example would be the exhaustion of a vein or a seam of metal, new investors entering the market with more resources than you, investment strategy, company's culture, employees' skills.

CAPM

In the search for individual assets that will minimize their portfolio's risk exposure at a given level of expected return, investors tend to focus on each asset's undiversifiable risk.

Definition (Capital Asset Pricing Model (CAPM))

The Capital Asset Pricing Model (CAPM) is an economic model that specifies what expected returns (and therefore prices) should be as a function of **systematic risk**.

The CAPM is a relationship in which the expected rate of return of the i-th asset is a linear function of that asset's systematic risk as represented by β_i:

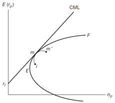
$$\mathbb{E}(r_i) = r_f + [\mathbb{E}(r_m) - r_f]\beta_i$$

CAPM

- The CAPM is a model for pricing an individual security or portfolio.
- The CAPM puts structure to mean-variance optimization theory.
- The CAPM assumes only one source of systematic risk: market risk.
- Investors are compensated for the market risk by a risk premium.
- Their compensation is proportional to the risk exposure.

CAPM - derivation

- Be i an **arbitrary** asset, and m denote the market portfolio.
- Let m' denote the market portfolio excluding the asset i
- Imagine a portfolio p that invests a proportion w_i in asset i and $(1 - w_i)$ in the portfolio m.
- Note that the market portfolio *already* contains some amount of i, thus w_i can be interpreted as the excess demand of i.
- (i) if $w_i = 1$, then $p = i$, (ii) if $w_i = 0$, then $p = m$, (iii) if nothing at all is invested in i, neither directly, nor within m, then $p = m'$ (w_i is some negative weight).
- the portfolio p lies anywhere on the line im' (as a special case m is on im' too), the exact location on the curve im' of p depends on the specific value of w_i.



CAPM - derivation

For the portfolio p (defined above to be a combination of i and m):

$$\mathbb{E}(r_p) = w_i\,\mathbb{E}(r_i) + (1-w_i)\,\mathbb{E}(r_m)$$
$$\sigma_p = \sqrt{w_i^2\sigma_i^2 + (1-w_i)^2\sigma_m^2 + 2w_i(1-w_i)\sigma_{im}}$$

As w_i changes, $\mathbb{E}(r_p)$ and σ_p change as follows:

$$\frac{d\,\mathbb{E}(r_p)}{dw_i} = \mathbb{E}(r_i) - \mathbb{E}(r_m)$$
$$\frac{d\sigma_p}{dw_i} = \left(w_i^2\sigma_i^2 - (1-w_i)^2\sigma_m^2 + (1-2w_i)\sigma_{im}\right)/\sigma_p$$

For the change in $\mathbb{E}(r_p)$ relative to a change in σ_p:

$$\frac{d\,\mathbb{E}(r_p)}{d\sigma_p} = \frac{\frac{d\,\mathbb{E}(r_p)}{dw_i}}{\frac{d\sigma_p}{dw_i}}$$

$$= \frac{(\mathbb{E}(r_i) - \mathbb{E}(r_m))\sigma_p}{w_i\sigma_i^2 - (1-w_i)\sigma_m^2 + (1-2w_i)\sigma_{im}}$$

This can be interpreted as a slope of the curve im' in the $(\mathbb{E}(r_p), \sigma_p)$ space (see previous plot). Thus, this is the *general* equation of the slope of *any* point on im'.

What is the slope exactly at the point m?

CAPM - derivation

When $w_i = 0$, $p \equiv m$ (p and m coincide), and thus $\sigma_p = \sigma_m$:

$$\left.\frac{d\,\mathbb{E}(r_p)}{d\sigma_p}\right|_{w_i=0} = \frac{(\mathbb{E}(r_i) - \mathbb{E}(r_m))\sigma_p}{\sigma_{im} - \sigma_m^2} = \frac{(\mathbb{E}(r_i) - \mathbb{E}(r_m))\sigma_m}{\sigma_{im} - \sigma_m^2}$$

and furthermore, the slope of the curve should be that of the CML, as the CML is tangent to the efficient frontier at m:

$$\frac{\mathbb{E}(r_m) - r_f}{\sigma_m} = \frac{(\mathbb{E}(r_i) - \mathbb{E}(r_m))\sigma_m}{\sigma_{im} - \sigma_m^2}.$$

After solving for $\mathbb{E}(r_i)$:

$$\mathbb{E}(r_i) = r_f + \left[\frac{\mathbb{E}(r_m) - r_f}{\sigma_m^2}\right]\sigma_{im} = r_f + [\mathbb{E}(r_m) - r_f]\beta_{im} \tag{26}$$

Eq.(26) is referred to as the **Security Market Line (SML)**

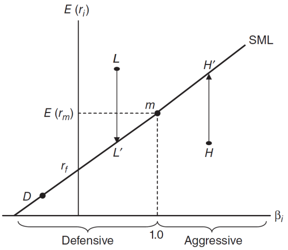
Implications:

- ! Every asset has a market return that is related to its covariance with the market portfolio
- the relationship is positive
- ! if an asset has zero-covariance with m then it expected return must be equal to r_f even if it has non-zero standard deviation
- According to eq.(26) the expected return is determined by the covariance and not by asset's variance
- thus the covariance σ_{im} is an appropriate measure of asset's risk

With $\beta_i = \sigma_{im}/\sigma_m^2$,

$$\mathbb{E}(r_i) = r_f + [\mathbb{E}(r_m) - r_f]\beta_i$$

Security Price Implications



All securities positioned above the SML are underpriced:

- Points between the CAPM and the vertical axis, such as point L, represent securities whose current prices are lower than they would be in equilibrium.
- Because they have unusually high expected returns, there will be strong demand for them
- This means that investors will bid their purchase prices up until their equilibrium rate of return is driven down onto the CAPM at L'.

Security Price Implications

Example

In equilibrium,

$$\mathbb{E}(r) = P_1/P_0 - 1 = r_f + [\mathbb{E}(r_m) - r_f]\beta_i$$

Having P_0 too low is like having an abnormal return above the equibilbium

$$\mathbb{E}(r) > r_f + [\mathbb{E}(r_m) - r_f]\beta_i$$

By lowering P_0 (underpricing the security), the expected return is higher than the equilibrium one. Therefore abnormally high returns correspond to securities that are currently underpriced.

Excess return

Definition

In the following, we denote the **excess return** by

$$Z_{it} = R_{it} - R_{ft}$$

for and asset i (or the market portfolio), w.r.t. a reference rate R_{ft}, i.e. the risk-free rate.

Note: Z_{it} is a random variable since R_{it} is stochastic.

CAPM (cont'd)

- The Sharpe-Lintner version can be most compactly expressed in terms of returns in excess of this risk-free rate or in terms of excess returns.
- With Z_i (Z_m) representing the return on the i-th asset (market portfolio) in excess of the risk-free rate.
- Then, for the Sharpe-Lintner CAPM;

$$\mathbb{E}(Z_i) = \beta_{im}\,\mathbb{E}(Z_m),$$
$$\beta_{im} = \frac{\mathbb{C}\text{ov}(Z_i, Z_m)}{\mathbb{V}(Z_m)},$$

- Empirically, proxies for the risk-free rate are stochastic and thus the betas can differ.

Beta

Definition (Beta coefficient)

The **beta** parameter β_{im} represents the sensitivity of the expected excess asset returns to the expected excess market returns, with

$$\beta_{im} = \frac{\mathbb{C}\text{ov}(Z_i, Z_m)}{\mathbb{V}(Z_m)}.$$

Beta (cont'd)

Interpretation:

- Empirical tests of the Sharpe-Lintner CAPM have focused on three implications:
- If $\beta_i = 0$, asset i is not exposed to market risk. Thus, the investor is not compensated with higher return:

$$\mathbb{E}(Z_i) = R_f$$

- If $\beta_i > 0$, asset i is exposed to market risk and $\mathbb{E}(R_i) > R_f$, provided that $\mathbb{E}(R_m) > R_f$.
- If $\beta_i = 1$, the expected return of asset i is equal to the expected market return

$$\mathbb{E}(R_i) = \mathbb{E}(R_m)$$

- **Question**: how to interpret $0 < \beta_i < 1$ and $\beta < 0$?

Section 5,
Empirical CAPM

Towards the empirical CAPM

- The CAPM is a single-period model; i.e. it does not have a time dimension.
- So far the CAPM is entirely theoreical, we need a workable model for estimation and testing.
- It is necessary to add an assumption concerning the time-series behavior of returns to estimate the model over time. I.e. for a testable and estimable model we would need some data, that is multiple values of the excess returns. These values are in practice the observed values across different days, thus the need to include a time-dimension in the model and set some hypotheses on the time-series of the relevant variables.
- We assume returns are i.i.d. and (jointly multivariate) normal (this applies to excess returns for the Sharpe-Lintner version and to real returns for the Black version).

Empirical CAPM

From a theoretical CAPM to a linear regression model

Definition (the CAPM as a regression model)

The **empirical CAPM** model for an asset i at all time t can defined as

$$R_{it} - R_{ft} = \alpha_i + \beta_i(R_{mt} - R_{ft}) + \varepsilon_{it}$$

where α_i is a constant term, β_i denotes the slope parameter and ε_{it} is an error term with $\mathbb{E}(\varepsilon_{it}) = 0$ and $\mathbb{V}(\varepsilon_{it}) = \sigma^2$.

Note: if α_i is null, we have

$$\mathbb{E}(R_{it}) = R_{ft} + \beta_i(\mathbb{E}(R_{mt} - R_{ft}))$$

Estimation - step 1

- The usual estimator of the beta of the equity is the OLS estimator of the slope coefficient in the excess-return market model, that is, the β in the regression equation

$$Z_{it} = \alpha_{it} + \beta_i Z_{mt} + \varepsilon_{it}$$

where i denotes the asset and t denotes the time period, $t = 1, \ldots, T$. Z_{it} and Z_{mt} are the *realized* excess returns in the time period t for asset i and the market portfolio.

- If CAPM holds, α_{it} is expected to be zero.

Estimation - step 1

- A stock's beta can be calculated in two ways:
  1. Direct calculation
  2. Through a time-series regression, separately for each stock

$$Z_{it} = \alpha_i + \beta_i Z_{mt} + \varepsilon_{it} \quad i = 1, \ldots, N \quad t = 1, \ldots, T$$

  with N the total number of stocks in the sample and T the number of time series observations.
- The intercept is the 'Jensen's alpha': measures how much the stock unperformed or outperformed what would have been expected given its level of market risk.

Estimation - step 2

- Suppose $N = 100$ and $T = 60$ months (5-years of monthly data, $60 = 12 \times 5$).
  1. Run 100 time-series regressions with the sixty monthly data points.
  2. Run a single cross-sectional regression of the average over time of the stock returns on a constant and the betas from the first step:

$$\bar{R}_i = \lambda_0 + \lambda_1 \beta_i + v_i, \quad i = 1, \dots, N$$

where $\bar{R}_i$ is the return for stock i averaged over T (sixty months) (the second stage involves actual returns, not excess ones).

Hypothesis testing: if the CAMP is valid:

1. $\lambda_0 = R_f$
2. $\lambda_1 = [\mathbb{E}(R_m) - R_f] > 0$

CAPM (cont'd)

- Furthermore, CAPM implies that:
  1. There is a linear relationship between a stock's return and its beta.
  2. No other variables should help to explain the cross-sectional variation in returns.
- (Fama and French, 1993) show that empirically CAPM is not a complete model.
- Development of French-Fama factor and multi-factor CAPM models.

CAPM (cont'd)

- The aggregate alpha over a portfolio is a measure of the performance of a fund's manager performance!
- The beta of a fund, is an indicator of how much it is expected to move relative to movements in the overall market. A beta greater than 1 suggests that the stock is more volatile than the broader market, and a beta less than 1 indicates a stock with lower volatility.

Want to hire a portfolio manager?

- Alpha: high.
- Beta: it depends. A high beta may be preferred by an investor in growth stocks but shunned by investors who seek steady returns and lower risk

T-bill (cont'd)

T-Bill:

- IRX is indeed the official **discount rate** d of the US Treasury.
- IRX is quoted as an **annualized** interest rate.
- To compute the daily excess logreturn, we have to convert this annual rate **into a daily return**.
- The official way of calculating the discount rate d is

$$d = \frac{100 - P}{100}\frac{360}{n}$$

  where P is the price per \$ 100 of par (face) value and n is the number of days until expiration.
- For 13 weeks, $n \approx 91$. The 5 missing days are bankers' holidays.
- In order to get the d of this formula we would divide the IRX by 100 because it is stated as a percent.

T-bill (cont'd) I

- We want $1 + r_f = \left(\frac{100}{P}\right)^{\frac{1}{n}}$, which is the daily risk-free return ratio (gross return).
- We solve the above equation for P,

$$d = \frac{100 - P}{100}\frac{360}{n} \qquad \rightarrow \qquad P = 100\left[1 - \frac{dn}{360}\right]$$

and substitute P in the equation for the gross return $1 + r_f$:

$$1 + R = \left(\frac{100}{P}\right)^{\frac{1}{n}} = \left(\frac{1}{1 - \frac{dn}{360}}\right)^{\frac{1}{n}}$$

T-bill (cont'd) II

- Since dn is small and it is furthermore divided by 360:

$$\frac{1}{1-\frac{dn}{360}} \approx 1 + \frac{dn}{360}$$

in fact,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \approx 1 + x$$

- $1 + \frac{dn}{360}$ is also the leading term in d in $\left(1+\frac{d}{360}\right)^n$:

$$1 + \frac{dn}{360} \approx \left(1 + \frac{d}{360}\right)^n$$

T-bill (cont'd) III

in fact,

$$
\begin{aligned}
(1+x)^n &= \sum_{k=0}^{\infty} x^k \binom{n}{k} \\
&= 1 + nx + \frac{1}{2!}(n-1)nx^2 + \frac{1}{3!}(n-2)(n-1)nx^3 + \dots \\
&\approx 1 + nx
\end{aligned}
$$

Therefore:

$$
1 + r_f = \left(\frac{100}{P}\right)^{\frac{1}{n}} \approx \left(1 + \frac{dn}{360}\right)^{\frac{1}{n}} \approx \left(\left(1 + \frac{d}{360}\right)^{n}\right)^{\frac{1}{n}} \approx 1 + \frac{d}{360}
$$

for which the corresponding net return is $r_f \approx d/360$.

T-bill (cont'd) IV

- **In practice:**
  - get the data for the ^IRX symbol from Yahoo Finance
  - use the *AdjClosing* column as IRX_t, and compute

$$r_{f,t} = \frac{1}{360}\text{IRX}_t$$

  - keep in mind that $r_{f,t}$ is expressed in percent.
- As a reference, see: `www.treasury.gov`, and `https://www.treasurydirect.gov/instit/annceresult/press/preanre/2004/ofcalc6decbill.pdf`

Example

Example

See the CAPM example in Matlab!

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