A Objective : compute $\frac{\partial X\beta}{\partial \beta}$ and show is equal to X'

PROOF THAT

$$\frac{\partial X\beta}{\partial \beta} = X^T$$

Used in OLS Derivation when Computing $\frac{\partial L}{\partial \beta}$

B What is $X\beta$?

$$X\beta = \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} \beta \end{bmatrix} = \begin{bmatrix} X_{11}\beta_1 + X_{12}\beta_2 + \text{---} + X_{1K}\beta_K \\ X_{21}\beta_1 + X_{22}\beta_2 + \text{---} + X_{2K}\beta_K \\ \\ X_{M1}\beta_1 + X_{M2}\beta_2 + \text{---} X_{MK}\beta_K \end{bmatrix} *$$

$M \times 1$

C So $\frac{\partial X\beta}{\partial \beta}$ is the deriv of a column vector ($X\beta$) by a column vector β

D $X\beta$ can be seen as a function $f : \beta \to X\beta$ from a vector β to a vector $X\beta$

E GENERAL RESULT :

E1: $\frac{\partial f(\underline{x})}{\partial \underline{x}^T}$ = meaning this matrix

This notation is defined as

$$\begin{array}{c|cccc} & & & j & {**} \\ \hline & \frac{\partial f_1(\underline{x})}{\partial x_1} & \frac{\partial f_1(\underline{x})}{\partial x_2} & \text{---} & \frac{\partial f_1(x)}{\partial x_K} \\ & \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(\underline{x})}{\partial x_2} & \text{---} & \frac{\partial f_2(x)}{\partial x_K} \\ i & & & \frac{\partial f_i(\underline{x})}{\partial x_j} & \\ & \frac{\partial f_K(\underline{x})}{\partial x_1} & \frac{\partial f_K(\underline{x})}{\partial x_2} & \text{---} & \frac{\partial f_K(x)}{\partial x_K} \end{array}$$

Meaning :

$$\frac{\partial f_i(\underline{x})}{\partial x_j} = \frac{\partial\ \text{i-th element of } f(\underline{x})}{\partial\ \text{j-th element of } x}$$

- Columns are the elements of $f(x)$
- Rows are the elements of x^T

E2: $\frac{\partial f(\underline{x})}{\partial \underline{x}}$ is *** defined as $\left[\frac{\partial f(\underline{x})}{\partial \underline{x}^T}\right]^T$

F take $*$ and compute $\frac{\partial X\beta}{\partial \beta^T}$ $\overset{**}{=}$ $\begin{bmatrix} x_{11} & x_{12} & \text{---} & x_{1K} \\ x_{21} & x_{22} & \text{---} & x_{2K} \\ \\ x_{M1} & x_{M2} & \text{---} & x_{MK} \end{bmatrix} = X$ $\overset{***}{\Rightarrow}$ $\frac{\partial X\beta}{\partial \beta} = \left[\frac{\partial X\beta}{\partial \beta^T}\right]^T = X^T$

replacing in E1 $f(\underline{x})$ with $X\beta$ and $\underline{x}$ with β^T