# Homework 1 : Md Mahin, 1900421

1. **(Question:10 points) Let B be a $4 \times 4$ matrix to which we apply the following operations**

   **(a) double column 1**
   **(b) halve row 3**
   **(c) add row 3 to row 1**
   **(d) interchange columns 1 and 4**
   **(e) subtract row 2 from each of the other rows**
   **(f) replace column 4 by column 3**
   **(g) delete column 1**

   - **Write the result as a product of eight matrices.**
   - **Write it again as a product ABC of three matrices.**

   **Answer:**

   Let a 4*4 matrix $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix}$

   and It's identity matrix $I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ as every operation on $B$ can be represented as multiplication with B, then

   (a) **double column 1** $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (b) **halve row 3** $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix}$

(c) **add row 3 to row 1** $= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix}$

(d) **interchange columns 1 and 4** $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(e) **subtract row 2 from each of the other rows** $= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix}$

(f) **replace column 4 by column 3**$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(g) **delete column 1** $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- **(Question)Write the result as a product of eight matrices.**
  **Answer:** To write the result as the product of eight matrices, row operations will be in the left of our matrix B and column operations will be in right.

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 9 & 6 \\ 1 & 5 & 8 & 7 \end{pmatrix} \cdot$$
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **(Question)Write it again as a product ABC of three matrices.**
$$\begin{pmatrix}1 & -1 & 1/2 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 1/2 & 0\\ 0 & -1 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 3 & 2 & 9 & 6\\ 1 & 5 & 8 & 7\end{pmatrix} \cdot \begin{pmatrix}0 & 2 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}-3 & -5 & 1/2\\ 6 & 10 & 7\\ -5 & -7 & 5/2\\ -1 & -8 & 1\end{pmatrix}$$

2. **(4 points) (Question) What is the rank of the matrix $M = \begin{pmatrix}1 & 2 & 4 & 4\\ 3 & 4 & 8 & 0\end{pmatrix}$ ?**
   **Answer:**
   To calculate the rank, first we need to calculate the reduced row echelon form.
   First Step: subtracting -3 times row 1 from row 2:
   $\begin{pmatrix}1 & 2 & 4 & 4\\ 0 & -2 & -4 & -12\end{pmatrix}$
   Second Step: adding row 2 with row 1:
   $\begin{pmatrix}1 & 0 & 0 & -8\\ 0 & -2 & -4 & -12\end{pmatrix}$
   Third Step: multiplying row 2 with -1/2:
   $\begin{pmatrix}1 & 0 & 0 & -8\\ 0 & 1 & 2 & 6\end{pmatrix}$
   As there are two non zero diagonal of the matrix, the rank of the matrix is 2.

3. **(Question:)(6 points) The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,**

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

   (a) **(Question:)Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$**
   **Answer:**
   From vector definition we know that,
   $\|x\| = \sqrt{x.x}$ So, we can write,
   $\|x_1 + x_2\|^2 = (x_1 + x_2).(x_1 + x_2)$
   $\|x_1 + x_2\|^2 = x_1.x_1 + x_1.x_2 + x_2.x_1 + x_2.x_2$
   $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + 2(x_1.x_2)$

$\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + 2.x_1.x_2 \cos(\theta)$
Now, as the vectors are orthogonal, $\theta = 90°$ and so, $\cos(\theta) = 0$
So, we can write,
$\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$

(b) **(Question)Show that this computation also establishes the general case, by induction.**
**Answer:** Previously we have proved, for $n = 2$, $\{x_i\}$,

$$\left\|\sum_{i=1}^{2} x_i\right\|^2 = \sum_{i=1}^{2} \|x_i\|^2$$

So, for $n = 3$,

$$\left\|\sum_{i=1}^{3} x_i\right\|^2 = \sum_{i=1}^{3} \|x_i\|^2$$

or,
$\|x_1 + x_2 + x_3\|^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2$
we can write,
$\|x_1 + x_2 + x_3\|^2 = (x_1 + x_2 + x_3).(x_1 + x_2 + x_3)$
$\|x_1 + x_2 + x_3\|^2 = x_1.x_1 + x_1.x_2 + x_1.x_3 + x_2.x_1 + x_2.x_2 + x_2.x_3 + x_3.x_1 + +x_3.x_2 + x_3.x_3$
$\|x_1 + x_2 + x_3\|^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_2\|^2 + 2(x_1.x_2) + 2(x_2.x_3) + 2(x_3.x_1)$
$\|x_1 + x_2 + x_3\|^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2$
So, it is proven that if the equation is true for 2 and 3. Now, let's assume the equation is true for $n = n - 1$. It is,

$$\left\|\sum_{i=1}^{n-1} x_i\right\|^2 = \sum_{i=1}^{n-1} \|x_i\|^2$$

for $x_1....x_{n-1}$ orthogonal vectors. Now, for another orthogonal vector $x_n$ to the vectors $x_1....x_{n-1}$, let assume, $x_1 = \left\|\sum_{i=1}^{n-1} x_i\right\|^2$ and $x_2 = x_n$, then,
$\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$
or,
$\|x_1 + x_2\|^2 = \left\|\sum_{i=1}^{n-1} x_i\right\|^2 + \|x_n\|^2$

So, we can write,

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

[Proved]

4. **(Question)(6 points) Let $A \in \mathbb{R}^{m\times m}$ be symmetric. An eigenvector of A is a nonzero vector $x \in \mathbb{R}^m$ such that $Ax = \lambda x$ for some $\lambda \in R$, the corresponding eigenvalue.**

   (a) **(Question:)Prove that all eigenvalues are real.**
   **Answer:**
   Here given that, $A \in \mathbb{R}^{m\times m}$ is a symmetric matrix and $Ax = \lambda x$ for some $\lambda \in R$. For symmetric matrix we know, $A = A^T = A^*$
   Given,
   $Ax = \lambda x$
   $Ax^* = \lambda x^*$ [Complex conjugating both sides]
   $x^*A^* = \lambda^* x^*$
   $x^*A = \lambda^* x^*$[As $A = A^*$]
   $x^*Ax = \lambda^* x^* x$[Multiplying x in both sides]
   $x^*\lambda x = \lambda^* x^* x$[As,$Ax = \lambda x$]
   $x^*\lambda x - \lambda^* x^* x = 0$
   $(\lambda - \lambda^*)x^* x = 0$
   Now, as, $x^*x$ can not become zero, as x is a non zero vector,
   $(\lambda - \lambda^*) = 0$
   So, we can write,
   $\lambda = \lambda^*$
   So, hence all eigenvalues are real. [Proved]

   (b) **(Question:)Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.**
   **Answer:**
   Let's, for two distinct eigenvectors x and y

$$Ax = \lambda_1 x \tag{1}$$

   and

$$Ay = \lambda_2 y \tag{2}$$

where, $\lambda_1$ and $\lambda_2$ are distinct eigenvalues for two vectors.
Multiplying $y^*$ on the left, first equation becomes,
$y^*Ax = y^*\lambda_1 x$
$y^*Ax = \lambda_1 y^*x$
Taking, complex conjugated of the equation,
$y^*Ax^* = \lambda_1 y^*x^*$
$x^*A^*y = \lambda_1 x^*y$
So,

$$x^*Ay = \lambda_1 x^*y \tag{3}$$

Multiplying $x^*$ on the left, second equation becomes,
$x^*Ay = x^*\lambda_2 y$
So,

$$x^*Ay = \lambda_2 x^*y \tag{4}$$

From equation three and four,
$\lambda_1 x^*y = \lambda_2 x^*y$
$\lambda_1 x^*y - \lambda_2 x^*y = 0$
$(\lambda_1 - \lambda_2)x^*y = 0$
Since, $(\lambda_1 != \lambda_2)$, so, $x^*y = 0$, and thus they are orthogonal. [Proved]

5. **(Question)6 points) If $u$ and $v$ are $m$-vectors, the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the identity.* Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar $\alpha$, and give an expression for $\alpha$. For what $u$ and $v$ is A singular? If it is singular, what is null(A)?**
   **Answer:**
   Given that $A$ is non singular and $A = I + uv^*$. If, $A$ is non singular then $A^{-1}$ exists and the determinate of $A != 0$.
   Lets consider, $A^{-1} = I + \alpha uv^*$
   Now, we know, $AA^{-1} = I$[Where, I is the identity matrix]
   So, $(I + uv^*)(I + \alpha uv^*) = I$
   or, $I + uv^* + \alpha uv^* + uv^*\alpha uv^* = I$
   or, $uv^* + \alpha uv^* + \alpha u(v^*u)v^* = 0$
   or, $uv^* + \alpha uv^* + \alpha u(v^*u)v^* = 0$
   or, $uv^*(1 + \alpha + \alpha v^*u) = 0$

That implies that, $(1 + \alpha + \alpha v^* u) = 0$
or, $\alpha = -\frac{1}{1+v^*u}$ **[Answer 1]**
Here, we can see that, if $A^{-1}$ exists, there exists a constant $\alpha$. However, if $v^*u = -1$, $\alpha$ becomes undefined. So, at that point $A$ becomes singular. So, $A$ is singular if, $v^*u = -1$.**[Answer 2]**
Now, if $A$ is singular, let for the vector $u$,
$Au = (I + uv^*)u$
or, $Au = Iu + uv^*u$
or, $Au = u + u(-1)$
or, $Au = u - u$
or, $Au = 0$
Thus, $u$ vector is the null space of $A$ if it is singular[**Answer 3**].

6. **(Answer 6:)(5 points) Let $\|.\|_w$ denote any norm on $\mathbb{R}^m$ and also the induced matrix norm on $\mathbb{R}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of A, that is largest absolute value of $|\lambda|$ of an eigenvalue $\lambda$ of A.**
   **Answer:**
   We know, $Ax = \lambda x$
   or, $\|Ax\| = \|\lambda x\|$
   We know, from the properties of metrics norm,
   $\|\lambda A\| = |\lambda| \, \|A\|$
   and
   $\|Ax\| <= \|A\| \, \|x\|$
   from here we can write,
   $\|\lambda x\| <= \|A\| \, \|x\|$
   or, $|\lambda| \, \|x\| <= \|A\| \, \|x\|$
   as, $x$ is eigenvector, $x! = 0$,
   So, $|\lambda| <= \|A\|$
   Given, $\rho(A) = \lambda$
   or, $\rho(A) \leq \|A\|$[Proved]

7. **(Question)(15 points) Determine SVDs of the following matrices(by hand calculations)**
   (a)**SVD of :** $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$
   **Answer:**
   Let, $A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$
   For SVD, we know, $A = U\Sigma V^T$

Now, $A^TA = V\Sigma^T\Sigma V$

$= \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$= \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$

Now,

$det(A^TA - \lambda I) = \begin{pmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$

We can write,

$(9-\lambda)(4-\lambda) = 0$

or,$\lambda = 9$ or $\lambda = 4$

For, $\lambda = 4$

$A^TA - 4I = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} = 0$

So, $5x_1 + 0x_2 = 0$

So, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For, $\lambda = 9$

$A^TA - 4I = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix} = 0$

So, $0x_1 - 5x_2 = 0$

So, $v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

So, our $V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = V^*$

Now,

$\Sigma = \begin{pmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

Again, $AV = U\Sigma$

$AV = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

So, $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So,Finally,

$A = U\Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b)**SVD of :** $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

**Answer:**

Let, $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

For SVD, we know, $A = U\Sigma V^T$

Now, $A^T A = V\Sigma^T \Sigma V$

$= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$

Now,

$det(A^T A - \lambda I) = \begin{pmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$

We can write,

$(-\lambda)(4-\lambda) = 0$

or,$\lambda = 0$ or $\lambda = 4$

For, $\lambda = 4$

$A^T A - 4I = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} = 0$

So, $-4x_1 + 0x_2 = 0$

So, $v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

For, $\lambda = 0$

$A^T A - 0I = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = 0$

So, $0x_1 + 4x_2 = 0$

So, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $v = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

and, $v^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Now,

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Again, $AV = U\Sigma$

$$AV = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AV = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So,Finally,

$$A = U\Sigma V^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c)**SVD of:**$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

**Answer:**

Let, $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

For SVD, we know, $A = U\Sigma V^T$

Now, $A^T A = V\Sigma^T \Sigma V$

$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Now,

$det(A^T A - \lambda I) = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

We can write,

$(1-\lambda)(1-\lambda) - 1 = 0$

$1 - 2\lambda + \lambda^2 - 1 = 0$

$\lambda(\lambda - 2) = 0$

or,$\lambda = 0$ or $\lambda = 2$

For, $\lambda = 0$

$A^T A - 0I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$

Using ow echelon form: $= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

So,$x_1 + x_2 = 0$
or, $x_1 = -x_2$

So, $v_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
For, $\lambda = 2$
$A^T A - 2I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = 0$

Using ow echelon form: $= \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

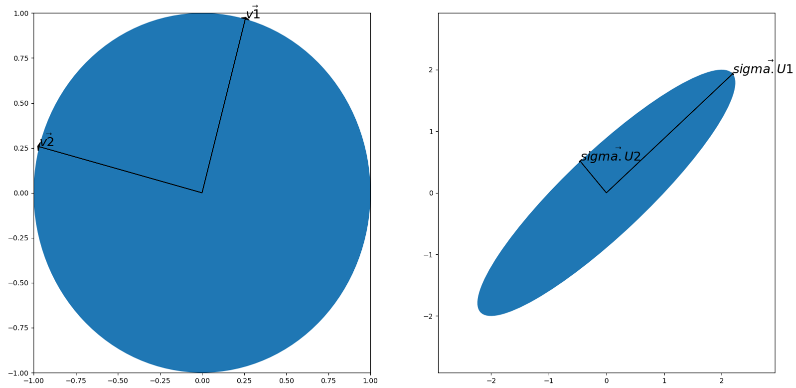
So,$-x_1 + x_2 = 0$
or, $x_1 = x_2$
So, $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
So, $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Now,
$\Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$

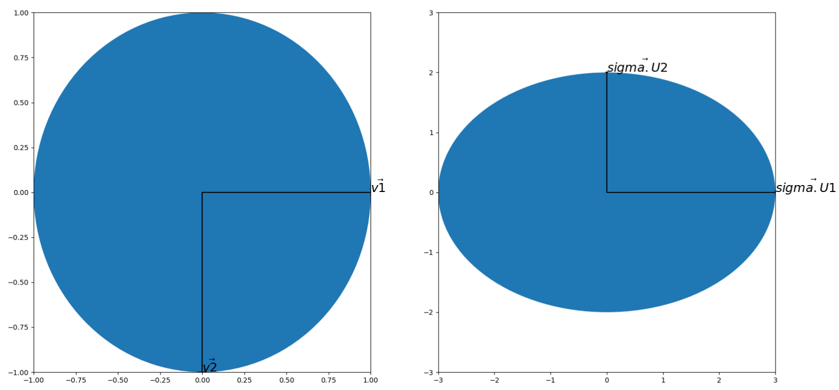
Again, $AV = U\Sigma$
$AV = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
$= \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$ So, $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
So,Finally,
$A = U\Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

8. **(Question:)(12 points) Write a Matlab program which, given a real $2 \times 2$ matrix A, plots the right singular vectors $v_1$ *and* $v_2$ in the unit circle and also the left singular vectors $u_1$ *and* $u_2$ in the appropriate ellipse, as in Figure 4.1. Apply your program to the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 2 \end{smallmatrix}\right]$ and also to the $2 \times 2$ matrices of Exercise 7.**

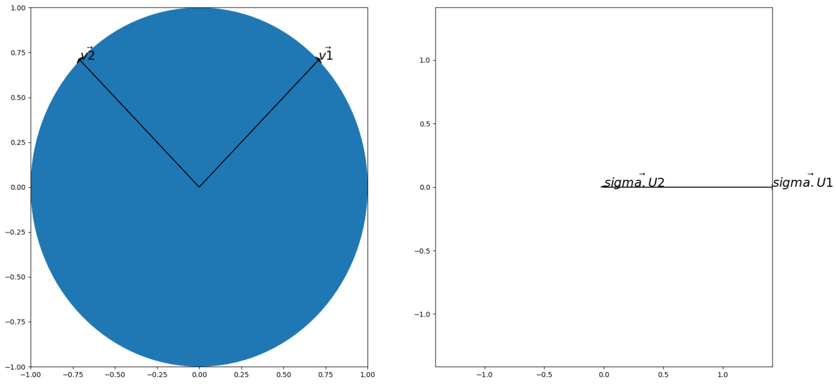
**Answer:**
For Matrix: $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 2 \end{smallmatrix}\right]$




**Answer:**
For Matrix: $\left[\begin{smallmatrix} 3 & 0 \\ 0 & -2 \end{smallmatrix}\right]$



**Answer:**
For Matrix: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$




9. **(Answer)(6 points) Consider the matrix**

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

**Using the SVD, work out(on paper) the exact values of $\sigma_{min}(A)$ and $\sigma_{max}(A)$ for this matrix.**
**Answer:**
Given,

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

So,

$$A^T A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

So,

$$A^T A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$$

Now, $det(A^T A - \lambda I) = \begin{pmatrix} 1-\lambda & 2 \\ 4 & 8-\lambda \end{pmatrix}$
or, $(1-\lambda)(8-\lambda) - 4 = 0$

or, $8 - 8\lambda - \lambda + \lambda^2 - 4 = 0$
or, $\lambda^2 - 9\lambda + 4 = 0$
or, $\lambda = \frac{9+\sqrt{65}}{2} or \frac{9-\sqrt{65}}{2}$
So, Largest value $\sigma_{max}(A)$ is $\frac{9+\sqrt{65}}{2}$ and the smallest value $\sigma_{min}(A)$ is $\frac{9-\sqrt{65}}{2}$.

10. **(Question:)(8 points) Suppose $A \in \mathbb{R}^{m\times m}$ has an SVD $A = U\Sigma V^*$. find an eigenvalue decomposition of the $2m \times 2m$ symmetric matrix**

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

**Please note: If the coloumns of a matrix $X \in \mathbb{R}^{m\times m}$ contain linearly independent eigen vectors of $A \in \mathbb{R}^{m\times m}$, the eigenvalue decomposition of A is**

$$A = X\Sigma X^{-1}$$

**where $\Sigma$ is an $m \times m$ diagnol matrix whose entries are the eigenvalue of A.**

**Answer:**
Given, $A = U\Sigma V^*$.
So, we can write, $AV = U\Sigma$
and $A^*U = V\Sigma$
By solving the linear equation, we can write,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \cdot \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U\Sigma \\ V\Sigma \end{pmatrix}$$

Same way, it can be written that,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \cdot \begin{pmatrix} U \\ -V \end{pmatrix} = \begin{pmatrix} -U\Sigma \\ V\Sigma \end{pmatrix}$$

From above two eqaution we can write,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} = \begin{pmatrix} U & -U \\ V & V \end{pmatrix}$$

or, $\begin{pmatrix} U\Sigma & U\Sigma \\ V\Sigma & -V\Sigma \end{pmatrix} = \begin{pmatrix} U & -U \\ V & V \end{pmatrix} \cdot \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}$

So, the EVD of A is,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} = \begin{pmatrix} U & -U \\ V & V \end{pmatrix} \cdot \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \begin{pmatrix} U & -U \\ V & V \end{pmatrix}^{-1}$$

Here, $X = \begin{pmatrix} U & -U \\ V & V \end{pmatrix}$ and $XX^T = 2I$

So, $\frac{1}{\sqrt{2}}X$ is unitary.

So, Eigen decomposition is,

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}X \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \frac{1}{\sqrt{2}}X^T$$

11. **(Question:)(8 points) Consider the matrices**

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

**Answer the following questions by hand calculation.**
**(Question:)What is the orthogonal projector P onto range(A), and what is the image under P of the vector** $(1, 2, 3)^*$**?**
**Answer:**
Given,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The orthogonal projector P onto range(A) is,

$P = A(A^T A)^{-1} A$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

So, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

**(Question:)What is the orthogonal projector P onto range(B), and what is the image under P of the vector $(1, 2, 3)^*$?**

Given,

$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$

The orthogonal projector P onto range(A) is,

$P = B(B^T B)^{-1} B$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} . \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{6} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

So, $\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}\frac{5}{6} & \frac{1}{3} & \frac{1}{6}\\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6}\end{pmatrix} . \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\0\\2\end{pmatrix}$

12. **(Question)(6 points) Let A be a matrix with the property that coloumns 1,3,5,7,... are orthogonal to columns 2,4,6,8,.... In a reduced QR fatorization $A = \hat{Q}\hat{R}$, what special structure does $\hat{R}$ possess? You may assume that A has full rank.**
**Answer:**
For a given matrix with vectors $a_1, a_2...$ we can construct orthogonal vectors $q_1, q_2...$ using the Gram-Schmidt Orthogonalization. Here we know,
$v_j = a_j - (q_1^* a_j)q_1 - (q_2^* a_j)q_2 - ..... - (q_{j-1}^* a_j)q_{j-1}$
If we normalize, our new orthogonal vectors takes the following form,
$q_1 = \frac{a_1}{r_{11}}$
$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}}$
$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}}$
$q_4 = \frac{a_4 - r_{14}q_1 - r_{24}q_2 - r_{34}q_3}{r_{44}}$
...
$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in}q_n}{r_{nn}}$

Without normalization, we can write,
$q_1 = a_1$
so, $q_2 = a_2 - r_{12}q_1 = a_2 - \frac{span(a_2,q_1)}{\|q_1\|}q_1 = a_2 - \frac{a_2.a_1}{\|a_1\|}a_1$
Here, as $a_2, a_1$ orthogonal, their dot product is zero. So, $r_{12} = 0$ and,
$q_2 = a_2$
similarly, $q_3 = a_3 - r_{13}q_1 - r_{23}q_2 = a_3 - \frac{span(a_3,q_1)}{\|q_1\|}q_1 - \frac{span(a_3,q_2)}{\|q_2\|}q_2$
$= a_3 - \frac{span(a_3,a_1)}{\|a_1\|}a_1 - \frac{span(a_3,a_2)}{\|a_2\|}a_2$
Here, as $a_3, a_2$ orthogonal, their dot product is zero. So, $r_{23} = 0$ and,
$q_3 = a_3 - r_{13}q_1$
or, $r_{13} = \frac{a_3.a_1}{\|a_1\|}$

Similarly, for $q_4 = a_4 - r_{24}q_2$ and the terms $r_{14} = r_{34} = 0$ and $r_{24} = \frac{a_4.a_2)}{\|a_2\|}$ and $r_{44} = \left\| a_4 - \frac{a_4.a_2)}{\|a_2\|} \right\|$
Here we can easily see that, $q1 \in (a1), q3 \in span(a1, a3), ..., q2k1 \in span(a1, ..., a2k1)$, and similarly $q2 \in (a2), q4 \in span(a2, a4), ..., q2k \in span(a2, ..., a2k)$. So, here we can write, on the other hand, if we consider the normalization,
for $q_1$, $r_{11} = \|a_1\|$, $q_2$, $r_{22} = \|a_2\|$, $q_2$, $r_{33} = \left\| a_3 - \frac{a_3.a_1}{\|a_1\|} \right\|...$ and $r_{nn} = \left\| a_n - \sum_{i=1}^{n-1} \frac{a_n.a_i}{\|a_i\|} \right\|$, where $n$ and $i$ have to be odd or even at the

same time.
As a result our matrix $R$ takes an upper triangular check-board metrics form like this,

$$R = \begin{pmatrix} \|a_1\| & 0 & \frac{a_3.a_1}{\|a_1\|} & 0 & ... \\ 0 & \|a_2\| & 0 & \frac{a_4.a_2}{\|a_2\|} & ... \\ 0 & 0 & \left\| a_3 - \frac{a_3.a_1}{\|a_1\|} \right\| & 0 & ... \\ 0 & 0 & 0 & \left\| a_4 - \frac{a_4.a_2)}{\|a_2\|} \right\| & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

where,
if $i = j : r_{ij}! = 0$
if $i < j$ and both $i$ or $j$ are odd or even: $r_{ij}! = 0$
if $i > j : r_{ij} = 0$ and
else, $r_{ij} = 0$

13. **(Answer:)(8 points) Write a MATLAB function $[Q, R] = mgs(A)$ that computes a reduced QR factorization $A = \hat{Q}\hat{R}$ of an $m \times n$ matrix A with $m \geq n$ using modified Gram-Schmidt orthogonalization. The output variables are a matrix $Q \in \mathbb{R}^{m \times n}$ with orthonormal coloumns and a triangular matrix $R \in \mathbb{R}^{n \times n}$. Use your MGS QR factorization to solve the linear system $Ax = b$ where**

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \end{pmatrix}, b = \begin{pmatrix} 13 \\ 32 \\ 46 \end{pmatrix}$$

```
Matrix A =
[[1 3 2]
 [4 5 6]
 [9 8 7]]

Vector B =
[[13]
 [32]
 [46]]

Q Matrix=
 [[ 0.10101525  0.83541209 -0.54026156]
 [ 0.40406102  0.46178558  0.78961305]
 [ 0.90913729 -0.2980616  -0.29091007]]

R Matrix=
 [[9.89949494 9.59644917 8.99035765]
 [0.         2.43067136 2.35510645]
 [0.         0.         1.62078469]]

Dot of QR=
 [[1. 3. 2.]
 [4. 5. 6.]
 [9. 8. 7.]]

Solution of X=
 [[1.]
 [2.]
 [3.]]

Process finished with exit code 0
```