IN3140 - Assignment 1

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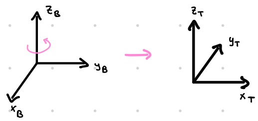
1. Transformations



$\underline{T_T^B}$:

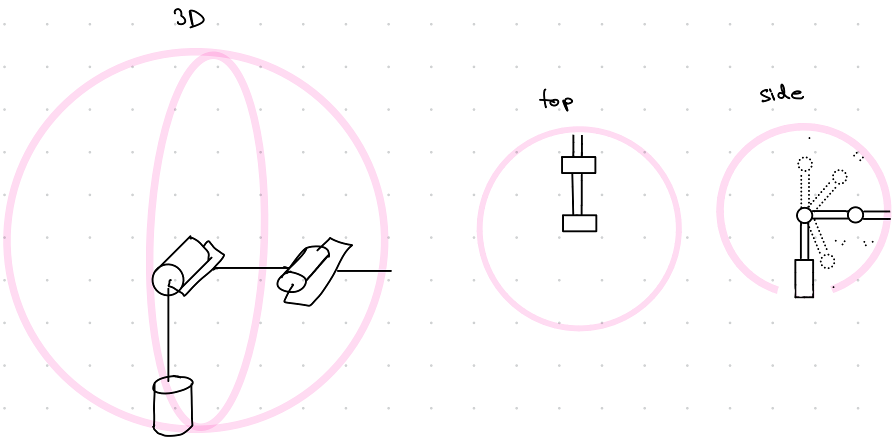
$$R_T^B = Rot_{z,90°} = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_T^B = O_T - O_B = \begin{bmatrix} 1000-250 \\ 400-650 \\ 900-1000 \end{bmatrix} = \begin{bmatrix} 750 \\ -250 \\ -100 \end{bmatrix}$$

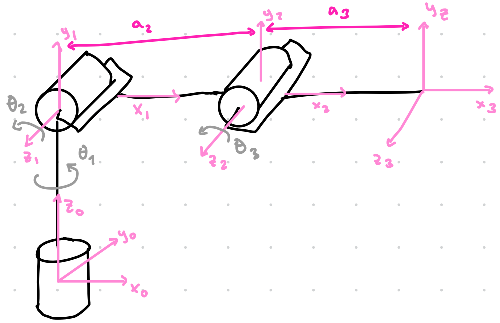
$$\Rightarrow T_T^B = \begin{bmatrix} R_T^B & d_T^B \\ 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & -1 & 0 & 750 \\ 1 & 0 & 0 & -250 \\ 0 & 0 & 1 & -100 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}$$

2. Forward kinematics I

a)



b)



DH-table

links		a_i	α_i	d_i	θ_i
	1	0	90°	0	θ_1
	2	a_2	0	0	θ_2
	3	a_3	0	0	θ_3

the origins are placed at the joints. the z-axes are aligned with the rotation axes of their joints, as each joint is revolute and rotates about its corresponding z-axis. The x_i-axes are chosen to be perpindicular to the z_{i-1}-axes, and intersect.

c)

$$A_i = \begin{bmatrix} C_{\theta i} & -S_{\theta i} C_{\alpha i} & S_{\theta i} S_{\alpha i} & a_i C_{\theta i} \\ S_{\theta i} & C_{\theta i} C_{\alpha i} & -C_{\theta i} S_{\alpha i} & a_i S_{\theta i} \\ 0 & S_{\alpha i} & C_{\alpha i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2 C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2 S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_3 C\theta_3 \\ S\theta_3 & C\theta_3 & 0 & a_3 S\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A_1 A_2 A_3$:

$$A_1 A_2 = \begin{bmatrix} C_1 C_2 & -C_1 S_2 & S_1 & a_2 C_1 C_2 \\ C_2 S_1 & -S_1 S_2 & -C_1 & a_2 C_2 S_1 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(A_1 A_2) A_3 = \begin{bmatrix} C_1 C_2 C_3 - C_1 S_2 S_3 & -C_1 C_2 S_3 - C_1 C_3 S_2 & S_1 & a_2 C_1 C_2 - a_3 C_1 S_2 S_3 + a_3 C_1 C_2 C_3 \\ C_2 C_3 S_1 - S_1 S_2 S_3 & -C_2 S_1 S_3 - C_3 S_1 S_2 & -C_1 & a_2 C_2 S_1 - a_2 S_1 S_2 S_3 + a_3 C_2 C_3 S_1 \\ C_2 S_3 + C_3 S_2 & C_2 C_3 - S_2 S_3 & 0 & a_2 S_2 + a_3 C_2 S_3 + a_3 C_3 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow T_t^B = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & a_2 C_1 C_2 + a_3 C_1 C_{23} \\ C_{23} S_1 & -S_{23} S_1 & -C_1 & a_2 C_2 S_1 + a_3 S_1 C_{23} \\ S_{23} & C_{23} & 0 & a_2 S_2 + a_3 S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Forward kinematics II

Find p^T coordinates in φ given T

From task 1: $T_T^B = \begin{bmatrix} R_T^B & d_T^B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 750 \\ 1 & 0 & 0 & 250 \\ 0 & 0 & 1 & -100 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

From task 2:

$$T_t^B = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & \underbrace{a_2 c_1 c_2 + a_3 c_1 c_{23}} \\ c_{23} s_1 & -s_{23} s_1 & -c_1 & a_2 c_2 s_1 + a_3 s_1 c_{23} \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

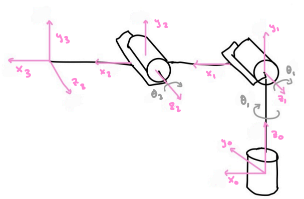
p^B

apply the homogeneous transform to transform p^B coordinates to task frame:

$$p^T = T_T^B \cdot p^B$$

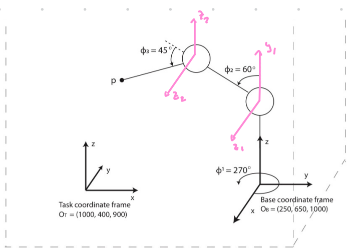
$$= \begin{bmatrix} 0 & -1 & 0 & 750 \\ 1 & 0 & 0 & 250 \\ 0 & 0 & 1 & -100 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 c_1 c_2 + a_3 c_1 c_{23} \\ a_2 c_2 s_1 + a_3 s_1 c_{23} \\ a_2 s_2 + a_3 s_{23} \\ 1 \end{bmatrix} = \begin{bmatrix} 750 - a_3 c_{23} s_1 - a_2 c_2 s_1 \\ a_2 c_1 c_2 + a_3 c_1 c_{23} + 250 \\ a_2 s_2 + a_3 s_{23} - 100 \\ 1 \end{bmatrix}$$

to find the DH angles θ_1, θ_2 and θ_3, we compare them with how the DH-frames are defined:



→ Each z_i-axis is aligned with the rotation axis of its corresponding joint.

→ Each x_i-axis extends along the length of the links, making the rotation angles measured in a way that matches the given angles.



Since the angles $\varphi_1, \varphi_2, \varphi_3$ describe the rotation of each joint the same way the DH coordinate frames are defined, they match the DH joint parameters:

$\theta_1 = \phi_1 = 270°$ $\quad \theta_2 = \Phi_2 = 60°$ $\quad \theta_3 = \phi_3 = 45°$

Now, insert these angles and given link lengths to evaluate the coordinates of p^T:

$L_1 = a_1 = 100.9\,mm$, $L_2 = a_2 = 222.1\,mm$, $L_3 = a_3 = 136.2\,mm$

$$p^T = \begin{bmatrix} 750 - a_3 c_{23} s_1 - a_2 c_2 s_1 \\ a_2 c_1 c_2 + a_3 c_1 c_{23} + 250 \\ a_2 s_2 + a_3 s_{23} - 100 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 750 - 136.2 \cdot \cos(60^\circ + 45^\circ) \cdot \sin(270^\circ) - 222.1 \cdot \cos(60^\circ) \cdot \sin(270^\circ) \\ 222.1 \cdot \cos(270^\circ) \cdot \cos(60^\circ) + 136.2 \cdot \cos(270^\circ) \cdot \cos(60^\circ + 45^\circ) + 250 \\ 222.1 \cdot \sin(60^\circ) + 136.2 \cdot \sin(60^\circ + 45^\circ) - 100 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 825.8 \\ 250 \\ 223.9 \\ 1 \end{bmatrix} \to p^T$$

4. Inverse kinematics

a) The two most common ways of deriving inverse kinematics are the analytical method and the geometric method.

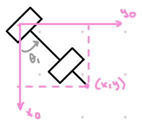
The analytic method involves finding the robots kinematic equations and then manipulating these algebraically to solve for the joint variables.

The idea of the geometric method is to project the manipulator onto planes and analyse and apply geometric principles to the different "2D"-views of the robot. From this, one can derive equations based on the geometric relationship and solve for the joint variables.

In task b. I will be using the geometrical approach.

b)

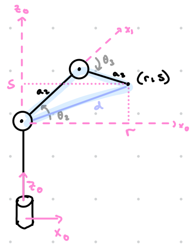
i)



$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \quad \theta_1 = \operatorname{atan2}(y, x)$$

(ii)

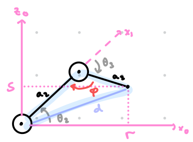


pythagoras to find d:

$$d^2 = s^2 + r^2$$

$$d = \sqrt{s^2 + r^2}$$

(iii)



* we can look at the plane formed by the second and third link when calculating θ_2, θ_3

we apply the law of cosines to express $\cos\theta_3$:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$d^2 = a_2^2 + a_3^2 - 2a_2a_3\cos\varphi$$

$$\Rightarrow \quad \cos\varphi = \frac{d^2 - a_2^2 - a_3^2}{-2a_2a_3} = \frac{a_2^2 + a_3^2 - (\overbrace{r^2+s^2}^{d^2})}{2a_2a_3}$$

$$\theta_3 = \pi - \varphi \quad \Rightarrow \quad \varphi = \pi - \theta_3$$

$$\Rightarrow \quad \cos\varphi = \cos(\pi - \theta_3) = -\cos\theta_3$$

$$\Rightarrow \quad \cos\theta_3 = -\left(\frac{a_2^2 + a_3^2 - (r^2+s^2)}{2a_2a_3}\right) = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

we define:

$$D = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

We can manipulate the trigonometric expression to get this:

$$\cos^2\theta_3 + \sin^2\theta_3 = 1$$

$$\sin^2\theta_3 = 1-\cos^2\theta_3$$

$$\sin\theta_3 = \pm\sqrt{1-\cos^2\theta_3} \Rightarrow \sin\theta_3 = \pm\sqrt{1-D^2}$$

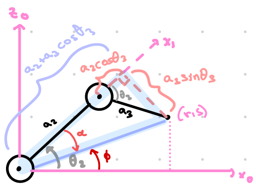
the $\pm$ determines wether θ_3 is elbow up or elbow down configuration.

Because of this, we can find θ_3 by finding $\tan\theta_3$:

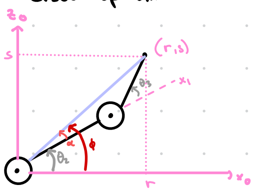
$$\tan\theta_3 = \frac{\sin\theta_3}{\cos\theta_3} = \frac{\pm\sqrt{1-D^2}}{D}$$

$$\Rightarrow \theta_3 = \text{atan2}\left(\pm\sqrt{1-D^2}, D\right)$$

(iv) elbow down ex.:



elbow up ex.:



We can find θ_2 using $\theta_2 = \phi - \alpha$, where ϕ gives the direction from the base to end-effector, while α adjusts for how the links are positioned.

$\theta_2 = \phi - \alpha$

$\phi = \text{atan2}(s, r)$

$\alpha = \text{atan2}(a_3\sin\theta_3, a_2 + a_3\cos\theta_3)$

$$\Rightarrow \theta_2 = \text{atan2}(s, r) - \text{atan2}(a_3\sin\theta_3, a_2 + a_3\cos\theta_3)$$

* the subtraction remains consistent for both elbow up and elbow down configurations because flipping the elbow direction also flips the sign of θ_3 and α.

Final solution:

$$\theta_1 = \text{atan2}(y, x)$$

$$\theta_2 = \text{atan2}(s, r) - \text{atan2}(a_3\sin\theta_3, a_2 + a_3\cos\theta_3)$$

$$\theta_3 = \text{atan2}\left(\pm\sqrt{1-D^2}, D\right), \quad D = \frac{r^2+s^2-a_2^2-a_3^2}{2a_2a_3}$$

c) The first joint θ_1 can rotate freely, since there are no restrictions. This means that the arm can be rotated in infinetely many ways while keeping the same end-effector position. For the θ_2 and θ_3 there are two possible ways to move: elbow up or elbow down.
So, for each orientation of θ_1, there are two possible configurations for θ_2 and θ_3, leading to an infinite number of valid joint angle combinations, and therefore an infinite amount of solutions.