MAT1120 - Obligatorisk oppgave 1

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Oppgave 1

a) kan skrive vektoren som en lineærkombinasjon av a og b:

$$\begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

de to vektorene er lineært uavhengige dersom

den eneste løsningen på $c_1 v_1 + c_2 v_2 = 0$ er $c_1 = 0,\ c_2 = 0$

$$c_1\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

I $c_1 + c_2 = 0 \rightarrow c_1 = -c_2$

II $2c_1 = 0$ II $-2c_2 = 0 \rightarrow c_2 = 0$

III $3c_1 - c_2 = 0$ III $3c_1 - 0 = 0 \rightarrow c_1 = 0$

IV $-c_2 = 0$

vektorene er lineært uavhengige. de utgjør derfor en basis for underrommet.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$$

b)

$$\begin{bmatrix} a+b+2c \\ 2a-b+c \\ 3a-b+2c \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

sjekker om vektorene er lineært uavhengige ved å radredusere matrisen bestående av dem:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 3 & -1 & 2 \end{bmatrix} \overset{\text{II}+(-2\text{I})}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 3 & -1 & 2 \end{bmatrix} \overset{\text{III}+(-3\text{I})}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & -4 & -4 \end{bmatrix} \underset{\text{III}\cdot(-\frac{1}{4})}{\overset{\text{II}\cdot(-\frac{1}{3})}{\sim}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & -4 & -4 \end{bmatrix} \underset{\text{III}\cdot(-\frac{1}{4})}{\overset{\text{II}\cdot(-\frac{1}{3})}{\sim}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overset{\text{III}+\text{II}}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\text{I}+(-\text{II})}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

kun vektor 1 og 2 er lineært uavhengige. en basis for underrommet kan være:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Oppgave 2

a) $\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}$

Sjekker hvike av vektorene som er lineært vavhengige:

$$\begin{bmatrix}3&2&4\\2&1&3\end{bmatrix} \overset{\frac{1}{3}\mathrm{I}}{\sim} \begin{bmatrix}1&\frac{2}{3}&\frac{4}{3}\\2&1&3\end{bmatrix} \overset{\mathrm{II}+(-2\mathrm{I})}{\sim} \begin{bmatrix}1&\frac{2}{3}&\frac{4}{3}\\0&-\frac{1}{3}&\frac{1}{3}\end{bmatrix} \overset{\mathrm{II}\cdot(-3)}{\sim} \begin{bmatrix}1&\frac{2}{3}&\frac{4}{3}\\0&1&-1\end{bmatrix} \overset{\mathrm{I}+(-\frac{2}{3}\mathrm{II})}{\sim} \begin{bmatrix}1&0&2\\0&1&-1\end{bmatrix}$$

vektor 1 og 2 utgjør en basis for rommet

$$\underline{\underline{\mathcal{B} = \left\{ \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}}}$$

b) $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$\begin{bmatrix}1&0&0\\0&1&1\\1&1&0\end{bmatrix} \overset{\mathrm{III}+(-\mathrm{I})}{\sim} \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix} \overset{\mathrm{II}+(-\mathrm{III})}{\sim} \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix} \overset{\mathrm{III}\leftrightarrow\mathrm{II}}{\sim} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

alle vektorene er lineært vavhengige. en basis for rommet er:

$$\underline{\underline{\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}}}$$

c) $\begin{pmatrix}7\\4\\-5\end{pmatrix}, \begin{pmatrix}4\\-7\\5\end{pmatrix}, \begin{pmatrix}1\\-5\\4\end{pmatrix}$

$$\begin{bmatrix}7&4&1\\4&-7&-5\\-5&5&4\end{bmatrix} \overset{\mathrm{I}\leftrightarrow\mathrm{III}}{\sim} \begin{bmatrix}-5&5&4\\4&-7&-5\\7&4&1\end{bmatrix} \overset{\mathrm{I}\cdot\frac{1}{5}}{\sim} \begin{bmatrix}-1&1&\frac{4}{5}\\4&-7&-5\\7&4&1\end{bmatrix} \underset{\mathrm{III}+7\mathrm{I}}{\overset{\mathrm{II}+4\mathrm{I}}{\sim}} \begin{bmatrix}-1&1&\frac{4}{5}\\0&-3&-\frac{9}{5}\\0&11&\frac{33}{5}\end{bmatrix} \underset{\mathrm{III}\cdot\frac{1}{11}}{\overset{\mathrm{II}\cdot(-\frac{1}{3})}{\sim}} \begin{bmatrix}-1&1&\frac{4}{5}\\0&1&\frac{3}{5}\\0&1&\frac{3}{5}\end{bmatrix} \underset{\mathrm{I}\cdot(-1)}{\overset{\mathrm{III}+(-\mathrm{II})}{\sim}} \begin{bmatrix}1&-1&-\frac{4}{5}\\0&1&\frac{3}{5}\\0&0&0\end{bmatrix} \overset{\mathrm{I}+\mathrm{II}}{\sim} \begin{bmatrix}1&0&-\frac{1}{5}\\0&1&\frac{3}{5}\\0&0&0\end{bmatrix}$$

basis for rommet:

$$\underline{\underline{\mathcal{B} = \left\{ \begin{pmatrix}7\\4\\-5\end{pmatrix}, \begin{pmatrix}4\\-7\\5\end{pmatrix} \right\}}}$$

d) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}$

$$\begin{bmatrix}1&1&2\\2&-1&1\\3&-1&2\end{bmatrix} \underset{\mathrm{III}+(-3\mathrm{I})}{\overset{\mathrm{II}+(-2\mathrm{I})}{\sim}} \begin{bmatrix}1&1&2\\0&-3&-3\\0&-4&-4\end{bmatrix} \underset{\mathrm{III}\cdot(-\frac{1}{4})}{\overset{\mathrm{II}\cdot(-\frac{1}{3})}{\sim}} \begin{bmatrix}1&1&2\\0&1&1\\0&1&1\end{bmatrix} \overset{\mathrm{III}+(-\mathrm{II})}{\sim} \begin{bmatrix}1&1&2\\0&1&1\\0&0&0\end{bmatrix} \overset{\mathrm{I}+(-\mathrm{II})}{\sim} \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\underline{\underline{\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix} \right\}}}$$

Oppgave 3

a) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

basis Col A:

$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \overset{II+(-2I)}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix} \overset{II\cdot\frac{1}{5}}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \overset{I+2II}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{\underline{\mathcal{B}_{ColA} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}}}$

basis Nul A:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = 0 \quad y = 0$

$\underline{\underline{Nul\,A = \{0\}, \ \mathcal{B}_{NulA} = \{\emptyset\}}}$

b) $A = (1)$

basis for ColA:

$Col(A) = Span\{(1)\}$

$\underline{\underline{\mathcal{B}_{ColA} = \{(1)\}}}$

basis for NulA:

$1 \cdot x = 0$

$\underline{\underline{\mathcal{B}_{NulA} = \{\emptyset\}}}$

c) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

basis for Col A:

$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \overset{I+(-3II)}{\sim} \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

$\underline{\underline{\mathcal{B}_{ColA} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}}}$

basis for NulA:

$\begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$x - 7z + 6p = 0 \quad \rightarrow \quad x = 7z - 6p$

$y + 4z - 2p = 0 \qquad y = -4z + 2p$

$v = z \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + p \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\underline{\mathcal{B}_{NulA} = \left\{ \begin{pmatrix} 7 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}}}$

d) $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

basis for ColA:

$Col A = Span\{0\} = \{0\} < \mathbb{R}^3$

$\underline{\underline{\mathcal{B}_{ColA} = \{\emptyset\}}}$

basis for NulA:

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

x, y, z, p kan være hvilket som helst tall.

En basis kan f.eks være:

$\underline{\underline{\mathcal{B}_{NulA} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}}$

Oppgave 4

a) En 5x6 matrise har 6 kolonner.

Vet at antall kolonner = $\text{Rank}(A) + \text{Dim}(\mathcal{N}(A))$, så får:

$$6 = \text{Rank}(A) + 4 \Rightarrow \underline{\underline{\text{Rank}(A) = 2}}$$

b) Den største mulige dimensjonen til $\mathcal{N}(A)$ vil forekomme når rangen til matrisen er så liten som mulig (0).

Hvis A er 6x8 (antall kolonner = 8)

$$\dim(\mathcal{N}(A))_{max} = 8 - 0 = \underline{\underline{8}}$$

Hvis A er 8x6 (antall kolonner = 6)

$$\dim(\mathcal{N}(A))_{max} = 6 - 0 = \underline{\underline{6}}$$

c) Størst mulig rang er begrenset av antall rader og antall kolonner.

Dersom A er en 6x8 matrise:

kan maksimalt få 6 lineært uavhengige rader. $\underline{\underline{\text{Rank}(A)_{max} = 6}}$

Hvis A er en 8x6 matrise:

kan maksimalt få 8 lineært uavhengige kolonner. $\underline{\underline{\text{Rank}(A)_{max} = 6}}$

Oppgave 5

a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda I)$$
$$= \det\left(\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & -1 \\ 2 & -\lambda \end{bmatrix}\right)$$
$$= (1-\lambda)(-\lambda) - (-1 \cdot 2)$$
$$\underline{\underline{= \lambda^2 - \lambda + 2}}$$

b) $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$

$$\chi_A(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 3 & -1 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & -\lambda \end{bmatrix}\right) = (1-\lambda)\big((1-\lambda)(-\lambda) - (-1 \cdot 2)\big) - 3\big((-1 \cdot 0) - (0 \cdot (-\lambda))\big) + (-1)\big((0 \cdot 2) - (1-\lambda) \cdot 0\big)$$

$$= (1-\lambda)(\lambda^2 - \lambda + 2) = \lambda^2 - \lambda + 2 - \lambda^3 + \lambda^2 - 2\lambda = \underline{\underline{-\lambda^3 + 2\lambda^2 - 3\lambda + 2}}$$

c) $A = \begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}$

$$\chi_A(\lambda) = \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix}\right) = (2-\lambda)(-1-\lambda) - (3\cdot 2)$$

$$= -2 - 2\lambda + \lambda + \lambda^2 - 6$$

$$\underline{\underline{= \lambda^2 - \lambda - 8}}$$

d) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\chi_A(\lambda) = \det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}\right) = (-\lambda)\left((1-\lambda)(-\lambda) - 0\cdot 0\right) - 0\left(0\cdot 1 - 0\cdot(-\lambda)\right) + 1\left(0\cdot 0 - (1-\lambda)\cdot(1)\right)$$

$$= (-\lambda)(-\lambda + \lambda^2) - (1-\lambda)$$

$$= \underline{\underline{-\lambda^3 + \lambda^2 + \lambda - 1}}$$

Oppgave 6

a) $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 2 & -\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(-\lambda) - (-1\cdot 2) = 0$$

$$-\lambda + \lambda^2 + 2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4\cdot 1\cdot 2}}{2} = \frac{1 \pm \sqrt{-7}}{2} \Rightarrow \underline{\underline{\lambda = \frac{1}{2} \pm i\frac{\sqrt{7}}{2}}}$$

b) $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1-\lambda & 3 & -1 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & -\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)\left((1-\lambda)(-\lambda) - (-1\cdot 2)\right) - 3\left(-1\cdot 0 - 0\cdot(-\lambda)\right) + (-1)\left(0\cdot 2 - (1-\lambda)\cdot 0\right) = 0$$

$$(1-\lambda)(-\lambda + \lambda^2 + 2) = 0$$

I $\quad 1 - \lambda = 0$ $\qquad$ II $\quad \lambda^2 - \lambda + 2 = 0$

$\underline{\underline{\lambda_1 = 1}}$ $\qquad$ fra oppgave a:

$$\underline{\underline{\lambda_2 = \frac{1}{2} + i\frac{\sqrt{7}}{2}, \quad \lambda_3 = \frac{1}{2} - i\frac{\sqrt{7}}{2}}}$$

c)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 1 \\ 2 & -1 & 4 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 1 \\ 2 & -1 & 4-\lambda \end{bmatrix}\right) = 0$$

$$(2-\lambda)\big((2-\lambda)(4-\lambda) - 1\cdot(-1)\big) - 0\cdot\big(1\cdot 2 - 4\cdot(4-\lambda)\big) + 0\cdot\big((2-\lambda)\cdot 2 - 4\cdot(-1)\big) = 0$$

$$(2-\lambda)\left(8 - 2\lambda - 4\lambda + \lambda^2 + 1\right) = 0$$

I $2-\lambda = 0$ II $\lambda^2 - 6\lambda + 9 = (\lambda-3)^2 = 0$

$\underline{\underline{\lambda_1 = 2}}$ $\underline{\underline{\lambda_2 = 3,\ \text{multiplisitet} = 2}}$

d)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)\big((1-\lambda)(1-\lambda) - 1\cdot 0\big) - (-1)\big(1\cdot 1 - 0\cdot(1-\lambda)\big) + 0\big((1-\lambda)\cdot 1 - 0\cdot 0\big) = 0$$

$$(1-\lambda)\left(1-\lambda-\lambda+\lambda^2\right) + 1 = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 1) + 1 = 0$$

$$(1-\lambda)(1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^3 = -1$$

$$1-\lambda = \sqrt[3]{-1}$$

I $1-\lambda = -1$ II $1-\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ III $1-\lambda = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\underline{\underline{\lambda_1 = 0}}$ $\underline{\underline{\lambda_2 = \frac{3}{2} + \frac{\sqrt{3}}{2}i}}$ $\underline{\underline{\lambda_3 = \frac{3}{2} - \frac{\sqrt{3}}{2}i}}$