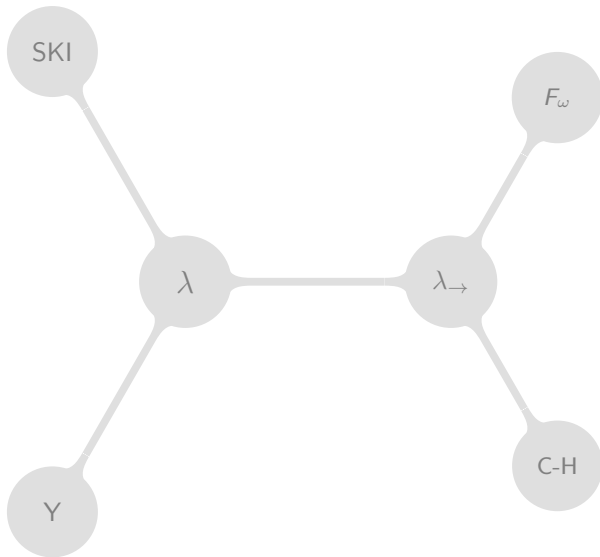


Introduction to Lambda Calculus

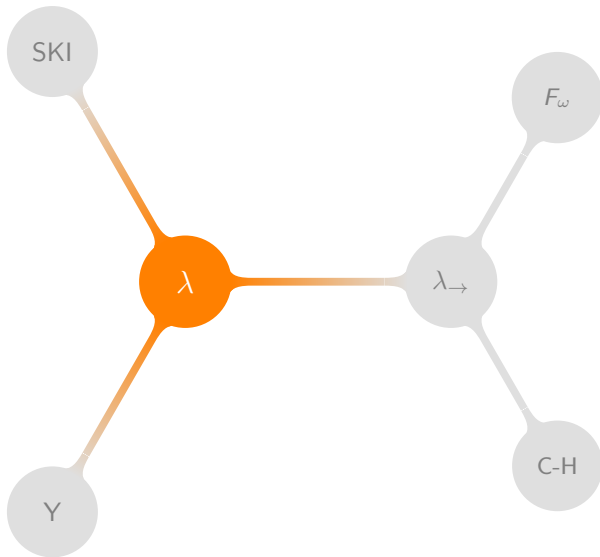
Maciek Makowski (@mmakowski)

21st October 2014

The Plan



Basic Lambda Calculus



Syntax

$\langle term \rangle ::= x$	(variable)
$(\lambda x. \langle term \rangle)$	(abstraction)
$(\langle term \rangle \langle term \rangle)$	(application)

where $x \in \mathbb{X}$ – the set of variables

Syntax

v_1

Syntax

v_1

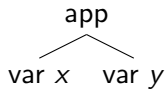
var v_1

Syntax

$x\ y$

Syntax

$x\ y$



Syntax

$\lambda a.b$

Syntax

$\lambda a.b$

abs a
|
var b

Syntax

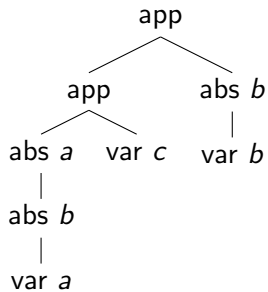
$(\lambda a. \lambda b. a) \ c \ (\lambda b. b)$

Syntax

$\langle term \rangle ::= x$	(variable)
$(\lambda x. \langle term \rangle)$	(abstraction)
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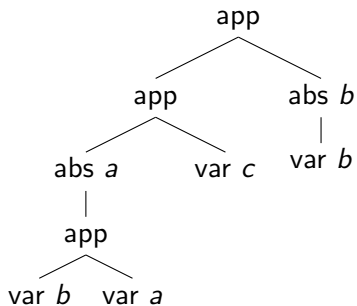
Syntax

$(\lambda a. \lambda b. a) c (\lambda b. b)$



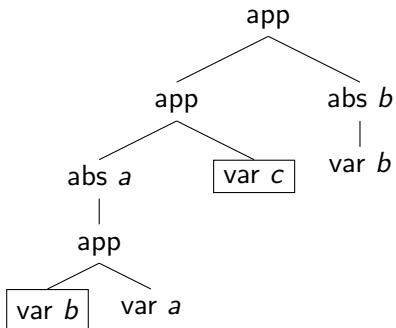
Syntax

$(\lambda a. b \ a) \ c \ (\lambda b. b)$



Syntax

$(\lambda a. \underline{b} \ a) \ \underline{c} \ (\lambda b. b)$



Syntax

- ▶ terms: trees consisting of
 - ▶ variables
 - ▶ abstractions
 - ▶ applications
- ▶ variables are *bound* by abstraction; otherwise *free*

Rewriting

α -conversion

$$(\lambda x.x\ y)\ (\lambda x.x) \longleftrightarrow_{\alpha} (\lambda a.a\ y)\ (\lambda b.b)$$

Rewriting

β -reduction

$$(\lambda x.M) N \longrightarrow_{\beta} M[x/N]$$

Rewriting

β -reduction

$$(\lambda x.M) N \longrightarrow_{\beta} M[x/N]$$

$$(\lambda x.x y) (\lambda z.z) \longrightarrow_{\beta} (\lambda z.z) y \longrightarrow_{\beta} y$$

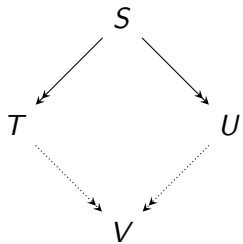
Rewriting

β -reduction

- ▶ *call-by-value*: start with innermost redex, do not reduce under abstraction
- ▶ *call-by-name*: start with outermost redex, do not reduce under abstraction

Rewriting

Church-Rosser



Semantics

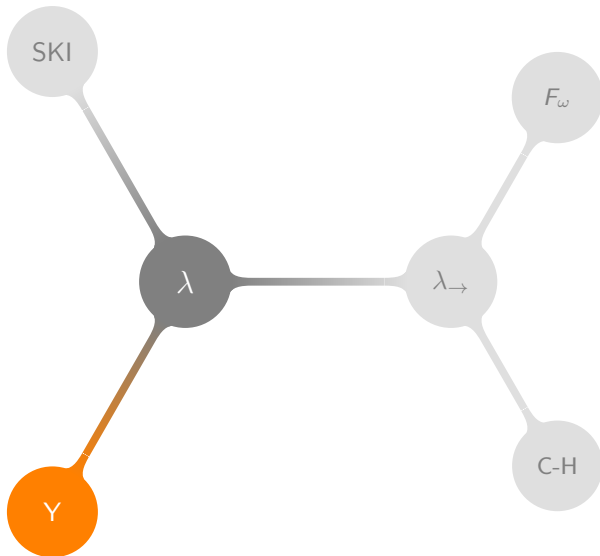
$$f(x) = 3 * x + 2$$

Semantics

$$f(x) = 3 * x + 2$$

$$\lambda x. + (* 3 x) 2$$

Programming in Lambda Calculus



Conditionals

if C then T else F

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$\text{true} = \lambda t. \lambda f. t$

$\text{false} = \lambda t. \lambda f. f$

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if C then T else F

$\text{true} = \lambda t. \lambda f. t$

$\text{false} = \lambda t. \lambda f. f$

$\text{test} = \lambda c. \lambda t. \lambda f. c \ t \ f$

$\text{if } C \text{ then } T \text{ else } F = \text{test } C \ T \ F$

Conditionals

if C then T else F

$\text{true} = \lambda t. \lambda f. t$

$\text{false} = \lambda t. \lambda f. f$

$\text{test} = \lambda c. \lambda t. \lambda f. c \ t \ f$

$\text{if } C \text{ then } T \text{ else } F = \text{test } C \ T \ F$

Numbers

$$0 = \lambda s. \lambda z. z$$

$$\text{succ} = \lambda n. \lambda s. \lambda z. s (n s z)$$

Numbers

$$0 = \lambda s. \lambda z. z$$

$$\text{succ} = \lambda n. \lambda s. \lambda z. s (n s z)$$

$$0 = \lambda s. \lambda z. z$$

$$1 = \text{succ } 0 = \lambda s. \lambda z. s z$$

$$2 = \text{succ } 1 = \lambda s. \lambda z. s (s z)$$

$$3 = \text{succ } 2 = \lambda s. \lambda z. s (s (s z))$$

\vdots

$$n = \lambda s. \lambda z. \underbrace{s (\dots s (s z) \dots)}_n$$

Numbers

$$0 = \lambda s. \lambda z. z$$

$$\text{succ} = \lambda n. \lambda s. \lambda z. s (n s z)$$

$$\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m s (n s z)$$

$$\text{times} = \lambda m. \lambda n. m (\text{plus } n) 0$$

Recursion

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n * (n - 1)! & \text{otherwise.} \end{cases}$$

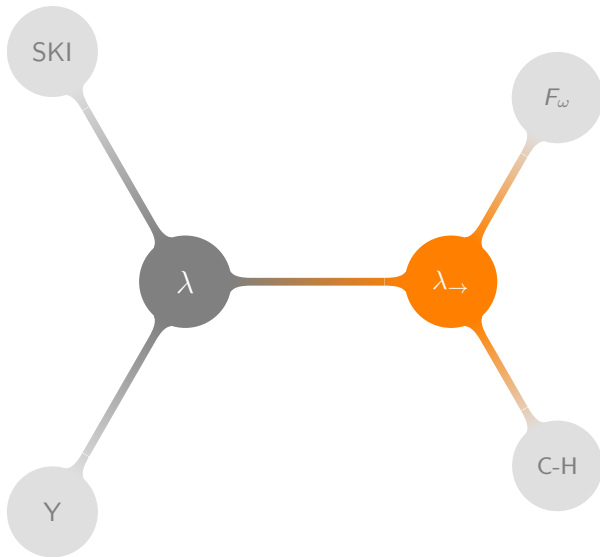
Recursion

$$Y = \lambda f.(\lambda x.f(x\ x))(\lambda x.f(x\ x))$$

$$g = \lambda f.\lambda n.\text{if eq } n\ 0 \text{ then } 1 \text{ else } (\text{times } n\ (f\ (\text{pred } n)))$$

$$\text{factorial} = Y\ g$$

Simple Types



Simple Types

$$\begin{array}{l} \langle type \rangle ::= \sigma \\ \quad | \quad (\langle type \rangle \rightarrow \langle type \rangle) \end{array}$$

Simple Types

TODO: strike-out

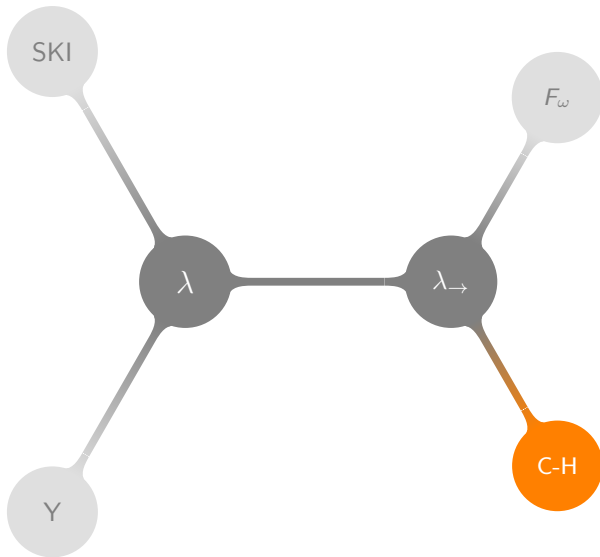
$$\frac{M : \sigma \rightarrow \tau \quad N : \sigma}{M N : \tau}$$

(application)

$$\frac{\begin{array}{c} x : \sigma \\ \vdots \\ M : \tau \end{array}}{\lambda x. M : \sigma \rightarrow \tau}$$

(abstraction)

Curry-Howard Correspondence



Curry-Howard Correspondence

$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

(implication elimination)

$$\frac{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}{\phi \Rightarrow \psi}$$

(implication introduction)

Curry-Howard Correspondence

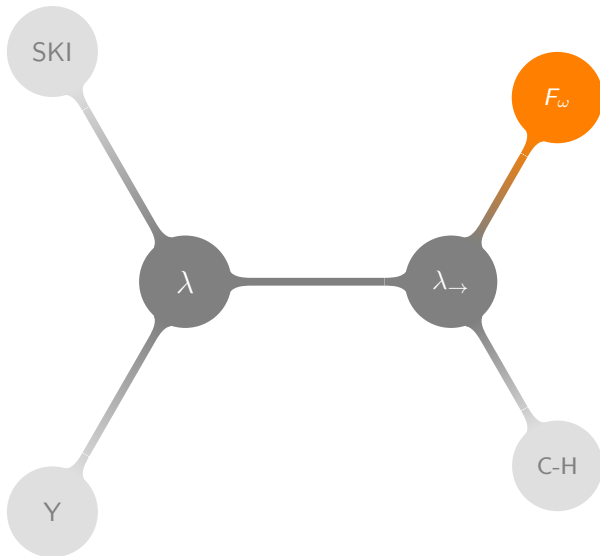
$$\frac{\phi \Rightarrow \psi \quad \phi}{\psi}$$

$$\frac{M : \sigma \rightarrow \tau \quad N : \sigma}{M N : \tau}$$

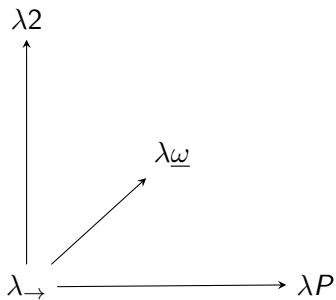
$$\frac{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}{\phi \Rightarrow \psi}$$

$$\frac{\begin{array}{c} x : \sigma \\ \vdots \\ M : \tau \end{array}}{\lambda x. M : \sigma \rightarrow \tau}$$

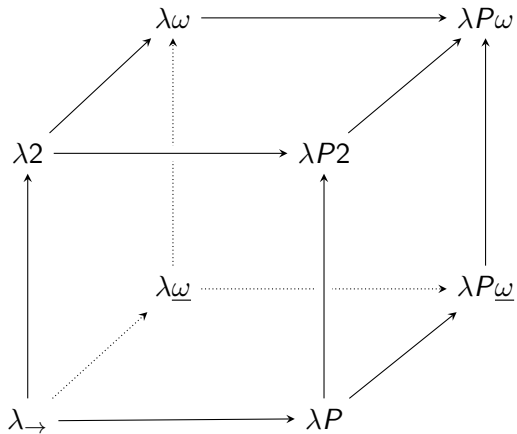
More Types



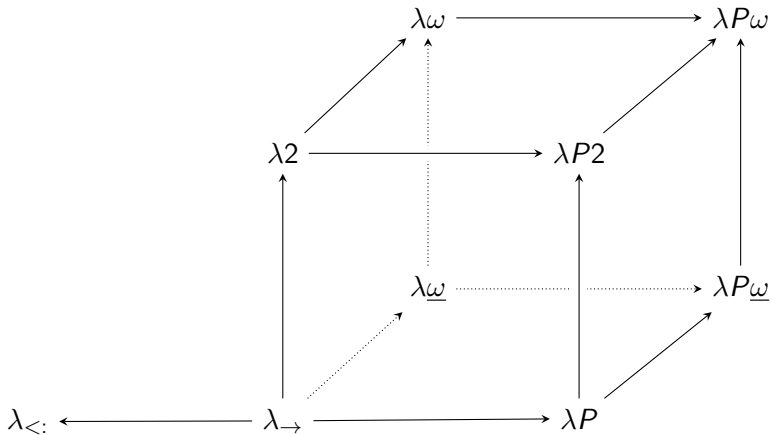
The Lambda Cube



The Lambda Cube



Subtyping



Subtyping

