

# Introduction to Lambda Calculus

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## 1 Motivation

Software is pervasive in the modern world and has influence over many aspects of our lives. In some cases, such as avionics or medical equipment control, human life depends on the correctness of software. Yet, high profile cases of bugs<sup>1</sup> do not inspire confidence in the state of software engineering. The "software crisis" is a phenomenon recognised by practitioners of the field. A number of ways to address the reliability issue has been proposed, from reliance on programmer's discipline[9][10], through tools that analyse programs written in popular languages for suspicious patterns[11], to languages that restrict valid programs to ones whose properties can be formally proven. The latter approach relies on a body of theoretical knowledge that can appear intimidating. It turns out, however, that much of the required insight is built on systematic extensions of a very simple formal system – the lambda calculus. Familiarity with the fundamentals of lambda calculus is a prerequisite for proficiency with modern software engineering tools. These notes aim to introduce lambda calculus in a very informal manner. The only background assumed is that of general programming experience.

## 2 Syntax

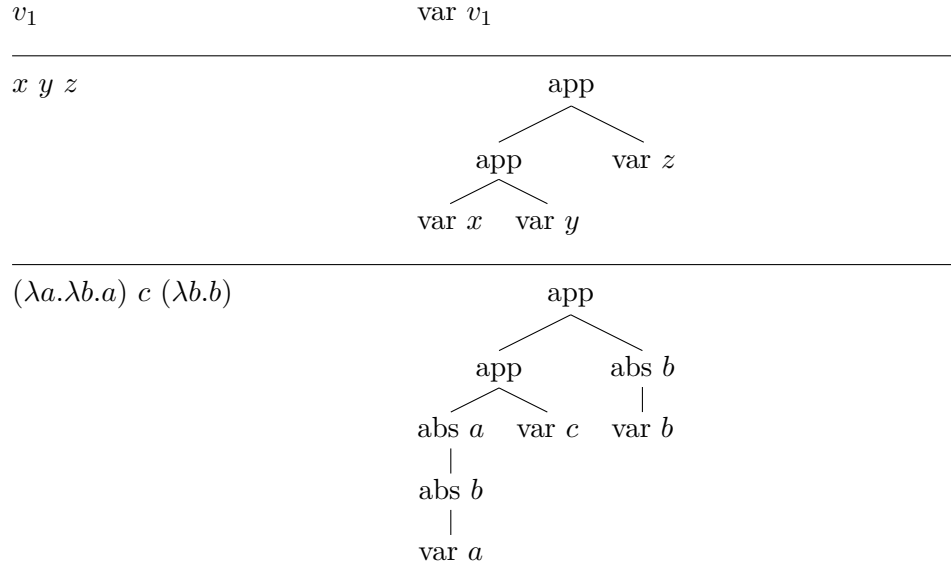
Given a set of variables  $X$ , terms of lambda calculus are generated by the following grammar:

$\langle term \rangle ::= x$	(variable)
$(\lambda x. \langle term \rangle)$	(abstraction)
$(\langle term \rangle \langle term \rangle)$	(application)

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<sup>1</sup>Infamous historical examples include Mars Climate Orbiter's inconsistent usage of units of measurement[5] and Therac-25 radiation therapy overdoses[6]. Recent faults such as security-related Apple goto fail[7] and OpenSSL Heartbleed bug[8], while not life-threatening, had wide-ranging implications for the security of e-commerce and privacy of internet users.

where  $x \in X$ . Notational convention is that application binds to the left, the full-stop can be treated as an opening parenthesis that extends until the end of the sub-term and redundant parentheses are omitted. Examples of lambda terms as they are typically written, together with ASTs they represent:



Within a term, occurrences of variables that are not bound by enclosing abstraction – i.e. where the variable name does not appear in any `abs` node on the path to the root of the AST – are called *free*. In the examples below free occurrences are underlined:

- $\lambda x. \underline{y}$
- $(\lambda a. \lambda \underline{b}. a) \ \underline{c} \ (\lambda b. b)$

Note that the same variable name might have both bound and free occurrences within a term, as in the second example. Terms with no free occurrences are known as *closed terms*, or *combinators*.

**Summary:** we have defined what lambda terms look like. The grammar has three productions – variable, abstraction and application – and generates abstract syntax trees.

### 3 Rewriting Rules

Lambda terms are not of much use without some operations that can be performed on them. Two<sup>2</sup> operations we will use going forward are presented below.

#### 3.1 Renaming of bound variables

Intuitively,  $\lambda x.x$  is similar to  $\lambda y.y$  – the "shape" of these two terms is the same, they differ only in the choice of variable used. In practice we will often want to identify terms of the same structure. This intuitive similarity is captured by an operation called  $\alpha$ -conversion that consistently renames variables. While the operation appears trivial, there are subtleties around bound vs. free variables – for instance, nested abstractions can use the same variable name – but these difficulties manifest themselves mostly in the implementation<sup>3</sup>, so we will not worry about them in this presentation. The intuition about variable renaming is correct in most cases we are interested in. In particular, we can assume that all the variables in the terms we are working with have been chosen so that they are distinct. It is always possible to  $\alpha$ -convert any term so that this is true.

**Example:**  $(\lambda x.x y) (\lambda x.x) \longleftrightarrow_{\alpha} (\lambda a.a y) (\lambda b.b)$

#### 3.2 Removal of abstraction under application

TODO:  $\beta$ -reduction

**Example:**  $(\lambda x.x y) (\lambda z.z) \longrightarrow_{\beta} (\lambda z.z) y \longrightarrow_{\beta} y$

In the first reduction,  $\lambda x.$  is eliminated by substituting  $\lambda z.z$  for  $x$ , in the second,  $\lambda z.$  is eliminated by substituting  $y$  for  $z$ .

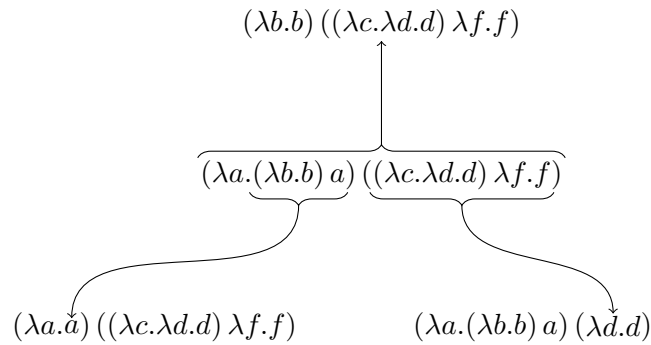
This is the key operation of lambda calculus, as it represents a *computation*. It allows us to view a lambda term as a program that can be evaluated to a final value (i.e. a term that cannot be further reduced). That final value is called a *normal form* of a term.

An application that can be  $\beta$ -reduced is called a *redex*. There can be multiple redexes in a given term:

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<sup>2</sup>The third frequently applied operation is introduction/removal of abstraction ( $\eta$ -conversion):  $\lambda x.M x \longleftrightarrow_{\eta} M$  for any term  $M$ . We will not require it in this presentation of lambda calculus.

<sup>3</sup>For that reason a convenient way to represent lambda terms when implementing evaluation is positional (de Bruijn) encoding which replaces variable names with numerical index of the lambda that binds given variable occurrence. For example, de Bruijn representation of  $\lambda a.\lambda b.a b c$  would be  $\lambda.\lambda.1\ 0\ 2$  under naming context  $\{c \mapsto 2\}$ . The naming context is required to map free variables to indices.



The choice of the redex to be reduced next is up to us. Prescribing which redex should be chosen gives rise to a reduction strategy. For example:

- TODO: call-by-name
- TODO: call-by-value

**Summary:** lambda-terms can be syntactically transformed according to two rules:  $\alpha$ -conversion, which renames the variables, and  $\beta$ -reduction, which "applies" one term to another.

## 4 Programming

We claimed that a lambda term can be treated as a computer program. To substantiate this, let us see how familiar elements of programming languages can be represented in lambda calculus.

### 4.1 Conditionals

TODO

### 4.2 Numbers

$$\begin{aligned}
 0 &= \lambda s. \lambda z. z \\
 1 &= \lambda s. \lambda z. s z \\
 2 &= \lambda s. \lambda z. s (s z) \\
 3 &= \lambda s. \lambda z. s (s (s z))
 \end{aligned}$$

In general,  $n = \lambda s. \lambda z. \underbrace{s (\dots s (s z) \dots)}_n$ .

TODO<sup>4</sup>

### 4.3 Loops

TODO

## 5 Model of Computation

TODO

## 6 Church-Rosser Theorem

TODO

## 7 Curry-Howard Correspondence

TODO

## 8 Further Reading

A direct inspiration for this talk was the presentation of lambda calculus in [1]. The book is very well written and builds a sophisticated type system in easy to follow steps, starting from untyped lambda calculus.

For a succinct but rigorous introduction to lambda calculus see [3].

TODO

## References

- [1] Benjamin C. Pierce, *Types and Programming Languages*, <http://www.cis.upenn.edu/~bcpierce/tapl/>
- [2] Morten Heine B. Sørensen, Paweł Urzyczyn, *Lectures on the Curry-Howard Isomorphism*, <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.17.7385>

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<sup>4</sup>Seeing how Peano arithmetic can be defined in lambda calculus, and drawing parallels with Zermelo-Fraenkel set-theoretical model of natural numbers and the build-out of other mathematical constructions on this basis, one could ask: can untyped lambda calculus be treated as a foundational theory in which all known mathematical concepts can be stated? Alonso Church had that in mind when he conceived lambda calculus, but it was soon proven by his students, Kleene and Rosser[12], that in fact the calculus as we present it here is inconsistent, i.e. any proposition can be proven in it.

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- [4] Klaus Grue, *Lambda calculus as a foundation of mathematics*, <http://www.diku.dk/~grue/papers/church/church.html>
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- [12] Wikipedia, *Kleene-Rosser paradox*, [https://en.wikipedia.org/wiki/Kleene-Rosser\\_paradox](https://en.wikipedia.org/wiki/Kleene-Rosser_paradox)