Introduction to Lambda Calculus

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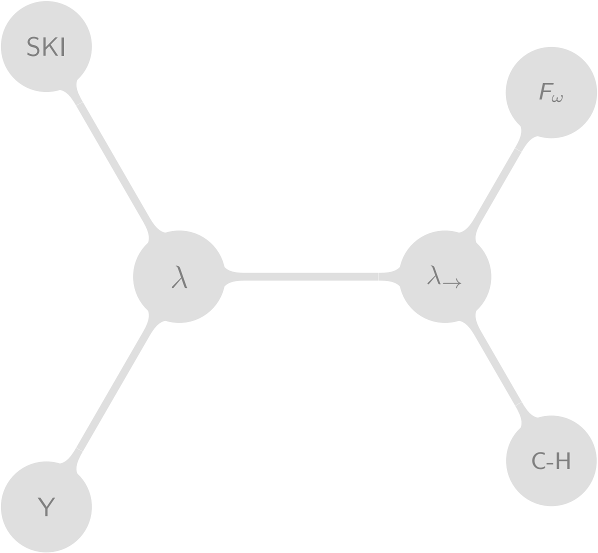
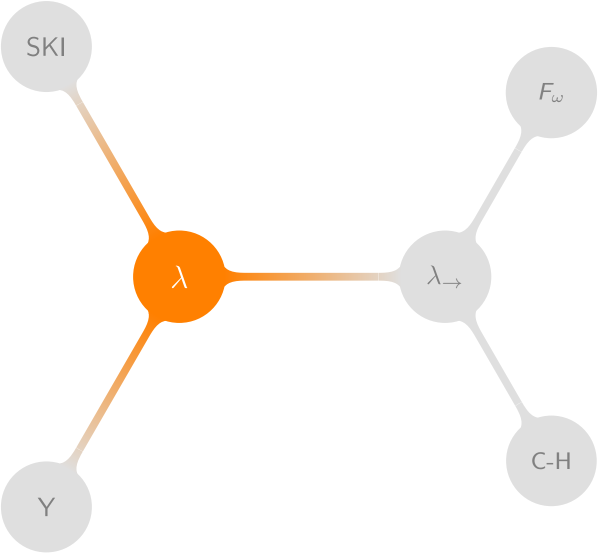
6th November 2014

The Plan



Basic Lambda Calculus



Semantics

$$f(x) = 3 * x + 2$$

Semantics

$$
\begin{aligned}
f(x) &= 3 * x + 2 \\
& \lambda x. + \; (* \, 3 \, x) \, 2
\end{aligned}
$$

Syntax

$$
\begin{array}{llll}
\langle \textit{term} \rangle & ::= & x & \text{(variable)} \\
& | & (\lambda x.\langle \textit{term} \rangle) & \text{(abstraction)} \\
& | & (\langle \textit{term} \rangle\ \langle \textit{term} \rangle) & \text{(application)}
\end{array}
$$

where $x \in \mathbb{X}$ – the set of variables

Syntax

v_1

Syntax

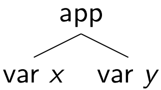
v_1 var v_1

Syntax

$x \ y$

Syntax

$x \; y$



Syntax

$\lambda a.b$

Syntax

$\lambda a.b$

$$\begin{array}{c} \text{abs } a \\ | \\ \text{var } b \end{array}$$

Syntax

$(\lambda a.\lambda b.a)\ c\ (\lambda b.b)$

Syntax

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\begin{array}{rcll}
\langle \mathit{term} \rangle & ::= & x & \text{(variable)} \\
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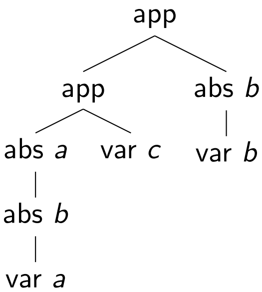
& \mid & (\langle \mathit{term} \rangle\ \langle \mathit{term} \rangle) & \text{(application)}
\end{array}
$$

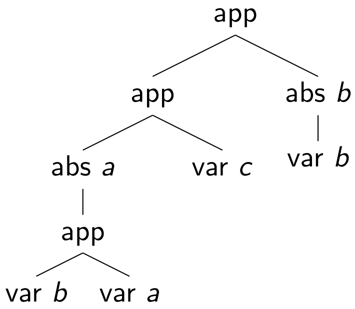
Syntax

$(\lambda a.\lambda b.a)\ c\ (\lambda b.b)$



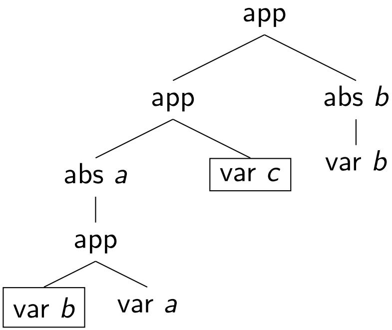
Syntax

$(\lambda a.b\ a)\ c\ (\lambda b.b)$



Syntax

$(\lambda a.\underline{b}\ a)\ \underline{c}\ (\lambda b.b)$



Syntax

- terms: trees consisting of
  - variables
  - abstractions
  - applications
- variables are *bound* by abstraction; otherwise *free*

Rewriting

α-conversion

$$(\lambda x.x\, y)\ (\lambda x.x) \longleftrightarrow_\alpha (\lambda a.a\, y)\ (\lambda b.b)$$

Rewriting

β-reduction

$$(\lambda x.M)\ N \longrightarrow_{\beta} M[x/N]$$

Rewriting

β-reduction

$$(\lambda x.M)\ N \longrightarrow_\beta M[x/N]$$

$$(\lambda x.x\,y)\ (\lambda z.z) \longrightarrow_\beta (\lambda z.z)\ y \longrightarrow_\beta y$$

Rewriting

β-reduction

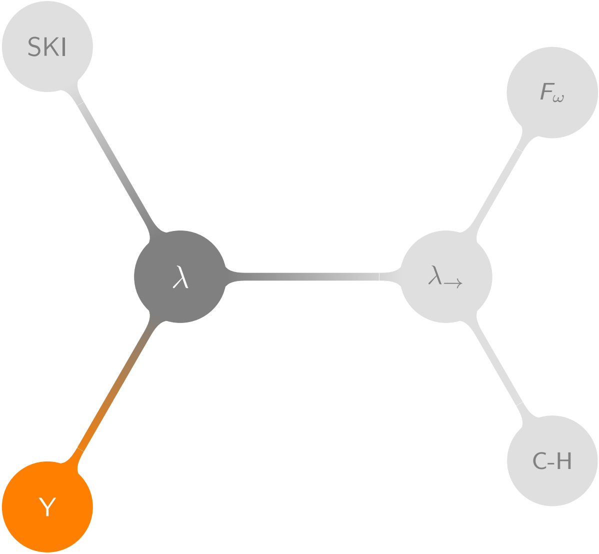
- *call-by-value*: start with innermost redex, do not reduce under abstraction
- *call-by-name*: start with outermost redex, do not reduce under abstraction

Rewriting

Church-Rosser

$$\begin{tikzcd}
 & S \arrow[dl, ->>] \arrow[dr, ->>] & \\
T \arrow[dr, ->>, dotted] & & U \arrow[dl, ->>, dotted] \\
 & V &
\end{tikzcd}$$

Programming in Lambda Calculus



Conditionals

`if` C `then` T `else` F

Conditionals

$$\texttt{if } C \texttt{ then } T \texttt{ else } F$$

$$\begin{aligned} \texttt{true} &= \lambda t.\lambda f.t \\ \texttt{false} &= \lambda t.\lambda f.f \end{aligned}$$

Conditionals

$$\texttt{if } C \texttt{ then } T \texttt{ else } F$$

$$\begin{aligned}
\texttt{true} &= \lambda t.\lambda f.t \\
\texttt{false} &= \lambda t.\lambda f.f \\
\texttt{test} &= \lambda c.\lambda t.\lambda f.c\,t\,f \\
\texttt{if}\,C\,\texttt{then}\,T\,\texttt{else}\,F &= \texttt{test}\;C\;T\;F
\end{aligned}$$

Numbers

$$
\begin{aligned}
0 &= \lambda s.\lambda z.z \\
\texttt{succ} &= \lambda n.\lambda s.\lambda z.s\,(n\,s\,z)
\end{aligned}
$$

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$$
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\end{aligned}
$$

$$
\begin{aligned}
&0 = && \lambda s.\lambda z.z \\
&1 = \mathtt{succ}\ \ 0 = && \lambda s.\lambda z.s\,z \\
&2 = \mathtt{succ}\ \ 1 = && \lambda s.\lambda z.s\,(s\,z) \\
&3 = \mathtt{succ}\ \ 2 = && \lambda s.\lambda z.s\,(s\,(s\,z)) \\
&\vdots && \\
&\mathtt{n} = && \lambda s.\lambda z\,\underbrace{s\,(\ldots s\,(s}_{n}\ z)\ldots)
\end{aligned}
$$

Numbers

$$\mathtt{0} = \lambda s.\lambda z.z$$
$$\mathtt{succ} = \lambda n.\lambda s.\lambda z.s\,(n\,s\,z)$$

$$\mathtt{plus} = \lambda m.\lambda n.\lambda s.\lambda z.m\ s\ (n\ s\ z)$$
$$\mathtt{times} = \lambda m.\lambda n.m\ (\mathtt{plus}\ n)\ \mathtt{0}$$

Recursion

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n * (n-1)! & \text{otherwise.} \end{cases}$$

Recursion

$$\mathtt{Y} = \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$$

Recursion

$$\mathtt{Y} = \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$$

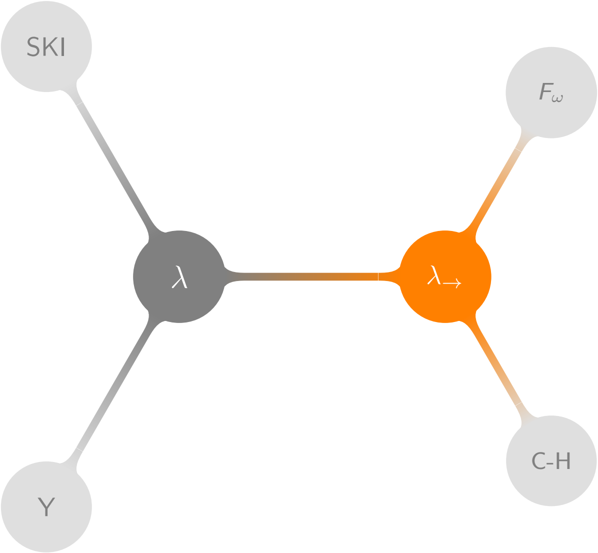
$$\begin{aligned}\mathtt{g} &= \lambda f.\lambda n.\mathtt{if\ eq}\ n\ 0\ \mathtt{then}\ 1\ \mathtt{else}\ (\mathtt{times\ n}\,(f\,(\mathtt{pred\ n})))\\ \mathtt{factorial} &= \mathtt{Y\ g}\end{aligned}$$

Recursion

$$
\begin{aligned}
\mathtt{Y} &= \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x)) \\
\mathtt{g} &= \lambda f.\lambda n.\mathtt{if\ eq}\ n\ 0\ \mathtt{then}\ 1\ \mathtt{else}\ (\mathtt{times\ n}\,(f\,(\mathtt{pred\ n}))) \\
\mathtt{factorial} &= \mathtt{Y\ g}
\end{aligned}
$$

$$
\begin{array}{ll}
\mathtt{factorial}\ 3 & \\
\mathtt{Y\ g}\ 3 & \\
(\mathtt{h\ h})\ 3 & \text{where } \mathtt{h} = \lambda x.\mathrm{g}\ (x\ x) \\
\mathtt{g}\ (\mathtt{h\ h})\ 3 & \\
\mathtt{g\ fct}\ 3 & \text{where } \mathtt{fct} = \mathtt{h\ h} \\
\mathtt{if\ eq}\ 3\ 0\ \mathtt{then}\ 1\ \mathtt{else}\ (\mathtt{times}\ 3\,(\mathtt{fct}\,(\mathtt{pred}\ 3))) &
\end{array}
$$

Simple Types



Simple Types

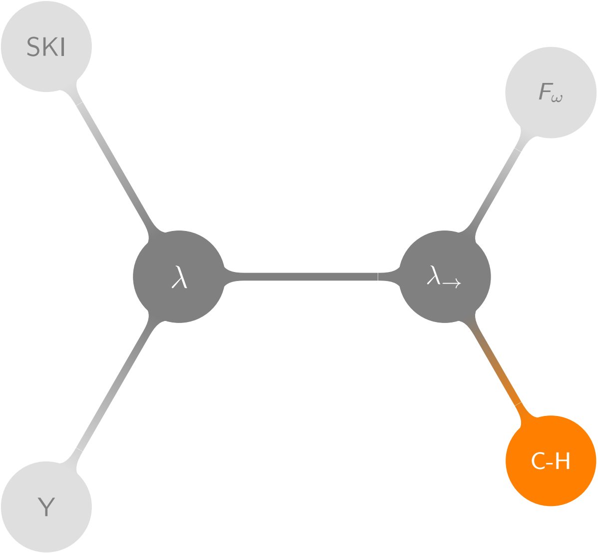
$$
\begin{aligned}
\langle \textit{type} \rangle &::= \sigma \\
&\mid \quad (\langle \textit{type} \rangle \to \langle \textit{type} \rangle)
\end{aligned}
$$

Simple Types

$$\frac{M : \sigma \to \tau \qquad N : \sigma}{M\,N : \tau} \qquad \text{(application)}$$

$$\frac{\begin{array}{c} \cancel{x : \sigma} \\ \vdots \\ M : \tau \end{array}}{\lambda x.M : \sigma \to \tau} \qquad \text{(abstraction)}$$

Curry-Howard Correspondence



Curry-Howard Correspondence

$$\frac{\phi \Rightarrow \psi \qquad \phi}{\psi}$$

(implication elimination)

$$\frac{\begin{array}{c}\cancel{\phi} \\ \vdots \\ \psi\end{array}}{\phi \Rightarrow \psi}$$

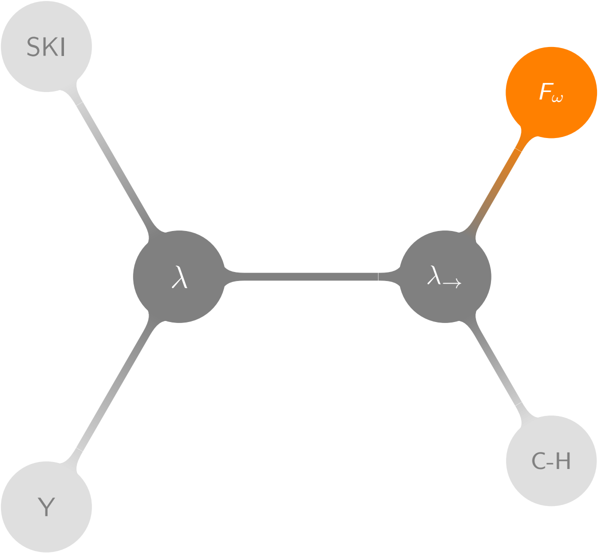
(implication introduction)

Curry-Howard Correspondence

$$\frac{\phi \Rightarrow \psi \qquad \phi}{\psi} \qquad\qquad \frac{M : \sigma \to \tau \qquad N : \sigma}{M\,N : \tau}$$

$$\frac{\begin{array}{c}\cancel{\phi} \\ \vdots \\ \psi\end{array}}{\phi \Rightarrow \psi} \qquad\qquad \frac{\begin{array}{c}\cancel{x : \sigma} \\ \vdots \\ M : \tau\end{array}}{\lambda x.M : \sigma \to \tau}$$

More Types



The Lambda Cube

$$
\begin{tikzcd}
\lambda 2 & & \\
& \lambda\underline{\omega} & \\
\lambda_{\to} \arrow[uu] \arrow[ur] \arrow[rr] & & \lambda P
\end{tikzcd}
$$

The Lambda Cube

$$
\begin{tikzcd}
 & \lambda\omega \arrow[rr] & & \lambda P\omega \\
\lambda 2 \arrow[ur] \arrow[rr] & & \lambda P2 \arrow[ur] & \\
 & \lambda\underline{\omega} \arrow[uu, dotted] \arrow[rr, dotted] & & \lambda P\underline{\omega} \arrow[uu] \\
\lambda_{\to} \arrow[uu] \arrow[ur, dotted] \arrow[rr] & & \lambda P \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

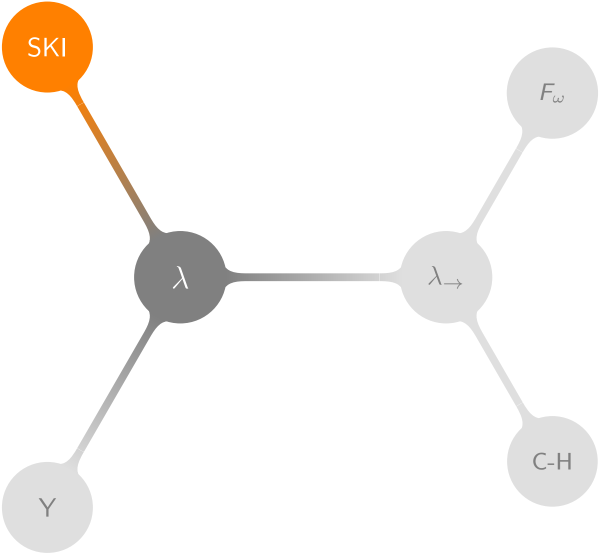
Subtyping

$$\begin{tikzcd}
 & & \lambda\omega \arrow[rr] & & \lambda P\omega \\
 & \lambda 2 \arrow[ur] \arrow[rr] & & \lambda P2 \arrow[ur] & \\
 & & \lambda\underline{\omega} \arrow[uu, dotted] \arrow[rr, dotted] & & \lambda P\underline{\omega} \arrow[uu] \\
\lambda_{<:} & \lambda_{\to} \arrow[l] \arrow[rr] \arrow[uu] \arrow[ur, dotted] & & \lambda P \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Subtyping

$$
\begin{tikzcd}
 & F^{\omega}_{<:} & & \lambda\omega \arrow[ll] \arrow[rr] & & \lambda P\omega \\
F_{<:} \arrow[ur] & & \lambda 2 \arrow[ll] \arrow[rr] \arrow[ur] & & \lambda P2 \arrow[ur] & \\
 & & & \lambda\underline{\omega} \arrow[uu, dotted] \arrow[rr, dotted] & & \lambda P\underline{\omega} \arrow[uu] \\
\lambda_{<:} \arrow[uu] & & \lambda_{\rightarrow} \arrow[ll] \arrow[uu] \arrow[rr] \arrow[ur, dotted] & & \lambda P \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Combinatory Logic



Combinatory Logic

$$\mathtt{K} = \lambda x.\lambda y.x$$
$$\mathtt{S} = \lambda x.\lambda y.\lambda z.xz(yz)$$
$$\mathtt{I} = \mathtt{S}\,\mathtt{K}\,\mathtt{K}$$

Combinatory Logic

$$\begin{aligned} \mathtt{K} &= \lambda x.\lambda y.x \\ \mathtt{S} &= \lambda x.\lambda y.\lambda z.xz(yz) \\ \mathtt{I} &= \mathtt{S}\,\mathtt{K}\,\mathtt{K} \end{aligned}$$

$$\lambda x.\lambda y.yx = \mathtt{S}\,(\mathtt{K}\,(\mathtt{S}\,\mathtt{I}))\,(\mathtt{S}\,(\mathtt{K}\,\mathtt{K})\,\mathtt{I})$$

Combinatory Logic

$$\mathtt{X} = \lambda x.x\,\mathtt{S}\,\mathtt{K}$$

Combinatory Logic

$$\mathtt{X} = \lambda x.x\,\mathtt{S}\,\mathtt{K}$$

$$\mathtt{K} = \mathtt{X}\,(\mathtt{X}\,(\mathtt{X}\,\mathtt{X}))$$
$$\mathtt{S} = \mathtt{X}\,(\mathtt{X}\,(\mathtt{X}\,(\mathtt{X}\,\mathtt{X})))$$

Further Reading

- Benjamin C. Pierce, *Types and Programming Languages*
- Morten Heine B. Sørensen, Paweł Urzyczyn, *Lectures on the Curry-Howard Isomorphism*
- Henk Berendregt, Erik Barendsen, *Introduction to Lambda Calculus*
- Henk Berendregt *The Lambda Calculus, its Syntax and Semantics*

Notes

- https://github.com/mmakowski/introlambda/blob/baml/notes.pdf
- https://github.com/mmakowski/introlambda/blob/baml/slides.pdf