Project II

Finite difference & uncertain volatility

Financial Engineering I

Makowski Michał, Osmelak Tomasz, Prokš Martin

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Introduction

In this project the main task is to implement the finite difference method for pricing more complex options such as European and American ones with barriers.

Some improvements were included, namely discrete dividend payment in the middle of lifetime of the option and uncertain volatility that was initially assumed to lie in an interval $[20\%, 30\%]$. We are supposed to compare prices of the option for different intervals as well as a variety of dividends amounts.

To justify correctness of our implementation the results were compared with binomial tree method which is presented at the end of the report.

1 Finite difference method

We are supposed to hedge options and to do this we use standard Black-Scholes formula:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

This partial differential equation can be analytically solved for simple option contract as Vanilla call/put or Binary (Digital) options. However, for other options as American and incorporation of barriers it may become very difficult to find a solution of such an equation and in this situations numerical methods come in handy.

When using finite difference approach we start with rewriting differential equation, which should be solved, into difference equation. For this purpose we use approximation of derivatives (the 'Greeks'). For Θ we utilize **forward difference** which then leads to explicit finite-difference scheme.

$$\Theta = \frac{\partial V}{\partial t}(S,t) \approx \frac{V_i^k - V_i^{k+1}}{\delta t} \tag{2}$$

In this equation we introduced notation

$$V_i^k = V(i\delta S, T - k\delta t) \tag{3}$$

which means that we have option values only on the grid and we have to use interpolation methods to obtain values between nodes. Notice inversion of time, i.e. we start at expiration time T and by increasing k we go back to present. This is, in particular, convenient when using finite difference method for option pricing and we will see the reason for this later. dS and dt denotes the smallest steps on asset and time axis respectively. They cannot be chosen arbitrarily because of stability issue that is mentioned later.

Next we approximate 'delta' by so called **central difference**

$$\Delta = \frac{\partial V}{\partial S}(S,t) \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S} \tag{4}$$

This approximation is slightly different than one used in 2 for theta but it has lower error ($O(\delta S^2)$). Finally, gamma may be expressed by formula

$$\Gamma = \frac{\partial^2 V}{\partial S^2}(S,t) \approx \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2} \tag{5}$$

When we have expressions for all Greeks we can insert them to Black-Scholes formula 1 and obtain desired difference equation.

$$\frac{V_i^k - V_i^{k+1}}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2} + rS\frac{V_{i+1}^k - V_{i-1}^k}{2\delta S} - rV_i^k = 0 \tag{6}$$

General idea in finite difference method is to start from given boundary conditions and gradually fill whole grid. In options pricing we know the value at the end, i.e. at maturity time T, and it is just payoff of the option. Therefore we express V_i^{k+1} from 6 and it gives us rule how to obtain option values for time just before maturity time. Here we see that when evaluating the options we go back in time and that is the reason for inversion in 3.

$$\begin{aligned} V_i^{k+1} = \; & V_i^k \frac{2\delta S^2 - \sigma^2 S^2 2\delta t - r2\delta t\delta S^2}{2\delta S^2} \\ & + V_{i+1}^k \frac{\sigma^2 S^2 \delta t + rS\delta t\delta S}{2\delta S^2} \\ & + V_{i-1}^k \frac{\sigma^2 S^2 \delta t - rS\delta t\delta S}{2\delta S^2} \end{aligned} \tag{7}$$

Unfortunately, only this equation is not enough for filling all the grid because the grid has finite dimensions and with 7 we are not able to determine the value at the lowest and highest edge. Here we have to incorporate another boundary conditions.

At the highest edge two cases can happen:

- *infinity condition* is used either when we have no barriers or we are evaluating put option
$$V_I^k = 2V_{I-1}^k - V_{I-2}^k$$
- *boundary condition* is used for call option with barriers and this means that the highest possible value for stock is set to barrier level and here the value of the option is zero

Similarly, we distinguish two cases at the lowest edge:

- $S = 0$ boundary condition is used always but for put option with barriers
$$V_0^k = (1 - r\delta t)V_0^{k-1}$$
r is interest rate and this equation means that since once the value of the stock is zero it remains zero forever and therefore the option value is just discounting the payoff
- for put option with barriers there is no need to fill the grid below the barrier level at which option value is zero

1.1 Stability

Finite difference method do not have to converge. To ensure convergence it can be derived that the following condition must hold.

$$\delta t \leq \frac{\delta S}{\sigma^2 S^2}$$

Since this constrain is the most restrictive for the highest S which is $I\delta S$ we get

$$\delta t \leq \frac{1}{\sigma^2 I^2}$$

and to satisfy this condition we use

$$\delta t = \frac{0.9}{\sigma^2 I^2}$$

1.2 Uncertain volatility

We were supposed to consider the fact that in reality we do not know the future value of volatility and pure estimation from historical data is not the best approach. Better solution is assuming that future volatility lies in some interval $\sigma \in [\sigma^-, \sigma^+]$, i.e. uncertain volatility. We should note the difference between uncertain and random parameter. Uncertain means that we do not have any probability distribution so we cannot consider, for example, expected value.

When we have uncertain parameter it is reasonable to adopt the worst scenario which can happen because this practice restricts unpleasant surprise in real evolution. It is not difficult to find that the worst case happens only for limiting values of volatility, i.e. σ^- and σ^+. Let have a look at derivation.

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} \right) \mathrm{d}t \tag{8}$$

is the increase of delta hedged portfolio containing one option and for long position the worst scenario corresponds to minimum, i.e. portfolio will grow as slow as possible. Since minimum is taken only over volatility we can arrive to slightly modified Black-Scholes formula

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(\Gamma)^2 S^2 \frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{9}$$

where

$$\sigma(\Gamma) = \begin{cases} \sigma^+ & \text{if } \Gamma < 0 \\ \sigma^- & \text{if } \Gamma > 0 \end{cases} \tag{10}$$

In this project we assume we are always writer of the option, hence, our portfolio is with negative sign but unfortunately the only changes are opposite inequalities in 10. For the implementation of finite difference method it means that at each time step we calculate gamma and accordingly set volatility as a lower or upper bound of the given interval.

1.3 Dividend payment

The dividend is presumed to be paid once in the middle of life-time of the option. Since dividend is an income for holder of the asset and not for buyer or seller of the option we may mistakenly expect that the price of the option does not change.

However, the dividend payment affects the stock price, which just drops down by the value of dividend, and this consequently influences option value. It is interesting that even though at time of dividend payment stock price undergoes jump the option value stay continuous as a function of time.

2 Main Analysis

The analysis will cover several types of options - American or European, call or put, with or without barriers and dividend.

Unless otherwise indicated the following parameters are used:

- Volatility: $\sigma \in [0.2, 0.3]$,
- Risk free interest rate: $r = 2.37$ (what is equal to WIBOR ON at 1 January 2015),
- Time to expiry: 0.5,
- Strike price: $K = 2400$,
- Barrier for call options: $B_{call} = 2600$,
- Barrier for put options: $B_{put} = 2200$,

2.1 European call option

The idea of pricing European knock-and-out option was introduced earlier, in the figure 1 we see the result of numeric evaluation. At time equal to 0.5 the price obviously looks exactly as payoff, at asset price equals to 2600 the price is equal to zero, due to barrier. The price on other points of the grid was calculated with the help of the formulas which were introduced earlier, not surpassingly they are forming the surface, which seems to be as much smooth as possible, but still meets boundary values. The figure shows the option price with assumptions that we get an dividend during life of an asset - it should be treated in special way - the values of the asset shift at the moment of dividend payment, so that we use different scales before and after moment of payment. The difference is exactly the amount of dividend, the shift is done toward zero, because the price of asset is being lowered by amount of cash we get at the moment of dividend.

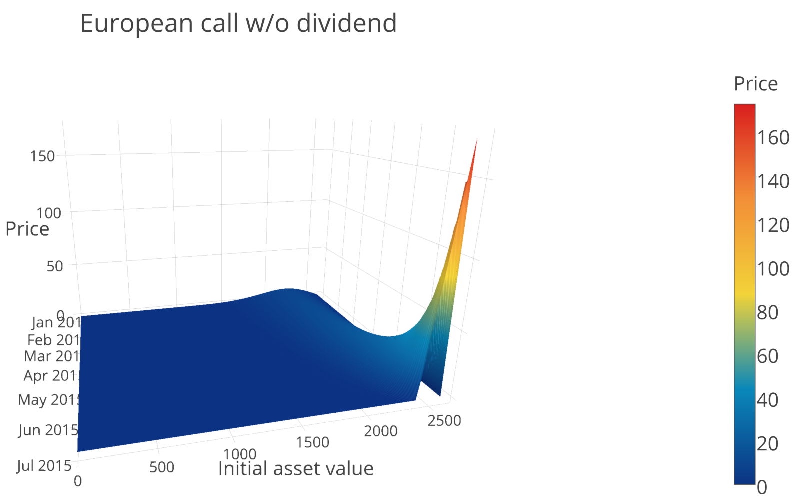

Figure 1: European call w/o dividend

The next figures will show dependencies between volatility and option price. As we can see, the longer period of volatility we take, the higher price we get, but the maximum is taken at the same point (Fig. 2).

Second analysis covers the shift in volatility, the length of period remains the same. In this case, the closer to zero volatility period is taken, the higher price is given. Moreover, the maximum also shifts closer to the option barrier with the period which is closer to zero (Fig. 3). After such observations the differences in shapes between the worst and the best case scenarios are obvious - more interesting is the difference between those two - the BCS is almost 16 times cheaper to hedge than the WCS (Fig. 4).

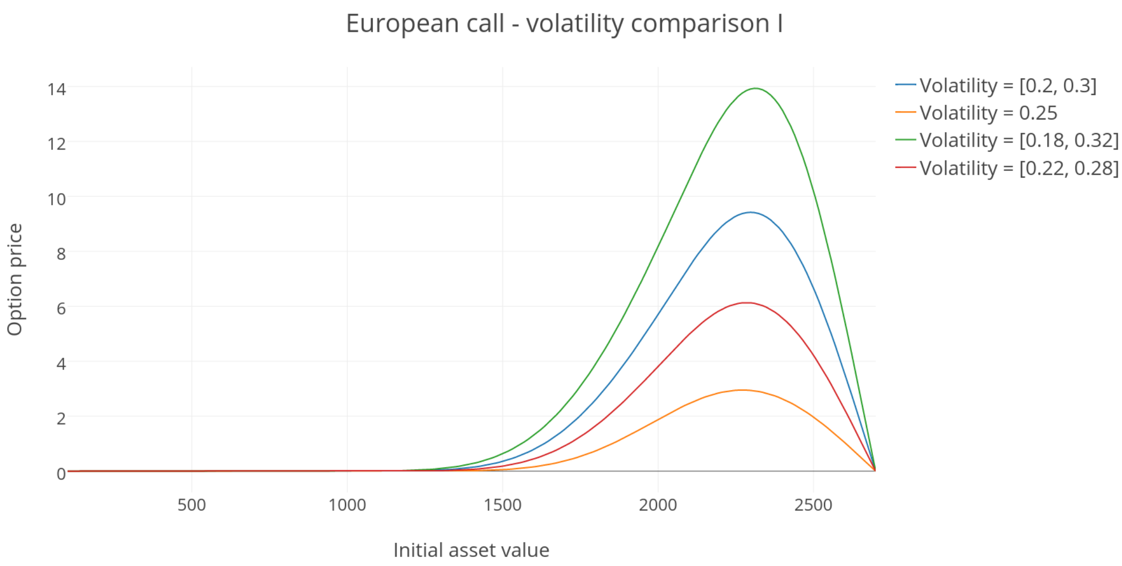

Figure 2: European call- volatility comparison I

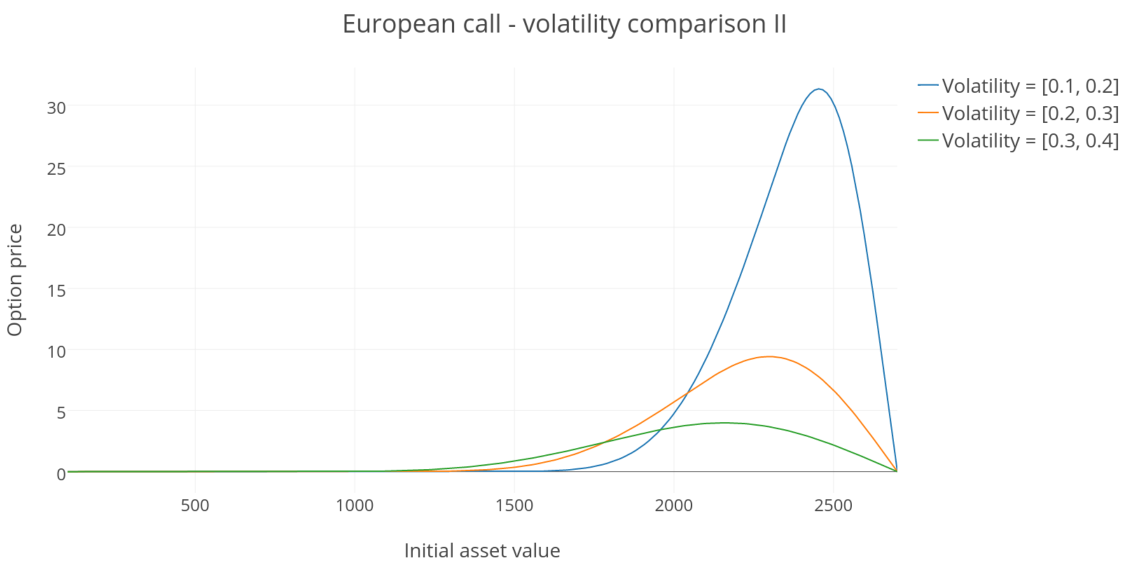

Figure 3: European call- volatility comparison II

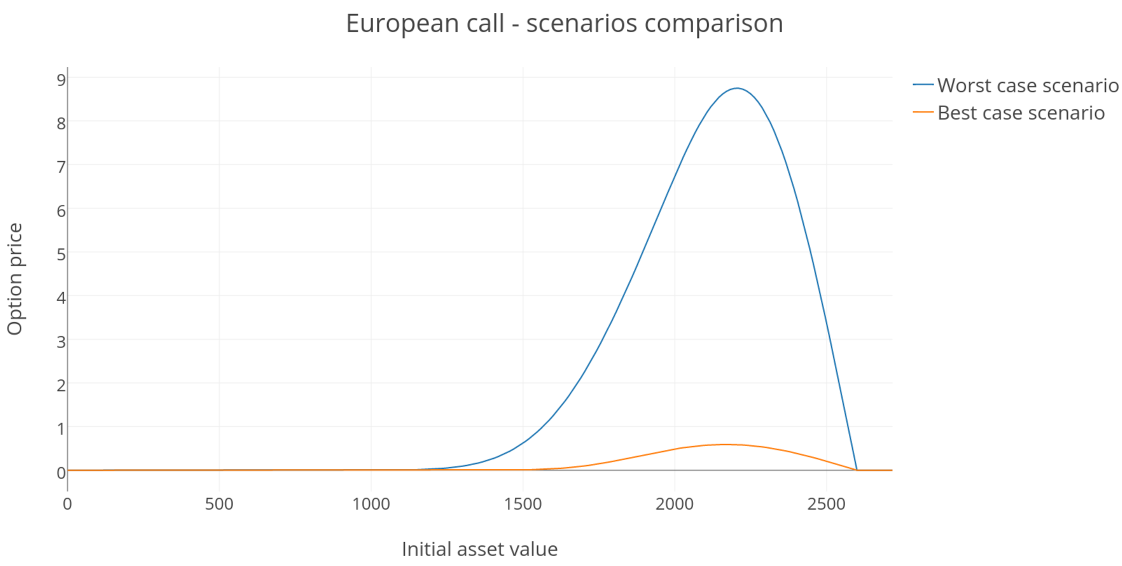

Figure 4: European call- scenario comparison

Which volatility we use depends on the delta calculated at this point. Following figure 5 shows the delta value on the whole grid. The dark blue color represents the points at which delta is negative, then we us there a lover volatility in our model (Fig. 5).

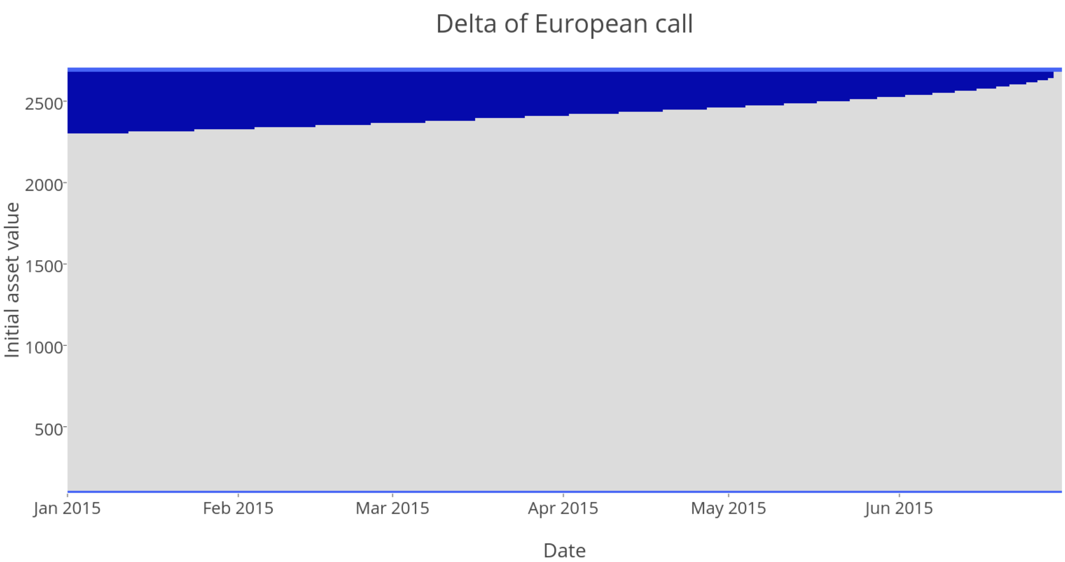

Figure 5: Delta of European call

2.2 European put option

Following figure 6 just shows how the price looks like, but the whole idea is the same as in call option.

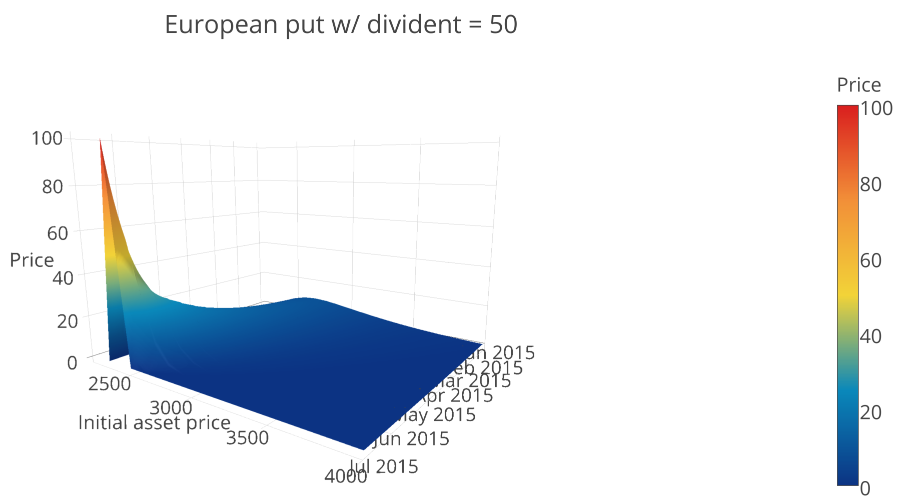

Figure 6: European put w/ dividend $= 50$

Following figure shows the dependencies between the price and the barrier level - this comparison brings obvious results, - the higher barrier we take, the higher price we get, also the maximum is shifted towards the barrier.

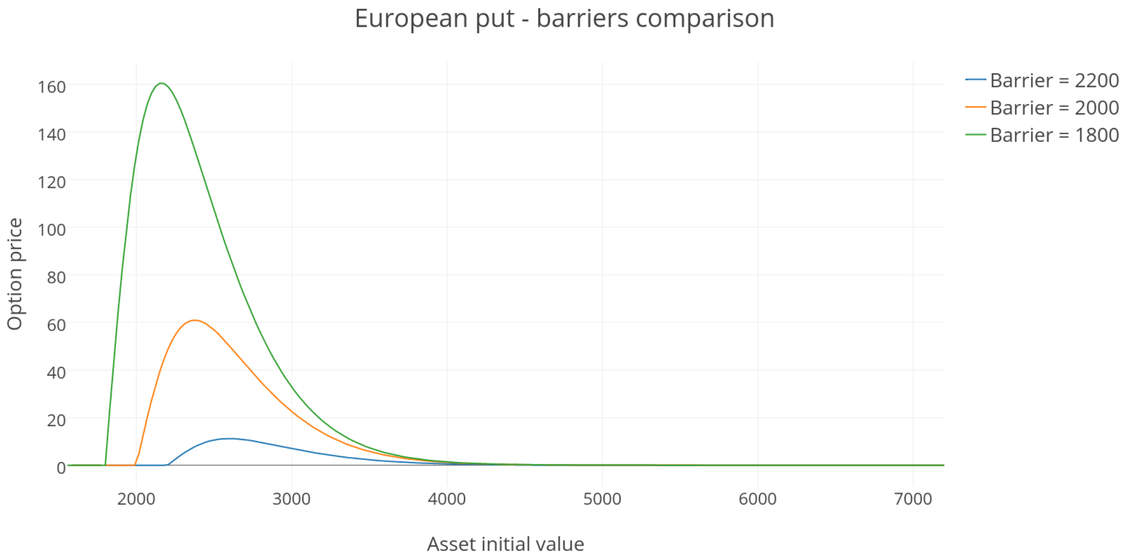

Figure 7: European put - barriers comparison

2.3 American call option

Two following figures compares American options, first with barrier and dividend, another without barrier and dividend. The figures shows that in case of American option the dividend have significant influence on option price during its life. The continuous nature of American options causes this (Fig. 8 and 9).

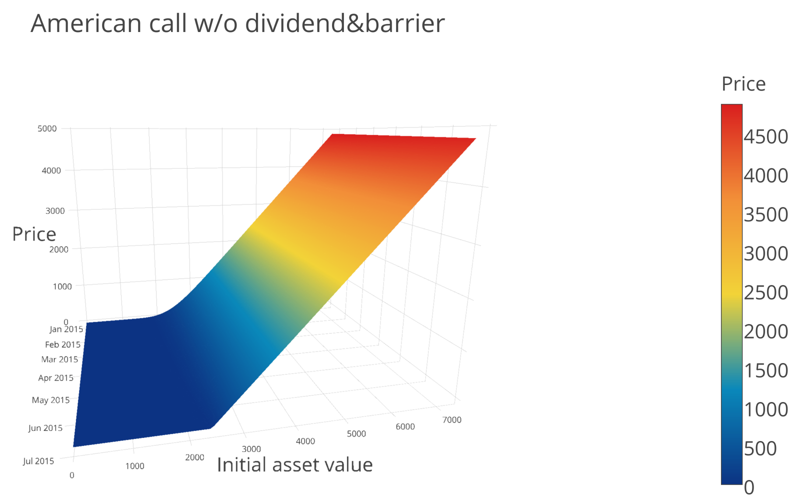

Figure 8: American call w/o dividend & barrier

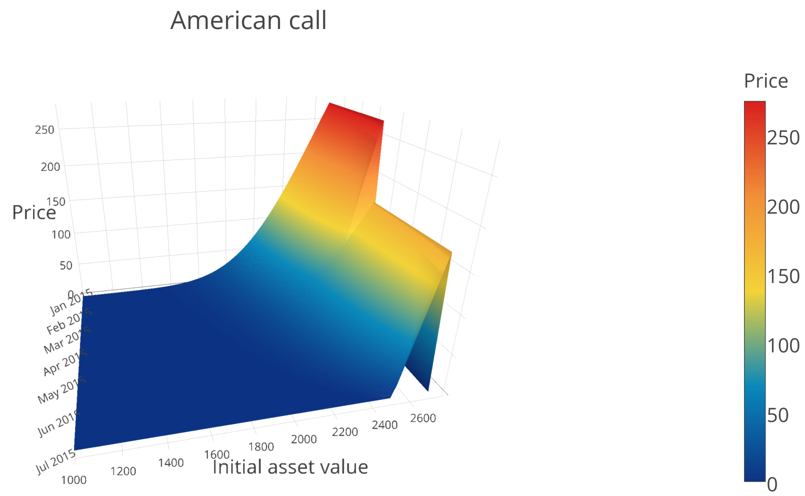

Figure 9: American call

The next figures will show dependencies between volatility and option price. As we can see, the longer period of volatility we take, the higher price we get. The same law could be applied to the second way of volatility period change, but in this case, the difference is much more significant. That happened because at the lower asset prices we more likely calculate the higher asset value rising what could cause higher payoff in the future. (Fig. 10 and 11).

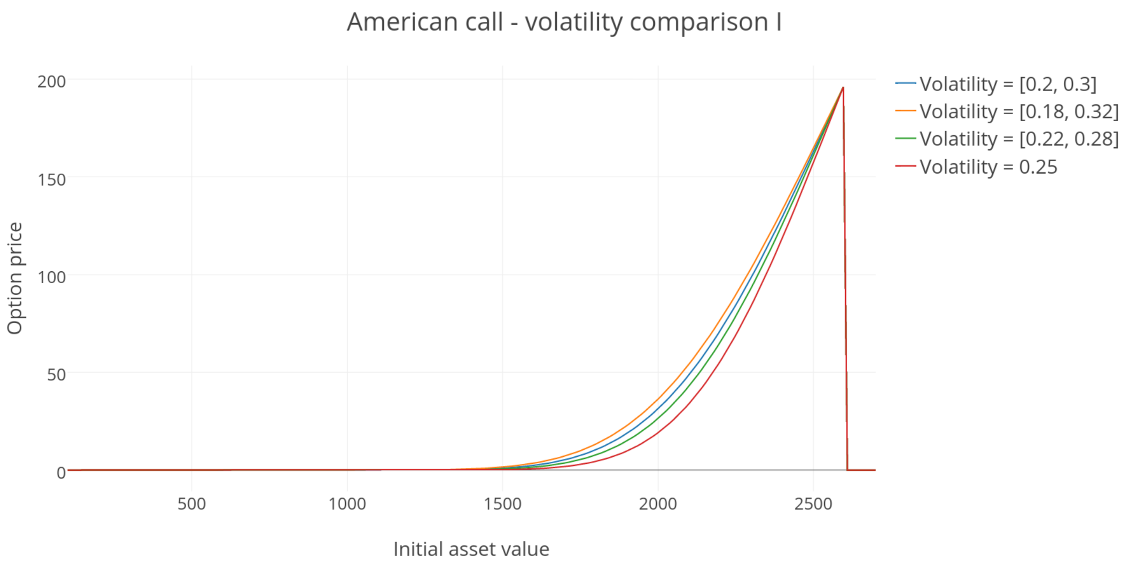

Figure 10: American call - volatility comparison I

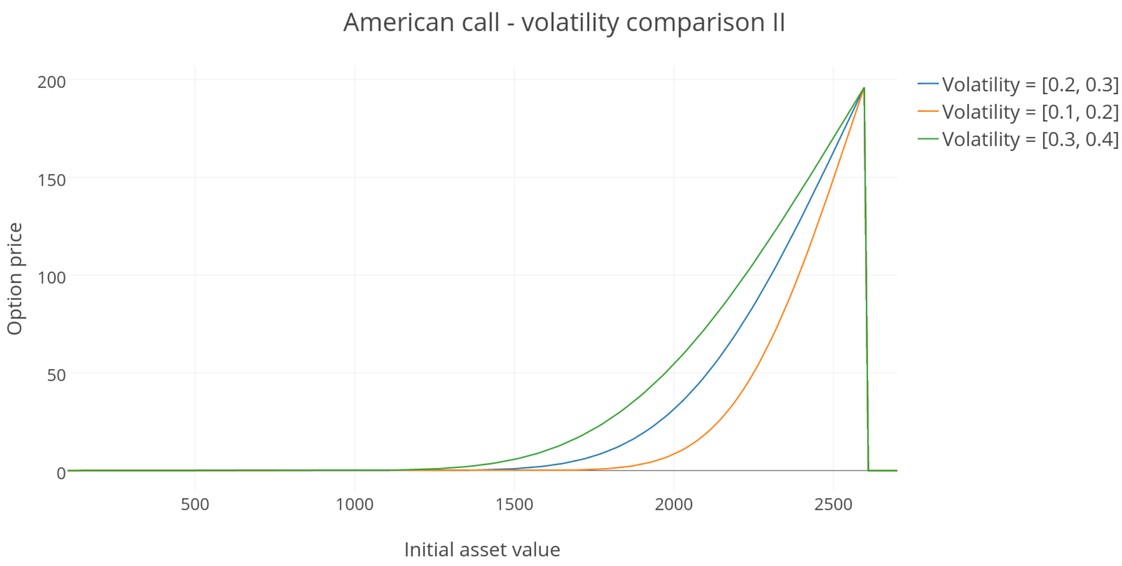

Figure 11: American call - volatility comparison II

Next figure shows dependency between highness of dividend and option price. It is logical, that the higher dividend we use to calculations the lower price we get at the initial values close to the strike. It is caused by the fact, that there exist scenario, in which we do not exercise our options just before dividend, and after dividend it is out-of-money till the end of its life, so we do not earn anything (as a owner of option) (Fig. 12).

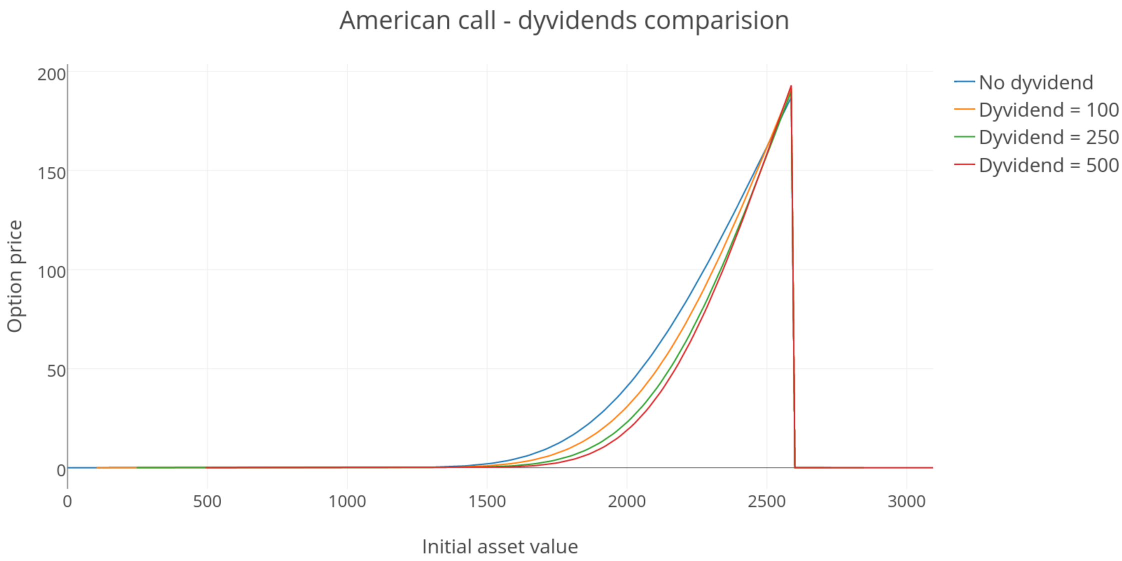

Figure 12: American call - dividends comparison

2.4 American put option

Following figure and 14 shows the price of American put option with dividend. Unfortunately, the scales on this figure are not proper and also, the shift caused by dividend is problematic to show. The saddle on this figure is caused by the fact, that there exist the scenario, in which the option could jump from middle position to the very high position (in terms of payoff). The two different scales on this figures could be useful to imagine what exactly happened there. What is missing there, I that, at the first part of the year, the payoff should not be cut at the level 100, but should be drawn up to 200.

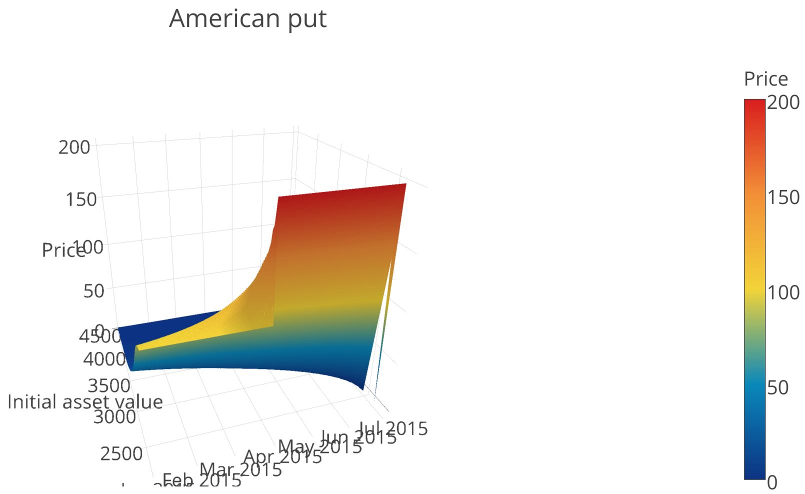

Figure 13: American put

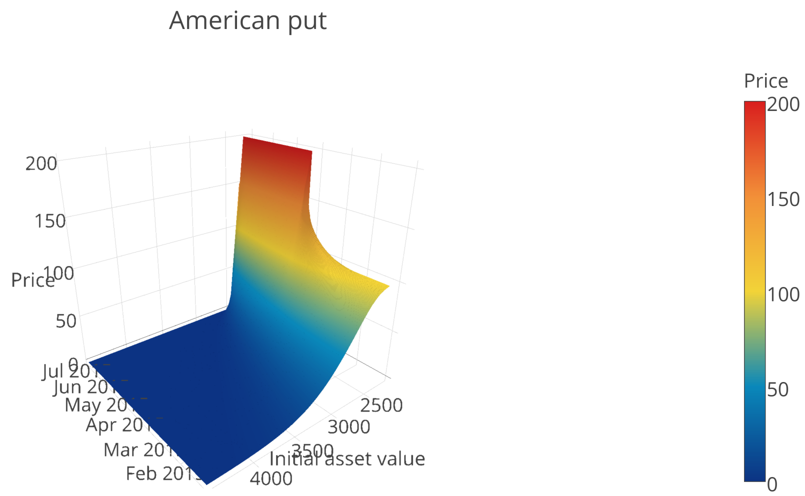

Figure 14: American put

3 Other method of valuation - Binomial Tree

3.1 Introduction to Binomial Tree Method

Binomial tree is graphical reprezentation of randomness of asset price in time. We assume the asset can only increase or decrease in adopted time step. In one time step the asset price can increase by u_1 times or decrease (by u_2 times).

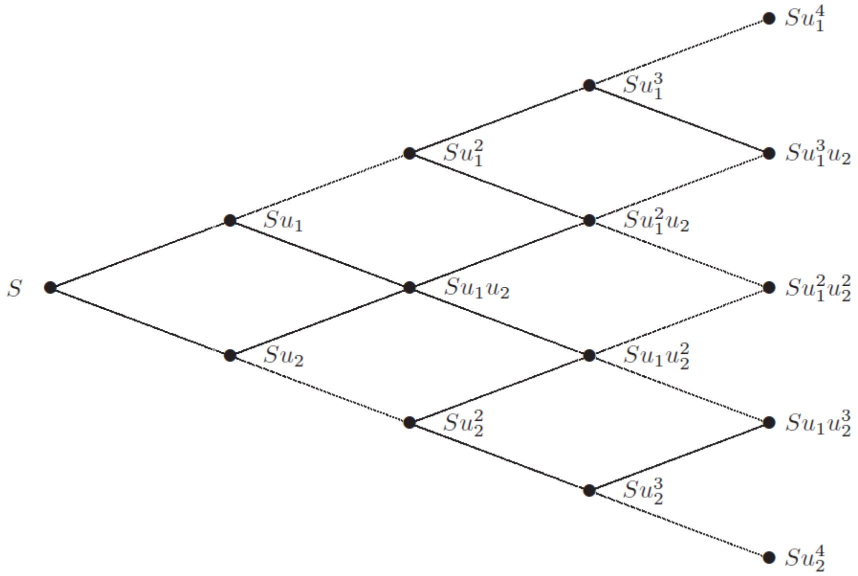

Figure 15: Construction of binomial tree.

In our model we assume:

$$u_1 = \frac{1}{u_2} = e^{r \cdot \sqrt{\delta t}}.$$

We define the arbitrage meassure (it is meassure, which does not allow to the arbitrage) by:

$$q_t = \frac{e^{r \cdot \delta t} - u_2}{u_1 - u_2}.$$

Then, the value of european option in every point of the binomial tree is given by:

$$V_t = e^{-r \cdot \delta t} \cdot \left[q_t \cdot V^+_{t+\delta t} + (1 - q_t) \cdot V^-_{t+\delta t} \right]$$

where $V^+_{t+\delta t}$ is the value of option in the next time step for the increased price of asset, $V^-_{t+\delta t}$ is the value of option in the next time step for the decreased price of asset.

For the last points of the binomial tree (where $t = maturity$) the option value equals the option payoff ($max\{S - K, 0\}$ for call and $max\{K - S, 0\}$ for put, where K is the strike price).

3.2 Binomial Tree Method for american and barrier options

For american options an option value is given by:

$$V_t = max\left\{e^{-r\cdot\delta t}\cdot\left[q_t\cdot V^{+}_{t+\delta t}+(1-q_t)\cdot V^{-}_{t+\delta t}\right], payoff(S_t)\right\}$$

Of course, for barrier options, in every point of the binomial tree, where the asset price is higher than the barrier (for call oprions) or lower than the barrier (for put options), the value of the option equals 0.

3.3 Binomial Tree Method with dividend

For the option with constant dividend we have a problem with "dissociating" binomial tree, which is shown below:

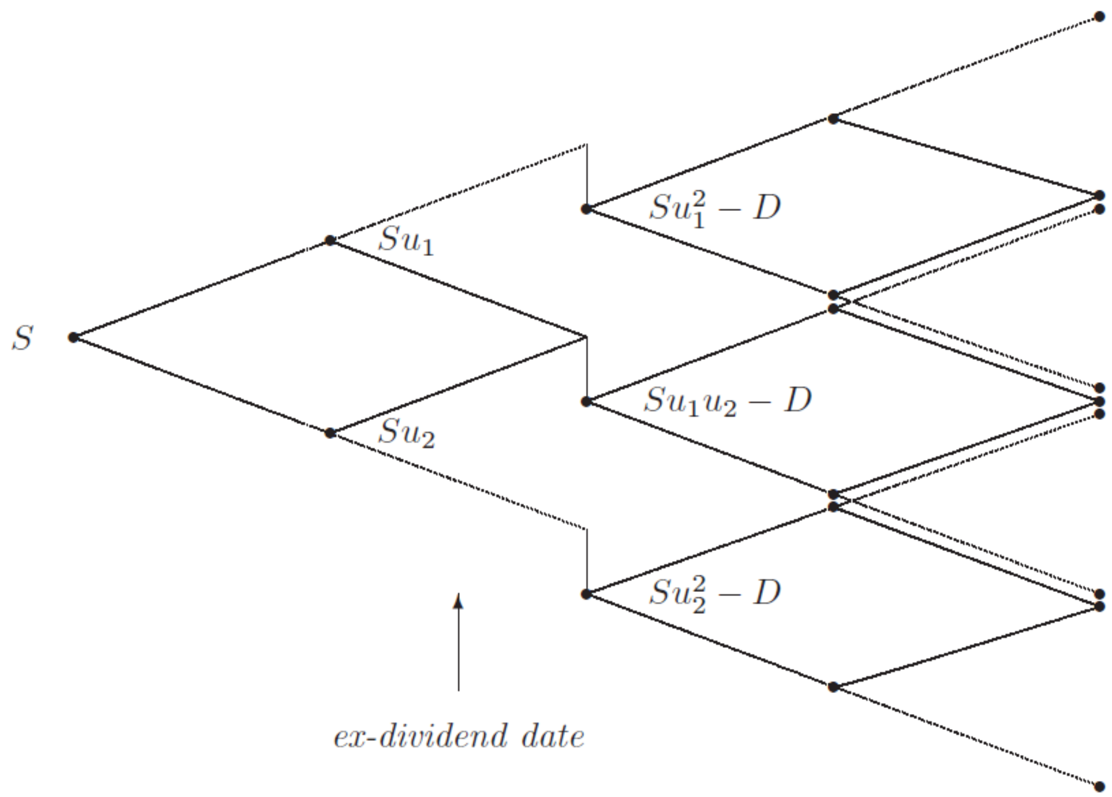

Figure 16: Binomial tree with constant dividend.

To avoid it, we construct our binomial tree in slightly different way. Before the payment of dividend (in time τ) the price of asset equals $S^* = S - D\cdot e^{-r(\tau - i\cdot\delta t)}$, where $i\cdot\delta t \le \tau$. For such defined S^* we create the binomial tree of asset price

$S^* \cdot u_1^j \cdot u_2^{i-j}$, $j = 0, 1, \ldots, i$, where $i \leq n$, $n \cdot \delta t = maturity$, and then for $i \cdot \delta t \leq \tau$ we have $S^* \cdot u_1^j \cdot u_2^{i-j} + D \cdot e^{-r(\tau - i \cdot \delta t)}$. This method allows us to bypass "dissociating" binomial tree and valuate the option for an asset with dividends.

3.4 Initial values

In our binomial tree model we adopted the following values:

- Maturity: $T = 0.5$,
- Step time: $\delta t = T/250$,
- Volatility: $\sigma = 0.25$,
- Risk free rate: $r = 0.0237$,
- Strike price: $K = 2400$,
- Dividend: $D = 50$,
- Barrier for call options: $B_{call} = 2600$,
- Barrier for put options: $B_{put} = 2200$,

3.5 Comparison of finite difference and binomial tree

With initial values from list above, we compared valuations of four options for the beginning time:

- european option call knock-and-out,
- european option put knock-and-out,
- american option call knock-and-out,
- american option put knock-and-out.

The following graphs show the comparision of these two methods:

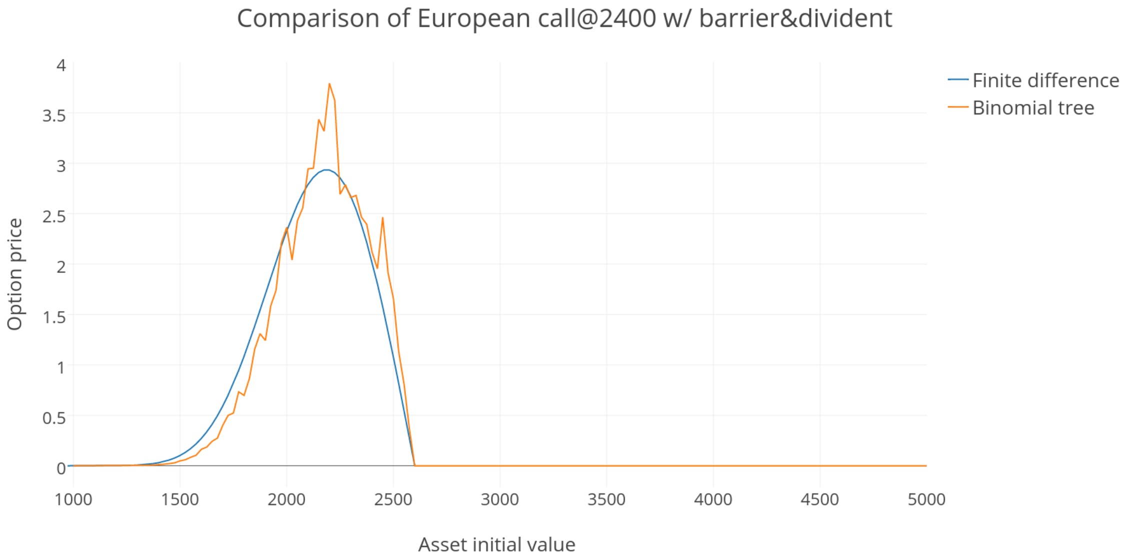

Figure 17: Comparison of European Call@2400 with barrier and dividend.

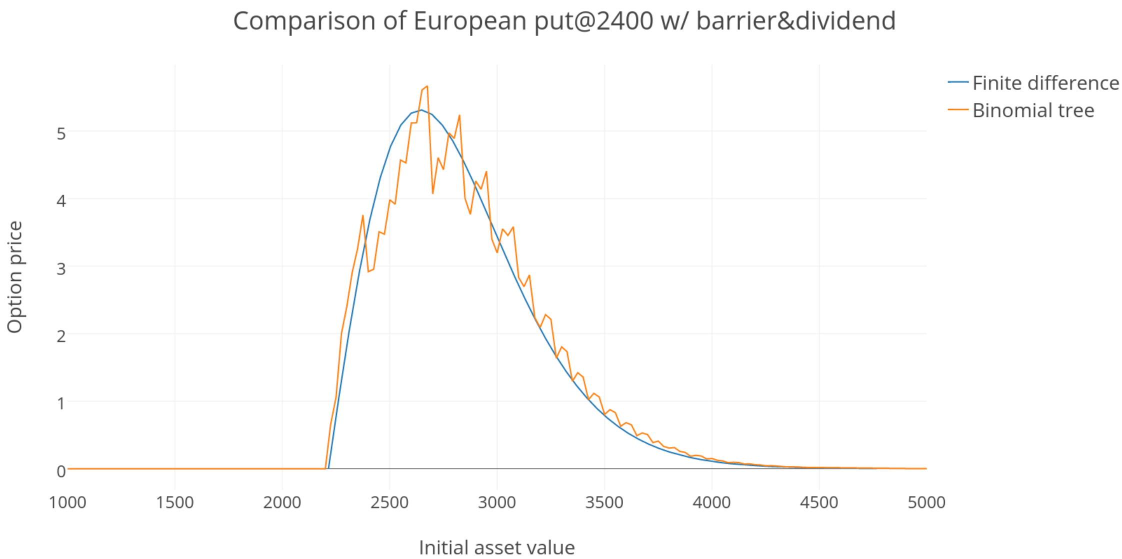

Figure 18: Comparison of European Put@2400 with barrier and dividend.

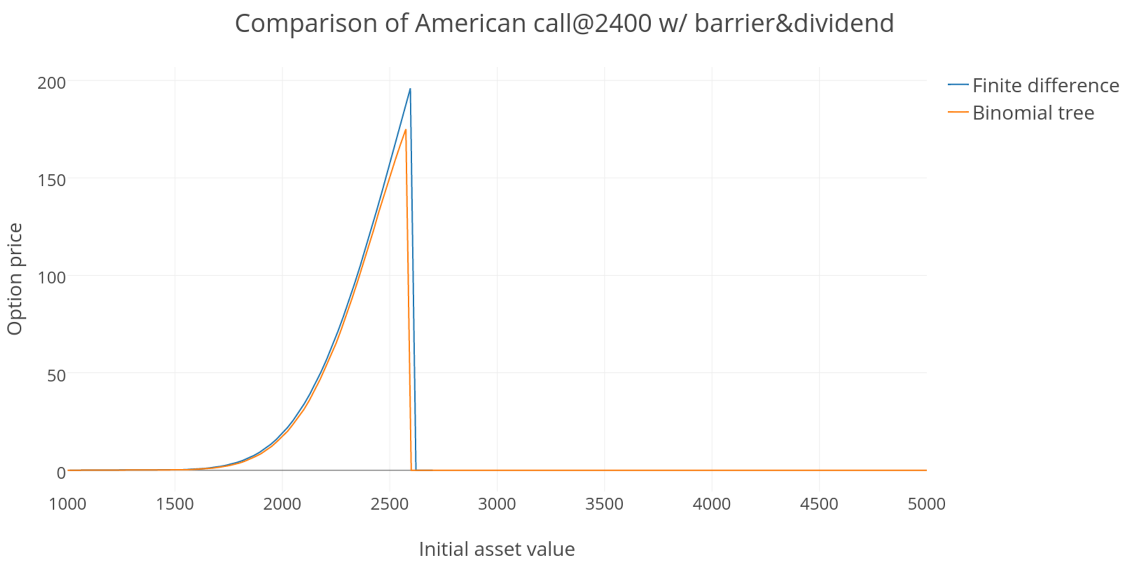

Figure 19: Comparison of American Call@2400 with barrier and dividend.

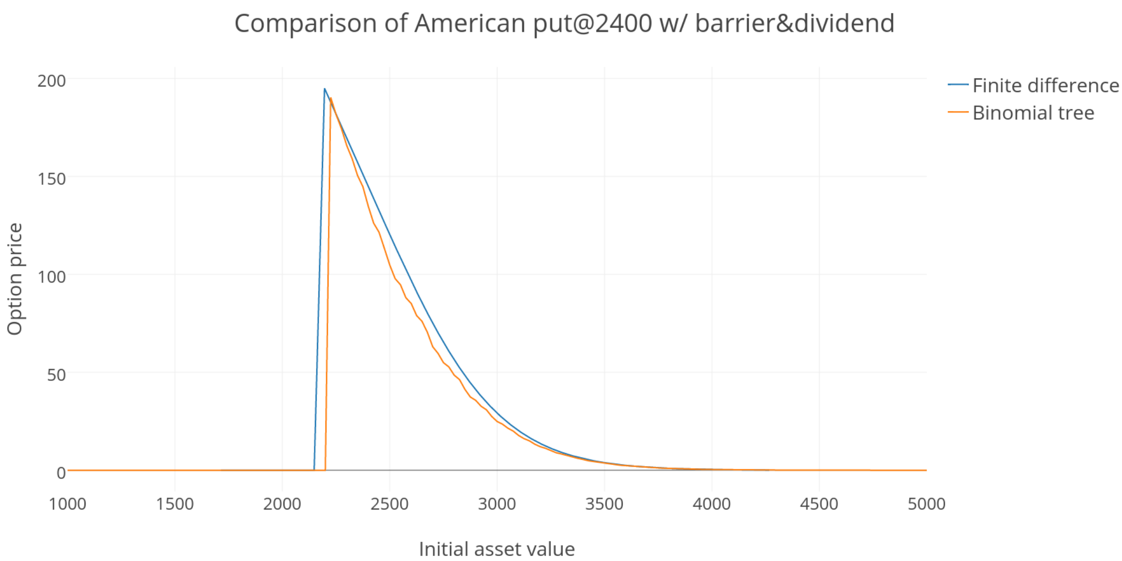

Figure 20: Comparison of American Put@2400 with barrier and dividend.

3.6 Conclusions from comparison

- For every option shapes of plots are generally similar for both methods.

- We can clearly see that the shape of the lines from binomial tree method is not smooth enough.
- The sharpness of lines probably comes from lack of number of steps (the number is as high as possible for acceptable time of computing).
- The other suspected reason is that there is no estimation for option values nearby the barrier.
- For american options we can see that plots do not focus in the same point. The reason of that is somewhere in scaling of plots - separated plots looked very good, but in the same graph they cant match exactly. Though many time spent on checking the code, we cannot find where is the mistake.