Deep Learning

Lecture 1: Linear Algebra

Dr. Mehrdad Maleki

Sets

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- Set $A = \{a, b, c\}$ is the same as the set $B = \{b, c, a\}$.
- There is no duplicate element in a set. So $\{a, b, b, c\}$ is the same as $\{a, b, c\}$.

Basic Concepts

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- If every element of a set A is also an element of set B we call A a **subset** of B and write $A \subseteq B$.

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- Irrational Numbers: $\mathbb{Q}^c = \{x : x \text{ is not rattional}\}$
- Real Numbers: $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{Q}^c$

Hierarchy of Sets

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

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- Looking for the roots of the equation like $x^2+1=0$ leads to invention of $\mathbb{C}$.

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- **Complement**: $\overline{A} = \{x : x \notin A\}$
- **Set Minus**: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
- Definition of complement is depend on a reference set.

Venn Diagram $A \cup B$

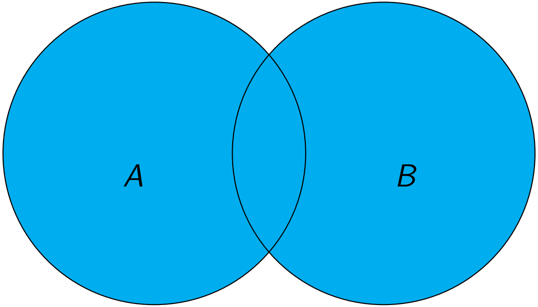

Figure: Union $A \cup B$

Venn Diagram $A \cap B$

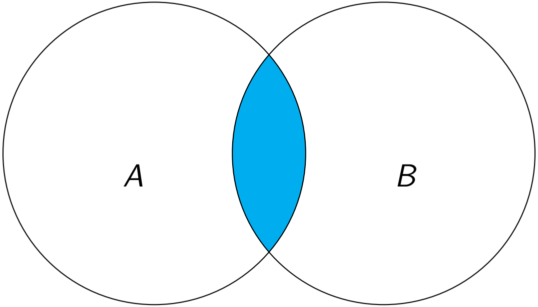

Figure: Intersection $A \cap B$

Venn Diagram $A \setminus B$

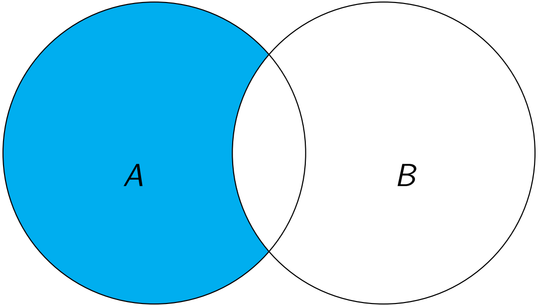

Figure: Set minus $A \setminus B$

Some Useful Properties

- $\emptyset \subseteq A$
- $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$
- **De Morgan's laws**:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$
$$\overline{\overline{A}} = A$$

Set in Python

```
empty_set=set()
```

```
x={'a','b','c'}
```

```
y={'c','d','d','e','f'}
```

```
y
```

```
{'c', 'd', 'e', 'f'}
```

Set in Python

```
x.union(y)
```

```
{'a', 'b', 'c', 'd', 'e', 'f'}
```

```
x.intersection(y)
```

```
{'c'}
```

```
x.difference(y)
```

```
{'a', 'b'}
```

Set in Python

```python
'a' in x
```

```
True
```

```python
'a' not in x
```

```
False
```

```python
x.issubset(y)
```

```
False
```

Cartesian Product

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- $A \times B \neq B \times A$
- A **relation** between A and B is a subset of $A \times B$.

Functions

- A **function** is a relation between two sets such that each element of the first set is related to at most one element of the second set.

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- Temperature of a cup of coffee is a function of time.
- Stock price with respect to time.
- Area of a circle with respect to its radius.

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- So if $a_1 = a_2$ then $f(a_1) = f(a_2)$.

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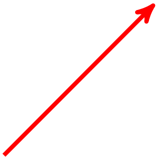

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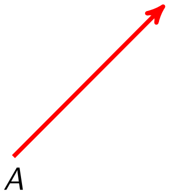

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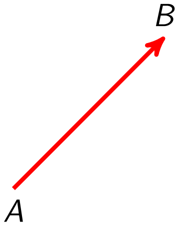

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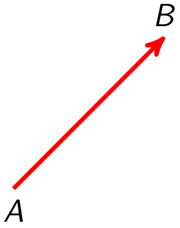

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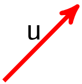

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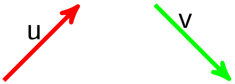

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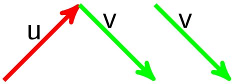

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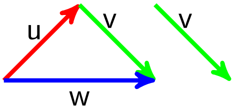

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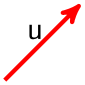

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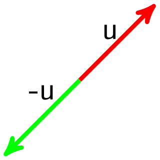

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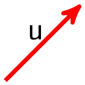

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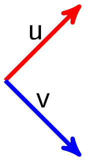

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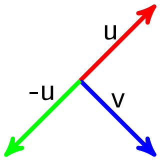

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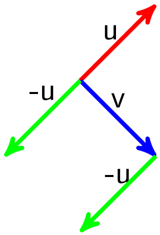

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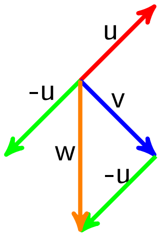

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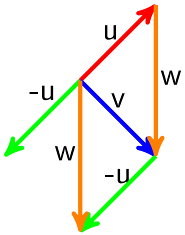
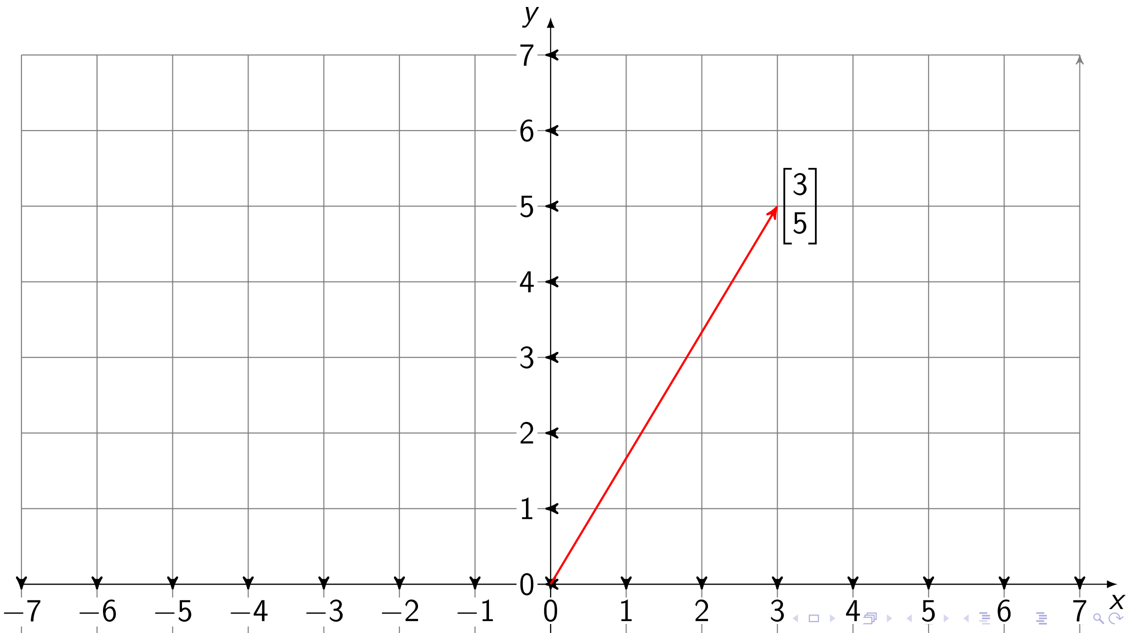


We can consider a reference and put the starting point of all vectors in this reference point, call it $\mathbf{O}$. So in two dimentions we will write a vector as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$


y
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7
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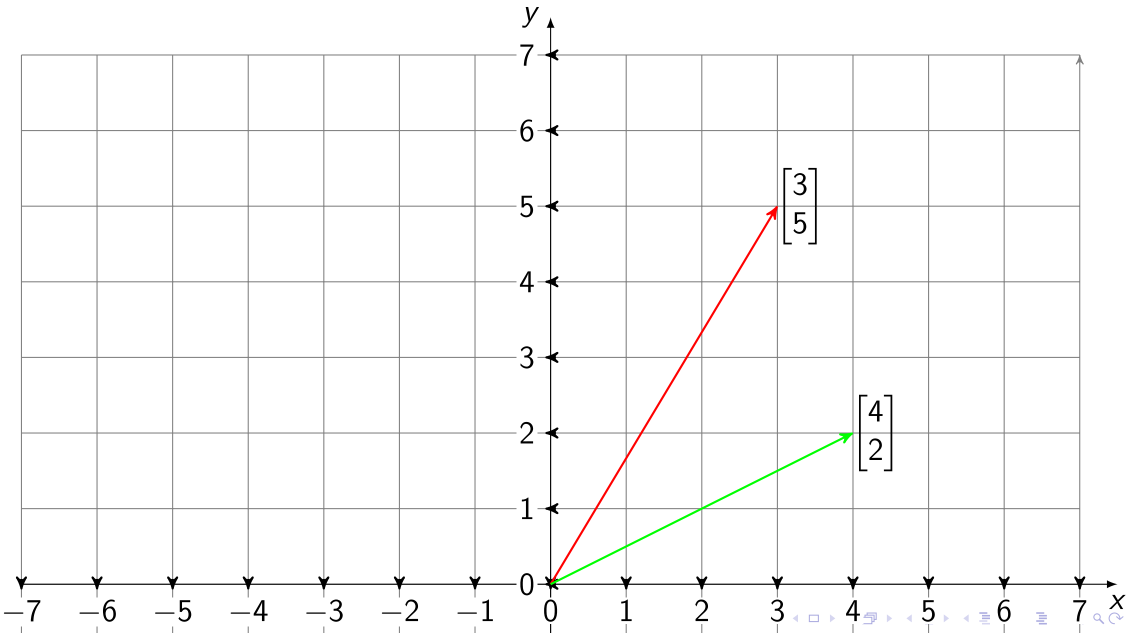
$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$



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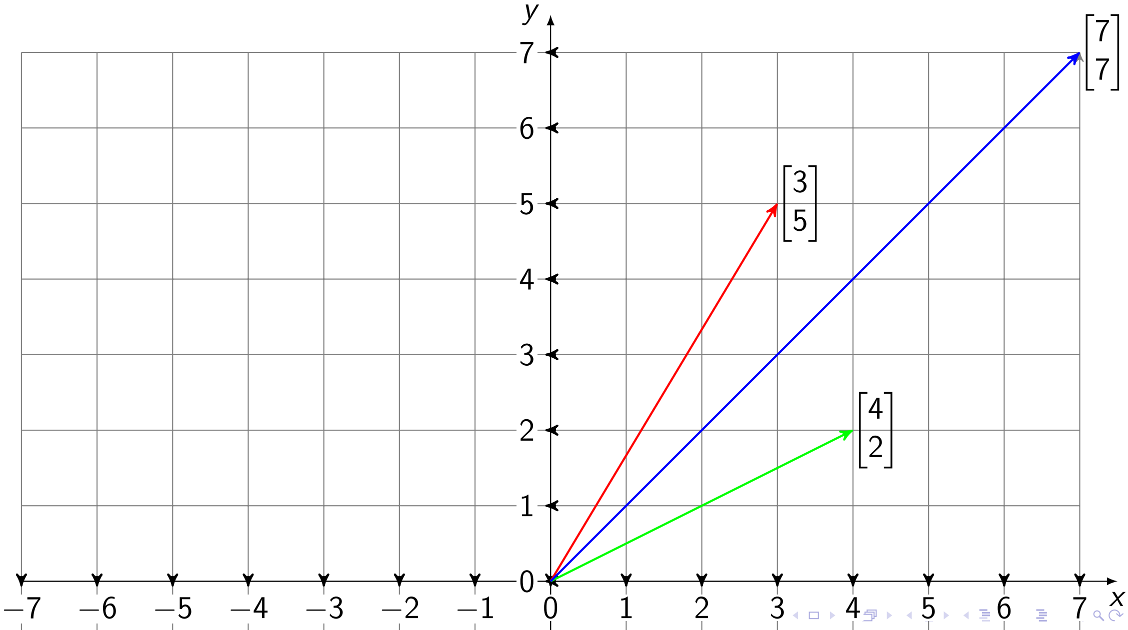
$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$
$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$



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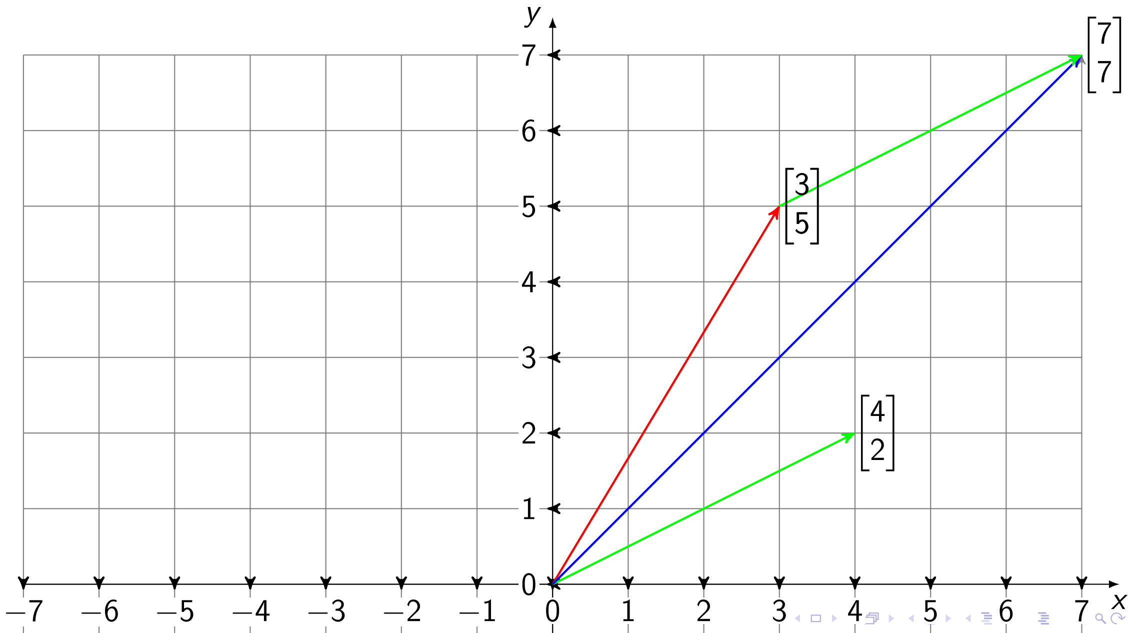
$\begin{bmatrix}3\\5\end{bmatrix}$
$\begin{bmatrix}4\\2\end{bmatrix}$



y
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$\begin{bmatrix} 7 \\ 7 \end{bmatrix}$
$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$
$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$
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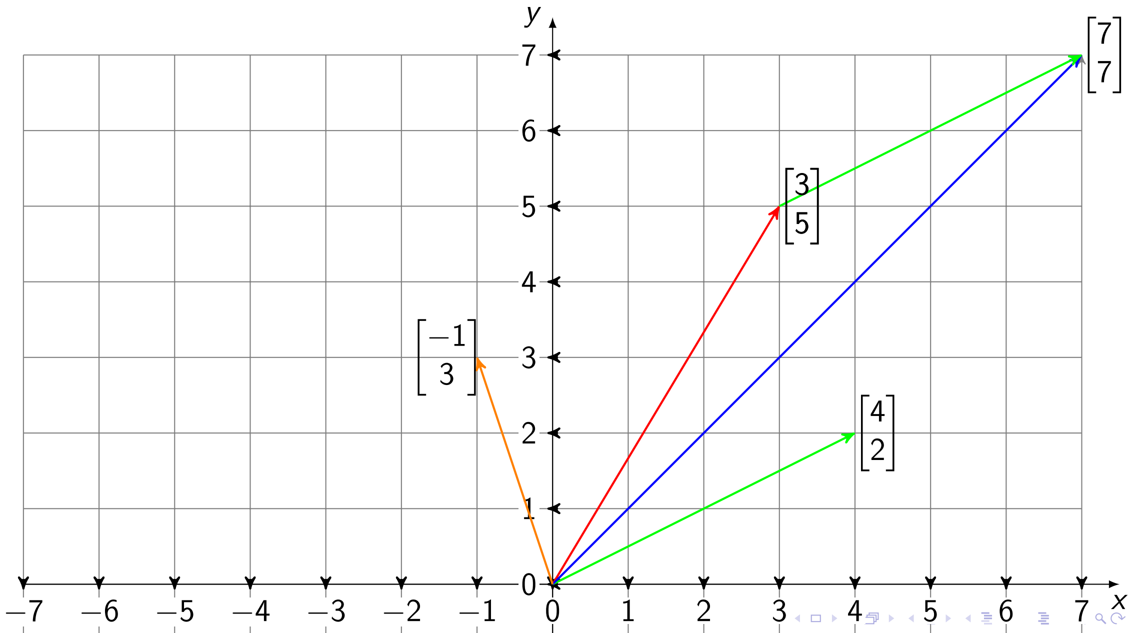

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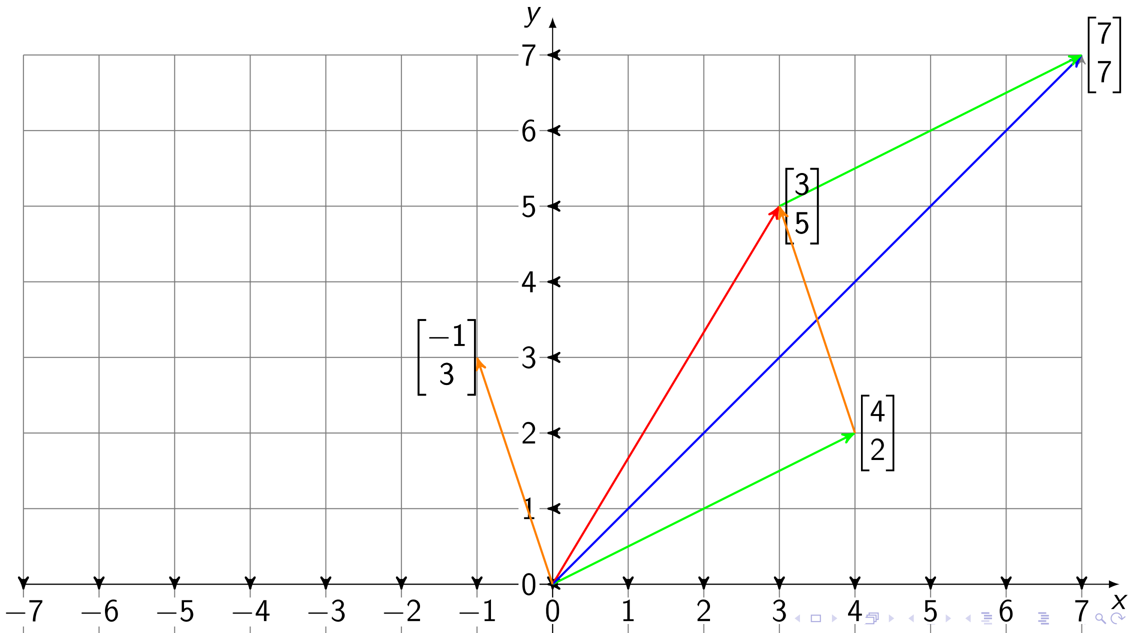

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Algebra of Vectors

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$$c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

n-dimensional vectors

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$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

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is denoted by $\mathbb{R}^n$.

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$$\|\mathbf{u}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

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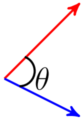
where θ is the angel between **u** and **v**.



If,

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If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u}$ is perpendicular with $\mathbf{v}$.

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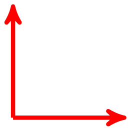

Let $\|\mathbf{u}\| = 1$ then $\mathbf{u} \cdot \mathbf{v}$ is the length of projection of $\mathbf{v}$ in the direction of $\mathbf{u}$,

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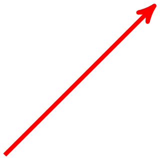

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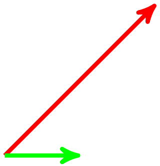

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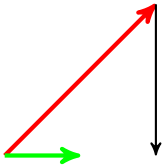

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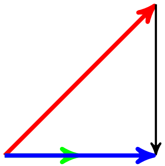

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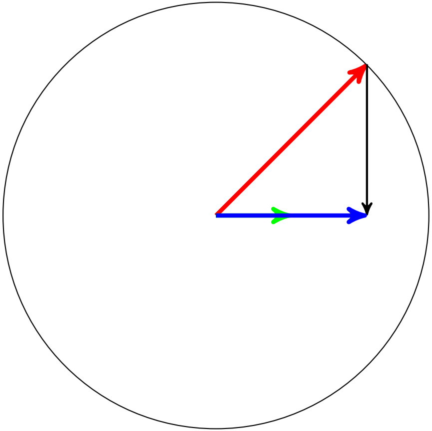

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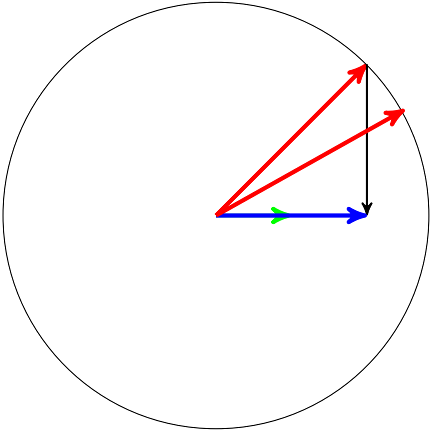

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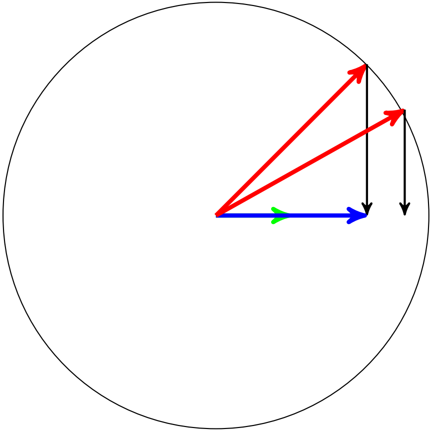

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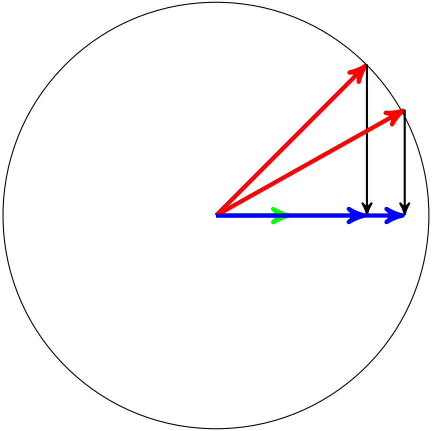

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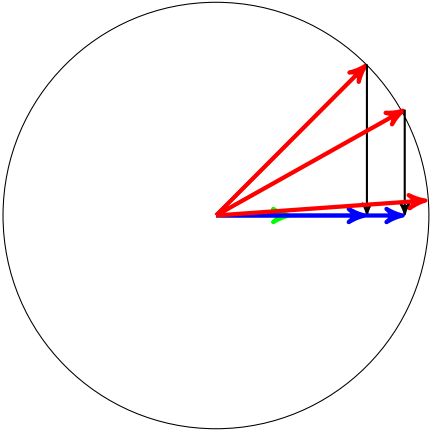

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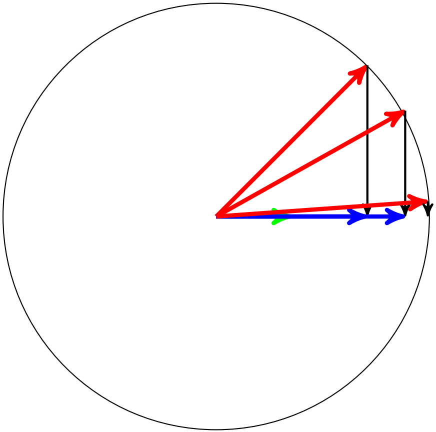

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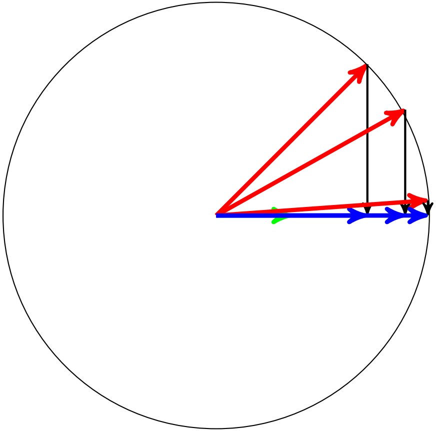

So, $\mathbf{u} \cdot \mathbf{v}$ is maximum if $\mathbf{v}$ is in the same direction as $\mathbf{u}$.

So, $\mathbf{u} \cdot \mathbf{v}$ is maximum if $\mathbf{v}$ is in the same direction as $\mathbf{u}$.

Vectors in Python

```
u=np.array([1,2,3])
```

```
v=np.array([4,5,6])
```

```
np.dot(u,v)
```

```
32
```

```
np.linalg.norm(u)
```

```
3.7416573867739413
```

Vectors in Python

```
t=np.inner(u,v)/(np.linalg.norm(u)*np.linalg.norm(v))
```

```
np.degrees(np.arccos(t))
```

```
12.933154491899135
```

Matrices

Matrix is a two-dimensional array-like data structure that represents a common quantity of two categories. For example, if you want to represent the number of students by gender in three different class of Math, CS, and Physic then you should use matrix representation as follow;

$$\begin{array}{cc} & \begin{array}{ccc} \color{red}{Math} & \color{red}{CS} & \color{red}{Physic} \end{array} \\ \begin{array}{c} \color{blue}{Men} \\ \color{blue}{Women} \end{array} & \begin{bmatrix} 23 & 37 & 18 \\ 28 & 14 & 16 \end{bmatrix} \end{array}$$

If a matrix has m rows and n columns then we say the **order** of the matrix is $m \times n$. So an n-dimensional vector is a special case of a matrix of order $n \times 1$.

A **square matrix** is a matrix that have the same number of rows and columns, i.e., $m = n$.

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An **identity matrix** is a square matrix with 1 in the main diagonal and 0 elsewhere. We denote the identity matrix of order n by $\mathbf{I}_n$.

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An **identity matrix** is a square matrix with 1 in the main diagonal and 0 elsewhere. We denote the identity matrix of order n by $\mathbf{I}_n$.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The **zero matrix** is a matrix such that every entries is 0. We denote the zero matrix of order $m \times n$ by $\mathbf{O}_{m\times n}$.

The **zero matrix** is a matrix such that every entries is 0. We denote the zero matrix of order $m \times n$ by $\mathbf{O}_{m\times n}$.

$$\mathbf{O}_{m\times n} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Transpose of an $m \times n$ matrix is an $n \times m$ matrix. The transpose of the matrix $\mathbf{A}$ is denoted by $\mathbf{A}^T$.

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$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

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$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} =$$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+0 & 3+1 \\ 4+2 & 5+(-1) & 6+2 \end{bmatrix}$$

If $\mathbf{A}_{1\times n}$ and $\mathbf{B}_{n\times 1}$ then we could define $\mathbf{AB}$ as follow,

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$$\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n b_n$$

So if $\mathbf{u}$ and $\mathbf{v}$ are n-dimensional vectors then,

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So if $\mathbf{u}$ and $\mathbf{v}$ are n-dimensional vectors then,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

If $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times k}$ then we could define $\mathbf{AB}$ as the following $m \times k$ matrix,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nk} \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \dots & \begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} & \dots \\ & \vdots & \end{bmatrix}$$

Inverse of Matrix

If for the square matrix $\mathbf{A}$ there exists another square matrix $\mathbf{B}$ such that,

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

then we say $\mathbf{A}$ is invertible and its inverse is $\mathbf{B}$. We denote $\mathbf{B}$ by $\mathbf{A}^{-1}$.

If $\mathbf{A}$ is invertible then the system of equations,

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has a unique solution, i.e.,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$\mathbf{I}\mathbf{A} = \mathbf{A}\mathbf{I} = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$
$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$
$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

A symmetric matrix is a square matrix such that $\mathbf{A}^T = \mathbf{A}$.

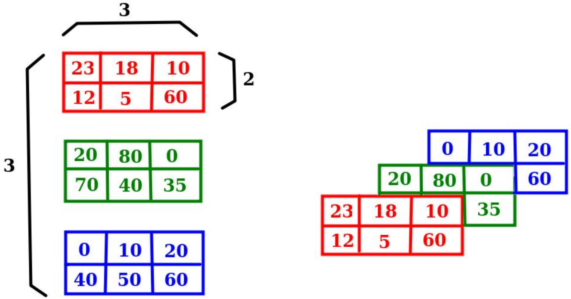
Tensors

Tensors are multi-dimensional array with a uniform type. A tensor of order $m \times n \times k$ (or shape (m, n, k)) is actually m matrices of order $n \times k$.

$$
\begin{bmatrix}
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\
\begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \\
\begin{bmatrix} 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix} \\
\begin{bmatrix} 19 & 20 & 21 \\ 22 & 23 & 24 \end{bmatrix}
\end{bmatrix}
$$

is a tensor of order $4 \times 2 \times 3$

$3 \times 2 \times 3$ tensor.



Kronecker(Tensor) Product

The **Kronecker product** of matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{k\times l}$ is a matrix of order $mk \times nl$ and define as follow,

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

$$
\begin{aligned}
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} x_1 \mathbf{I} & x_2 \mathbf{I} \end{bmatrix} \\
&= \begin{bmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_1 & 0 & x_2 \end{bmatrix}
\end{aligned}
$$

Eigenvalue

Let $\mathbf{A}$ be an square $n \times n$ matrix. If there exists n-dimensional vector $\mathbf{v}$ and a real number λ such that,

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

then we say $\mathbf{v}$ is an eigenvector and the corresponding eigenvalue is λ.

To obtain the eigenvalues of a matrix you need to find the roots of the following polynomal equation of order n,

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$$det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

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$$det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

eigenvalue could be complex number, i.e., a matrix may don't have any real valued eigenvalue but all eigenvalues of a symmetric matrix are real numbers.

Rotation Matrix

$$\mathbf{R}_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

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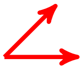
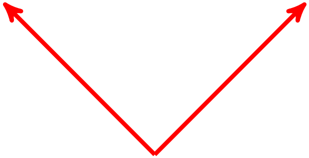


$\mathbf{R}_{\frac{\pi}{2}}$ has no eigenvalue but $\mathbf{R}^2_{\frac{\pi}{2}}$ has -1 as the only real eigenvalue.

$\mathbf{R}_{\frac{\pi}{2}}$ has no eigenvalue but $\mathbf{R}_{\frac{\pi}{2}}^2$ has -1 as the only real eigenvalue.



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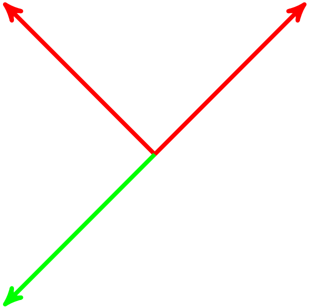

Matrices in Python

```
A=np.array([[1,2,3],
            [4,5,6]])
```

```
A.shape
```

```
(2, 3)
```

```
A.T
```

```
array([[1, 4],
       [2, 5],
       [3, 6]])
```

Matrices in Python

```
A=np.array([[1,2,3],
            [4,5,6]])
```

```
B=np.array([[1,0,1],
           [0,1,0],
           [1,1,1]])
```

```
np.dot(A,B).shape
```

```
(2, 3)
```

```
np.dot(A,B)
```

```
array([[ 4,  5,  4],
       [10, 11, 10]])
```

```
np.matmul(A,B)
```

```
array([[ 4,  5,  4],
       [10, 11, 10]])
```

Matrices in Python

```python
np.identity(3)
```

```
array([[1., 0., 0.],
       [0., 1., 0.],
       [0., 0., 1.]])
```

```python
np.zeros((3,4))
```

```
array([[0., 0., 0., 0.],
       [0., 0., 0., 0.],
       [0., 0., 0., 0.]])
```

Matrices in Python

```
Z=np.array([[1,1,0],
            [0,1,1],
            [1,0,1]])
```

```
np.linalg.inv(Z)
```

```
array([[ 0.5, -0.5,  0.5],
       [ 0.5,  0.5, -0.5],
       [-0.5,  0.5,  0.5]])
```

```
np.linalg.det(Z)
```

```
2.0
```

Matrices in Python

```
X=np.array([[2,3]])
Y=np.identity(2)
```

```
np.kron(X,Y)
```

```
array([[2., 0., 3., 0.],
       [0., 2., 0., 3.]])
```

Matrices in Python

```
np.kron(np.identity(3),np.zeros((4,5))+3)
```

```
array([[3., 3., 3., 3., 3., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [3., 3., 3., 3., 3., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [3., 3., 3., 3., 3., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [3., 3., 3., 3., 3., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 3., 3., 3., 3., 3., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 3., 3., 3., 3., 3., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 3., 3., 3., 3., 3., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 3., 3., 3., 3., 3., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3., 3., 3., 3., 3.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3., 3., 3., 3., 3.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3., 3., 3., 3., 3.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 3., 3., 3., 3., 3.]])
```

Matrices in Python

```
A=np.array([[0,1,1],
            [1,0,1],
            [1,1,0]])
```

```
np.linalg.eigvals(A)
```

```
array([-1.,  2., -1.])
```

eigenvalues

```
np.linalg.eig(A)
```

```
(array([-1.,  2., -1.]), array([[-0.81649658,  0.57735027,  0.19219669],
       [ 0.40824829,  0.57735027, -0.7833358 ],
       [ 0.40824829,  0.57735027,  0.59113912]]))
```

eigenvectors

Thank You