Deep Learning

Lecture 6: Gradient Descent

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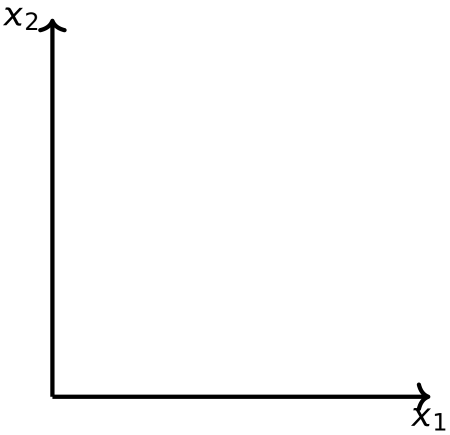
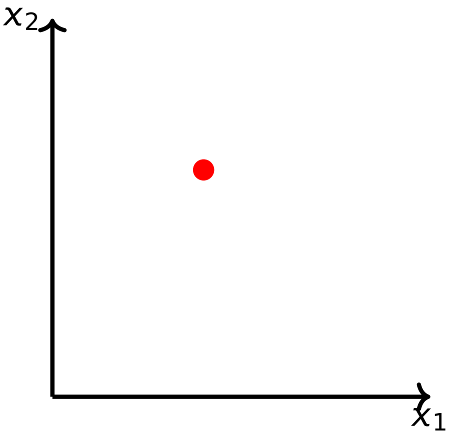
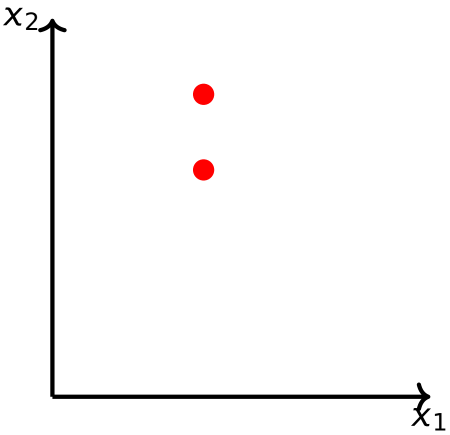
Motivation

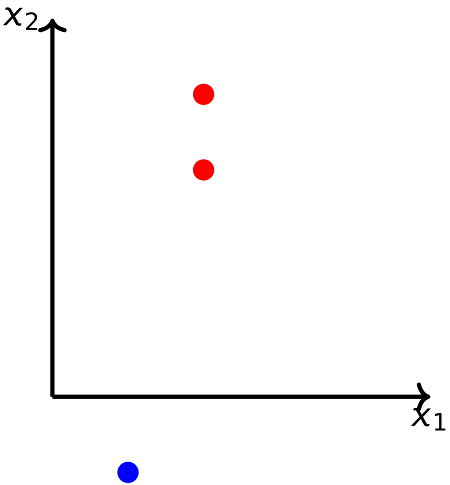
You have a NN that meant to compute desire function $g(\mathbf{X})$. But your NN actually compute $f(\mathbf{X}, \mathbf{W})$ and you need to find the value of $\mathbf{W}$ such that $||g(\mathbf{X}) - f(\mathbf{X}, \mathbf{W})||$ is minimum. But the value of $g(\mathbf{X})$ is not known for all $\mathbf{X}$. So we sample the function g. On input $\mathbf{X_i}$ if $\mathbf{y_i} = g(\mathbf{X_i})$ then for $i = 1, \ldots, N$, $\{(\mathbf{X_i}, \mathbf{y_i}) : 1 \leq i \leq N\}$ is the set of samples.

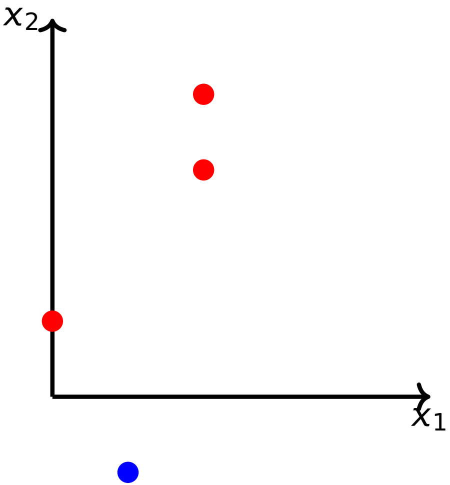

x_2
x_1



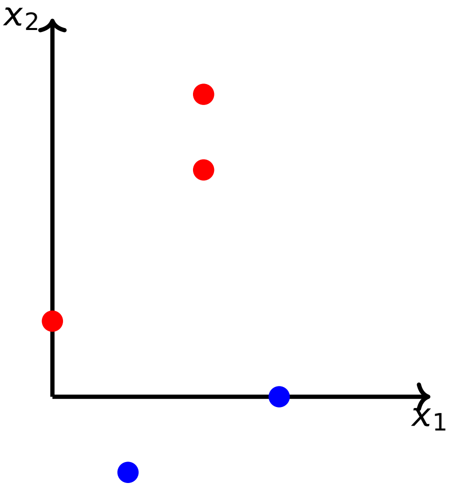
x_2
x_1

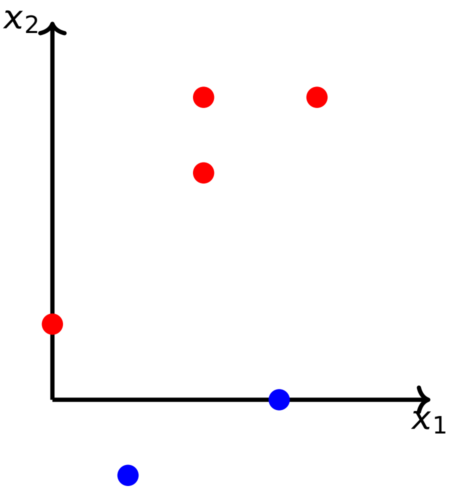

x_2
x_1

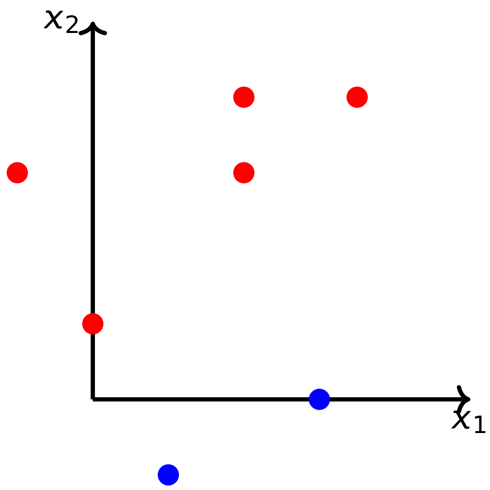

x_2
x_1

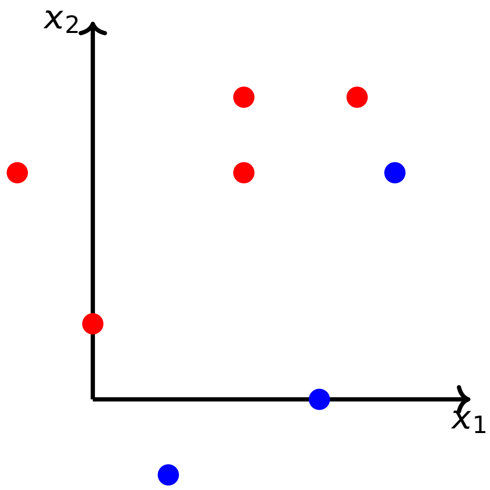


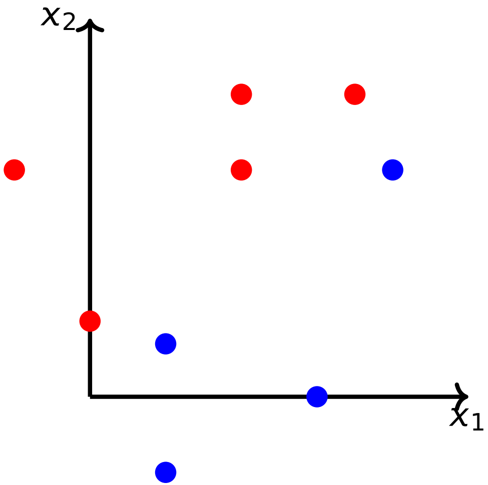
x_2
x_1

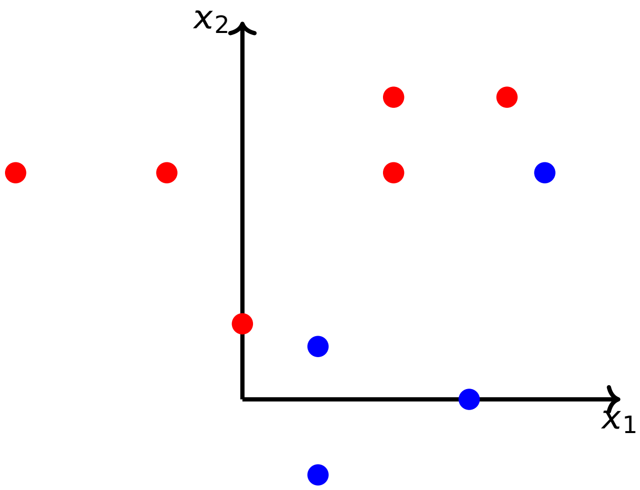

x_2
x_1

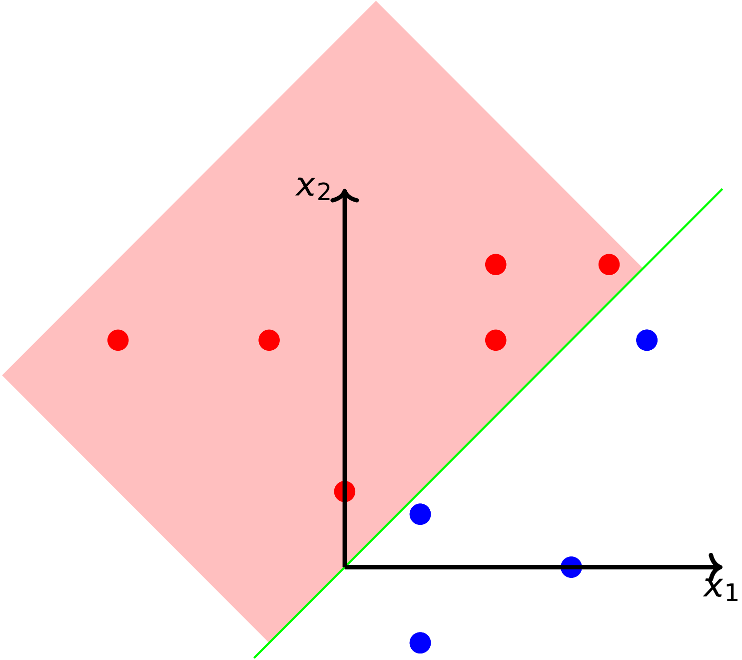

x_2
x_1

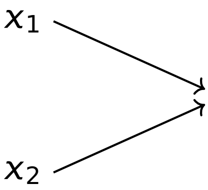

x_2
x_1

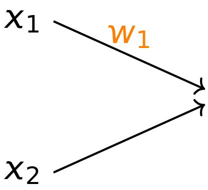

x_2
x_1

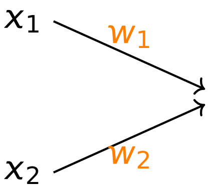

x_2
x_1

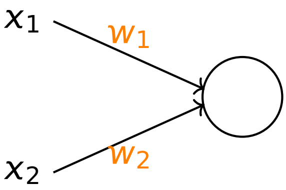

x_2
x_1


x_2
x_1

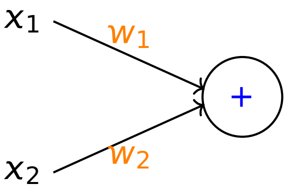

x_1
x_2

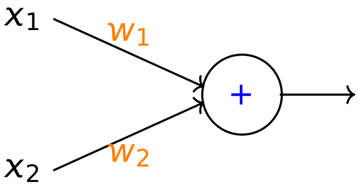

x_1
w_1
x_2

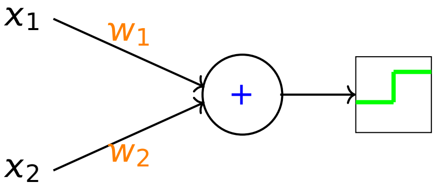

x_1
w_1
x_2
w_2

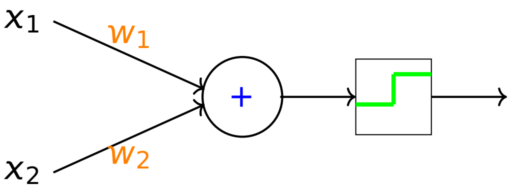


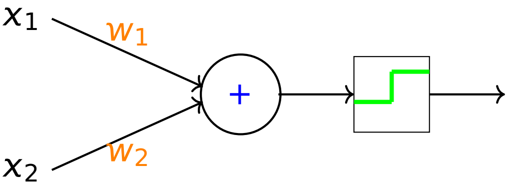
x_1
w_1
x_2
w_2


x_1
w_1
$+$
x_2
w_2

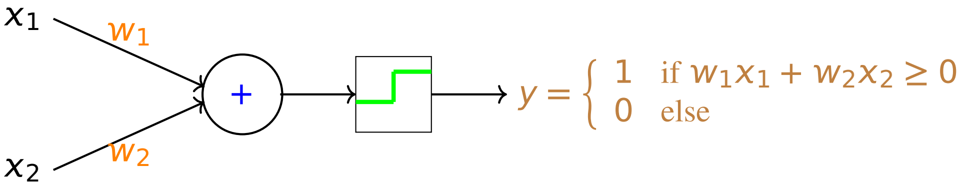

x_1
w_1
+
x_2
w_2


x_1
w_1
+
x_2
w_2


x_1
w_1
x_2
w_2
+



$$y = \begin{cases} 1 & \text{if } w_1 x_1 + w_2 x_2 \geq 0 \\ \end{cases}$$


x_1
w_1
x_2
w_2
$+$
$y = \begin{cases} 1 & \text{if } w_1x_1 + w_2x_2 \geq 0 \\ 0 & \text{else} \end{cases}$

We know that,

	Red		Blue
1	$g(2,3)=1$	7	$g(1,-1)=0$
2	$g(2,4)=1$	8	$g(3,0)=0$
3	$g(0,1)=1$	9	$g(4,3)=0$
4	$g(3.5,4)=1$	10	$g(1,0.7)=0$
5	$g(-1,3)=1$		
6	$g(-3,3)=1$		

we need to find the parameters w_1, w_2 such that if

$$f(x_1, x_2\,; w_1, w_2) = \begin{cases} 1 & \text{if } w_1 x_1 + w_2 x_2 \geq 0 \\ 0 & \text{else} \end{cases}$$

then,

$$\sum_{i=1}^{10} \| f(x_1^{(i)}, x_2^{(i)})\,; w_1, w_2) - g(x_1^{(i)}, x_2^{(i)}) \|$$

is minimum.

Suppose we only have one point with red label, i.e., 1. Then the optimal weight vector for this point is as follow,

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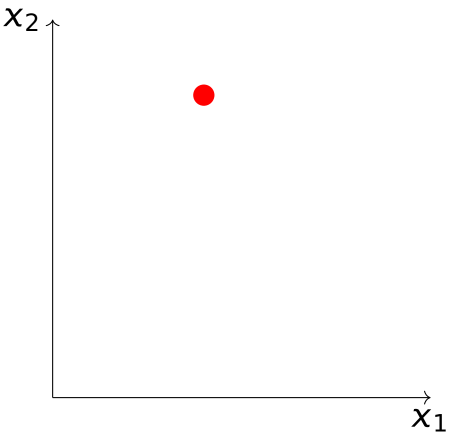

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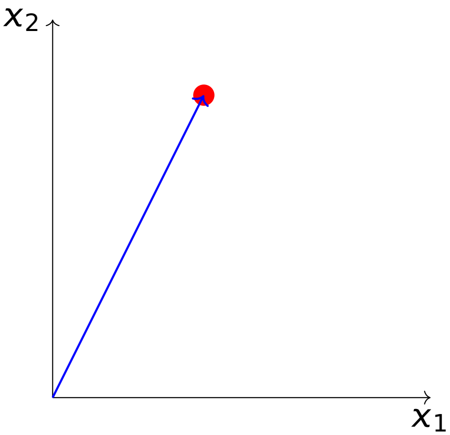

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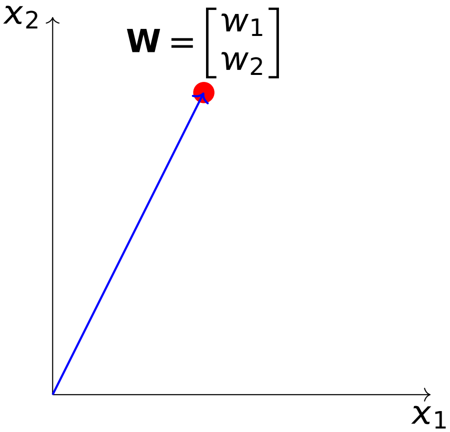

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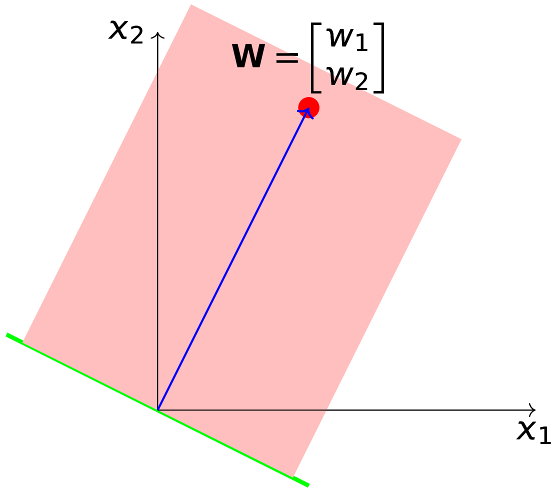
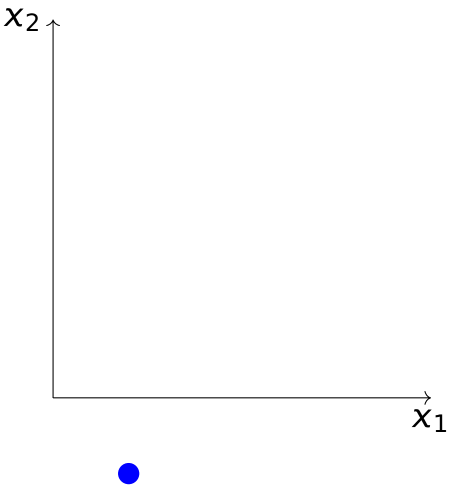
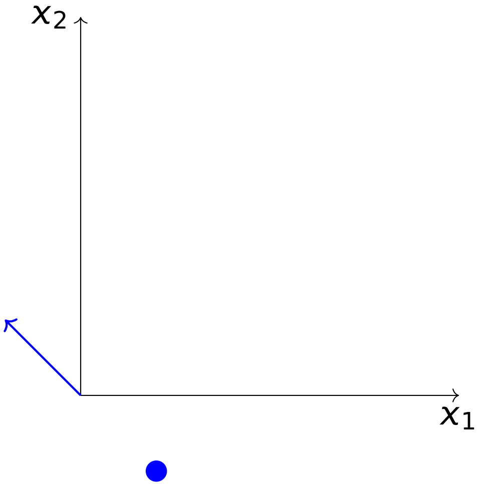


Suppose we only have one point with blue label, i.e., -1. Then the optimal weight vector for this point is as follow,

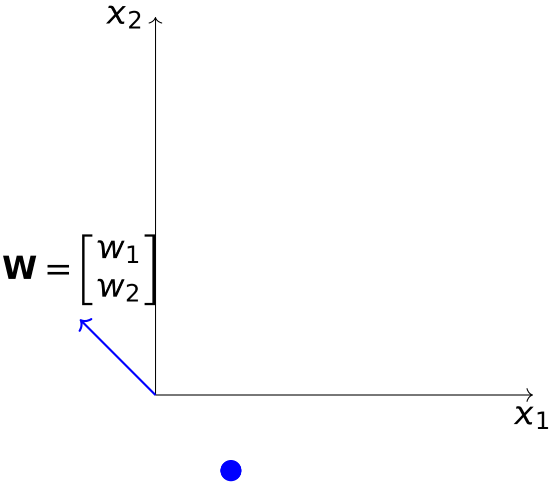
Suppose we only have one point with blue label, i.e., -1. Then the optimal weight vector for this point is as follow,



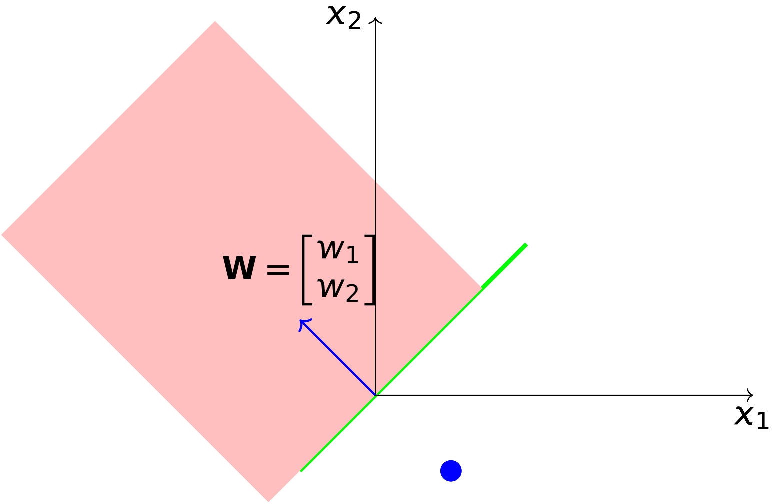
Suppose we only have one point with blue label, i.e., -1. Then the optimal weight vector for this point is as follow,

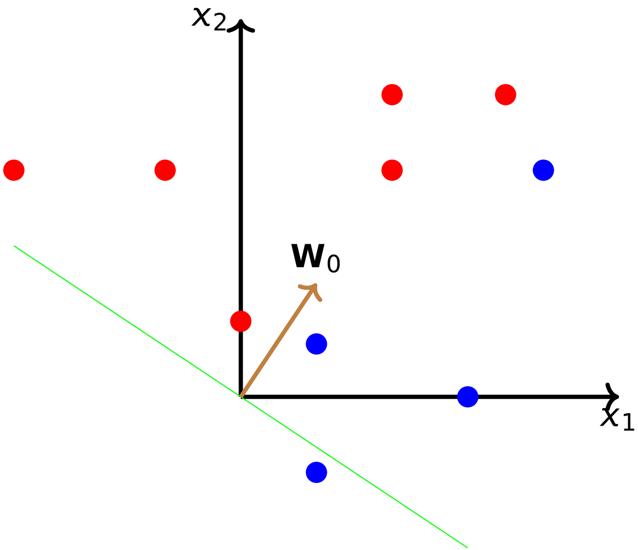


Suppose we only have one point with blue label, i.e., -1. Then the optimal weight vector for this point is as follow,

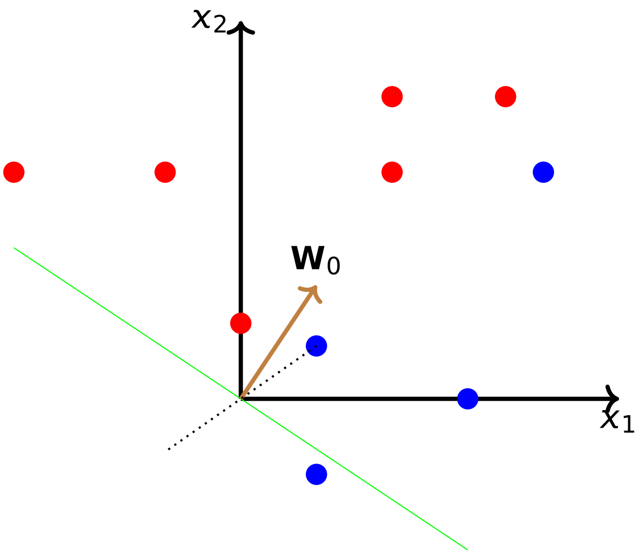


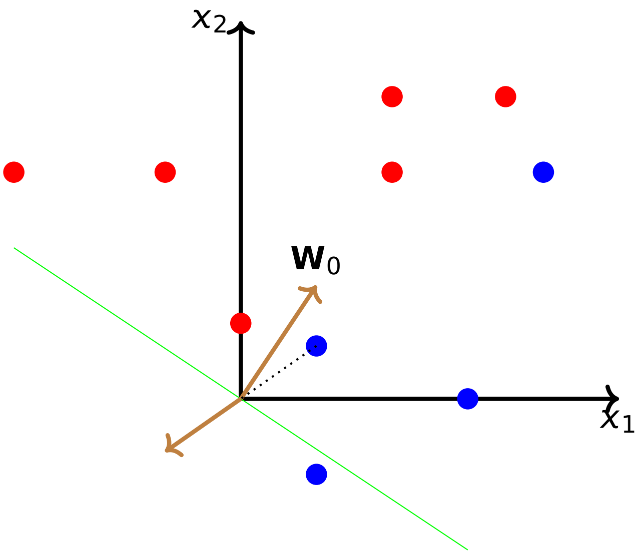
Suppose we only have one point with blue label, i.e., -1. Then the optimal weight vector for this point is as follow,



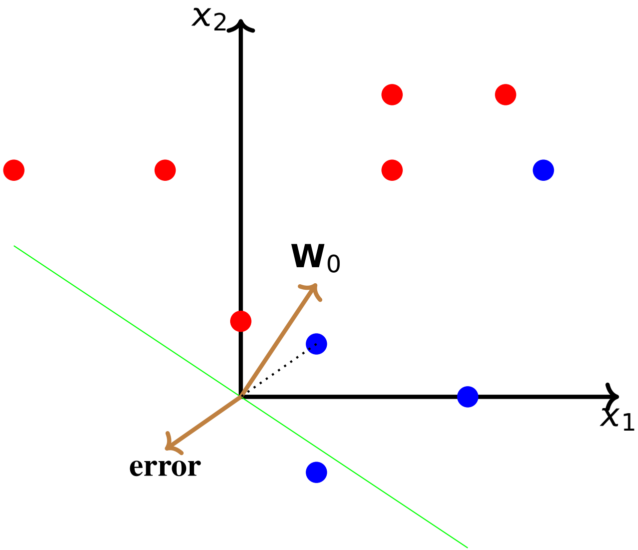


x_2
$\mathbf{w}_0$
x_1

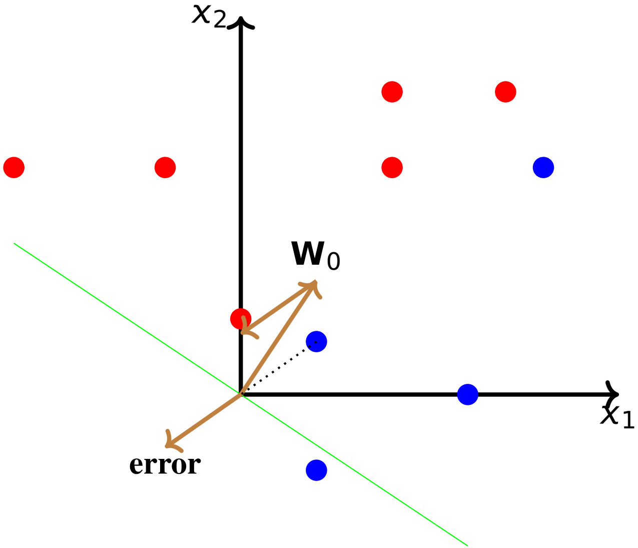

x_2
$\mathbf{w}_0$
x_1



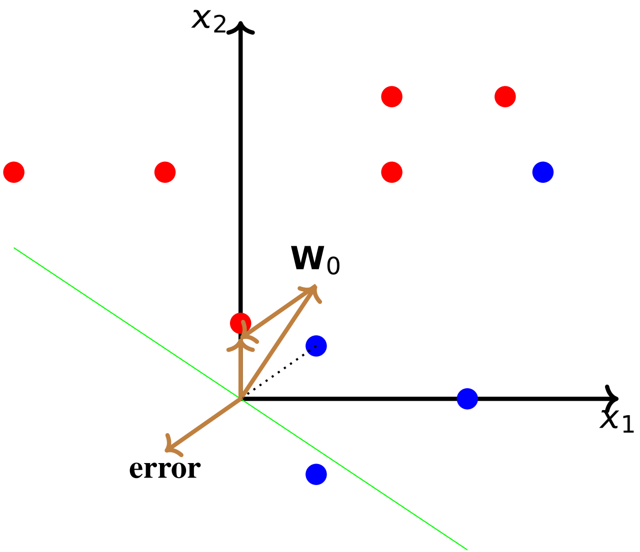
x_2
$\mathbf{w}_0$
x_1



x_2
$\mathbf{w}_0$
x_1
error



x_2
$\mathbf{w}_0$
x_1
error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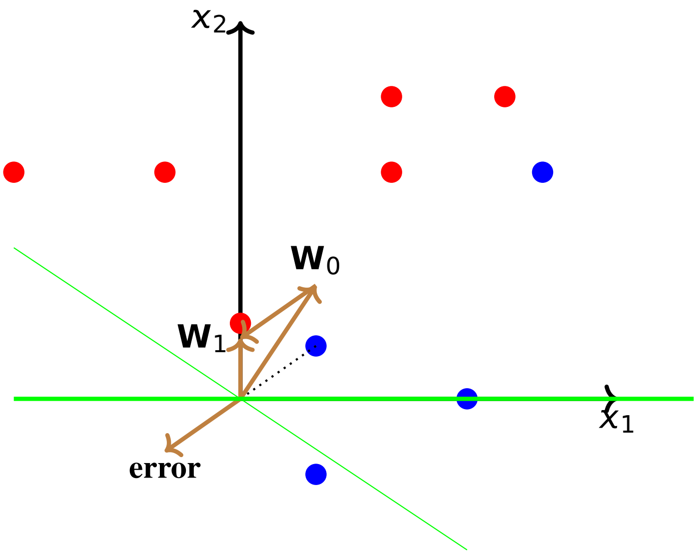
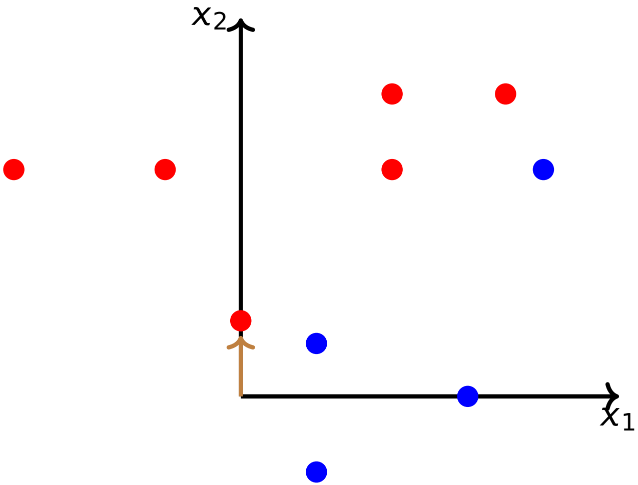
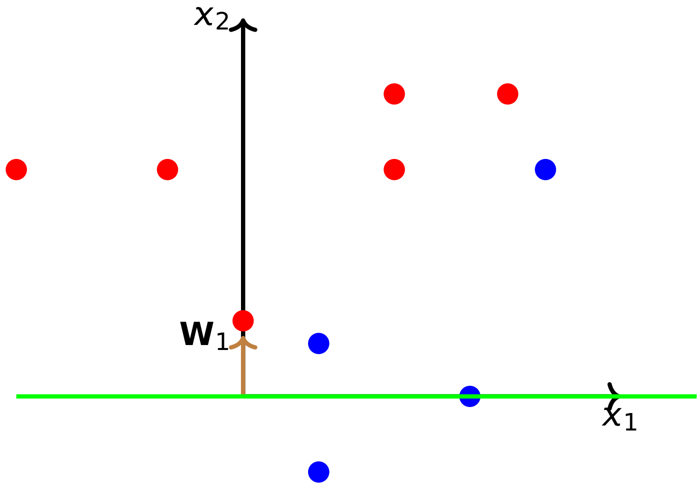
$\mathbf{w}_0$
x_1
error

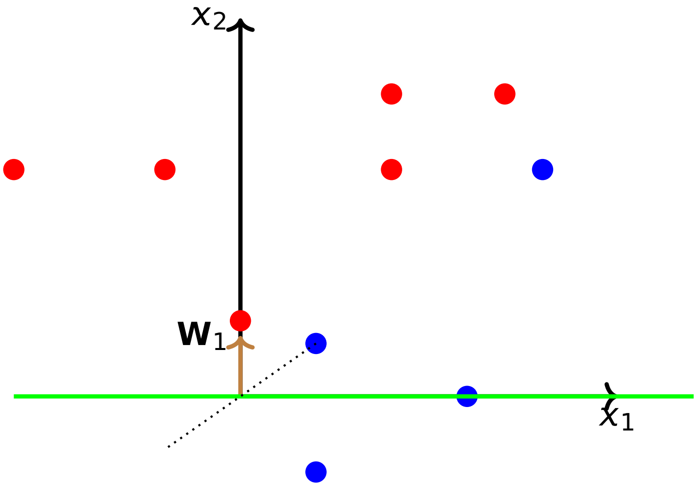


x_2
$\mathbf{w}_0$
$\mathbf{w}_1$
x_1
error


x_2
x_1



x_2
$\mathbf{w}_1$
x_1


x_2
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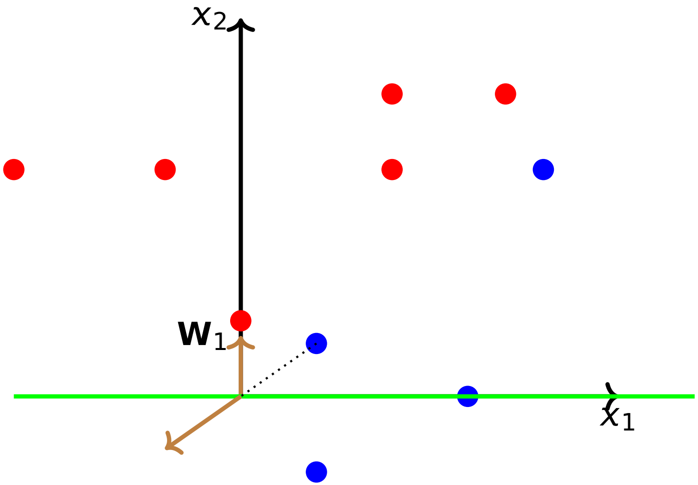
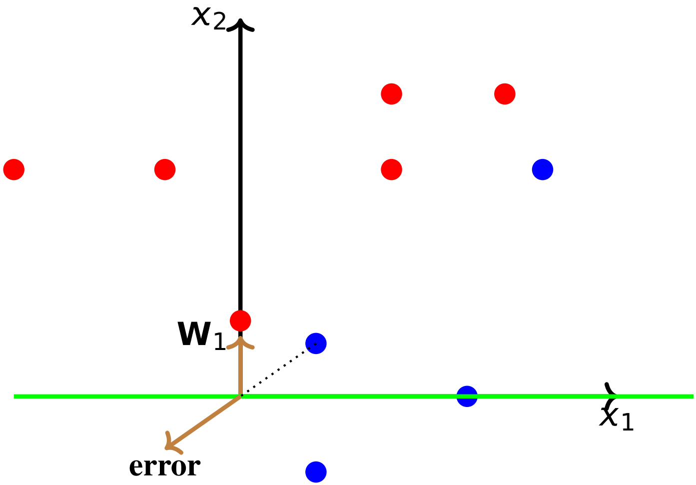
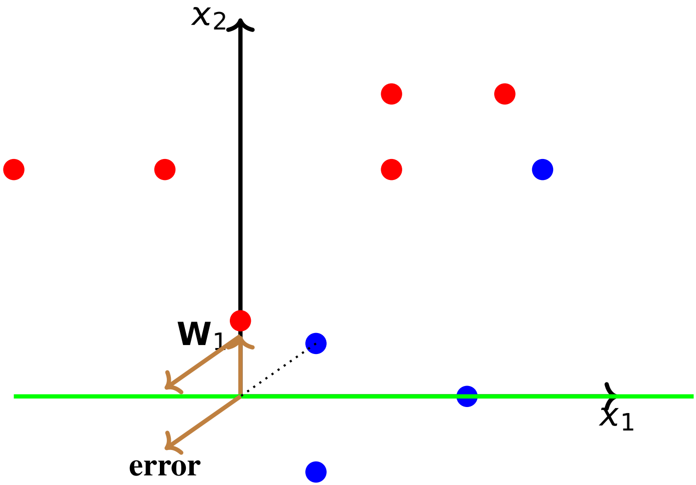
x_1



x_2
$\mathbf{w}_1$
x_1



x_2
$\mathbf{w}_1$
x_1
error



x_2
$\mathbf{w}_1$
x_1
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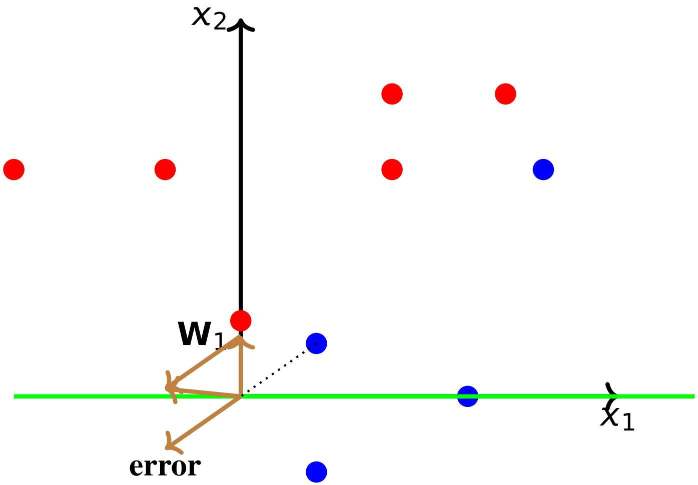
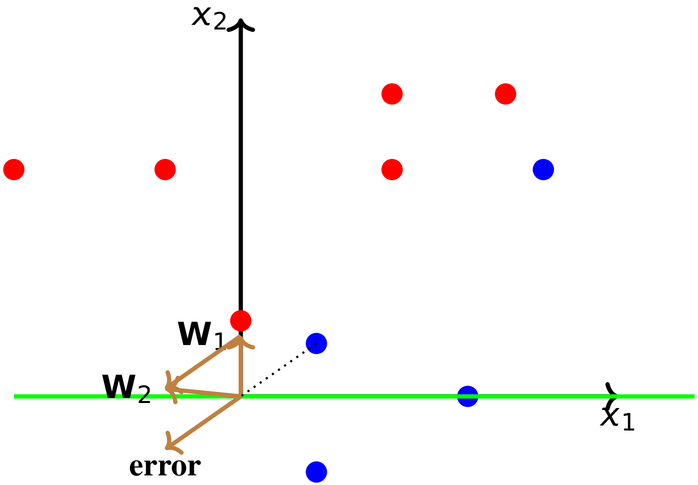
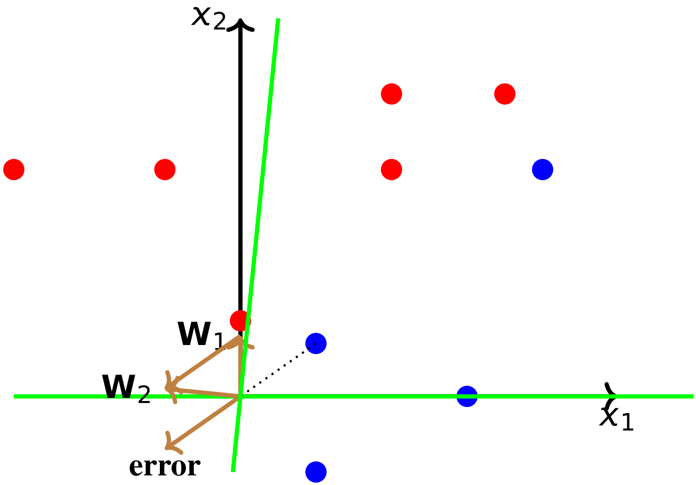


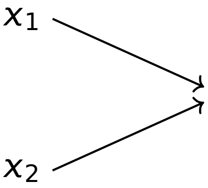
x_2
$\mathbf{w}_1$
x_1
error

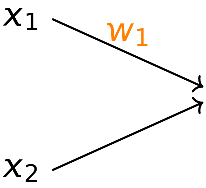


x_2
$\mathbf{w}_1$
$\mathbf{w}_2$
x_1
error

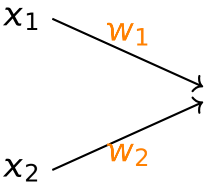


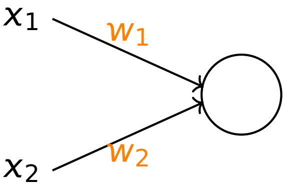
x_2
x_1
$\mathbf{w}_1$
$\mathbf{w}_2$
error

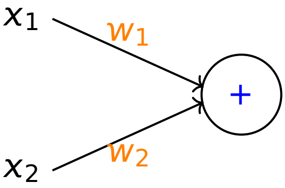

x_1
x_2

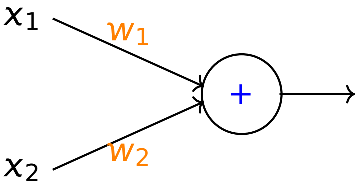


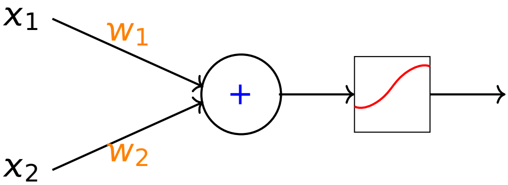
x_1
w_1
x_2


x_1
w_1
x_2
w_2

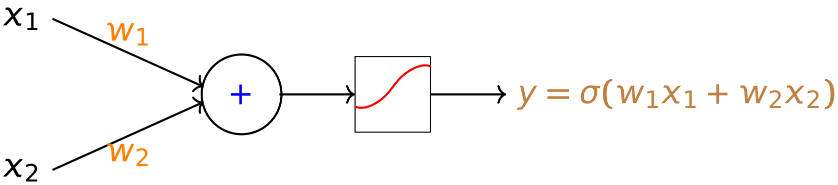

x_1
w_1
x_2
w_2


x_1
w_1
+
x_2
w_2


x_1
w_1
$+$
x_2
w_2



x_1
w_1
+
x_2
w_2


x_1
w_1
x_2
w_2
$+$
$y = \sigma(w_1 x_1 + w_2 x_2)$

y is the probability that the output labe is equal to 1 if x_1 and x_2 are given,

$$\sigma(w_1 x_1^{(i)} + w_2 x_2^{(i)}) = P(label = 1 | x_1^{(i)}, x_2^{(i)})$$

because,

$$w_1 x_1^{(i)} + w_2 x_2^{(i)} \geq 0 \Rightarrow \sigma(w_1 x_1^{(i)} + w_2 x_2^{(i)}) \geq \frac{1}{2}$$

$$w_1 x_1^{(i)} + w_2 x_2^{(i)} < 0 \Rightarrow \sigma(w_1 x_1^{(i)} + w_2 x_2^{(i)}) < \frac{1}{2}$$

So the pmf of the output $\hat{y}^{(i)}$ where $\hat{y}^{(i)} \in \{0, 1\}$ is the Bernoulli distribution, i.e.,

$$P(output = \hat{y}^{(i)} | x_1^{(i)}, x_2^{(i)}) = \sigma(z^{(i)})^{\hat{y}^{(i)}} (1 - \sigma(z^{(i)})^{1-\hat{y}^{(i)}}$$

where $z^{(i)} = w_1 x_1^{(i)} + w_2 x_2^{(i)}$. So we need to solve the optimization problem,

$$\max_{w_1, w_2} P(output = \hat{y}^{(i)} | x_1^{(i)}, x_2^{(i)})$$

but instead we could solve the following minimization problem,

$$\min_{w_1, w_2} -\hat{y}^{(i)} \log(\sigma(z^{(i)})) - (1 - \hat{y}^{(i)}) \log(1 - \sigma(z^{(i)}))$$

If we get average over all sample points we have definition of the **loss function**, i.e. ,

$$\mathcal{L}(w_1, w_2) = \frac{1}{N}\sum_{i=1}^{N} -\hat{y}^{(i)} \log(\sigma(z^{(i)})) - (1 - \hat{y}^{(i)}) \log(1 - \sigma(z^{(i)}))$$

So the goal of the learing is to solve the following minimization problem,

$$\min_{w_1, w_2} \mathcal{L}(w_1, w_2)$$

Gradient Descent

To solve unconstraint optimization problem like,

$$\min_{x_1, x_2} f(x_1, x_2)$$

we make a guss (x_1^0, x_2^0) that minimize the function f and we start improving this guess by moving in a direction that have a smaller value than $f(x_1^0, x_2^0)$. But this direction is in the opposite of the gradient at (x_1^0, x_2^0). So the next step is,

$$(x_1^1, x_2^1) = (x_1^0, x_2^0) - \alpha \mathbf{J}_f(x_1^0, x_2^0)$$

where α is the **learning rate**.

Gradient Descent

1. Make a guess: (x_1^0, x_2^0)
2. $n = 0$
3. Update: $(x_1^{n+1}, x_2^{n+1}) = (x_1^n, x_2^n) - \alpha \mathbf{J}_f(x_1^n, x_2^n)$
4. $n = n + 1$
5. If $\|(x_1^{n+1}, x_2^{n+1}) - (x_1^n, x_2^n)\| < \epsilon$ stop otherwise go to 3.

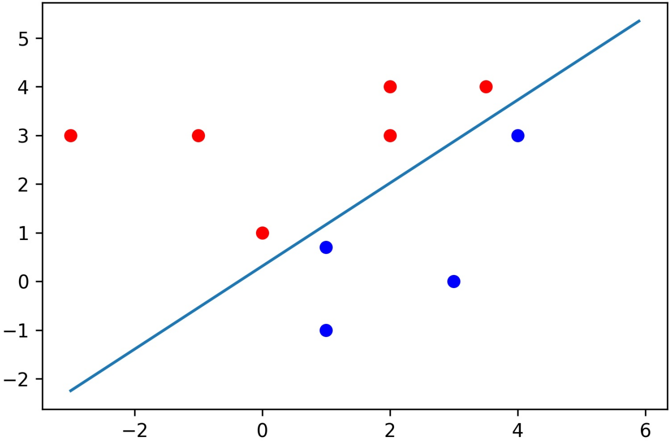


Figure: Gradient descent for Red/Blue example with 50000 iteration and learning rate $\alpha = 0.001$

Thank You