BM 593 Numerical Methods & C Programming

8th week **The Solution of Linear Systems $\mathbf{Ax} = \mathbf{b}$**

Upper Triangular Linear Systems

$$
\begin{aligned}
u_{11}x_1 + u_{12}x_2 + u_{13}x_3 + \ldots + u_{1N-1}x_{N-1} + u_{1N}x_N &= b_1 \\
u_{22}x_2 + u_{23}x_3 + \ldots + u_{2N-1}x_{N-1} + u_{2N}x_N &= b_2 \\
u_{33}x_3 + \ldots + u_{3N-1}x_{N-1} + u_{3N}x_N &= b_3 \\
\vdots \qquad\qquad &\quad \vdots \\
u_{N-1N-1}x_{N-1} + u_{N-1N}x_N &= b_{N-1} \\
u_{NN}x_N &= b_N
\end{aligned}
$$

Back Substitution

$x_N = b_N/u_{NN}$

$x_{N-1} = (b_{N-1} - u_{N-1N}x_N)/u_{N-1N-1}$

$x_k = (b_k - \sum_{j=k+1}^{N} u_{kj}x_j)/u_{kk}$

Lower Triangular Linear Systems

$$
\begin{aligned}
l_{11}x_1 &= b_1 \\
l_{21}x_1 + l_{22}x_2 &= b_2 \\
l_{31}x_1 + l_{32}x_2 + l_{33}x_3 &= b_3 \\
\vdots \qquad\qquad &\quad \vdots \\
l_{N1}x_1 + l_{N2}x_2 + \ldots\ldots + l_{NN}x_N &= b_N
\end{aligned}
$$

Back Substitution

$x_1 = b_1/l_{11}$

$x_2 = (b_2 - l_{21}x_1)/l_{22}$

$x_k = (b_k - \sum_{j=1}^{k-1} l_{kj}x_j)/l_{kk}$

LU Decomposition

$$
\begin{bmatrix}
1 & & & & \\
l_{21} & 1 & & & \\
l_{31} & l_{32} & 1 & & \\
\vdots & \vdots & & & \\
l_{N1} & l_{N2} & \ldots & & 1
\end{bmatrix}
\begin{bmatrix}
u_{11} & u_{12} & u_{13} & \ldots & u_{1N} \\
 & u_{22} & u_{23} & \ldots & u_{2N} \\
 & & u_{33} & \ldots & u_{3N} \\
 & & \vdots & & \vdots \\
 & & & & u_{NN}
\end{bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1N} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2N} \\
a_{31} & a_{32} & u_{33} & \ldots & u_{3N} \\
\vdots & & \vdots & \vdots & \\
a_{N1} & a_{N2} & a_{N3} & \ldots & a_{NN}
\end{bmatrix}
$$

1. $u_{11} = a_{11}$
2. Go with the 1st column by computing $l_{i1} = a_{i,1}/u_{11}$
3. Go with the 1st row by computing $u_{1j} = a_{1j}$

4. Go with the jth column by computing $l_{ij} = [a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}]/u_{ij}$

5. Go with the ith row by computing $u_{ij} = [a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}]/l_{ij}$

Gaussian Elimination and Pivoting

$$
\begin{aligned}
0x_1 + 2x_2 + 3x_3 &= 8 \\
4x_1 + 6x_2 + 7x_3 &= -3 \\
2x_1 + x_2 + 6x_3 + x_4 &= 5
\end{aligned}
$$

$$
\begin{array}{c}
0 \quad 2 \quad 3 \quad : \; 8 \\
(row \quad pivoting) \Updownarrow exchange \\
4 \quad 6 \quad 7 \quad : \; -3 \\
2 \quad 1 \quad 6 \quad : \; 5
\end{array}
$$

$$
\begin{array}{cccccc}
4 & 6 & 7 & : & -3 & \qquad 2 \quad 1 \quad 6 \;\; : 5 \qquad 4 \quad 6 \quad 7 \;\; : -3 \\
0 & 2 & 3 & : & 8 & -2/4 \;\; (4 \quad 6 \quad 7 \; : -3) \qquad 0 \quad 2 \quad 3 \;\; : 8 \\
2 & 1 & 6 & : & 5 & 0 - 2 - 2.5 : 6.5 \qquad 0 - 2 - 2.5 : 6.5
\end{array}
$$

$$
\begin{array}{ccccc}
2 & 3 & : & 8 & \qquad -2 - 2.5 : 6.5 \qquad 4 \quad 6 \quad 7 \;\; : -3 \\
-2 & -2.5 & : & 6.5 & -(-2/2)(2 \quad 3 \quad : 8) \qquad 0 \quad 2 \quad 3 \;\; : 8 \\
 & & & & 0 \; 0.5 : 14.5 \qquad 0 \quad 0 \quad 0.5 \;\; : 14.5
\end{array}
$$

Inverse of a Matrix Using Gauss-Jordan Method

$$
\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

By applying Gauss Elimination to the lower and upper directions we can find the inverse as:

$$
\begin{bmatrix} 0.335 & -0.005 & 0.007 \\ -0.005 & 0.143 & 0.004 \\ -0.01 & -0.003 & 0.100 \end{bmatrix}
$$

Cholesky Algorithm Applied to positive definite matrices

$L = U^T$

1. Set $u_{11} = \sqrt{a_{11}}$

2. For the 1st row compute $u_{1j} = a_{1j}/u_{11}$

3. Compute $u_{ii} = \{a_{ii} - \sum_{k=1}^{i-1}(u_{ki})^2\}^{1/2}$

4. Compute $u_{ij} = 1/u_{ii}\{a_{ij} - \sum_{k=1}^{i-1}(u_{ki})(u_{kj})\}$

Iterative Methods

Jacobi Method

$A = V + D + W$

A: Upper Diagonal elements, D: Diagonal elements , W: Lower Diagonal elements

$Ax = b \rightarrow (V + D + W)x = b \rightarrow x = D^{-1}[b - (V + W)x]$

$x_i^k = x_i^{k-1} + (1/d_{ii})[b_i - \sum_{j=1}^{N} a_{i,j}x_j^{k-1}]$

Gauss-Seidel Method

$Ax = b$

$x_1^k = (b_1 - a_{12}x_2^{k-1} - a_{13}x_3^{k-1} - \ldots - a_{1N}x_N^{k-1})/a_{11}$

$x_2^k = (b_2 - a_{21}x_1^k - a_{23}x_3^{k-1} - \ldots - a_{2N}x_N^{k-1})/a_{22}$

$x_i^k = (b_i - a_{i1}x_1^k - a_{i2}x_2^k - \ldots - a_{ik-1}x_{k-1}^k - a_{ik+1}x_{k+1}^{k-1} - \ldots - a_{iN}x_N^{k-1})/a_{kk}$

For convergence A must be diagonally dominant i.e.

$|a_{kk}| > |a_{k1}| + \ldots + |a_{kk-1}| + |a_{kk+1}| + \ldots + |a_{kN}|$ for $k = 1, 2, \ldots, N$

Singular Value Decomposition

$A = UWV^T$

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \ldots \vec{u}_n \\ \downarrow & \downarrow \quad \downarrow \end{bmatrix}_{mxn} \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_n \end{bmatrix}_{nxn} \begin{bmatrix} \vec{v}_1 \rightarrow \\ \vec{v}_2 \rightarrow \\ \vdots \\ \vec{v}_n \rightarrow \end{bmatrix}_{nxn}$$

$\vec{u}_i$: left eigenvectors of A (which span the Range Space of A),

$\vec{v}_i$: right eigenvectors of A (which span the Null Space of A),

w_{ii} : singular values of A.

Dimension of the Range is equal to the Rank of A.

$\vec{u}_i$ for which $w_{ii} \neq 0$, span the Range Space of A.

$\vec{v}_i$ for which $w_{ii} = 0$, span the Null Space of A.

$\vec{u}_i \perp \vec{u}_j$, $\vec{v}_i \perp \vec{v}_j$, $\vec{u}_i \perp \vec{v}_j$

A can be expressed as $A = \sum_{i=1}^{n} w_{ii}(\vec{u}_i\vec{v}_i^T)$

A^{-1} is called the Pseudo-inverse and determined as: $A^{-1} = \sum_{i=1}^{n} 1/w_{ii}(\vec{v}_i\vec{u}_i^T)$