

BM 593 Numerical Methods & C Programming**5th week****Interpolation and Approximation****Interpolation**

Taylor Approximation : A polynomial approximation

$$f(x - x_0) = f(x_0) + f'(x_0)(x - x_0)/1! + f''(x_0)(x - x_0)^2/2! + \dots = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

nth order polynomial fits to n+1 points

Newton's Interpolating Polynomials

Linear Interpolation

$$f(x) = a_0 + a_1x$$

$$f(x_1) = a_0 + a_1x_1 \text{ and } f(x_2) = a_0 + a_1x_2$$

$$a_0 = f(x_1) - x_1(f(x_2) - f(x_1))/(x_2 - x_1) \text{ and } a_1 = (f(x_2) - f(x_1))/(x_2 - x_1)$$

$$f(x) = f(x_1) + (x - x_1)(f(x_2) - f(x_1))/(x_2 - x_1)$$

Quadratic Polynomial

$$f(x) = a_0 + a_1x^2 + a_2x^2 = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0) \quad b_1 = (f(x_1) - f(x_0))/(x_1 - x_0) \text{ and } b_2 = \frac{(f(x_2) - f(x_0))/(x_2 - x_0) - (f(x_1) - f(x_0))/(x_1 - x_0)}{(x_2 - x_1)}$$

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \dots + b_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_1, x_0] = (f(x_1) - f(x_0))/(x_1 - x_0)$$

$$b_2 = f[x_2, x_1, x_0] = f[x_2, x_1] - f[x_1, x_0]/(x_2 - x_0)$$

⋮

$$b_n = f[x_n, x_{n-1}, \dots, x_1, x_0] = f[x_n, x_{n-1}, \dots, x_1] - f[x_{n-1}, x_{n-2}, \dots, x_0]/(x_n - x_0)$$

Lagrange Interpolating Polynomials

Linear Case

$$f(x) = \frac{(x - x_1)}{x_0 - x_1} f(x_0) + \frac{(x - x_0)}{x_1 - x_0} f(x_1)$$

Quadratic Case

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Interpolating with Cubic Splines :

 $i = 0, 1 \dots n : n + 1$ data points with n intervals

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i : 4n \text{ unknowns}$$

1. $f(x_i) = f_i(x_i)$ at all the knots. $(n + 1)$ conditions.
2. The $f_i(x_i) = f_{i+1}(x_i)$ at the internal knots. $(n - 1)$ conditions.
3. $f'_i(x_i) = f'_{i+1}(x_i)$ at the internal knots. $(n - 1)$ conditions.
4. $f''_i(x_i) = f''_{i+1}(x_i)$ at the internal knots. $(n - 1)$ conditions.
5. The $f''_i(x_0) = f''(x_n) = 0$ at the end nodes. 2 conditions.

Differentiate f twice and express it with 1st order Lagrange Interpolating Polynomials in the interval i :

$$f''_i(x) = 6a_i x + 2b_i = f''(x_{i-1}) \frac{(x-x_i)}{x_{i-1}-x_i} + f''(x_i) \frac{(x-x_{i-1})}{x_i-x_{i-1}}$$

Integrate it twice

$$f_i(x) = f''(x_{i-1}) \frac{(x-x_i)^2}{6(x_{i-1}-x_i)} + f''(x_i) \frac{(x-x_{i-1})^2}{6(x_i-x_{i-1})} + c_1 x + c_2$$

The spline functions are evaluated at the interval knot points i.e. $f_i(x_{i-1})$ and $f_i(x_i)$

c_1 and c_2 are solved in terms of $f''(x_i)$ and $f''(x_{i-1})$ as

$$f_i(x) = f''(x_{i-1}) \frac{(x-x_i)^3}{6(x_{i-1}-x_i)} + f''(x_i) \frac{(x-x_{i-1})^3}{6(x_i-x_{i-1})} + \left[\frac{f(x_{i-1})}{x_i-x_{i-1}} - \frac{f''(x_{i-1})(x_i-x_{i-1})}{6} \right] (x_i - x) + \left[\frac{f(x_i)}{x_i-x_{i-1}} - \frac{f''(x_i)(x_i-x_{i-1})}{6} \right] (x - x_{i-1})$$

To determine the $f''(x_i)$ and $f''(x_{i-1})$ $f_i(x)$ is differentiated once and the condition that

$f'_{i-1}(x_i) = f'_i(x_i)$ is used ($n-1$ equations):

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1}) = \frac{6}{(x_{i+1}-x_i)} [f(x_{i+1}) - f(x_i)] + \frac{6}{(x_i-x_{i-1})} [f(x_{i-1}) - f(x_i)]$$

The f'' is assumed to be 0 at the end knot points, therefore $n+1$ unknown $f''(x_i)$ s and $n+1$ equations.

Ex:

$$x_0 = 3, f(x_0) = 2.5$$

$$x_1 = 4.5, f(x_1) = 1$$

$$x_2 = 7, f(x_2) = 2.5$$

$$x_3 = 9, f(x_3) = 0.5$$

$$(4.5 - 3) f''(3) + 2(7 - 3) f''(4.5) + (7 - 4.5) f''(7) = \frac{6(2.5-1)}{(7-4.5)} + \frac{6(2.5-1)}{4.5-3}$$

$$(7 - 4.5) f''(4.5) + 2(9 - 4.5) f''(7) + (9 - 7) f''(9) = \frac{6(0.5-2.5)}{(9-7)} + \frac{6(1-2.5)}{(7-4.5)}$$

$$f''(3) = 0$$

$$f''(9) = 0$$

Linear Regression

$y_i = ax_i + b_i$ for $i = 1, \dots, n$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_1 & 1 \\ y_2 & 1 \\ \vdots & \vdots \\ y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}, \quad \mathbf{y} = \mathbf{A}\alpha + \epsilon$$

Least Squares Solution

$$\frac{\partial \|\epsilon\|^2}{\partial a} = \frac{\partial \|\epsilon\|^2}{\partial b} = 0$$

$$\frac{\partial \|\epsilon\|^2}{\partial a} = \mathbf{x}^T \mathbf{y} - \mathbf{y}^T \mathbf{x} + \mathbf{x}^T (\mathbf{A}\alpha) + (\mathbf{A}\alpha)^T \mathbf{x} = 0 \rightarrow \mathbf{x}^T (\mathbf{A}\alpha) = \mathbf{x}^T \mathbf{y}$$

$$\frac{\partial \|\epsilon\|^2}{\partial b} = \mathbf{1}^T \mathbf{y} - \mathbf{y}^T \mathbf{1} + \mathbf{1}^T (\mathbf{A}\alpha) + (\mathbf{A}\alpha)^T \mathbf{1} = 0 \rightarrow \mathbf{1}^T (\mathbf{A}\alpha) = \mathbf{1}^T \mathbf{y}$$

$$\mathbf{A}^T \mathbf{A}\alpha = \mathbf{A}^T \mathbf{y}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}, \quad \mathbf{z} = \mathbf{A}\alpha + \epsilon$$

Polynomial Regression

$$y_i = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \dots x_1^n \\ 1 & x_2 & x_2^2 \dots x_2^n \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \dots x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}, \quad \mathbf{y} = \mathbf{A}\alpha + \epsilon$$

Nonlinear Regression

$$y = f(x_i; a_0, a_1, \dots, a_m)$$

$$f(x_i)_{j+1} = f(x_i)_j + \partial f(x_i)_j / \partial a_0 \Delta a_0 + \partial f(x_i)_j / \partial a_1 \Delta a_1 + \dots$$

$$\begin{bmatrix} y_1 - f(x_1)_j \\ y_2 - f(x_2)_j \\ \vdots \\ y_n - f(x_n)_j \end{bmatrix} = \begin{bmatrix} \partial f(x_1)_j / \partial a_0 & \partial f(x_1)_j / \partial a_1 \\ \vdots & \vdots \\ \partial f(x_n)_j / \partial a_0 & \partial f(x_n)_j / \partial a_1 \end{bmatrix} \begin{bmatrix} \Delta a_0 \\ \Delta a_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}, \quad \mathbf{y} = \mathbf{A}\alpha + \epsilon$$

Ex:

$$f(x) = a_0(1 - e^{-a_1 x})$$

Initial Conditions

$$a_0 = a_1 = 0.1$$

$$\partial f(x) / \partial a_0 = (1 - e^{-a_1 x})$$

$$\partial f(x) / \partial a_1 = a_0 a_1 (e^{-a_1 x})$$

1. Start with j=0 and initial a_0 and a_1
2. Apply least squares to find Δa_0 and Δa_1
3. Update $(a_0)_{j+1} = (a_0)_j + (\Delta a_0)_j$
4. Iterate until convergence