BM 593 Numerical Methods & C Programming

9th week The Solution of 2nd Order Linear Partial Differential Equations

$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$

$B^2 - 4AC < 0 \rightarrow$ Elliptic

$B^2 - 4AC = 0 \rightarrow$ Parabolic

$B^2 - 4AC > 0 \rightarrow$ Hyperbolic

Finite Difference Method

$\frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$, and $\frac{\partial u}{\partial y} \approx \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$

Laplace Equation

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Poisson's Equation

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x,y)$ where $f(x,y)$ denotes the heat sources and sinks.

$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1j} - 2Tij + T_{i-1j}}{(\Delta x)^2}$, $\frac{\partial^2 T}{\partial y^2} \approx \frac{T_{i+1j} - 2Tij + T_{i-1j}}{(\Delta y)^2}$.

Laplace Difference Equation

For $\Delta x = \Delta y \Rightarrow$, $T_{i+1j} + T_{i-1j} + T_{ij+1} + T_{ij-1} - 4Tij = 0$

Ex: A grid of 3 by 3 with boundary temperatures as :

left:$75^0 C$, right:$50^0 C$, top: $100^0 C$ and Bottom:$0^0 C$.

$T_{21} + T_{01} + T_{12} + T_{10} - T_{11} = 0 \rightarrow -4T_{11} + T_{12} + T_{21} = -75$

A linear system of 9 equations

Because of dimensionality, the recursive methods are preferred.

Liebmann's Method

$Tij^k = (T_{i+1j}^{k-1} + T_{i-1j}^{k-1} + T_{ij+1}^{k-1} + T_{ij-1}^{k-1})/4$

Iterate

$T_{ij}^{k+1} = \lambda T_{ij}^k + (1-\lambda) T_{ij}^{k-1}$

until convergence.

Ex: $\lambda = 1.5$ $\epsilon = 1\%$

$T_{ij}^0 = 0$ $T_{11}^1 = (0 + 75 + 0 + 0)/4 = 18.75$

$T_{11} = 1.518.75 + (1 - 1.5)0 = 28..125$

After the 9th iteration $\max\{\epsilon_{ij}\} = 0.71 < 1\%$

Boundary Conditions given as $\frac{\partial T}{\partial x}$

$\frac{\partial T}{\partial x} = \frac{T_{1j} - T_{-1j}}{2\Delta x}$

$T_{-1j} = T_{1j} - 2\Delta x \frac{\partial T}{\partial x}$

$\frac{\partial T}{\partial x} = 0$: Insulated Boundary

$2T_{1j} - 2\Delta x \frac{\partial T}{\partial x} + T_{0j+1} + T_{0j-1} - 4T_{0j} = 0$

Ex: Lower Edge Insulated

$2T_{i+1\ 0} + T_{i-1\ 0} + 2T_{i1} - 2\Delta y \frac{\partial T}{\partial y} - 4T_{0j} = 0$

$\frac{\partial T}{\partial y} = 0$

$T_{i+1\ 0} + T_{i-1\ 0} + 2T_{i1} - 4T_{i0} = 0$

Irregular Boundary

$(\frac{\partial T}{\partial x})_{i-1\ \rightarrow\ i} = \frac{T_{ij} - T_{i-1j}}{\alpha_1 \Delta x}$, $(\frac{\partial T}{\partial x})_{i\ \rightarrow\ i+1} = \frac{T_{i+1\ j} - T_{ij}}{\alpha_2 \Delta x}$

$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial T}{\partial x}) \approx \frac{(\frac{\partial T}{\partial x})_{i-1 \rightarrow\ i} - (\frac{\partial T}{\partial x})_{i \rightarrow\ i+1}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}} \approx 2\frac{[\frac{T_{i-1j} - T_{ij}}{\alpha_1 \Delta x} + \frac{T_{i+1j} - T_{ij}}{\alpha_2 \Delta x}]}{(\alpha_1 + \alpha_2)\Delta x}$

$\frac{\partial^2 T}{\partial x^2} \approx \frac{2}{(\Delta x)^2}[\frac{T_{i-1j} - T_{ij}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{T_{i+1j} - T_{ij}}{\alpha_2(\alpha_1 + \alpha_2)}]$

$\frac{\partial^2 T}{\partial y^2} \approx \frac{2}{(\Delta x)^2}[\frac{T_{ij-1} - T_{ij}}{\beta_1(\beta_1 + \beta_2)} + \frac{T_{ij+1} - T_{ij}}{\beta_2(\beta_1 + \beta_2)}]$

Finite Difference for Parabolic Equations

$K\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

Time as well as space is involved.

Explicit Methods

$\frac{\partial^2 T}{\partial x^2} = \frac{T^l_{i+1} - 2T^l_i + T^l_{i-1}}{(\Delta x)^2}$, $\frac{\partial T}{\partial t} = \frac{T^{l+1}_i - T^l_i}{\Delta t} \rightarrow \qquad T^{l+1}_i = T^l_i + \lambda(T^l_{i+1} - 2T^l_i + T^l_{i-1})$,

where $\lambda = k\Delta t/(\Delta x)^2$.

This is written for all interior nodes.

Ex: A horizontal bar with length 10cm. $\Delta x = 2$cm , $\Delta t = 0.1$s.

At $t = 0$ $T(x) = 0$, $T(0) = 100^0 C$ and $T(100) = 50^0 C$

$K = 0.835$, $\lambda = K\Delta t/(\Delta x)^2 = 0.02$

$T^{i+1}_i = T^l_i + \lambda(T^l_{i+1} - 2T^l_i - T^l_{i-1})$

$T^1_1 = T^0_1 + 0.02(T^0_2 - 2T^0_1 - T^0_0) = 0 + 0.02(0 - 2 \times 0 - 100)$

Convergence $\Rightarrow$ As $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$

finite difference results should approach to the true solution.

Stability $\Rightarrow$ Errors should diminish rather than amplify as the iteration goes on if $\Delta t \leq \frac{1}{2}\frac{(\Delta x)^2}{K}$.

An Implicit Method

$\frac{\partial^2 T}{\partial x^2} = \frac{T^{l+1}_{i+1} - 2T^{l+1}_i + T^{l+1}_{i-1}}{(\Delta x)^2}$

K$\frac{\partial^2 T}{\partial x^2} = \frac{T^{l+1}_{i+1} - 2T^{l+1}_i + T^{l+1}_{i-1}}{(\Delta x)^2} = \frac{T^{l+1}_i - T^l_i}{\Delta t} \Rightarrow -\lambda T^{l+1}_{i-1} + (1 + 2\lambda T^{l+1}_i) - \lambda T^{l+1}_{i+1} = T^l_i$

Boundary temperature varies as $T^t_0 = f_0(t)$ and $T^t_m = f_m(t)$.

Ex:

$1 + 2 \times 0.2T^1_1 - 0.02T^1_2 = T^0_1 + 0.02 \times 100 \rightarrow 1.04T^1_1 - 0.02T^1_2 = 0 + 0.02 \times 100$

Crank Nicholson Method

$\frac{\partial^2 T}{\partial t} \approx \frac{T^{l+1}_i - T^l_i}{\Delta t}$, $\frac{\partial^2 T}{\partial t}$ is approximated at $t = l + 1/2$.

$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}[\frac{T^l_{i+1} - 2T^l_i + T^l_{i-1}}{(\Delta x)^2} + \frac{T^{l+1}_{i+1} - 2T^{l+1}_i + T^{l+1}_{i-1}}{(\Delta x)^2}]$

Boundary temperature varies as $T^t_0 = f_0(t)$ and $T^t_m = f_m(t)$.

1st interior node:

$2(1 + \lambda)T^{l+1}_1 - \lambda T^{l+1}_2 = \lambda f_0(l) + 2(1 - \lambda)T^l_1 + \lambda T^l_2 + \lambda f_0(l + 1)$

Last interior node:

$-\lambda T^{l+1}_{m-1} + 2(1 + \lambda)T^{l+1}_m - \lambda T^{l+1}_2 = \lambda f_{m+1}(l) + 2(1 - \lambda)T^l_m + \lambda T^l_{m+1} + \lambda f_m(l + 1)$

Parabolic Equations in 2 dimensions

K$(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) = \frac{\partial T}{\partial t}$, Stability Criterion $\Delta t \leq \frac{1}{8}\frac{(\Delta x)^2 + (\Delta y)^2}{K}$

Alternating Direction Implicit Scheme

Explicit

$$K[\frac{T^l_{i+1j}-2T^l_{ij}+T^l_{i-1j}}{(\Delta x)^2}+\frac{T^{l+1/2}_{ij+1}-2T^{l+1/2}_{ij}+T^{l+1/2}_{ij-1}}{(\Delta y)^2}]=\frac{T^{l+1/2}_{ij}-T^l_{ij}}{\Delta t/2}$$

$$\lambda T^{l+1/2}_{ij-1}+(1+\lambda)T^{l+1/2}_{ij}-\lambda T^{l+1/2}_{ij+1}=\lambda T^l_{i-1j}+2(1-\lambda)T^l_{ij}+\lambda T^l_{i+1j}$$

Implicit

$$K[\frac{T^{l+1}_{i+1j}-2T^{l+1}_{ij}+T^{l+1}_{i-1j}}{(\Delta x)^2}+\frac{T^{l+1/2}_{ij+1}-2T^{l+1/2}_{ij}+T^{l+1/2}_{ij-1}}{(\Delta y)^2}]=\frac{T^{l+1}_{ij}-T^{l+1/2}_{ij}}{\Delta t/2}$$

$$\lambda T^{l+1/2}_{i-1j}+2(1+\lambda)T^{l+1}_{ij}-\lambda T^{l+1}_{i+1j}=\lambda T^{l+1/2}_{ij-1}+2(1-\lambda)T^{l+1/2}_{ij}+\lambda T^{l+1/2}_{ij+1}$$

Ex: $\Delta t = 10s$ $K = 0.835\text{cm}^2/\text{s}$, $\Delta x = \Delta y = 10\text{cm}$. $\lambda = 10 \times 0.835/(10)^2$

At $t = 5$s, Apply the 1st equation for the 1st column i.e.

$$\begin{bmatrix} 2.1670 & -0.0835 & 0.0000 \\ -0.0835 & 2.1670 & -0.0835 \\ 0.0000 & -0.0835 & 2.1670 \end{bmatrix}\begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix}=\begin{bmatrix} 6.26 \\ 6.26 \\ 14.61 \end{bmatrix}$$

then repeat it for the 2nd and 3rd columns.

At $t = 10$s, Apply the 2nd equation for the 1st row i.e.

$$\begin{bmatrix} 2.1670 & -0.0835 & 0.0000 \\ -0.0835 & 2.1670 & -0.0835 \\ 0.0000 & -0.0835 & 2.1670 \end{bmatrix}\begin{bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{bmatrix}=\begin{bmatrix} 12.64 \\ 0.26 \\ 8.06 \end{bmatrix}$$

then repeat it for the 2nd and 3rd rows.