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HOMEWORK 3

Problem 10 - The RBF kernel

Consider the function:

$$k(x,x') = exp(-\frac{||x-x'||^2}{2\sigma^2}), x, x' \in R^d$$

popularly known as the RBF kernel. Prove that it is a valid kernel. Hint: expand the square and express the kernel as the product of three terms.

Theorem 1 *If k is a kernel, then $\gamma k(x,x')$ is also a kernel, for any $\gamma > 0$. (Problem 4)*

Theorem 2 *If k is a kernel, then $exp(k(x,x'))$ is also a kernel. (Problem 9)*

Theorem 3 *Let $f : \mathbb{X} \to \mathbb{R}$. If k is a kernel, then $g(x,x') = f(x)k(x,x')f(x')$ is also a kernel.*

$$k(x,x') = exp(-\frac{||x-x'||^2}{2\sigma^2}) = \tag{1}$$

$$exp(-\frac{||x||^2 - 2||x||||x'|| + ||x'||^2}{2\sigma^2}) = \tag{2}$$

$$exp(-\frac{||x||^2}{2\sigma^2} + \frac{||x||||x'||}{\sigma^2} - \frac{||x'||^2}{2\sigma^2}) = \tag{3}$$

$$exp(-\frac{1}{2\sigma^2}||x||^2)exp(\frac{1}{\sigma^2}||x||||x'||)exp(-\frac{1}{2\sigma^2}||x'||^2) \tag{4}$$

Let $f(x) = exp(-\frac{1}{2\sigma^2}||x||^2)$. Then:

$$k(x,x') = f(x)exp(\frac{1}{\sigma^2}||x||||x'||)f(x') = f(x)k'(x,x')f(x) \tag{5}$$

By *Theorem 3*, it's only necessary to prove that $k'(x,x') = exp(\frac{1}{\sigma^2}||x||||x'||)$ is a kernel. Start from a linear kernel (polynomial kernel with degree 1):

$$k_0(x,x') = xx' \tag{6}$$

Consider following kernel:

$$k_1(x,x') = \frac{1}{\sigma^2}xx' = \frac{1}{\sigma^2}k_0(x,x') = \gamma k_0(x,x') \tag{7}$$

By *Theorem 1*, k_1 is a kernel for $\gamma = \frac{1}{\sigma^2} > 0$. Now, construct k_2:

$$k_2(x,x') = exp(\frac{1}{\sigma^2}xx') = exp(k_1(x,x')) \tag{8}$$

By *Theorem 2*, k_2 is a kernel.