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HOMEWORK 1

Problem 1

Part I

Consider the real two-dimensional space $\mathbb{R}^2$; in this space planes correspond to lines. Find the point $\boldsymbol{p}$ on a given plane π that is closest to the origin; in other words, derive the formula

$$d(0, \pi) = \frac{|b|}{||w||}$$

where $\pi : \langle w, x \rangle + b = 0$ or $w_2 x_2 + w_1 x_1 + b = 0$.

Denote by $d(X, Y)$ the distance between points X and Y. In general, the distance $d(X, Y)$ between two points $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is equal to $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. Hence, the distance $d(P, Q)$, where $P, Q \in \mathbb{R}^2$, is equal to $\sqrt{\sum_{i=1}^{2} (p_i - q_i)^2}$.
Let $P = (x, y)$ and $Q = (0, 0)$. Then, $d(P, Q) = \sqrt{x^2 + y^2}$. Consider following system of equations:

$$\begin{aligned} w_1 x + w_2 y + b &= 0 \\ d(0, \pi) &= \sqrt{x^2 + y^2} \end{aligned}$$

Firstly, define the Lagrangian:

$$\alpha(x, y, \beta) = \sqrt{x^2 + y^2} + \beta(w_1 x + w_2 y + b)$$

Then, solve it:

$$\begin{cases} \frac{\partial \alpha}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} + \beta w_1 = 0 \\ \frac{\partial \alpha}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \beta w_2 = 0 \\ \frac{\partial \alpha}{\partial \beta} = w_1 x + w_2 y + b = 0 \end{cases}$$

$$\begin{cases} x = -\beta w_1 * \sqrt{x^2 + y^2} \\ y = -\beta w_2 * \sqrt{x^2 + y^2} \\ 0 = w_1 x + w_2 y + b \end{cases}$$

$$\begin{aligned} -\beta w_1 w_1 \sqrt{x^2 + y^2} - \beta w_2 w_2 \sqrt{x^2 + y^2} + b &= 0 \\ -\beta \sqrt{x^2 + y^2} (w_1^2 + w_2^2) + b &= 0 \\ \beta \sqrt{x^2 + y^2} (w_1^2 + w_2^2) &= b \\ \sqrt{x^2 + y^2} &= \frac{b}{w_1^2 + w_2^2} * \frac{1}{\beta} \\ d(0, \pi) &= \frac{b}{w_1^2 + w_2^2} * \frac{1}{\beta} \end{aligned}$$

Part II

In $\mathbb{R}^d$, find the point $\boldsymbol{p}$ on a given plane π that is closest to a point $\boldsymbol{q}$; in other words, derive the formula

$$d(q, \pi) = \frac{g(q)}{||w||}$$

where $g(p) = \langle w, p \rangle + b$. Note that the previous case corresponds to q = 0.

Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$. Then, $d(P, Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$. Consider following system of equations:

$$\begin{aligned} w_1p_1 + w_2p_2 + b = 0 \\ d(P, Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \end{aligned}$$

Firstly, define the Lagrangian:

$$\alpha(x, y, \beta) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} + \beta(w_1p_1 + w_2p_2 + b)$$

Then, solve it:

$$\begin{cases} \frac{\partial\alpha}{\partial p_1} = \frac{p_1 - q_1}{\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}} + \beta w_1 = 0 \\ \frac{\partial\alpha}{\partial p_2} = \frac{p_2 - q_2}{\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}} + \beta w_2 = 0 \\ \frac{\partial\alpha}{\partial \beta} = w_1p_1 + w_2p_2 + b = 0 \end{cases}$$

$$\begin{cases} p_1 = q_1 - \beta w_1\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \\ p_2 = q_2 - \beta w_2\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \\ 0 = w_1p_1 + w_2p_2 + b \end{cases}$$

$$w_1(q_1 - \beta w_1\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}) + w_2(q_2 - \beta w_2\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}) + b = 0$$

$$w_1q_1 + w_2q_2 - \beta\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}w_1^2w_2^2 + b = 0$$

$$\beta\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}w_1^2w_2^2 = b + w_1q_1 + w_2q_2$$

$$\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} = \frac{b + w_1q_1 + w_2q_2}{\beta w_1^2w_2^2}$$

$$d(q, \pi) = \frac{b + w_1q_1 + w_2q_2}{\beta w_1^2w_2^2}$$

Problem 3

Consider the circle formed by the intersection of the unit sphere with the plane $x+y+z=0.5$. Consider the three-dimensional optimization problem of finding the point on this circle that is closest to the point $(1,2,3)$. Solve it using Lagrange multipliers.

Let $f(x,y,z)=(x+x_0)^2+(y+y_0)^2+(z+z_0)^2=1$ be an equation of a unit sphere centered in (x_0,y_0,z_0). For simplicity assume that the sphere is centered in the origin, that is $(x_0,y_0,z_0)=(0,0,0)$. So, $f(x,y,z)=x^2+y^2+z^2=1$. Let $\pi(x,y,z)=x+y+z=0.5$ be an equation of a plane. Let $P=(1,2,3)$. The distance $d(Q,P)$, where $Q,P\in\mathbb{R}^3$, is equal to $\sqrt{\sum_{i=1}^{3}(q_i-p_i)^2}$.

The task is to find a point $Q=(x,y,z)$ such that $d(Q,P)=d((x,y,z),(1,2,3))=d(x,y,z)$ is minimal. Consider following system of equations:

$$\begin{cases} x^2+y^2+z^2=1 \\ x+y+z=0.5 \\ d(x{,}y{,}z)=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2} \end{cases}$$

Firstly, define the Lagrangian α:

$$\alpha(x,y,z,\beta,\gamma)=d(x,y,z)+\beta*(x^2+y^2+z^2-1)+\gamma*(x+y+z-0.5)$$
$$\alpha(x,y,z,\beta,\gamma)=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}+\beta*(x^2+y^2+z^2-1)+\gamma*(x+y+z-0.5)$$

Then, solve it:

$$\begin{cases} \frac{\partial\alpha}{\partial x}=\frac{(x-1)^2}{\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}+2\beta x+\gamma=0 \\ \frac{\partial\alpha}{\partial y}=\frac{(y-2)^2}{\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}+2\beta y+\gamma=0 \\ \frac{\partial\alpha}{\partial z}=\frac{(z-3)^2}{\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}+2\beta z+\gamma=0 \\ \frac{\partial\alpha}{\partial\beta}=x^2+y^2+z^2-1=0 \\ \frac{\partial\alpha}{\partial\gamma}=x+y+z-0.5=0 \end{cases}$$

$$\vdots$$