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на тему: Магнитуда некоторых гибридных многообразий

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1 Project description

As a continuation of my previous coursework which approximated magnitude function of a metric graph, as a summer project I decided to keep on working within the same theme by attempting to finish the calculations of magnitude of three types of hybrid manifolds from another students thesis.

Despite not obtaining any magnitude values as a result of calculations, this summer project had been a productive and valuable practice for me — for the first time I came across Riemannian manifolds and hybrid manifolds, tried using Laplace's method to evaluate magnitude function asymptotics, got familiar with another approach for finding magnitude and revised a few mathematical analysis and optimization methods topics.

This report leaves out the calculations attempts (since they are unsuccessful) and focuses on definitions and approaches which were new to me.

2 Preliminaries

2.1 Magnitude

The definition which I used for my coursework for approximating magnitude by computer simulations is given below (as it appears in Willerton [4]):

Definition 1.

Let X be a finite metric space with metric d. If $\forall x \in X\ \exists\ w_x$(also called weightening) :

$$\forall x \in X : \sum_{x' \in X} w_{x'} \cdot e^{-d(x,x')} = 1$$

then magnitude of X is

$$\mathcal{M}(X) := \sum_{x \in X} w_x$$

Remark.

If Z is a $|X| \times |X|$ matrix, with values $Z_{i,j} = e^{-d(x_i,x_j)}$, where $x_i, x_j \in X$ and Z is invertible, then:

$$\mathcal{M}(X) = \sum_{i,j} (Z^{-1})_{i,j}$$

The following definition, which I found out only during this project, is useful for more analytical (rather than computational) approach and was shown in Meckes [2].

Definition 2.

Let X be a finite metric space with metric d and a measure ν defined on X, such that:

$$\forall y \in X : \int_{x \in X} e^{-d(x,y)} d\nu(x) = 1$$

then magnitude of X is

$$\mathcal{M}(X) = \nu(X)$$

2.1.1 Possible algorithm for finding magnitude

As 2nd definition of magnitude states, we need to find a measure satisfying the condition given above. Since linear combination of measures is still a measure we can try to find suitable measure by undetermined coefficients method, using known ones: Lebesgue measures (length for intervals, area for hybrid manifolds) and Dirac measures (usually used for indication of gluing points).

After calculating integrals for different cases of $y \in X$ (as in 2nd definition), we can attempt finding coefficients from obtained equations and in case of success calculate the resulting measure of X.

For example for metric tree the resulting measure will be equal to

$$\nu = \frac{1}{2} \sum_{e \in E(X)} \mu_e + \sum_{v \in V(X)} (1 - \frac{1}{2}\deg(v))\delta_v$$

(μ_e is Lebesgue for each edge and δ_x is Dirac for each vertex)
meaning that magnitude equals to

$$\mathcal{M}(X) = \nu(X) = \frac{\sum w_e}{2} + 1$$

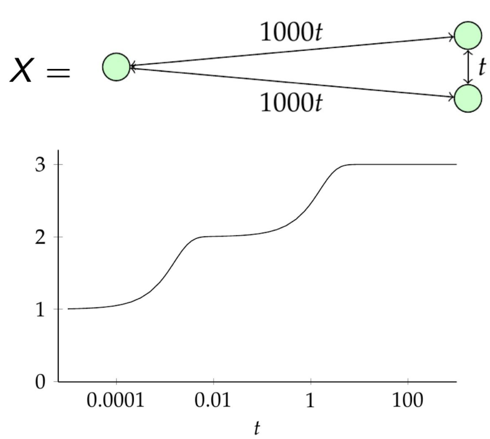
2.1.2 Magnitude function

Definition 3. If X is a metric space with metric $d(x,y)$ and $t \in (0,\infty)$ then tX is a metric space with metric $t \cdot d(x,y)$

Definition 4. $f_{\mathcal{M}} : (0,\infty) \to \mathbb{R}$ is a magnitude function of X, if

$$f_{\mathcal{M}}(t) = \mathcal{M}(tX)$$

Illustration of magnitude function for a three point space with the following distances:
Note how the values of the magnitude function correspond to the number of points seen if we change the perspective: first we see 1 point, then 2 as we get closer but still see two of them as one, then 3.



2.2 Hybrid manifolds

Definition 5. (coming from Pankrashkin, Roganova, and Yeganefar [3]) Let $M_1, M_2, ..., M_k$ be a set of compact Riemannian manifolds, and $L_1, L_2, ..., L_n$ a set of segments. On manifolds we choose $2n$ points, each of them we glue (construct a bijection) to one of the segments at one of its ends. The resulting metric space with shortest path metric is a hybrid manifold.

Note, that metric graph is a simplest hybrid manifold with each M_i being a point. In thesis, that this project was based on, author studies 3 simple hybrid manifolds, each with case $n = k = 1$, and M_1 being sphere, cylinder and torus respectively.

For each of the cases author reviews the value of $d(x, y)$ to find the undetermined coefficients of the measure expression, but due to big number of cases and complicated integrals the calculations are hard to compute.

2.3 Laplace's method

As mentioned earlier to calculate magnitude we need to calculate (or at least approximate) integrals of the form

$$\int_{x\in X} e^{t\cdot(-d(x,y))}dx$$

We added t since the idea was to calculate magnitude function (particularly its asymptotics) rather than magnitude. Besides x can be multidimensional, as in case of all three examined manifolds it is 2-dimensional.

To approximate this integral we can use multivariate case of Laplace's method (from Fedoruk [1]):

If $x = (x_1, ..., x_d)$; $f : \mathbb{R}^d \to \mathbb{R}$ and $x_0 \in X : f(x_0)$ - maximum on X, then

$$\int_{x\in X} e^{M\cdot f(x)}dx \approx \left(\frac{2\pi}{M}\right)^{\frac{d}{2}} \frac{e^{M\cdot f(x_0)}}{|-H(f)(x_0)|^{\frac{1}{2}}} = g(M) \text{ as } M \to \infty$$

where $|H|$ is Hessian determinant.

$\approx$ in this case means that $\lim_{M\to\infty} \frac{\int_{x\in X} e^{M\cdot f(x)}}{g(M)} = 1$ so we can evaluate asymptotics of magnitude as a result.

3 Conclusions and future work

As already mentioned above I did not obtain any results due to short duration of summer projects, but I do plan to continue this project in the future.

My plan is the following: using Laplace's method from the last section on different cases of integrals presented in the thesis this project is based on, for every scaling factor I hope to obtain a system of equations which solutions will approximate the undetermined coefficients as scaling factor goes to infinity. Thus, substituting these solutions into the measure integral which on infinity would be asymptotically equivalent to the magnitude function.

All in all, the project was interesting and challenging to me as it is the first time I was given a task of figuring out and continuing someone else's work, besides it dealt with a combination of new topics.

References

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