

**ПРАВИТЕЛЬСТВО РОССИЙСКОЙ ФЕДЕРАЦИИ**  
**ФГАОУ ВО НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ**  
**«ВЫСШАЯ ШКОЛА ЭКОНОМИКИ»**

Факультет компьютерных наук  
Образовательная программа «Прикладная математика и информатика»

УДК: 004.9

**Отчет об исследовательском проекте**

на тему: Приближение магнитудной функции метрического графа

(промежуточный, этап 1)

**Выполнил:**

Студент группы БПМИ204

14.02.2023

Дата

Подпись

М.М.Марченко

И.О.Фамилия

**Принял:**

Руководитель проекта

Всеволод Леонидович Чернышев

Имя, Отчество, Фамилия

Доцент

Должность, ученое звание

НИУ ВШЭ, ФКН, Департамент больших данных и информационного поиска

Место работы (Компания или подразделение НИУ ВШЭ)

Дата проверки 2023

Оценка (по 10-ти бальной шкале)

Подпись

Москва 2023

# Contents

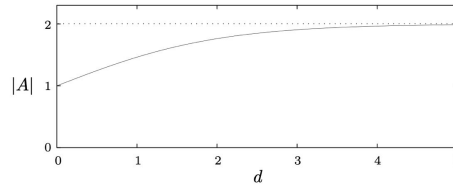
<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Background</b>	<b>3</b>
2.1	Definitions . . . . .	3
2.1.1	Metric Graph . . . . .	3
2.1.2	Magnitude . . . . .	3
2.1.3	Magnitude function . . . . .	4
2.1.4	Spread . . . . .	4
2.2	Metric graph as a compact metric space . . . . .	4
<b>3</b>	<b>Conjecture</b>	<b>5</b>
<b>4</b>	<b>Experiments and their interpretations</b>	<b>5</b>
4.1	Approximation of the magnitude function . . . . .	5
4.2	Overview of the relevant results . . . . .	7
4.3	Intervals . . . . .	8
4.4	Trees . . . . .	8
4.5	Cicles . . . . .	8
4.6	Cicles combined with trees . . . . .	8
4.7	Combinations of cicles . . . . .	8
4.8	Conjecture testing for a random graph . . . . .	8
<b>5</b>	<b>Methods</b>	<b>8</b>
5.1	Numeric computations . . . . .	8
5.2	Symbolic computations - unsuccessful attempt . . . . .	10
<b>6</b>	<b>Conclusion</b>	<b>10</b>

## 1 Introduction

Metric graph is a topological space which, simply put, represents a glued set of closed intervals. Unlike combinatorial weighted graph it is not discrete, but continuous, containing not only vertices, but also all points on the edges.

For a given metric graph we can calculate a magnitude function, object originally coming from category theory. Being a generalized notion of size, magnitude function is defined for enriched categories, and metric graph is a type of metric space, which, in turn, is a type of enriched category.

### Magnitude function of two points space:



As calculated by Leinster and Willerton [2] the magnitude function of an interval of length  $l$  equals  $\frac{lt}{2} + 1$ , while magnitude of two points space with distance  $l$  between them equals  $\frac{2}{1+e^{-lt}}$ . Calculation of magnitude function of the first object has some non-trivial mathematical steps, while calculation of the magnitude function of the second object is a simple algorithmic task of finding a reverse 4 element matrix. Moreover the resulting values of the magnitude function don't seem to have some sort of simple relation and magnitude of interval can not be simply calculated from magnitude of its endpoints.

But  $l \rightarrow \frac{l}{2} + 1$  is a rather simple function which might have been guessed from the computer simulations.

Magnitude of a combinatorial graph was already studied by LEINSTER [1], but as we see in the case of interval of length  $l$  and two discrete points with distance  $l$  between them, studying magnitude of the continuous space is a whole different problem.

In this work we present the results of computer simulations for different metric graphs and the conjecture about behavior of the magnitude function derived from them, which goes along with all the experiments so far.

The work is structured in the following way: in the Background section we will give definitions of magnitude function and metric graph, explain their relation and explain why magnitude of a metric graph is defined and how it can be calculated.

Then in the Conjecture section we explain the following suggestion: *magnitude function of a metric graph asymptotically satisfies the inclusion-exclusion principle.*

In the Experiments and their interpretations we explain how we approximate the magnitude function, and then go through all the graphs we experimented on. For each of them we give the simulations result and then explain how it aligns with the conjecture (so we decompose the graphs to simpler parts to use the inclusion-exclusion principle) and also show how the result aligns with already known magnitudes and theorems.

After that in Methods we go through details about the way we calculated magnitude function numerically and then explain the unsuccessful way how we tried to approximate magnitude.

## 2 Background

### 2.1 Definitions

#### 2.1.1 Metric Graph

As already mentioned in the introduction, we can intuitively understand metric graphs as weighted graphs which, besides vertices, also contain all the points on the edges. The other intuitive understanding would be a set of intervals joined by their ends, thus forming a graph.

A stricter definition, originally coming from Mugnolo [3], is given below.

$\mathcal{E} := \bigsqcup [0, l_e] \mid e \in E, l_e \in (0, \infty)$ , where  $E$  is a countable set and  $\bigsqcup$  is a disjointed topological union.

The notation for elements of  $\mathcal{E}$  is:  $(x, e) \in \mathcal{E} \iff e \in E \wedge x \in [0, l_e]$

$\mathcal{V} := \bigsqcup \{(0, e), (l_e, e)\}$

On  $\mathcal{V}$  we define an equivalency relation  $\sim_{\mathcal{V}}$  with classes of equivalency  $\{V_i\}$  (each represents a vertex)

On  $\mathcal{E}$  we define an equivalency relation  $\sim_{\mathcal{E}} \mid (x_1, e_1) \sim_{\mathcal{E}} (x_2, e_2) \iff x_1 = x_2; e_1 = e_2 \text{ or } (x_1, e_1) \sim_{\mathcal{V}} (x_2, e_2)$

**Definition 1.**  $\mathcal{G} = \mathcal{E} / \sim_{\mathcal{E}}$  is a **metric graph** with vertices  $\mathcal{V} / \sim_{\mathcal{V}}$

#### 2.1.2 Magnitude

First we will give a definition of magnitude for a finite metric space, then two equivalent definitions for compact positive definite metric spaces, taken from Leinster and Willerton [2].

**Definition 2.**

Let  $X$  be a finite metric space with metric  $d$ .

Magnitude of  $X$  is  $\mathcal{M}(X) := \sum_{x \in X} w_x$ , where  $w_x \in \mathbb{R}$  and  $\forall x \in X : \sum_{x' \in X} w_{x'} \cdot e^{-d(x, x')} = 1$

**Remark.**

If  $Z$  is a  $|X| \times |X|$  matrix, with values  $Z_{i,j} = e^{-d(x_i, x_j)}$ , where  $x_i, x_j \in X$  and  $Z$  is invertible, then:

$$\mathcal{M}(X) = \sum_{i,j} (Z^{-1})_{i,j}$$

**Definition 3.**

Let  $X$  be a positive definite metric space, then:

$$\mathcal{M}(X) = \sup\{\mathcal{M}(Y) : Y \subset X \text{ and } |Y| < \infty\}$$

**Definition 4.** (equivalent to definition 3)

Let  $X$  be a positive definite metric space and  $\{X_i\}$  its subsets :  $|X_i| < \infty$  and  $X_i \rightarrow X$  in Hausdorff metric then:

$$\mathcal{M}(X) = \lim_{i \rightarrow \infty} \mathcal{M}(X_i)$$

### 2.1.3 Magnitude function

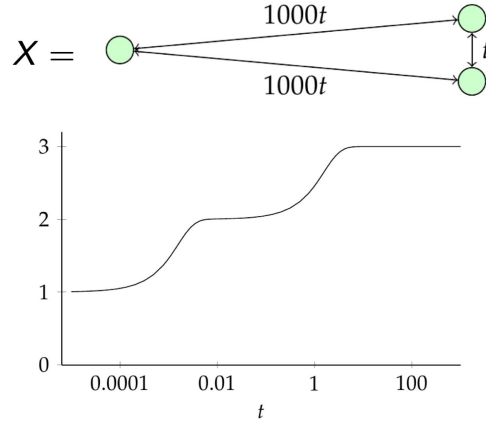
**Definition 5.** If  $X$  is a metric space with metric  $d(x, y)$  and  $t \in (0, \infty)$  then  $tX$  is a metric space with metric  $t \cdot d(x, y)$

**Definition 6.**  $f_{\mathcal{M}} : (0, \infty) \rightarrow \mathbb{R}$  is a magnitude function of  $X$ , if

$$f_{\mathcal{M}}(t) = \mathcal{M}(tX)$$

**Illustration of magnitude function for a three point space with the following distances:**

Note how the values of the magnitude function correspond to the number of points seen if we change the perspective: first we see 1 point, then 2 as we get closer but still see two of them as one, then 3.



#### 2.1.4 Spread

## 2.2 Metric graph as a compact metric space

Metric graphs are metric spaces with metric equivalent to shortest path between two points. Let's quickly define it as it appears in Mugnolo [3].

$$d_{\mathcal{E}}((x_1, e_1), (x_2, e_2)) = \begin{cases} |x_1 - x_2|, & \text{if } e_1 = e_2 \\ \infty, & \text{otherwise} \end{cases}$$

$$d_{\mathcal{G}}(\xi, \theta) = \inf\left\{\sum_{i=1}^k d_{\mathcal{E}}(\xi_i, \theta_i)\right\} \quad \xi_i, \theta_i \in \mathcal{E}$$

In our case we consider graphs with  $|E| < \infty$ , thus we can easily prove that it's a metric space and moreover a compact metric space.

## Metric graph is a metric space

$$1. d_{\mathcal{G}}(x, y) = 0 \iff x = y$$

$$2. d_{\mathcal{G}}(x, y) = d_{\mathcal{G}}(y, x)$$

1-2 are obvious from the definition of  $d_{\mathcal{E}}, d_{\mathcal{G}}$  as either distances (which is interval metric)

between points on the interval, either their sum.

$$3. d_{\mathcal{G}}(x, y) \leq d_{\mathcal{G}}(x, z) + d_{\mathcal{G}}(z, y)$$

3 is also obvious since  $x \rightarrow z \rightarrow y$  is also a valid path, thus can't be less than infimum.

## Metric graph is a compact metric space

Since  $|E| < \infty$  for any sequence of points we can find an edge which contains infinite subsequence and since edge is an interval which is a compact, it has a convergent subsequence, thus metric graph is a compact.

## 3 Conjecture

The pattern that we observed in the behavior of the magnitude function during computer simulations is formulated below:

**Theorem 1** (not proved). The magnitude of the metric graph asymptotically satisfies the inclusion-exclusion principle.

Similar conjecture about subsets of Euclidean space was made in Leinster and Willerton [2]. But since metric graphs are not always subsets of Euclidean space and to our knowledge this conjecture has not been proven, our result is a contribution.

More strictly conjecture states:

$$\mathcal{M}(A) = P(A) + q(A), \text{ where } \lim q(A) = 0 \text{ as } A \text{ is scaled and}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If the conjecture is right we can derive magnitude of any graph by decomposing it step by step to simpler elements: as shown in section Overview of the relevant results, we already know magnitude of finite number of points, magnitude of an interval, magnitude of a cycle and magnitude of a tree.

## 4 Experiments and their interpretations

### 4.1 Approximation of the magnitude function

Here we will briefly explain how the magnitude function was approximated. Pseudocode, relevant computation details and all links can be found in the Methods section.

For a fixed  $t$  and a fixed graph  $\mathcal{G}$  we approximate a continuous metric graph with it's discrete subsets - finite combinatorial graphs, which we get by dividing each edge into equal parts.

For each graph in this sequence  $\{\mathcal{G}_k\}$  we calculate magnitude ( $\mathcal{M}_k(t)$ ) as in remark 1: as sum of elements of  $Z^{-1}$ , where  $Z_{i,j} = e^{-t \cdot d_{\mathcal{G}_k}(v_i, v_j)} \mid v_i, v_j \in \mathcal{V}_k$ .

As follows from the 4th definition:

$$\mathcal{M}(t) = \lim_{k \rightarrow \infty} \mathcal{M}_k(t)$$

We run this algorithm on several  $t$  and draw a plot which with the measured magnitude. Range of  $k$  and  $t$  is specified and explained in the Methods section. Each experiment is represented by two plots below.

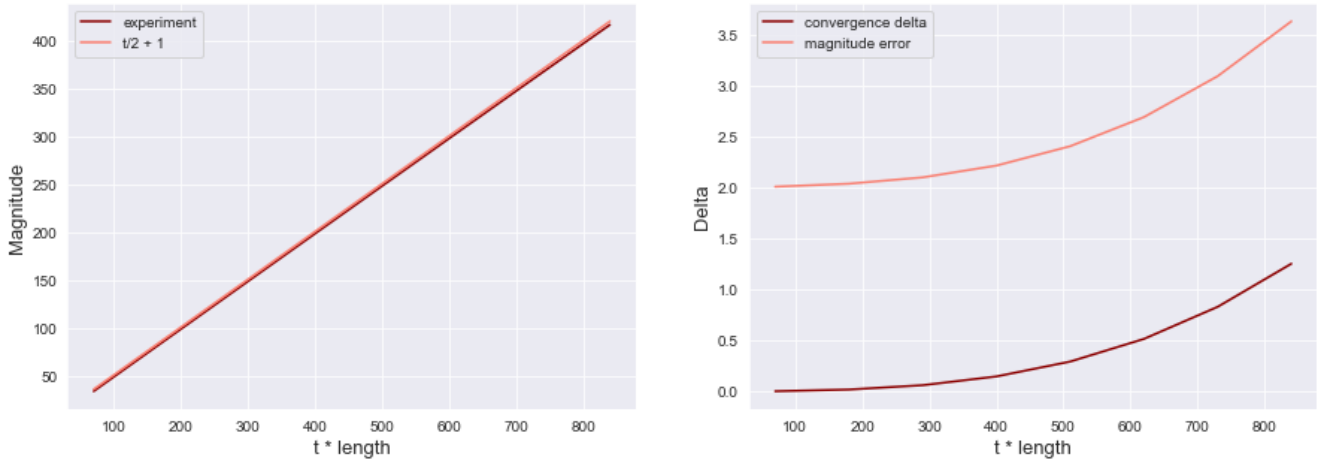
The first one compares the magnitude we got from the simulation with the guessed magnitude function. The *length* of the graph equals to the sum of its weighted edges.

The second plot represents the convergence delta - by which we mean  $|\mathcal{M}_k(t) - \mathcal{M}_{k-1}(t)|$ , compared to the difference between the experiment result and the guessed magnitude function -  $(|\mathcal{M}_{guessed}(t) - \mathcal{M}_k(t)|)$ , where  $\mathcal{M}_k(t)$  is the magnitude measured for the last finite graph in the sequence (the one with the biggest number of vertices).

As  $t$  gets bigger, the pairwise distances between the points of the combinatorial graph grow too. This leads to the superlinear growth of the numerical error, because the magnitude is calculated by inverting a matrix with elements equal to the exponent to the power of negated pairwise distances. That's why we need some reference to check that they grow proportionally. In our case as a reference we use convergence delta since it perfectly shows how close the magnitude is to the real value (similar to the Cauchy convergence).

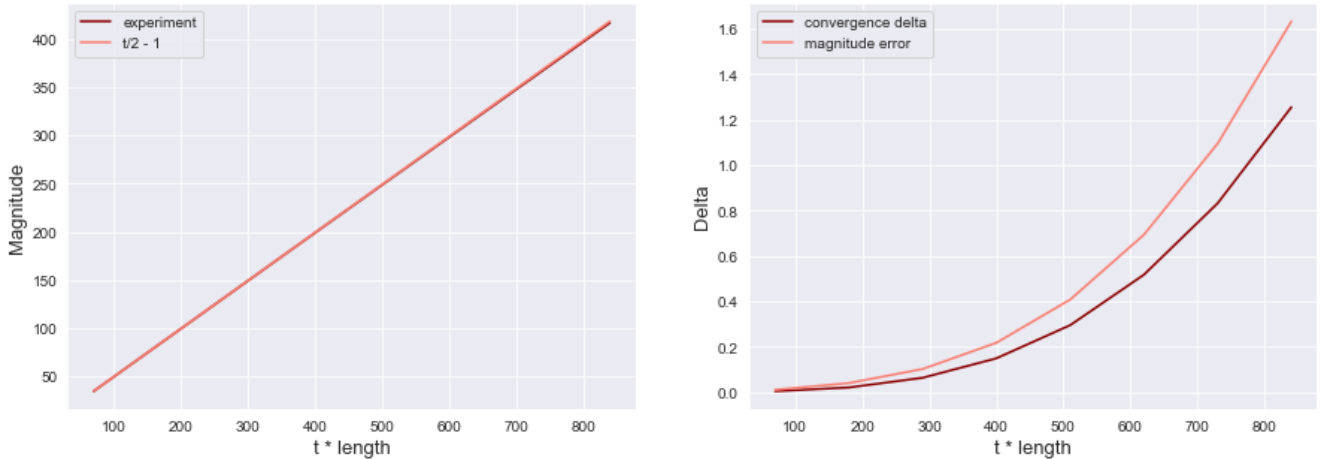
Note that all the magnitudes we studied ended up in the  $\frac{t \cdot \text{length}}{2} + \text{const}$  neighbourhood, so in the most of the cases the second plot was more useful. The next example shows the difference between the plot when the *const* was guessed correctly compared to the plot with an incorrect *const*.

Cicles glued by vertex magnitude, 4000 points



Example where the *const* was guessed wrong: the plot is still similar to  $\frac{t}{2}$  but in comparison to convergence delta we can see, that the *const* should be decreased by 2.

Cicles glued by vertex magnitude, 4000 points



This example illustrates the right guess - note how convergence delta correlates with difference between real

magnitude and experiment results.

## 4.2 Overview of the relevant results

To interpret the experiments results we will briefly overview the relevant results.

First of all, the magnitude of the most basic objects: let  $p_1$  be 1 point,  $p_1 \leftrightarrow p_2$  - 2 points on distance equal to  $l$ . Then:

$$\mathcal{M}_{p_1}(t) = 1$$

$$\mathcal{M}_{p_1 \leftrightarrow p_2}(t) = \frac{2}{1 + e^{-t \cdot l}}$$

These magnitudes, as well as magnitude of any finite metric space, are easily computed without any analytic steps by inverting one matrix and summing the elements of the result.

For the subsets of the Euclidean space known results are the magnitude of an interval of length  $l$  and a circle of circumference  $l$  (Leinster and Willerton [2]):

$$\mathcal{M}_{\text{interval of length } l}(t) = \frac{l \cdot t}{2} + 1$$

$$\mathcal{M}_{\text{circle of circumference } l}(t) = \frac{l \cdot t}{2} + q(t), \text{ where } \lim_{t \rightarrow \infty} q(t) = 0$$

Note that in the case of a circle the metric does not need to be euclidean - magnitude is the same for the arc length metric, which would be useful to us in the case of a cycle.

Both of these calculations require non-trivial steps (and asymptotic analysis in the case of circle) but as we can see the dependence is simpler than in the case of two points and could have been guessed from simulations, see sections Intervals and Cycles.

As already mentioned in the third section, in Leinster and Willerton [2] there is a similar conjecture to the one stated in this paper: "the magnitude of subsets of Euclidean space asymptotically satisfies the inclusion-exclusion principle".

The other important result and its two corollaries are taken from LEINSTER [1]. We are not giving definitions of subgraph *projections* and subgraph *convexity*, since it is not applicable for general case of the graphs (for example, biggest half of the cut cycle is not convex and does not projects the rest of the cycle), and two corollaries cover all of the studied cases where this theorem might have been applied. Both of this definitions (with examples) can be found in the original paper.

**Theorem 2.** Let  $X$  be a graph, with subgraphs  $G$  and  $H$  such that  $G \cup H = X$ . Suppose that  $G \cap H$  is *convex* in  $X$  and that  $H$  *projects* to  $G \cap H$ . Then:

$$\mathcal{M}_X = \mathcal{M}_G + \mathcal{M}_H - \mathcal{M}_{(G \cap H)}$$

**Corollary 1.** Let  $G$  and  $H$  be graphs and  $G \vee H$  is the graph obtained by gluing them together with one point. Then:

$$\mathcal{M}_{G \vee H} = \mathcal{M}_G + \mathcal{M}_H - 1$$

**Corollary 2.** Let  $X$  be a tree, with subtrees  $G$  and  $H$  such that  $G \cup H = X$ . Then:

$$\mathcal{M}_X = \mathcal{M}_G + \mathcal{M}_H - \mathcal{M}_{G \cap H}$$

### 4.3 Intervals

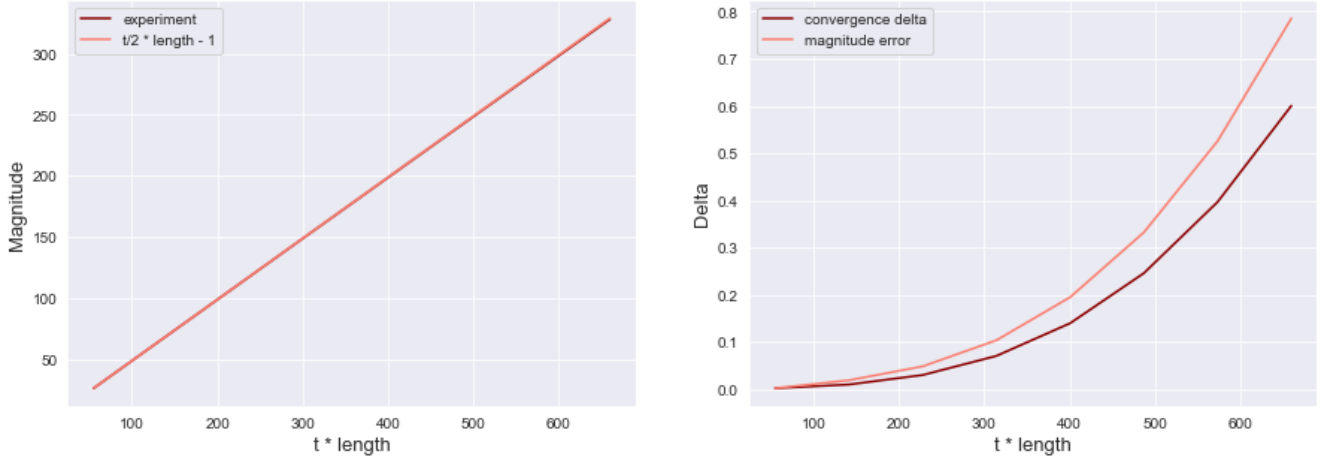
### 4.4 Trees

### 4.5 Cicles

### 4.6 Cicles combined with trees

### 4.7 Combinations of cicles

Cicles glued by edge magnitude, 4000 points



explanation

### 4.8 Conjecture testing for a random graph

## 5 Methods

### 5.1 Numeric computations

All the results in this paper were obtained with the following numeric algorithm:

---

**Algorithm 1** Magnitude function

---

**Require:**  $\mathcal{E}$  - list of edges of the graph  $\triangleright$  edge is represented as  $(v_1, v_2, weight)$

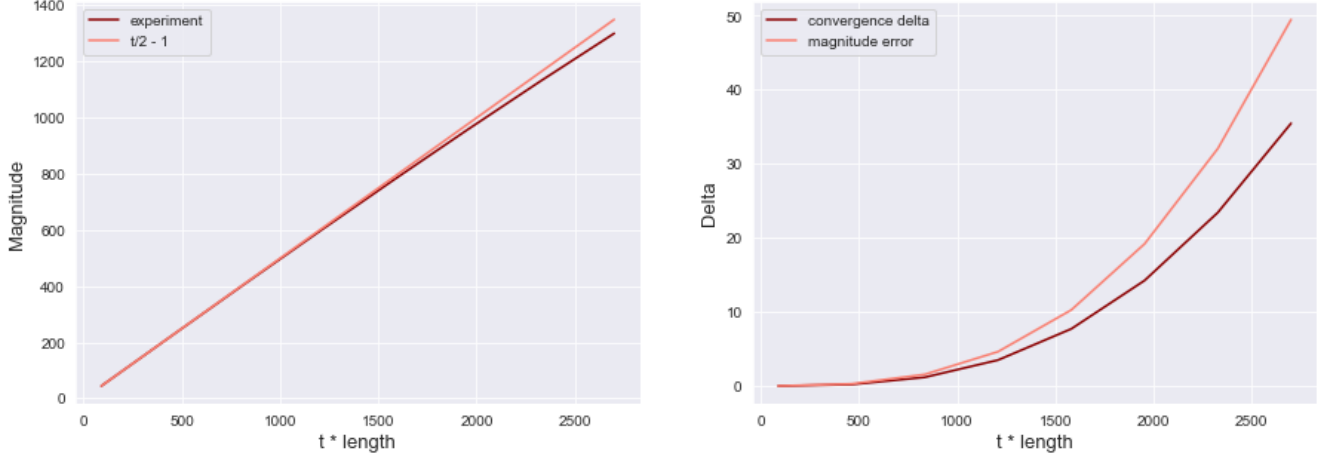
- 1: **procedure** MAGNITUDE FUNCTION( $\mathcal{E}$ )
- 2:   matrix  $D = \text{bellman-ford-algorithm}(\mathcal{E})$   $\triangleright$  precompute pairwise shortest distances for all vertices
- 3:   **for**  $t \in [10, 120]$  **do**
- 4:     **for** number of points  $\in [10, 4000]$  **do**
- 5:       points = [points obtained from breaking edges to equal parts]  $\triangleright$  lists length = number of points
- 6:       matrix  $M$ :  $M[i][j] = e^{-t \cdot \text{dist}(\text{points}[i], \text{points}[j])}$   $\triangleright$  dist uses  $D$  for vertices adjacent to points
- 7:       magnitude = sum(inverted( $M$ ))
- 8:       magnitude measures.append(magnitude)
- 9:     **end for**
- 10:   **end for**
- 11:   **return** magnitude measures  $\triangleright$  used to plot magnitude( $t$ ) and check that it converges
- 12: **end procedure**

---

Range of  $t$  and number of points can be varied but note that scaling number of points results in computation time (for  $[10, 4000]$  range it takes 10 min). Scaling  $t$  in respect to graphs length (sum of weights) will result in worse convergence (since  $e^{-t \cdot \text{dist}}$  becomes too small), in our experience to avoid that  $t \cdot \text{length}$  should be  $< 1000$ .



Cicles glued by edge magnitude, 4000 points



The plot above illustrates low precision of magnitude of too high  $t \cdot length$ . Note how errors go up to 40-50.

With  $1 < t \cdot length < 1000$  and  $10 < numpoints < 4000$  convergence rate is fast enough to expect real magnitude be in the neighbourhood:

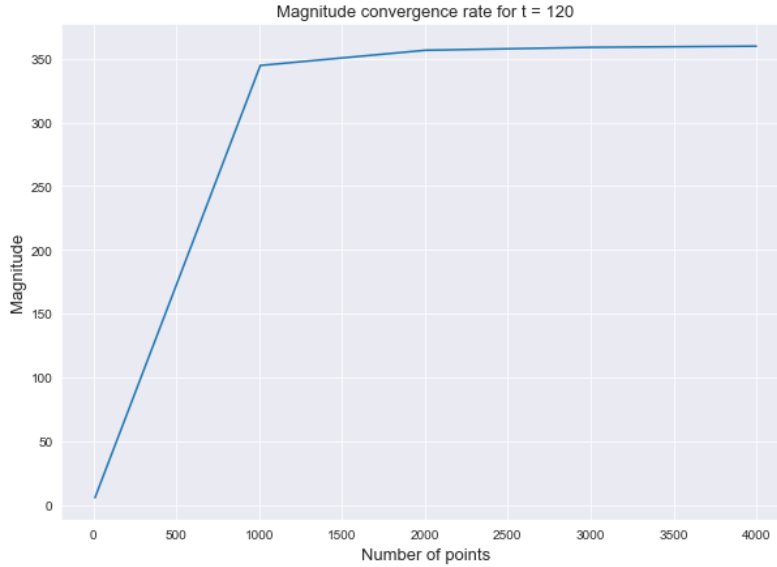


Illustration of the convergence rate, for an example graph. On the other plots in the work you can see that if  $t \cdot length < 1000$  convergence delta usually stays under 1 or 2

For different graph types we varied edges weights, so the plots in this report, although each of them shows magnitude of one type of graph with fixed weights, represents a set of graphs with the same configuration but different edges weights. Varying weights in all the cases didn't influence magnitude function.

The code is implemented in Python with the use of Numpy for matrix computations, for calculating inverted matrix we use `numpy.linalg.inv`. We tried using Scipy and other methods for matrix inversion: `linsolve`, methods specified for positive definite or symmetric matrix. All those methods didn't give any improvements nor with inversion precision, nor with the computational time.

Code can be found here (github) and here (colab), we tried to make it user-friendly so to check any hypothesis it is enough to list all edges of the graph and call one method to plot comparison of guessed magnitude function and simulation result.

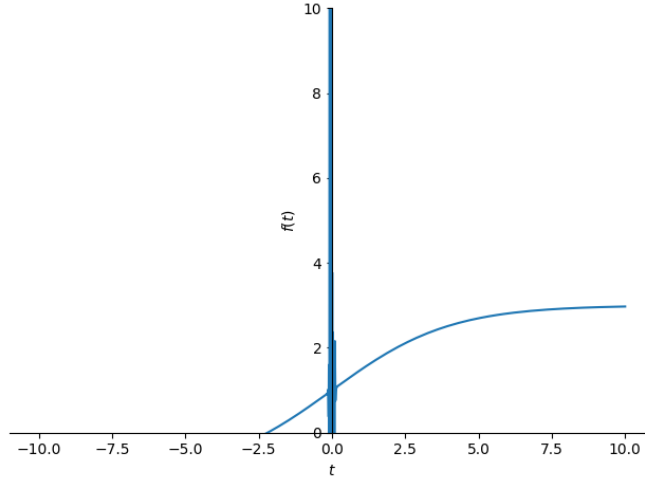
## 5.2 Symbolic computations - unsuccessful attempt

The other approach which we initially tried included symbolic computations. Instead of measuring series of magnitudes for finite set of  $t$  and then guessing the real magnitude function, we wanted to approximate it as **limit of function series over  $t$** .

For a fixed metric graph we generated a sequence of its combinatorial subgraphs, and then calculated magnitude directly as a function (not in small number of fixed  $t$ ), using Sympy.

Unfortunately this approach didn't work out since this kind of computations are way more complex than numerical ones and while symbolic computation for 3 points took around 4 sec, for 4 points it worked for more than 10 min (note, how for numeric computations we calculated magnitude for 4000 points). Also the results were not interpretable, since number of terms in the magnitude failed to simplify and had a lot of terms (this is the expression for only 3 points):

$$\begin{aligned} & \frac{-3e^{0.5t} + e^{1.5t} - e^{-1.5t} + 3e^{-0.5t}}{4e^{1.0t} - e^{2.0t} - 6 - e^{-2.0t} + 4e^{-1.0t}} + \frac{-e^{1.0t} + e^t - e^{-t} + e^{-1.0t}}{-e^{1.0t} + 3 + 2e^{-2.0t} - e^{-2t} - 3e^{-1.0t}} + \\ & \frac{-2e^{1.0t} - 2e^t + e^{2.0t} + 6 + e^{-2.0t} + 2e^{-t} - 6e^{-1.0t}}{3 \cdot (56e^{1.0t} - 28e^{2.0t} + 8e^{3.0t} - e^{4.0t} - 70 - e^{-4.0t} + 8e^{-3.0t} - 28e^{-2.0t} + 56e^{-1.0t})} + \\ & \frac{126e^{0.5t} - 84e^{1.5t} + 36e^{2.5t} - 9e^{3.5t} + e^{4.5t} - e^{-4.5t} + 9e^{-3.5t} - 36e^{-2.5t} + 84e^{-1.5t} - 126e^{-0.5t}}{28e^{1.0t} - 8e^{2.0t} + e^{3.0t} - 56 + e^{-5.0t} - 8e^{-4.0t} + 28e^{-3.0t} - 65e^{-2.0t} + 9e^{-2t} + 70e^{-1.0t}} + \\ & \frac{36e^{1.0t} - 9e^{2.0t} + e^{3.0t} - 84 - e^{-6.0t} + 9e^{-5.0t} - 36e^{-4.0t} + 84e^{-3.0t} - 145e^{-2.0t} + 19e^{-2t} + 126e^{-1.0t}}{132e^{1.0t} - 29e^{2t} - 136e^{2.0t} + 110e^{3.0t} - 44e^{4.0t} + 10e^{5.0t} - e^{6.0t} + e^{-6.0t} - 10e^{-5.0t} + 44e^{-4.0t} - 110e^{-3.0t} + 174e^{-2.0t} - 9e^{-2t} - 132e^{-1.0t}} + \\ & \frac{792e^{1.0t} - 126e^{2t} - 369e^{2.0t} + 220e^{3.0t} - 66e^{4.0t} + 12e^{5.0t} - e^{6.0t} - 924 - e^{-6.0t} + 12e^{-5.0t} - 66e^{-4.0t} + 220e^{-3.0t} - 504e^{-2.0t} + 9e^{-2t} + 792e^{-1.0t}}{1} \end{aligned}$$



Magnitude calculated as function in Sympy and its plot

## 6 Conclusion

## References

- [1] TOM LEINSTER. “The magnitude of a graph”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 166.2 (Nov. 2017), pp. 247–264. DOI: 10.1017/s0305004117000810. URL: <https://doi.org/10.1017/s0305004117000810>.
- [2] Tom Leinster and Simon Willerton. “On the asymptotic magnitude of subsets of Euclidean space”. In: *Geometriae Dedicata* 164.1 (Aug. 2012), pp. 287–310. DOI: 10.1007/s10711-012-9773-6. URL: <https://doi.org/10.1007/s10711-012-9773-6>.
- [3] Delio Mugnolo. *What is actually a metric graph?* 2019. DOI: 10.48550/ARXIV.1912.07549. URL: <https://arxiv.org/abs/1912.07549>.