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Образовательная программа «Прикладная математика и информатика»

УДК: 004.9

# Отчет об исследовательском проекте

на тему: Приближение магнитудной функции метрического графа

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# Contents

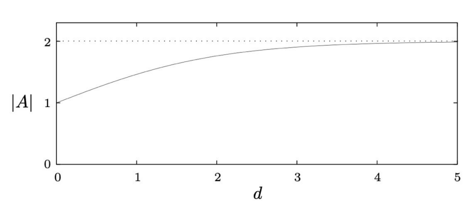


# 1 Introduction

Metric graph is a topological space which, simply put, represents a glued set of closed intervals. Unlike combinatorial weighted graph it is not discrete, but continuous, containing not only vertices, but also all points on the edges.

For a given metric graph we can calculate a magnitude function, object originally coming from category theory. Being a generalized notion of size, magnitude function is defined for enriched categories, and metric graph is a type of metric space, which, in turn, is a type of enriched category.

**Magnitude function of two points space:**



As calculated by Leinster and Willerton [3] the magnitude function of an interval of length $l$ equals $\frac{lt}{2}+1$, while magnitude of two points space with distance $l$ between them equals $\frac{2}{1+e^{-lt}}$. Calculation of magnitude function of the first object has some non-trivial mathematical steps, while calculation of the magnitude function of the second object is a simple algorithmic task of finding a reverse 4 element matrix. Moreover the resulting values of the magnitude function don't seem to have some sort of simple relation and magnitude of interval can not be simply calculated from magnitude of its endpoints.

But $l \to \frac{l}{2}+1$ is a rather simple function which might have been guessed from the computer simulations.

Magnitude of a combinatorial graph was already studied by LEINSTER [2], but as we see in the case of interval of length $l$ and two discrete points with distance $l$ between them, studying magnitude of the continuous space is a whole different problem.

In this work we present the results of computer simulations for different metric graphs and the conjecture about behavior of the magnitude function derived from them, which goes along with all the experiments so far.

The work is structured in the following way: in the Background section we will give definitions of magnitude function and metric graph, explain their relation and explain why magnitude of a metric graph is defined and how it can be calculated.

Then in the Conjectures section we explain the following 2 suggestions:

1. *magnitude function of a metric graph asymptotically satisfies the inclusion-exclusion principle*

2. *magnitude of a metric graph asymptotically is equal to* $\frac{t \cdot length}{2} + \chi(\mathcal{G})$

In the Experiments and their interpretations we prove 2nd hypothesis for particular cases of metric graphs, explain how we approximate magnitude and then go through all the graphs we experimented on. For each of them we give the simulations result and then explain how it aligns with the conjectures (so we decompose the graphs to simpler parts to use the inclusion-exclusion principle) and also show how the result aligns with already known magnitudes and theorems.

After that in Methods we go through details about the way we calculated magnitude function numerically and then explain the unsuccessful way how we tried to approximate magnitude.

# 2 Background

## 2.1 Definitions

### 2.1.1 Metric Graph

As already mentioned in the introduction, we can intuitively understand metric graphs as weighted graphs which, besides vertices, also contain all the points on the edges. The other intuitive understanding would be a set of intervals joined by their ends, thus forming a graph.

A stricter definition, originally coming from Mugnolo [4], is given below.

$$\mathcal{E} := \bigsqcup [0, l_e] \mid e \in E,\ l_e \in (0, \infty), \text{ where } E \text{ is a countable set and } \bigsqcup \text{ is a disjoined topological union.}$$

$$\text{The notation for elements of } \mathcal{E} \text{ is: } (x, e) \in \mathcal{E} \iff e \in E \wedge x \in [0, l_e]$$

$$\mathcal{V} := \bigsqcup \{(0, e), (l_e, e)\}$$

$$\text{On } \mathcal{V} \text{ we define an equivalency relation } \sim_{\mathcal{V}} \text{ with classes of equivalency } \{V_i\} \text{ (each represents a vertex)}$$

$$\text{On } \mathcal{E} \text{ we define an equivalency relation } \sim_{\mathcal{E}} \mid (x_1, e_1) \sim_{\mathcal{E}} (x_2, e_2) \iff x_1 = x_2; e_1 = e_2 \text{ or } (x_1, e_1) \sim_{\mathcal{V}} (x_2, e_2)$$

**Definition 1.** $\mathcal{G} = \mathcal{E} \ / \sim_{\mathcal{E}}$ is a **metric graph** with vertices $\mathcal{V} \ / \sim_{\mathcal{V}}$

### 2.1.2 Magnitude

First we will give a definition of magnitude for a finite metric space, then two equivalent definitions for compact positive definite metric spaces, taken from Leinster and Willerton [3].

**Definition 2.**
Let $X$ be a finite metric space with metric $d$.

$$\text{Magnitude of } X \text{ is } \mathcal{M}(X) \coloneqq \sum_{x \in X} w_x, \text{ where } w_x \in \mathbb{R} \text{ and } \forall x \in X : \sum_{x' \in X} w_x \cdot e^{-d(x,x')} = 1$$

**Remark.**
If $Z$ is a $|X| \times |X|$ matrix, with values $Z_{i,j} = e^{-d(x_i,x_j)}$, where $x_i, x_j \in X$ and Z is invertible, then:

$$\mathcal{M}(X) = \sum_{i,j} (Z^{-1})_{i,j}$$

**Definition 3.**
Let $X$ be a positive definite metric space, then:

$$\mathcal{M}(X) = \sup\{\mathcal{M}(Y) : Y \subset X \text{ and } |Y| < \infty\}$$

**Definition 4.** (equivalent to definition 3)
Let $X$ be a positive definite metric space and $\{X_i\}$ its subsets $: |X_i| < \infty$ and $X_i \to X$ in Hausdorff metric then:

$$\mathcal{M}(X) = \lim_{i \to \infty} \mathcal{M}(X_i)$$

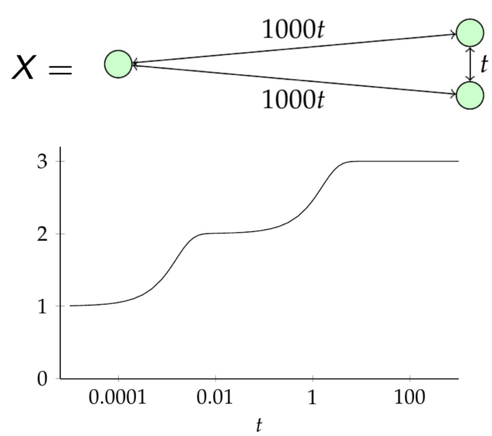
### 2.1.3 Magnitude function

**Definition 5.** If $X$ is a metric space with metric $d(x,y)$ and $t \in (0,\infty)$ then $tX$ is a metric space with metric $t{\cdot}d(x,y)$

**Definition 6.** $f_{\mathcal{M}} : (0,\infty) \to \mathbb{R}$ is a magnitude function of X, if

$$f_{\mathcal{M}}(t) = \mathcal{M}(tX)$$

**Illustration of magnitude function for a three point space with the following distances:**
Note how the values of the magnitude function correspond to the number of points seen if we change the perspective: first we see 1 point, then 2 as we get closer but still see two of them as one, then 3.



## 2.2 Metric graph as a compact metric space

Metric graphs are metric spaces with metric equivalent to shortest path between two points. Let's quickly define it as it appears in Mugnolo [4].

$$d_{\mathcal{E}}((x_1, e_1), (x_2, e_2)) = \begin{cases} |x_1 - x_2|, & \text{if } e_1 = e_2 \\ \infty, & \text{otherwise} \end{cases}$$

$$d_{\mathcal{G}}(\xi, \theta) = \inf\{\sum_{i=1}^{k} d_{\mathcal{E}}(\xi_i, \theta_i)\} \quad \xi_i, \theta_i \in \mathcal{E}$$

In our case we consider graphs with $|E| < \infty$, thus we can easily prove that it's a metric space and moreover a compact metric space.

**Metric graph is a metric space**

1. $d_{\mathcal{G}}(x, y) = 0 \iff x = y$
2. $d_{\mathcal{G}}(x, y) = d_{\mathcal{G}}(y, x)$

1-2 are obvious from the definition of $d_{\mathcal{E}}, d_{\mathcal{G}}$ as either distances (which is interval metric) between points on the interval, either their sum.

3. $d_{\mathcal{G}}(x, y) \leqslant d_{\mathcal{G}}(x, z) + d_{\mathcal{G}}(z, y)$

3 is also obvious since $x \to z \to y$ is also a valid path, thus can't be less than infimum.

**Metric graph is a compact metric space**

Since $|E| < \infty$ for any sequence of points we can find an edge which contains infinite subsequence and since edge is an interval which is a compact, it has a convergent subsequence, thus metric graph is a compact.

# 3 Conjectures

The pattern that we observed in the behavior of the magnitude function during computer simulations is formulated below:

**Hypothesis 1.** The magnitude of the metric graph asymptotically satisfies the inclusion-exclusion principle.

Similar conjecture about subsets of Euclidean space was made in Leinster and Willerton [3]. But since metric graphs are not always subsets of Euclidean space and to our knowledge this conjecture has not been proven, our result is a contribution.

More strictly the hypothesis states:

$$\forall A, B. \exists q_A, q_A : \mathcal{M}_A(t) = P_A(t) + q_A(t), \mathcal{M}_B(t) = P_B(t) + q_B(t), \text{ where } q_A(t) \to 0, q_B(t) \to 0$$

If the conjecture is right we can derive magnitude of any graph by decomposing it step by step to simpler elements: as shown in section Overview of the relevant results, we already know magnitude of finite number of points, magnitude of an interval, magnitude of a cycle and magnitude of a tree.

Moreover, stronger conjecture would be the following one:

**Hypothesis 2.** If $\mathcal{G}$ is a metric graph with edges $\mathcal{E}$, then $\mathcal{M}_{\mathcal{G}} = \frac{l \cdot t}{2} + const$, where $l = \sum_{e \in \mathcal{E}} length_e$ and $const =$ Euler characteristic of $\mathcal{G}$.

This conjecture is also consistent with all the simulation results. Besides, since Euler characteristic $\chi = |\mathcal{V}| - |\mathcal{E}|$ Ławniczak et al. [1] satisfies the inclusion-exclusion principle with some restrictions, and both cycle magnitude and edge magnitude satisfy 2nd theorem, it might be corollary of the 1st theorem.

**We did not obtain any proofs concerning the 1st hypothesis. 2nd hypothesis proofs for spectial cases of the metric graphs can be found in section 4.3.**

# 4 Experiments and their interpretations

## 4.1 Approximation of the magnitude function

Here we will briefly explain how the magnitude function was approximated. Pseudocode, relevant computation details and all links can be found in the Methods section. Code can be found on Github and in Colab.

For a fixed $t$ and a fixed graph $\mathcal{G}$ we approximate a continuous metric graph with it's discrete subsets — finite combinatorial graphs, which we get by dividing each edge into equal parts.

For each graph in this sequence $\{\mathcal{G}_k\}$ we calculate magnitude ($\mathcal{M}_k(t)$) as in remark 1: as sum of elements of $Z^{-1}$, where $Z_{i,j} = e^{-t \cdot d_{\mathcal{G}_k}(v_i,v_j)} \mid v_i, v_j \in \mathcal{V}_k$.

As follows from the 4th definition:

$$\mathcal{M}(t) = \lim_{k \to \infty} \mathcal{M}_k(t)$$

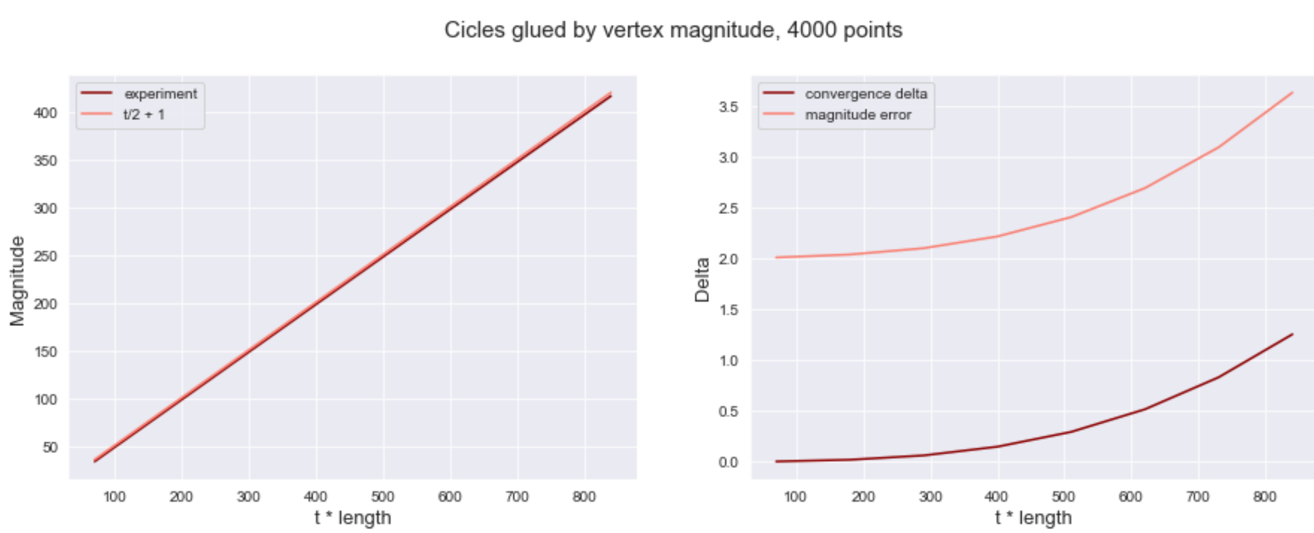
Range of $k$ and $t$ is specified and explained in the Methods section.

We run this algorithm on several $t = t_1, t_2, \ldots$ and draw a plot with the measured magnitude:
$\mathcal{M}_k = (t_1, \mathcal{M}_k(t_1)), (t_2, \mathcal{M}_k(t_2)), \ldots$.
After that we guess the real magnitude function: $\mathcal{M}_{guessed}(t)$, and compare the experiment result and our guess using two plots, described below.

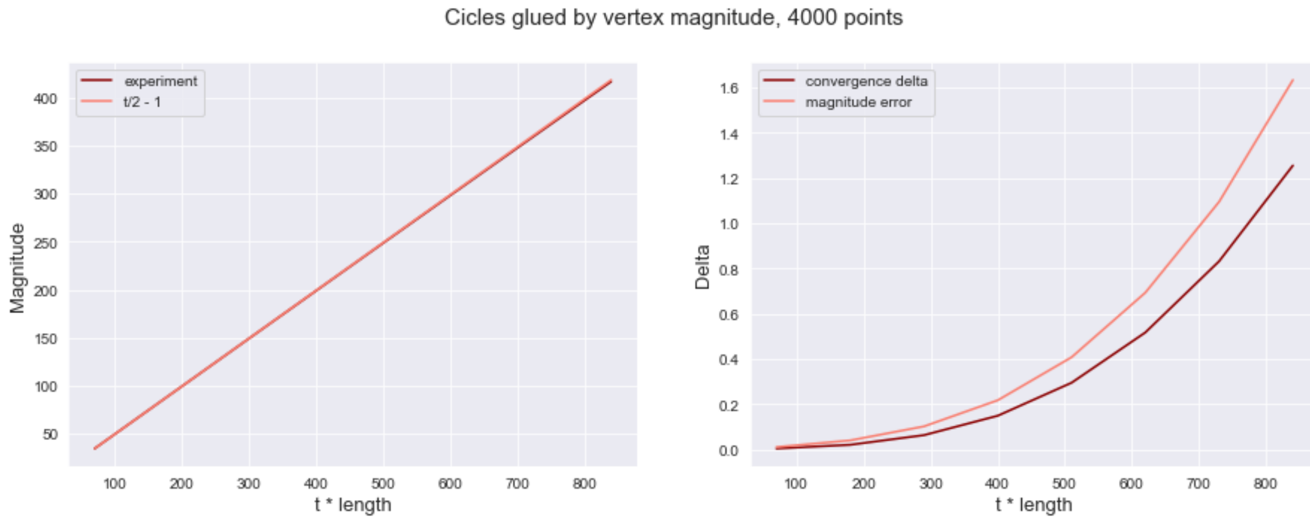
The first one draws $\mathcal{M}_{guessed}(t)$ and $\mathcal{M}_k$. The *length* of the graph equals to the sum of its weighted edges.

The second plot represents the convergence delta — by which we mean $|\mathcal{M}_k(t) - \mathcal{M}_{k-1}(t)|$, compared to the difference between the experiment result and the guessed magnitude function: ($|\mathcal{M}_{guessed}(t) - \mathcal{M}_k(t)|$), where $\mathcal{M}_k(t)$ is the magnitude measured for the last finite graph in the sequence (the one with the biggest number of vertices).
As t gets bigger, the pairwise distances between the points of the combinatorial graph grow too. This leads to the superlinear growth of the numerical error, because the magnitude is calculated by inverting a matrix with elements equal to the exponent to the power of negated pairwise distances. That's why we need some reference to check that they grow proportionally. In our case as a reference we use convergence delta since it perfectly shows how close the magnitude is to the real value (similar to the Cauchy convergence).

Note that all the magnitudes we studied ended up in the $\frac{t \cdot length}{2} + const$ neighbourhood, so in the most of the cases the second plot was more useful. The next example shows the difference between the plot when the *const* was guessed correctly compared to the plot with an incorrect *const*.




Example where the *const* was guessed wrong: the plot is still similar to $\frac{t}{2}$ but in comparison to convergence delta we can see, that the *const* should be decreased by 2.




This example illustrates the right guess - note how convergence delta correlates with difference between real magnitude and experiment results.

## 4.2 Overview of the relevant results

To interpret the experiments results we will briefly overview the relevant results.

First of all, the magnitude of the most basic objects: let $p_1$ be 1 point, $p_1 \leftrightarrow p_2$ — 2 points on distance equal to $l$. Then:

$$\mathcal{M}_{p_1}(t) = 1$$

$$\mathcal{M}_{p_1 \leftrightarrow p_2}(t) = \frac{2}{1 + e^{-t \cdot l}}$$

These magnitudes, as well as magnitude of any finite metric space, are easily computed without any analytic steps by inverting one matrix and summing the elements of the result.

For the subsets of the Euclidean space known results are the magnitude of an interval of length $l$ and a circle of circumference $l$ (Leinster and Willerton [3]):

$$\mathcal{M}_{\text{interval of length } l}(t) = \frac{l \cdot t}{2} + 1$$

$$\mathcal{M}_{\text{circle of circumference } l}(t) = \frac{l \cdot t}{2} + q(t), \text{ where } \lim_{t \to \infty} q(t) = 0$$

Another interesting result, Leinster and Willerton [3]:

**Theorem 1.** If X is a finite metric space with n points then $\lim_{t \to \infty} \mathcal{M}_X(t) = n$.

Note that in the case of a circle the metric does not need to be euclidean — magnitude is the same for the arc length metric, which would be useful to us in the case of a cycle.

Both of these calculations require non-trivial steps (and asymptotic analysis in the case of circle) but as we can see the dependence is simpler than in the case of two points and could have been guessed from simulations, see sections Intervals and Cycles.

As already mentioned in the third section, in Leinster and Willerton [3] there is a similar conjecture to the one stated in this paper: "the magnitude of subsets of Euclidean space asymptotically satisfies the inclusion-exclusion principle".

The other important result and its two corollaries are taken from LEINSTER [2].

**Theorem 2.** Let $X$ be a graph, with subgraphs $G$ and $H$ such that $G \cup H = X$. Suppose that $G \cap H$ is *convex* in $X$ and that $H$ *projects* to $G \cap H$. Then:

$$\mathcal{M}_X = \mathcal{M}_G + \mathcal{M}_H - \mathcal{M}_{(G \cap H)}$$

**Corollary 1.** Let $G$ and $H$ be graphs and $G \vee H$ is the graph obtained by gluing them together with one point. Then:

$$\mathcal{M}_{G \vee H} = \mathcal{M}_G + \mathcal{M}_H - 1$$

**Corollary 2.** Let $X$ be a tree, with subtrees $G$ and $H$ such that $G \cup H = X$. Then:

$$\mathcal{M}_X = \mathcal{M}_G + \mathcal{M}_H - \mathcal{M}_{G \cap H}$$

## 4.3 Hypothesis proofs for particular cases

### 4.3.1 Cycle and interval

Cycle and interval satisfy the 2nd hypothesis:

$$\chi_{cycle} = 0; \ \chi_{interval} = 1$$

From Leinster and Willerton [3] (circle with an arc distance metric):

$$\mathcal{M}_{circle} = \mathcal{M}_{cycle} = \frac{l \cdot t}{2} + q = \frac{l \cdot t}{2} + \chi_{cycle} + q, \text{ where } \lim_{t \to \infty} q = 0$$

$$\mathcal{M}_{interval} = \frac{l \cdot t}{2} + 1 = \frac{l \cdot t}{2} + \chi_{interval}$$

### 4.3.2 Metric graphs glued by vertex

**Theorem 3.** Let $X, G, H$ be metric graphs such that $X = G \vee H$ (glued by vertex $v_1$) and $\mathcal{M}_G, \mathcal{M}_H$ satisfy the 2nd hypothesis. Then $\mathcal{M}_X$ also satisfy the 2nd hypothesis.

**Proof:**

1. $X$ is a metric graph $\Rightarrow$

$$\Rightarrow \chi_X = |V_X| - |E_X| = |V_G \cup V_H| - |E_G \cup E_H| = |V_G| + |V_H| - |V_G \cap V_H| - |E_G| - |E_H| = \chi_G + \chi_H - 1$$

2. Let $\{G_i\}$, $\{H_i\}$ be sequences of combinatorial subgraphs of $G$ and $H$ accordingly, such that:

$$v_1 \in G_i, H_i; \ \lim_{i \to \infty} G_i = G; \ \lim_{i \to \infty} H_i = H$$

Then:

$$\lim_{i \to \infty} G_i \cup H_i = X \ \Rightarrow \ \mathcal{M}_X = \ \text{[by 4th definition]} \ = \lim_{i \to \infty} \mathcal{M}_{G_i \cup H_i} =$$

$$= \ \text{[by 2nd theorem since vertex is convex and projection of any graph]} \ =$$

$$= \lim_{i \to \infty} (\mathcal{M}_{G_i} + \mathcal{M}_{H_i} - \mathcal{M}_{G_i \cap H_i}) = \lim_{i \to \infty} \mathcal{M}_{H_i} + \lim_{i \to \infty} \mathcal{M}_{E_i} - 1 = \ \text{[by assumption]} \ =$$

$$= \frac{l_G \cdot t}{2} + \chi_G + q_G + \frac{l_H \cdot t}{2} + \chi_H + q_H - 1 = \frac{l_X \cdot t}{2} + (\chi_G + \chi_H - 1) + (q_G + q_H) =$$

$$= \frac{l_X \cdot t}{2} + \chi_X + q, \text{ where } q \to \infty$$

### 4.3.3 Graphs with $\leqslant 1$ cycle

All of the following corollaries immedeately follow from the 3rd theorem, since we can step by step construct these metric graphs from either a cycle or an edge by gluing new edges or cycles to the vertices of the existing graph.

**Corollary 3.** Any **metric tree** satisfy the 2nd hypothesis.

**Corollary 4.** Any **metric graph with 1 cycle** satisfy the 2nd hypothesis.

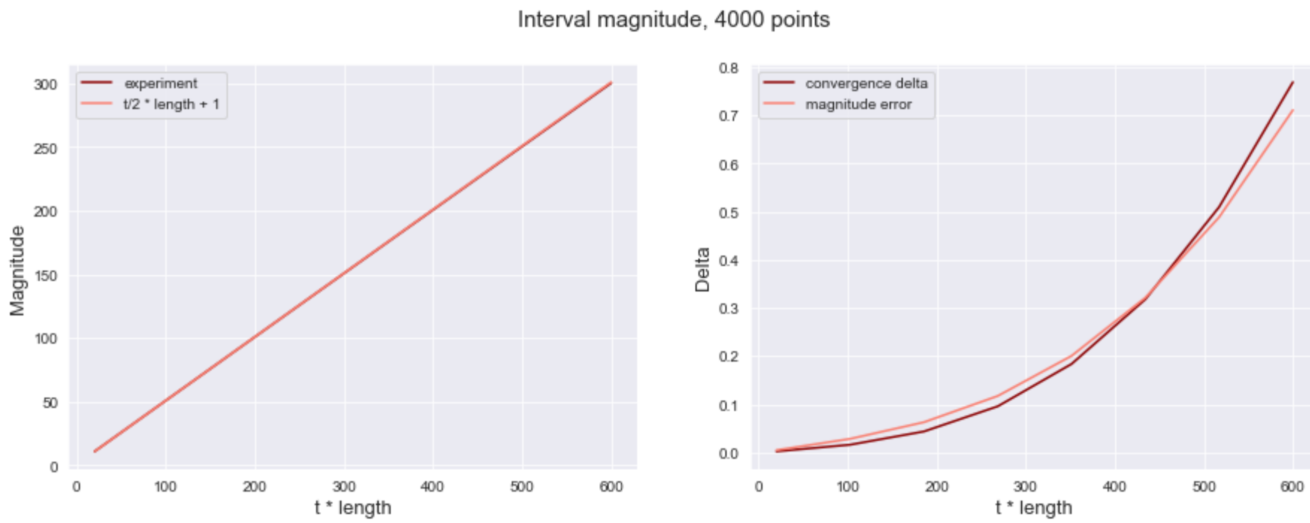
**Corollary 5.** Any metric graph where all cycles intersect in 1 point, satisfy the 2nd hypothesis.

## 4.4 One edge


**Asymptotic magnitude from simulations:**

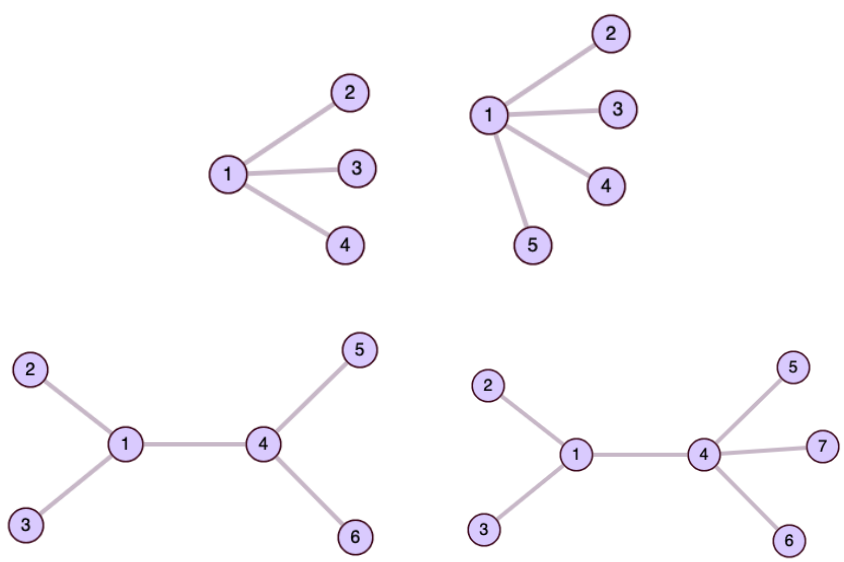
$$\mathcal{M}_{edge} = \frac{l \cdot t}{2} + 1$$

**Simulations plot:**




**Hypothesis testing:** consistent, see the 4.3.1 section.

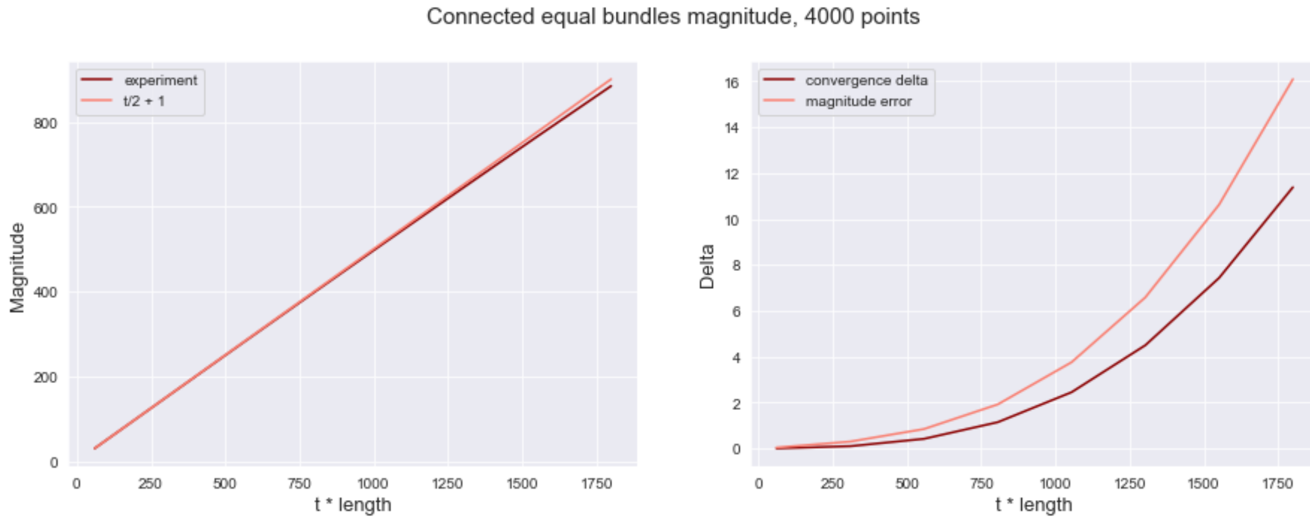
## 4.5 Trees



**Asymptotic magnitude from simulations (for all of the configurations):**

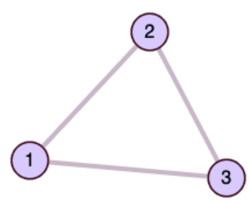
$$\mathcal{M}_{tree} = \frac{l \cdot t}{2} + 1$$

**Simulations plot:**
(For one of the configurations, other plots can be found in colab)




**Hypothesis testing:** consistent, see the 4.3.3 section.

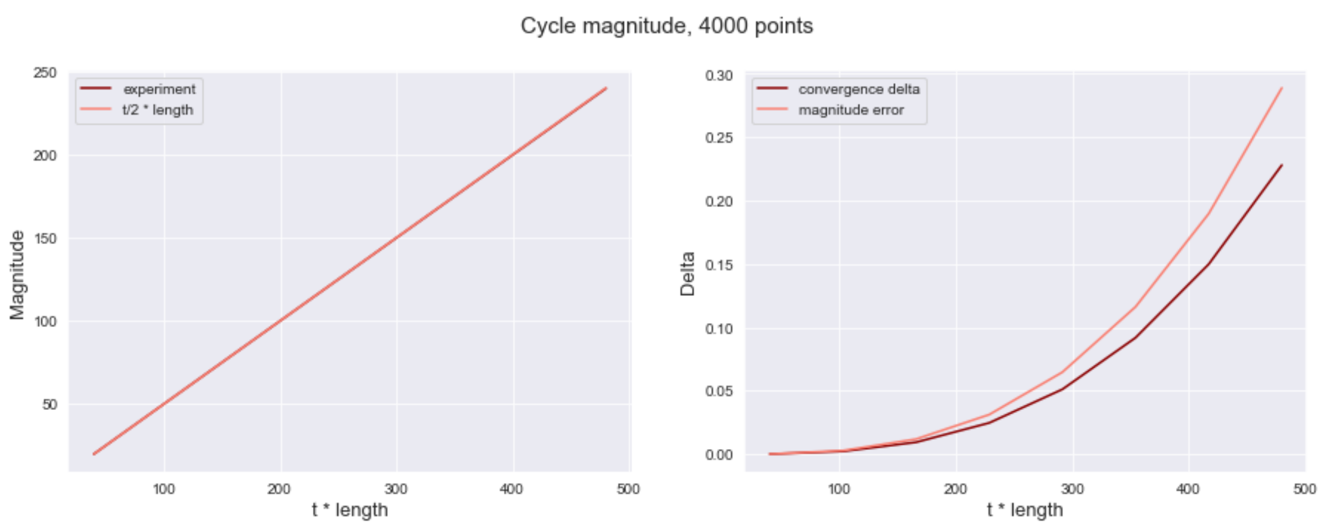
## 4.6 Cycle




**Asymptotic magnitude from simulations:**

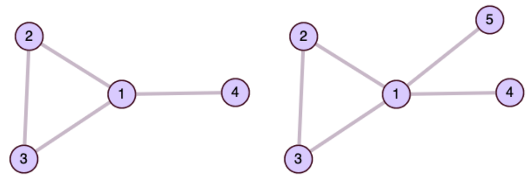
$$\mathcal{M}_{cycle} = \mathcal{M}_{circle} = \frac{l \cdot t}{2}$$

**Simulations plot:**




**Hypothesis testing:** consistent, see the 4.3.1 section.

### 4.7 Cycles combined with trees



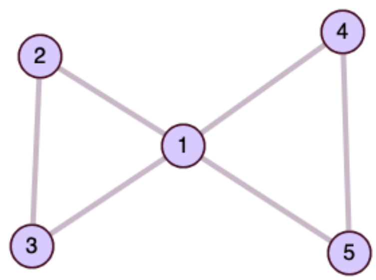

**Asymptotic magnitude from simulations:**

$$\mathcal{M}_{cycle\ +\ tree} = \frac{l \cdot t}{2}$$

**Hypothesis testing:** consistent, see the 4.3.3 section.

### 4.8 Combinations of 2 cycles

#### 4.8.1 Glued by vertex




**Asymptotic magnitude from simulations:**

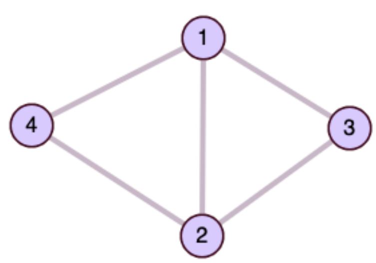
$$\mathcal{M}_{2cycles-1vertex} = \frac{l \cdot t}{2} - 1$$

**Consistent with the 1st hypothesis:** when we glue 2 cycles with 1 vertex by inclusion-exclusion principle:

$$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle1} + \mathcal{M}_{cycle2} - \mathcal{M}_{connecting\ point} = \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} - 1 = \frac{l \cdot t}{2} - 1$$

**Consistent with the 2nd hypothesis:** see the 4.3.3 section.

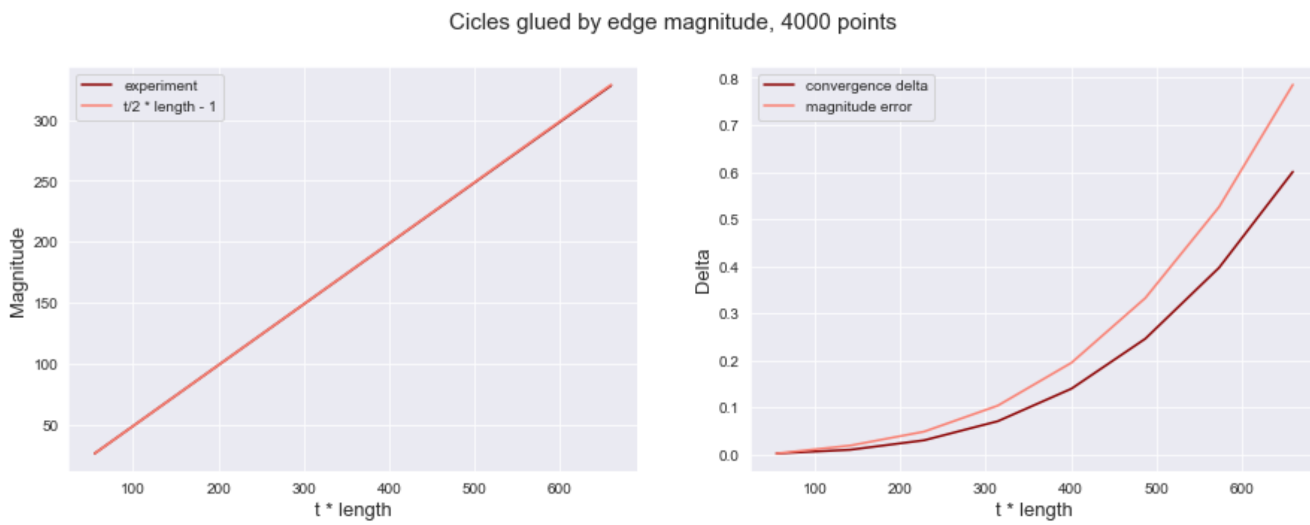
#### 4.8.2 Glued by edge



**Asymptotic magnitude from simulations:**

$$\mathcal{M}_{2cycles-1edge} = \frac{l \cdot t}{2} - 1$$

**Simulations plot:**




**Consistent with the 1st hypothesis:**

1. When we glue 2 cycles by edge by inclusion-exclusion principle:

$$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle1} + \mathcal{M}_{cycle2} - \mathcal{M}_{connecting\ edge} = \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} - (\frac{l_3 \cdot t}{2} + 1) = \frac{l \cdot t}{2} - 1$$

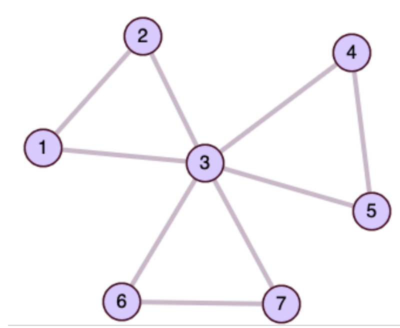
2. Another way to apply conjecture is to view this graph as a tree glued to a cycle by two points (1-2-3 as a cycle, 1-4-2 as a tree, and 1,2 as their intersection). Then:
$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle} + \mathcal{M}_{tree} - \mathcal{M}_{p_1 \leftrightarrow p_2} = \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} + 1 - \mathcal{M}_{p_1 \leftrightarrow p_2} = \frac{l \cdot t}{2} - 1 + q$, where $q \to 0$, since from Theorem 3: $\mathcal{M}_{n\ points} \to n$

Thus in both cases we get asymptotically the same magnitude as in simulations (note, that inclusion-exclusion conjecture is asymptotic).

**Consistent with the 2nd hypothesis:** $\chi = |\mathcal{V}| - |\mathcal{E}| = 4 - 5 = -1$.

## 4.9 Combinations of 3 cycles

### 4.9.1 Glued by vertex



**Asymptotic magnitude from simulations:**

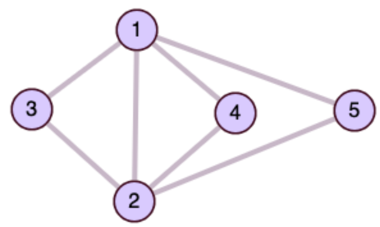
$$\mathcal{M}_{3cycles-1vertex} = \frac{l \cdot t}{2} - 2$$

**Consistent with the 1st hypothesis:** when we glue 3 cycles with 1 vertex by inclusion-exclusion principle:

$$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle1} + \mathcal{M}_{cycle2} + \mathcal{M}_{cycle3} - 2 \cdot \mathcal{M}_{connecting\ point} = \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} + \frac{l_3 \cdot t}{2} - 2 = \frac{l \cdot t}{2} - 2$$

**Consistent with the 2nd hypothesis:** see 4.3.3 section.

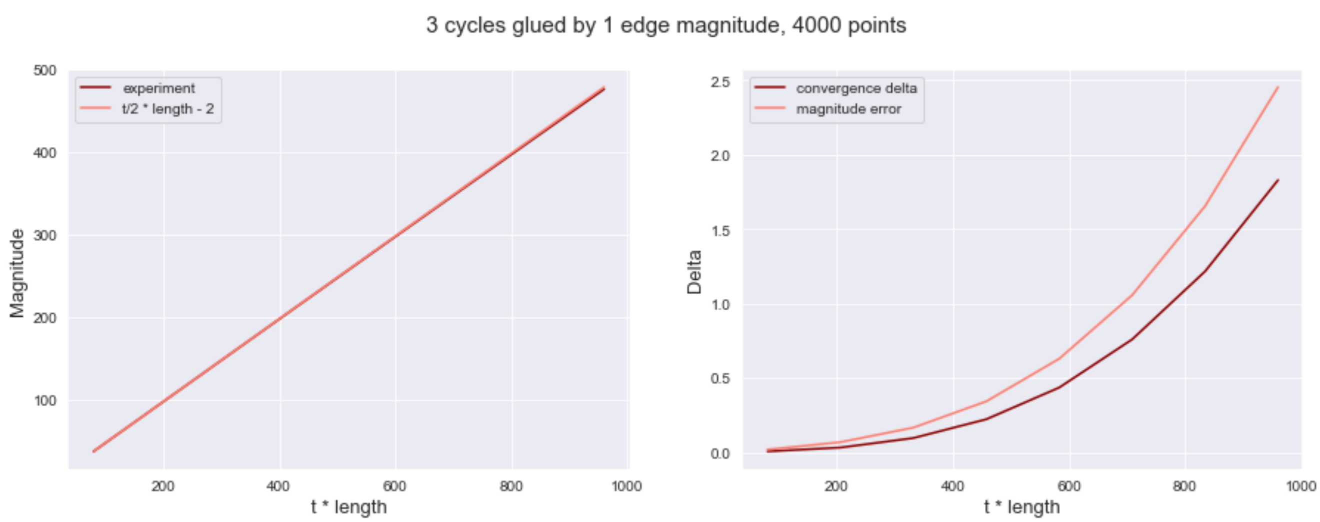
### 4.9.2 Glued by 1 edge




**Asymptotic magnitude from simulations:**

$$\mathcal{M}_{3cycles-1edge} = \frac{l \cdot t}{2} - 2$$

**Simulations plot:**

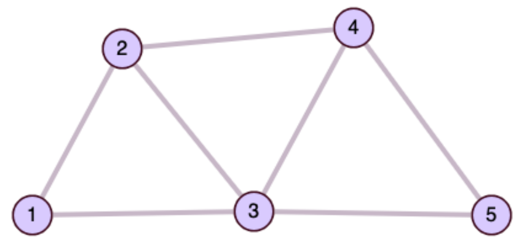



**Consistent with the 1st hypothesis:** when we glue 3 cycles with 1 edge by inclusion-exclusion principle:

$$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle1} + \mathcal{M}_{cycle2} + \mathcal{M}_{cycle3} - 2 \cdot \mathcal{M}_{connecting\ edge} = \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} + \frac{l_3 \cdot t}{2} - 2 \cdot (\frac{l_4 \cdot t}{2} + 1) = \frac{l \cdot t}{2} - 2$$

**Consistent with the 2nd hypothesis:** $\chi = |\mathcal{V}| - |\mathcal{E}| = 5 - 7 = -2$.

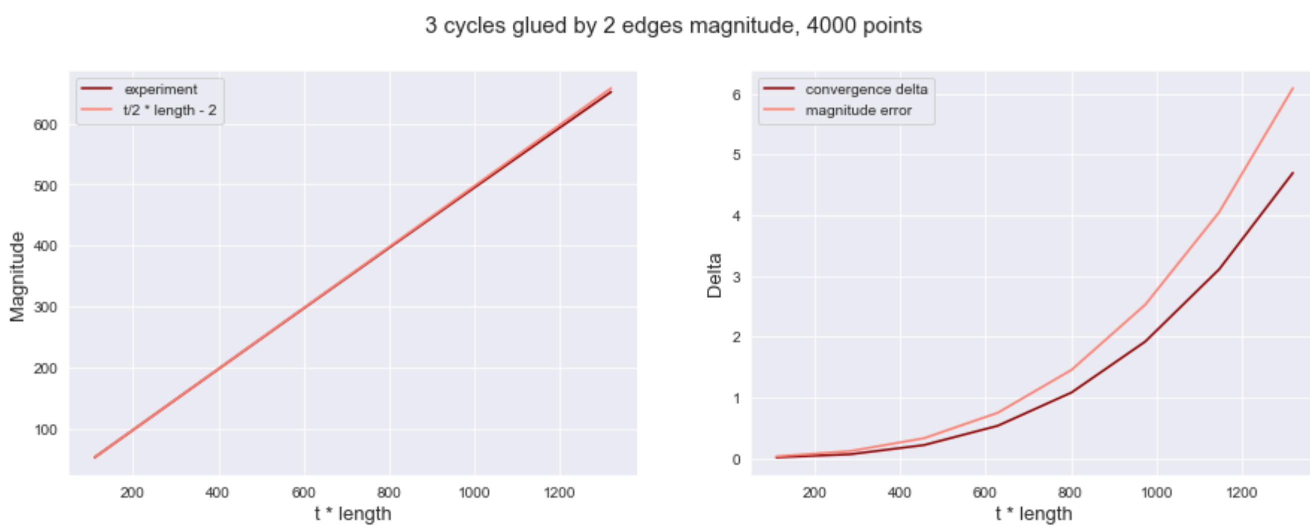
### 4.9.3 Glued by 2 edges




**Asymptotic magnitude from simulations:**

$$\mathcal{M}_{3cycles-2edges} = \frac{l \cdot t}{2} - 2$$

**Simulations plot:**

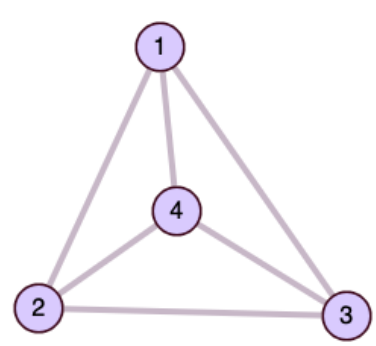



**Consistent with the 1st hypothesis:** when we glue 3 cycles with 2 edges by inclusion-exclusion principle:

$$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle1} + \mathcal{M}_{cycle2} + \mathcal{M}_{cycle3} - 2 \cdot \mathcal{M}_{connecting\ edges} = \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} + \frac{l_3 \cdot t}{2} - 2 \cdot (\frac{l_4 \cdot t}{2} + 1) = \frac{l \cdot t}{2} - 2$$

**Consistent with the 2nd hypothesis:** $\chi = |\mathcal{V}| - |\mathcal{E}| = 5 - 7 = -2$.

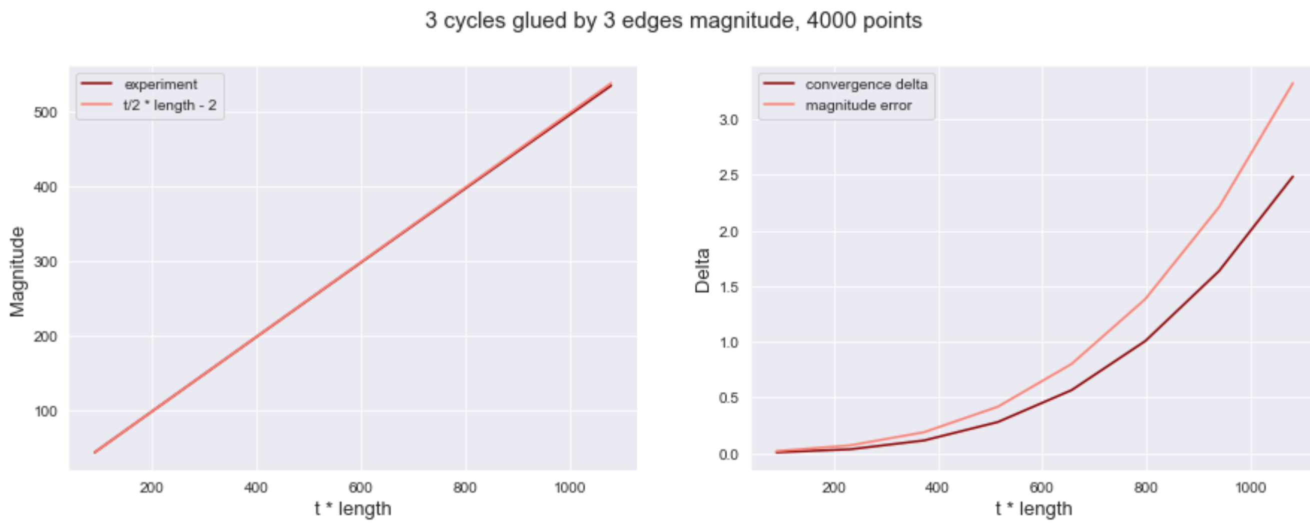
### 4.9.4 Glued by 3 edges



**Asymptotic magnitude from simulations:**

$$\mathcal{M}_{3cycles-3edges} = \frac{l \cdot t}{2} - 2$$

**Simulations plot:**



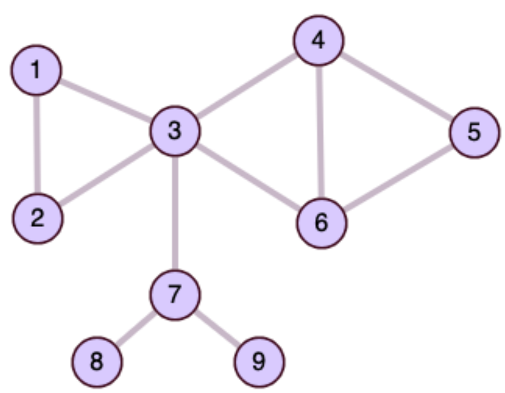

**Consistent with the 1st hypothesis:** when we glue 2 cycles with 1 vertex by inclusion-exclusion principle:

$$\mathcal{M}_{new\ graph} = \mathcal{M}_{cycle1} + \mathcal{M}_{cycle2} + \mathcal{M}_{cycle3} - 3 \cdot \mathcal{M}_{connecting\ edges} + \mathcal{M}_{central\ point} =$$

$$= \frac{l_1 \cdot t}{2} + \frac{l_2 \cdot t}{2} + \frac{l_3 \cdot t}{2} - 3 \cdot (\frac{l_4 \cdot t}{2} + 1) + 1 = \frac{l \cdot t}{2} - 2$$

**Consistent with the 2nd hypothesis:** $\chi = |\mathcal{V}| - |\mathcal{E}| = 4 - 6 = -2$.

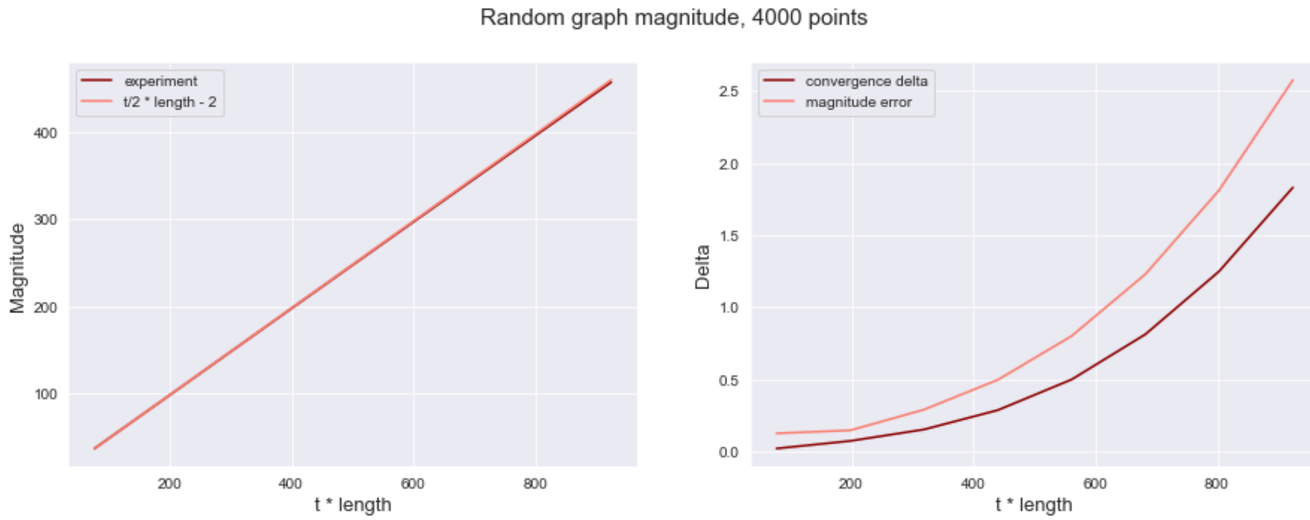
## 4.10 Random graph




**Asymptotic magnitude from simulations:**

$$\mathcal{M}_{random} = \frac{l \cdot t}{2} - 2$$

**Simulations plot:**




**Consistent with the 1st hypothesis** (easy to check by decomposing to cycles and trees).

**Consistent with the 2nd hypothesis:** $\chi = |\mathcal{V}| - |\mathcal{E}| = 9 - 11 = -2$,

# 5 Methods

## 5.1 Numeric computations

Brief explanation of how we approximated magnitude functions (similar code can be found in Willerton [5]): first, we choose a few fixed $t = t_1, t_2, \ldots$; for each of them we approximate magnitude as $\lim \mathcal{M}_k(t)$ (see the pseudocode below).

After that we plot connected $(t_1, \mathcal{M}_k(t_1)), (t_2, \mathcal{M}_k(t_2)), \ldots$ and try to guess the real $\mathcal{M}_{real}(t)$, to check the guess we plot $|\mathcal{M}_{real}(t_i) - \mathcal{M}_k(t_i)|$ (right subplot).

If it's close to 0 and behaves similar to the convergence delta, we conclude that we asymptotically guessed the $\mathcal{M}_{real}$.

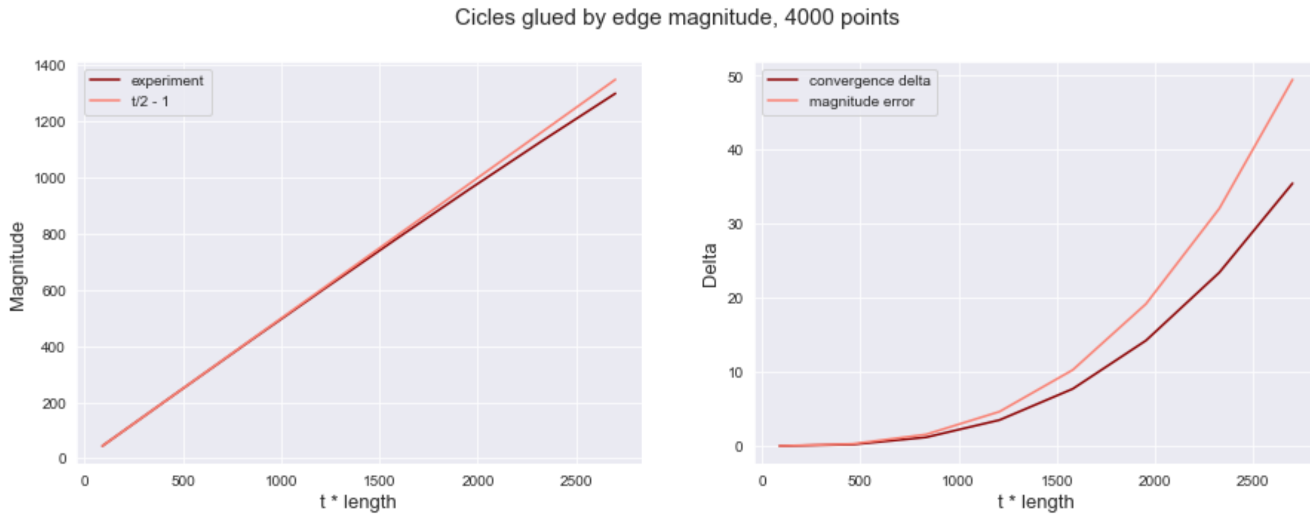
All the results in this paper were obtained with the following numeric algorithm:

**Algorithm 1** Magnitude function

**Require:** $\mathcal{E}$ - list of edges of the graph ▷ edge is represented as $(v_1, v_2, weight)$

**procedure** MAGNITUDE FUNCTION($\mathcal{E}$)
  matrix D = bellman-ford-algorithm($\mathcal{E}$) ▷ precompute pairwise shortest distances for all vertices
  **for** t $\in [10, 120]$ **do**
    **for** number of points $\in [10, 4000]$ **do**
      points = [points obtained from breaking edges to equal parts] ▷ lists length = number of points
      matrix M: M[i][j] = $e^{-\text{t} \cdot \text{dist(points[i], points[j])}}$ ▷ dist uses D for vertices adjacent to points
      magnitude = sum(inverted(M))
      magnitude measures.append(magnitude)
    **end for**
  **end for**
  **return magnitude measures** ▷ used to plot magnitude(t) and check that it converges
**end procedure**

Range of t and number of points can be varied but note that scaling number of points results in computation time (for [10, 4000] range it takes 10 min). Scaling t in respect to graphs length (sum of weights) will result in worse convergence (since $e^{-t \cdot dist}$ becomes too small), in our experience to avoid that $t \cdot length$ should be $< 1000$.




The plot above illustrates low precision of magnitude of too high $t \cdot length$. Note how errors go up to 40-50.

With $1 < t \cdot length < 1000$ and $10 < numpoints < 4000$ convergence rate is fast enough to expect real magnitude be in the neighbourhood:

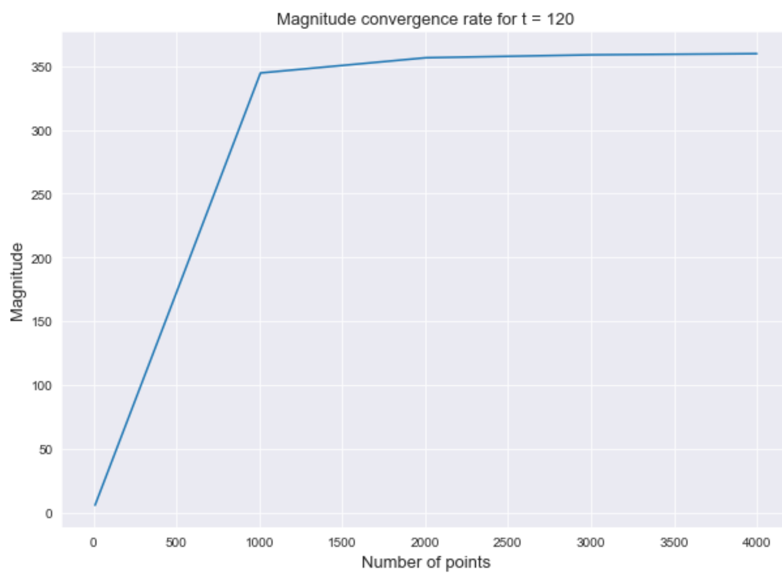



Illustration of the convergence rate, for an example graph. On the other plots in the work you can see that if $t \cdot length < 1000$ convergence delta usually stays under $1 - 2$.

For different graph types we varied edges weights, so the plots in this report, although each of them shows magnitude of one type of graph with fixed weights, represents a set of graphs with the same configuration but different edges weights. Varying weights in all the cases didn't influence magnitude function.

The code is implemented in Python with the use of Numpy for matrix computations, for calculating inverted matrix we use numpy.linalg.inv. We tried using Scipy and other methods for matrix inversion: linsolve, methods specified for positive definite or symmetric matrix. All those methods didn't give any improvements nor with inversion precision, nor with the computational time.

Code can be found on Github and in Colab, we tried to make it user-friendly so to check any hypothesis it is enough to list all edges of the graph and call 2 methods to calculate and plot comparison of guessed magnitude function and simulation result.

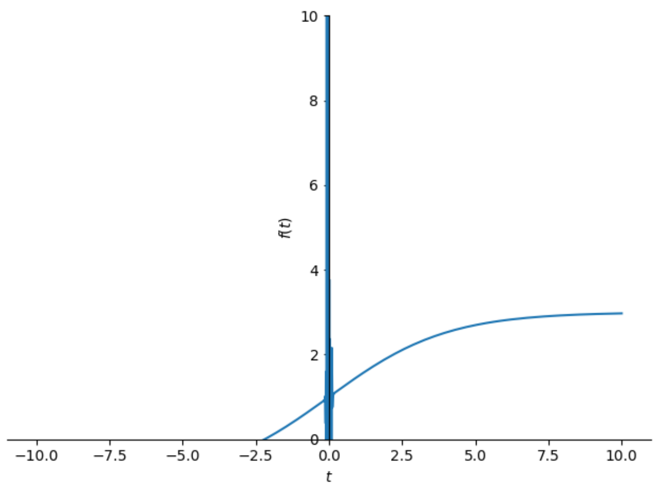
### 5.2 Symbolic computations

The other approach which we initially tried is symbolic computations. Instead of measuring series of magnitudes for finite set of t and then guessing the real magnitude function, we wanted to approximate it as **limit of function series over t**.

For a fixed metric graph we generated a sequence of it combinatorial subgraphs, and then calculated magnitude directly as a function (not in small number of fixed t), using Sympy.

Unfortunately this approach didn't work out since this kind of computations are way more complex than numerical ones and while symbolic computation for 3 points took around 4 sec, for 4 points it worked for more than 10 min (note, how for numeric computations we calculated magnitude for 4000 points). Also the results were not interpretable, since number of terms in the magnitude failed to simplify and had a lot of terms (this is the expression for only 3 points):

$$\frac{-3e^{0.5t}+e^{1.5t}-e^{-1.5t}+3e^{-0.5t}}{4e^{1.0t}-e^{2.0t}-6-e^{-2.0t}+4e^{-1.0t}}+\frac{-e^{1.0t}+e^{t}-e^{-t}+e^{-1.0t}}{-3e^{1.0t}-3e^{t}+e^{2.0t}+15+e^{-4.0t}-6e^{-3.0t}+15e^{-2.0t}+3e^{-t}-23e^{-1.0t}}+$$
$$\frac{-e^{1.0t}+e^{t}-e^{-t}+e^{-1.0t}}{-2e^{1.0t}-2e^{t}+e^{2.0t}+6+e^{-2.0t}+2e^{-t}-6e^{-1.0t}}+\frac{-e^{1.0t}+3+2e^{-2.0t}-e^{-2t}-3e^{-1.0t}}{-e^{1.0t}+4-e^{-3.0t}+6e^{-2.0t}-2e^{-2t}-6e^{-1.0t}}+$$
$$\frac{3\cdot\left(56e^{1.0t}-28e^{2.0t}+8e^{3.0t}-e^{4.0t}-70-e^{-4.0t}+8e^{-3.0t}-28e^{-2.0t}+56e^{-1.0t}\right)}{126e^{0.5t}-84e^{1.5t}+36e^{2.5t}-9e^{3.5t}+e^{4.5t}-e^{-4.5t}+9e^{-3.5t}-36e^{-2.5t}+84e^{-1.5t}-126e^{-0.5t}}+$$
$$\frac{28e^{1.0t}-8e^{2.0t}+e^{3.0t}-56+e^{-5.0t}-8e^{-4.0t}+28e^{-3.0t}-65e^{-2.0t}+9e^{-2t}+70e^{-1.0t}}{36e^{1.0t}-9e^{2.0t}+e^{3.0t}-84-e^{-6.0t}+9e^{-5.0t}-36e^{-4.0t}+84e^{-3.0t}-145e^{-2.0t}+19e^{-2t}+126e^{-1.0t}}+$$
$$\frac{132e^{1.0t}-29e^{2t}-136e^{2.0t}+110e^{3.0t}-44e^{4.0t}+10e^{5.0t}-e^{6.0t}+e^{-6.0t}-10e^{-5.0t}+44e^{-4.0t}-110e^{-3.0t}+174e^{-2.0t}-9e^{-2t}-132e^{-1.0t}}{792e^{1.0t}-126e^{2t}-369e^{2.0t}+220e^{3.0t}-66e^{4.0t}+12e^{5.0t}-e^{6.0t}-924-e^{-6.0t}+12e^{-5.0t}-66e^{-4.0t}+220e^{-3.0t}-504e^{-2.0t}+9e^{-2t}+792e^{-1.0t}}$$




Magnitude calculated as function in Sympy and its plot

# 6 Conclusion

We experimented with calculation of magnitude function of the metric graph and found some patterns in its behavior. In this work we suggested two new hypothesis:

1. Magnitude function of the metric graph asymptotically satisfies inclusion-exclusion rule.

2. $\mathcal{M}_G(t) = \frac{t \cdot length}{2} + \chi(G) + q$, where $q \to 0$ and $\chi$ is Euler characteristic.

The second hypothesis was proven for particular cases (e.g trees, graphs with $\leqslant 1$ cycles, graphs glued by 1 vertex). Both of the hypothesis were confirmed by all of the computer simulations.

Further work on the project might include searching for counterexample for one of the two hypothesis or finding generalized proof for any of them.

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[5] Simon Willerton. *Heuristic and computer calculations for the magnitude of metric spaces.* 2009. DOI: 10.48550/ARXIV.0910.5500. URL: https://arxiv.org/abs/0910.5500.