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Отчет об исследовательском проекте

на тему: Приближение магнитудной функции метрического графа

(промежуточный, этап 1)

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1 Introduction

Metric graph is a topological space which, simply put, represents a glued set of closed intervals. Unlike combinatorial weighted graph it is not discrete, but continuous, containing not only vertices, but also all points on the edges.

For a given metric graph we can calculate a magnitude function, object originally coming from category theory. Being a generalized notion of size, magnitude function is defined for enriched categories, and metric graph is a type of metric space, which, in turn, is a type of enriched category.

The aim of the following work is to study the relation between metric graphs and magnitude function by generating metric graphs and approximating their magnitude function using symbolic computations. After the experiment results are ready, the final step would be to make a suggestion of its value for a general case of a metric graph.

2 Background

2.1 Definitions

2.1.1 Metric Graph

As already mentioned in the introduction, we can intuitively understand metric graphs as weighted graphs which, besides vertices, also contain all the points on the edges. The other intuitive understanding would be a set of intervals joined by their ends, thus forming a graph.

A stricter definition, originally coming from Mugnolo [3], is given below.

$$\mathcal{E} := \bigsqcup [0, l_e] \mid e \in E,\ l_e \in (0, \infty), \text{ where } E \text{ is a countable set and } \bigsqcup \text{ is a disjoined topological union.}$$

$$\text{The notation for elements of } \mathcal{E} \text{ is: } (x, e) \in \mathcal{E} \iff e \in E \land x \in [0, l_e]$$

$$\mathcal{V} := \bigsqcup \{(0, e), (l_e, e)\}$$

$$\text{On } \mathcal{V} \text{ we define an equivalency relation } \sim_{\mathcal{V}} \text{ with classes of equivalency } \{V_i\} \text{ (each represents a vertex)}$$

$$\text{On } \mathcal{E} \text{ we define an equivalency relation } \sim_{\mathcal{E}} \mid (x_1, e_1) \sim_{\mathcal{E}} (x_2, e_2) \iff x_1 = x_2; e_1 = e_2 \text{ or } (x_1, e_1) \sim_{\mathcal{V}} (x_2, e_2)$$

Definition 1. $\mathcal{G} = \mathcal{E} \,/\sim_{\mathcal{E}}$ is a **metric graph** with vertices $\mathcal{V} \,/\sim_{\mathcal{V}}$

2.1.2 Magnitude

First we will give a definition of magnitude for a finite metric space, then two equivalent definitions for compact positive definite metric spaces, taken from Leinster and Willerton [2].

Definition 2.
Let X be a finite metric space with metric d.

$$\text{Magnitude of } X \text{ is } \mathcal{M}(X) \coloneqq \sum_{x \in X} w_x, \text{ where } w_x \in \mathbb{R} \text{ and } \forall x \in X : \sum_{x' \in X} w_x \cdot e^{-d(x,x')} = 1$$

Remark.
If Z is a $|X| \times |X|$ matrix, with values $Z_{i,j} = e^{-d(x_i,x_j)}$, where $x_i, x_j \in X$ and Z is invertible, then:

$$\mathcal{M}(X) = \sum_{i,j} (Z^{-1})_{i,j}$$

Definition 3.
Let X be a positive definite metric space, then:

$$\mathcal{M}(X) = \sup\{\mathcal{M}(Y) : Y \subset X \text{ and } |Y| < \infty\}$$

Definition 4. (equivalent to definition 3)
Let X be a positive definite metric space and $\{X_i\}$ its subsets $: |X_i| < \infty$ and $X_i \to X$ in Hausdorff metric then:

$$\mathcal{M}(X) = \lim_{i \to \infty} \mathcal{M}(X_i)$$

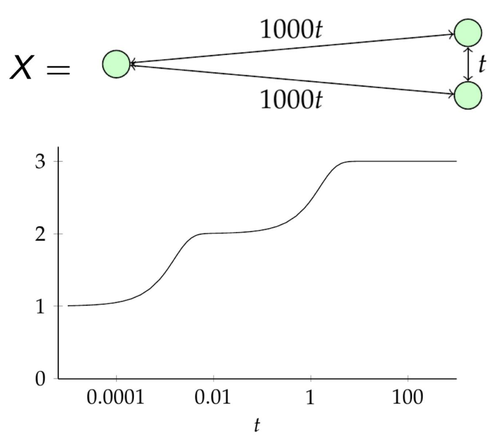
2.1.3 Magnitude function

Definition 5. If X is a metric space with metric $d(x, y)$ and $t \in (0, \infty)$ then tX is a metric space with metric $t \cdot d(x, y)$

Definition 6. $f_{\mathcal{M}} : (0, \infty) \to \mathbb{R}$ is a magnitude function of X, if

$$f_{\mathcal{M}}(t) = \mathcal{M}(tX)$$

Illustration of magnitude function for a three point space with the following distances:
Note how the values of the magnitude function correspond to the number of points seen if we change the perspective: first we see 1 point, then 2 as we get closer but still see two of them as one, then 3.



2.1.4 Spread

2.2 Metric graph as a compact metric space

Metric graphs are metric spaces with metric equivalent to shortest path between two points. Let's quickly define it as it appears in Mugnolo [3].

$$d_{\mathcal{E}}((x_1, e_1), (x_2, e_2)) = \begin{cases} |x_1 - x_2|, & \text{if } e_1 = e_2 \\ \infty, & \text{otherwise} \end{cases}$$

$$d_{\mathcal{G}}(\xi, \theta) = \inf\{\sum_{i=1}^{k} d_{\mathcal{E}}(\xi_i, \theta_i)\} \quad \xi_i, \theta_i \in \mathcal{E}$$

In our case we consider graphs with $|E| < \infty$, thus we can easily prove that it's a metric space and moreover a compact metric space.

Metric graph is a metric space

1. $d_{\mathcal{G}}(x, y) = 0 \iff x = y$
2. $d_{\mathcal{G}}(x, y) = d_{\mathcal{G}}(y, x)$

1-2 are obvious from the definition of $d_{\mathcal{E}}, d_{\mathcal{G}}$ as either distances (which is interval metric) between points on the interval, either their sum.

3. $d_{\mathcal{G}}(x, y) \leqslant d_{\mathcal{G}}(x, z) + d_{\mathcal{G}}(z, y)$

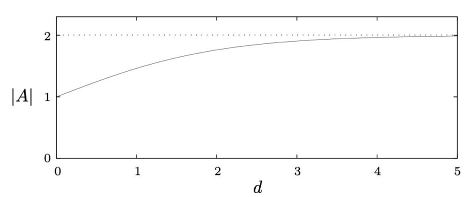
3 is also obvious since $x \to z \to y$ is also a valid path, thus can't be less than infimum.

Metric graph is a compact metric space

Since $|E| < \infty$ for any sequence of points we can find an edge which contains infinite subsequence and since edge is an interval which is a compact, it has a convergent subsequence, thus metric graph is a compact.

3 Calculation of magnitude function of a metric graph

Magnitude function of two points space:



As calculated by Leinster and Willerton [2] the magnitude function of an interval of length l equals $\frac{lt}{2} + 1$, while magnitude of two points space with distance l between them equals $\frac{2}{1+e^{-lt}}$. Calculation of magnitude function of the first object has some non-trivial mathematical steps, while calculation of the magnitude function of the second object is a simple algorithmic task of finding a reverse 4 element matrix. Moreover the resulting values of the magnitude function don't seem to have some sort of simple relation and magnitude of interval can not be simply calculated from magnitude of its endpoints.

But $l \to \frac{l}{2} + 1$ is a rather simple function which might have been guessed from the computer simulations.

Magnitude of a combinatorial graph was already studied by LEINSTER [1], but as we see in the case of interval of length l and two discrete points with distance l between them, studying magnitude of the continuous space is a whole different problem.

To do so we plan to generate metric graphs and by calculating magnitude function of sequence of its finite subspaces, we can approximate magnitude function of a metric graph as the limit of their magnitudes (due to 4th definition).

As already mentioned calculating magnitude function of a finite object is a simpler task and could be done with code similar to Willerton [4].

After the results of the experiments are ready we hope to find some kind of pattern in behavior of magnitude function and to make some provable conclusions.

References

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[2] Tom Leinster and Simon Willerton. "On the asymptotic magnitude of subsets of Euclidean space". In: *Geometriae Dedicata* 164.1 (Aug. 2012), pp. 287–310. DOI: `10.1007/s10711-012-9773-6`. URL: `https://doi.org/10.1007%2Fs10711-012-9773-6`.

[3] Delio Mugnolo. *What is actually a metric graph?* 2019. DOI: `10.48550/ARXIV.1912.07549`. URL: `https://arxiv.org/abs/1912.07549`.

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