Ph20 Problem Set 3

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1 Part 1

1.1 Explicit Euler Simulation

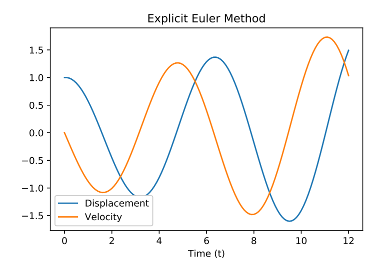

Figure 1: Plot of position and velocity over time generated with the Explicit Euler Method and a time step of 0.1

1.2 Explicit Euler Error

The analytic solution to this problem is $x = cos(x)$ and $v = -sin(x)$.

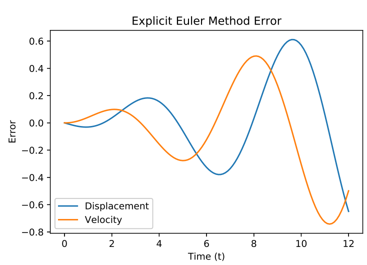

Figure 2: Plot of error in position and velocity over time of the Explicit Euler Method as compared to the analytic solution.

1.3 Explicit Euler Error vs Time Step

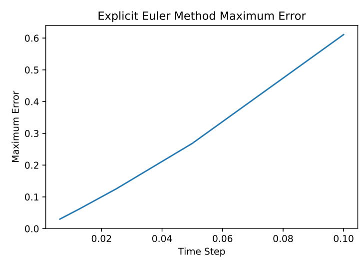

Figure 3: Plot of error maximum error in the Explicit Euler Method as a function of time step. The plot is very close to linear, suggesting that error is proportional to the time step for small h.

1.4 Explicit Euler Energy

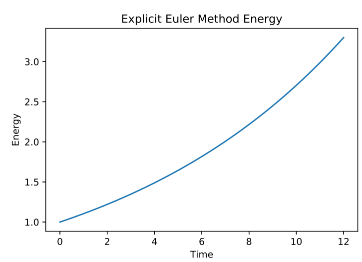

Figure 4: Plot of error energy over time as calculated by the Explicit Euler Method. The energy clearly increases over time, on the same scale as the error.

1.5 Implicit Euler Method

We are given the equation

$$\begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix} \begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$$

We can easily solve this by inverting the leftmost matrix, which we'll call A, and left-multiplying it on both sides. $det(A) = 1 + h^2$, so

$$\frac{A}{det(A)} = \begin{pmatrix} \frac{1}{1+h^2} & \frac{-h}{1+h^2} \\ \frac{h}{1+h^2} & \frac{1}{1+h^2} \end{pmatrix}$$

Flipping the diagonal elements and negating the off-diagonal elements, we have

$$A^{-1} = \begin{pmatrix} \frac{1}{1+h^2} & \frac{h}{1+h^2} \\ \frac{-h}{1+h^2} & \frac{1}{1+h^2} \end{pmatrix} = \frac{1}{1+h^2}\begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix}$$

Left multiplying this on both sides of the equation, we find

$$\begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} \frac{x_i+hv_i}{1+h^2} \\ \frac{v_i-hx_i}{1+h^2} \end{pmatrix}$$

And therefore our equations are

$$x_{i+1} = \frac{x_i + hv_i}{1+h^2} and v_{i+1} = \frac{v_i - hx_i}{1+h^2}$$

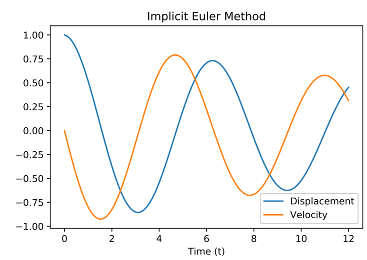

Figure 5: Plot of position and velocity over time generated with the Implicit Euler Method and a time step of 0.1

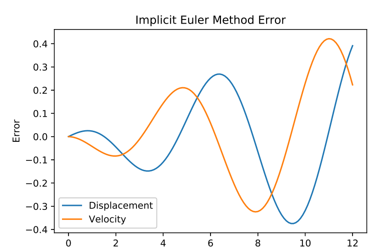

Figure 6: Plot of error in position and velocity over time of the Implicit Euler Method as compared to the analytic solution.

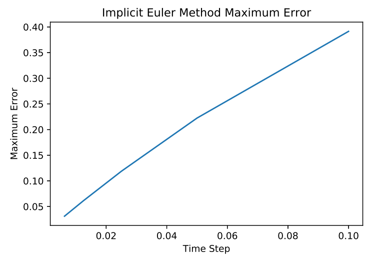

Figure 7: Plot of error maximum error in the Implicit Euler Method as a function of time step. The plot is very close to linear, suggesting that error is proportional to the time step for small h.

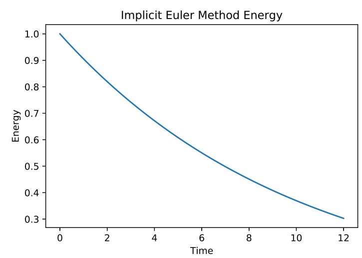

Figure 8: Plot of error energy over time as calculated by the Implicit Euler Method. The energy clearly decreases over time, on the same scale as the error increases.

2 Part 2

2.1 Phase Space Geometry of Explicit and Implicit Euler

2.2 Phase Space Geometry of Symplectic Euler Method

2.3 Symplectic Euler Method Energy

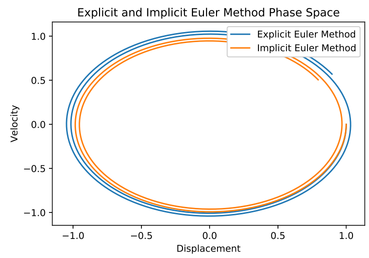

Figure 9: Phase space plot of Explicit and Implicit Euler Methods with a time step of 0.01. Note that neither method produces a closed figure.

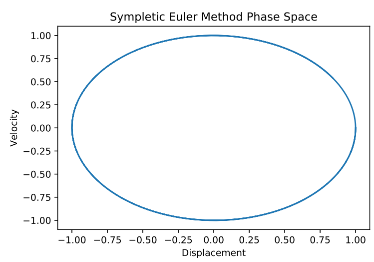

Figure 10: Phase space plot of Symplectic Euler Method with the same time step of 0.01 as the previous plot. Unlike the previous plot, this figure is closed.

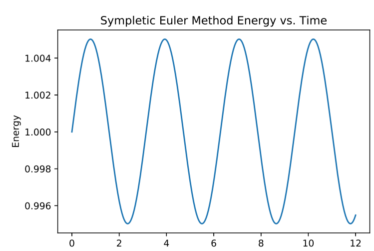

Figure 11: Plot of energy of the Symplectic Euler Method over time. The energy oscillates with very small amplitude around the actual value with no large-scale increase or decrease. This is consistent with this approximation being much closer to the true solution as demonstrated by the closed figure in the previous phase-space plot.