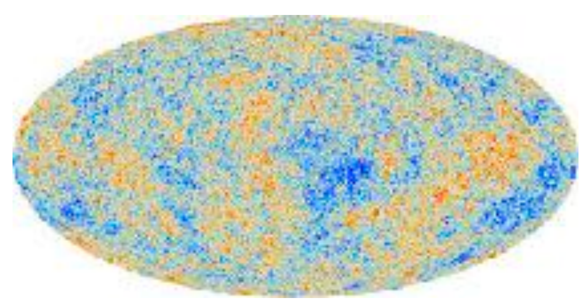
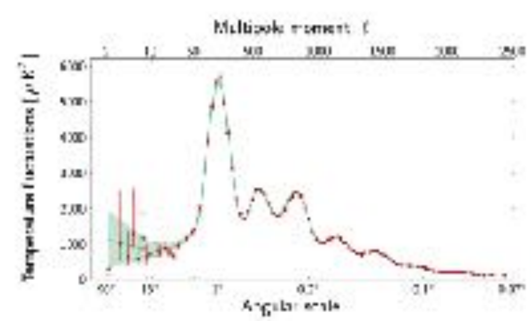
CMB Perturbations





J. Alberto Vázquez

ICF - UNAM

Perturbaciones

June 11-13, 2018

VI TALLER DE GRAVITACIÓN Y COSMOLOGÍA
ICF-UNAM, Cuernavaca, Morelos, 2 y 3 de agosto, 2018
Segunda Circular

26/05/2018

El objetivo del presente taller es servir como foro para discutir características y consecuencias de los trabajos recientes en los temas de gravitación, cosmología y áreas afines. Este taller también pretende establecer nuevas colaboraciones entre los participantes que trabajan en dichas áreas ampliando con ello las redes de investigación dentro de la comunidad científica mexicana.

PARTICIPANTES

La inscripción al evento sigue abierta para investigadores y estudiantes que contribuyan al taller. Por el momento tenemos confirmada la participación de los siguientes investigadores.

MIGUEL ASPEITIA
NORA BRETON
KAREN CABALLERO MORA
JOSÉ ANTONIO GONZÁLEZ CERVERA
FRANCISCO S. GUZMÁN
ALFREDO HERRERA AGUILAR
GERMAN IZQUIERDO
ANDRÉS PLAZAS

ESTUDIANTES

Los estudiantes interesados en participar deberán estar inscritos en un programa de posgrado, ser estudiantes regulares y enviar carta de apoyo (recomendación) de su supervisor.

Enviar resumen de plática (dado el caso) y carta de recomendación a más tardar el día viernes 8 de junio de 2018 a Juan Carlos Hidalgo al correo hidalgo@fis.unam.mx

HOSPEDAJE PARA ESTUDIANTES Y POSTDOCS

Habrá apoyo en forma de hospedaje y alimentos para estudiantes y postdocs. Se dará preferencia a quienes presenten plática corta.

SEDE DEL EVENTO

La sede del Taller será el auditorio del Instituto de Ciencias Físicas de la Universidad Nacional Autónoma de México, ubicado en el Campus de la Universidad Autónoma del Estado de Morelos, en la ciudad de Cuernavaca, Morelos.

Outline

Cosmic Microwave Background

The Hot Big Bang:

Recombination, Decoupling, Last Scattering

Black body radiation

Boltzmann equation

Temperature, Polarization,

Line of sight strategy

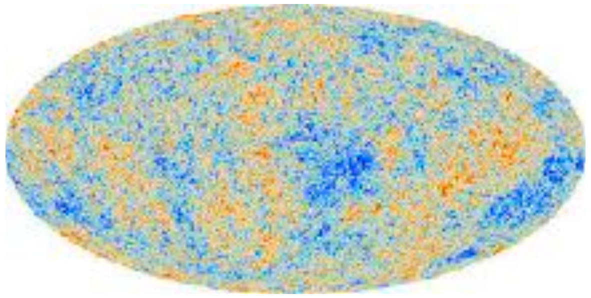
Perturbations — Talacha —

CMB Power Spectrum

Acoustic peaks
Codes
Observations

What else, Running, Non-gaussianity, Primordial Gravitational waves …

Motivation



A must do!

- The cosmic microwave background (CMB) is the **thermal radiation left over** from the "Big Bang", also known as **"relic radiation"**.

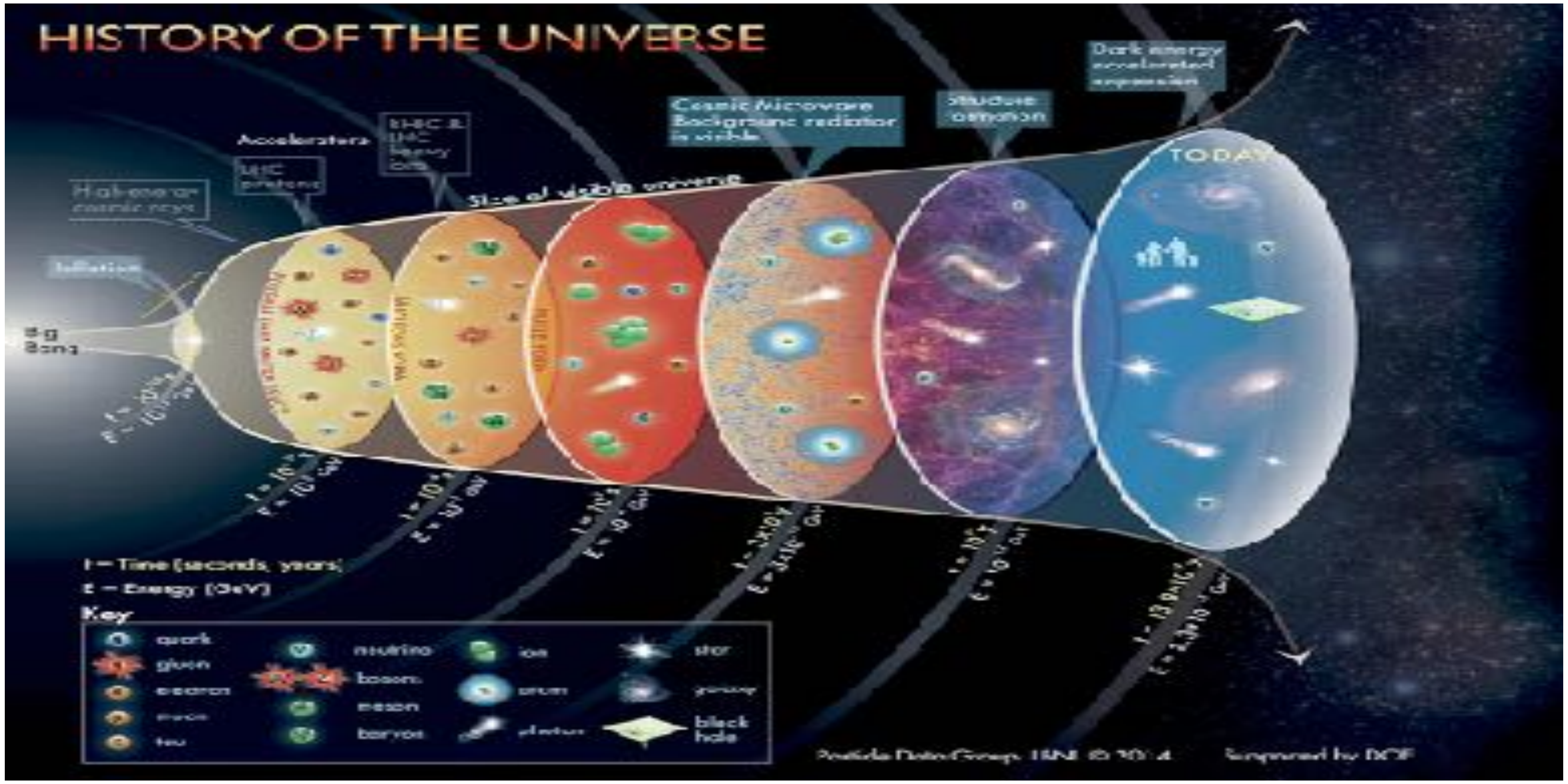
- The CMB is a **snapshot of the oldest light** in our Universe, imprinted on the sky when the Universe was just **380,000 years old,** dating to the epoch of **recombination**.

- With a traditional **optical telescope**, the space between stars and galaxies is **completely dark**. However, a sufficiently sensitive radio telescope shows a faint background glow, almost exactly the **same in all directions**. This glow is strongest in the **microwave region**.

Motivation

It shows **tiny temperature fluctuations** that correspond to regions of slightly different densities, **representing the seeds of all future structure**:

the stars and galaxies of today.



The Hot Big Bang

The Hot Big Bang

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

The Hot Big Bang

- Once Big Bang Nucleosynthesis is over, at time t ~ 300s and temperature T ~ 8×10^8K, the **Universe is a thermal bath** of photons, protons, electrons, in addition to neutrinos and the unknown dark matter particle(s).

The key to understanding the thermal history of the universe is the comparison between the **rate of interactions Γ** and the **rate of expansion H**.

- $\Gamma \gg H,$ Local thermal equilibrium is then reached **before** the effect of **the expansion becomes relevant.**
- As the universe cools, the **rate of interactions may decrease** faster than the expansion rate
- At $\Gamma \sim H$ the **particles decouple** from the thermal bath.

Different particle species may have different interaction rates and so may **decouple at different times**.

Fermi-Dirac (+) and Bose-Einstein (-)

$$f(p) = \frac{1}{e^{(E-\mu)/T \pm 1}}$$

For T<E?

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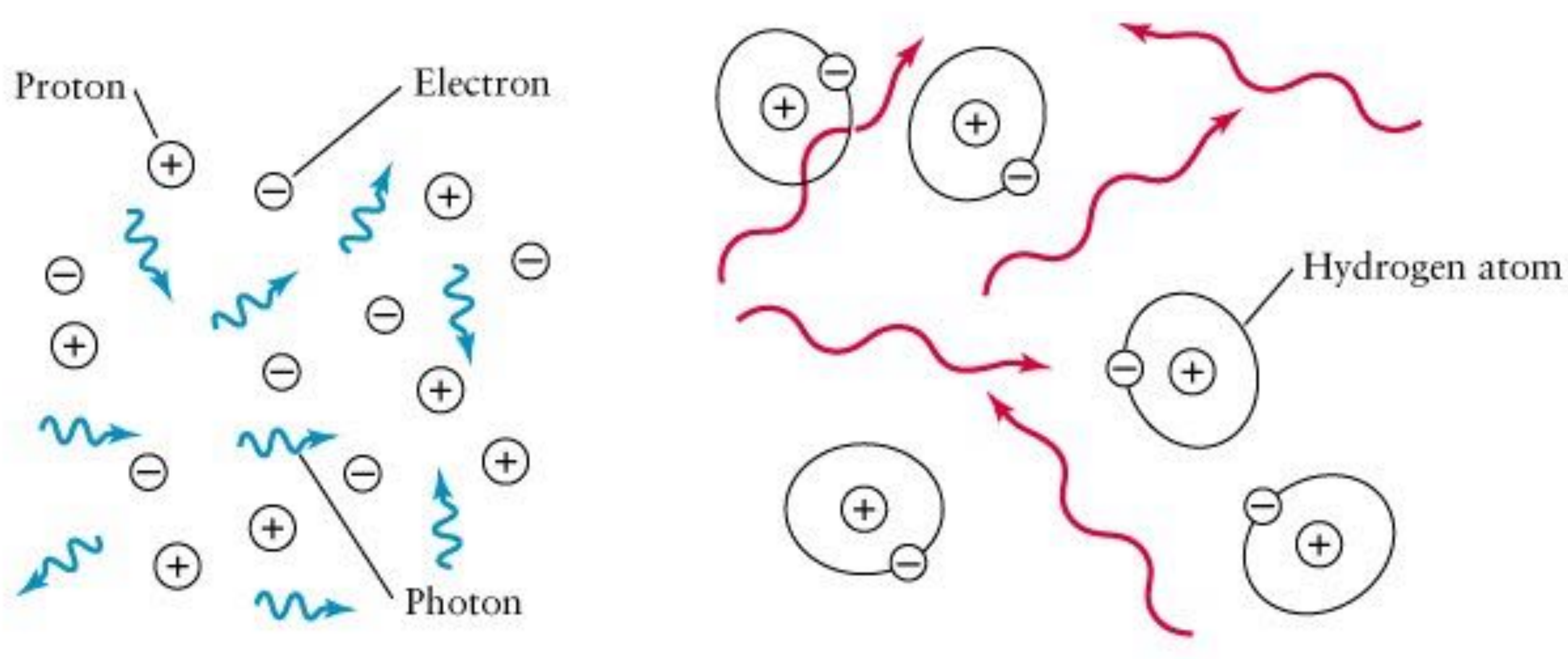
Recombination

Photons were tightly **coupled to the electrons** via Compton scattering, which in turn strongly interacted with protons via Coulomb scattering.

When the temperature became low enough, **the electrons and nuclei combined** to form neutral atoms (**recombination**), and the **density of free electrons fell sharply**.



a Before recombination b After recombination

[Diff Compton Vs Thompson ?]

[Why recom?]

Saha equation

T >1eV, when **baryons and photons were still in equilibrium** through electromagnetic reactions such as

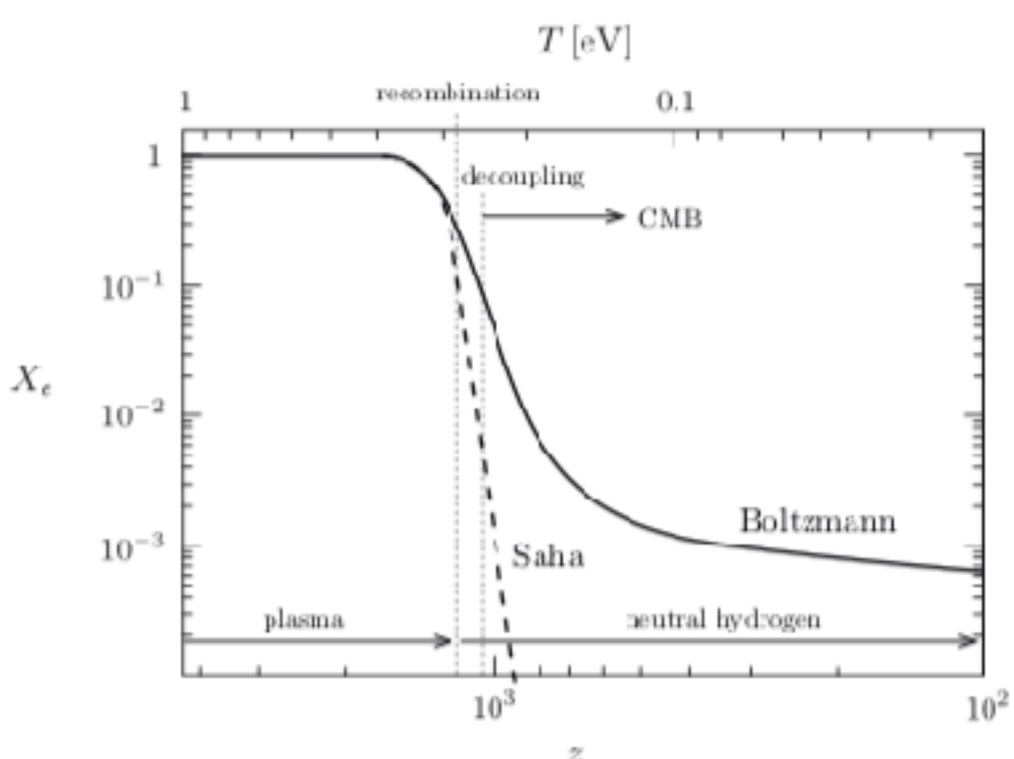
$$e^- + p^+ \rightleftharpoons H + \gamma$$

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$$

We wish to follow the **free electron fraction** defined as the ratio

$$X_e \equiv \frac{n_e}{n_b}$$

$$\left(\frac{1-X_e}{X_e^2}\right)_{\rm eq} = \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}$$



The Saha approximation **correctly identifies the onset of recombination**, but it is clearly **insufficient** if the aim is to determine the relic density of electrons after freeze-out.

m_i ? μ_γ ?

$$\left(\frac{1-X_e}{X_e^2}\right)_{\rm eq} = \frac{2\zeta(3)}{\pi^2}\eta\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}$$

Recombination

Let us define the **recombination temperature Trec** as the temperature where $X_e = 10^{-1}$, i.e. when **90% of the electrons have combined** with protons to form hydrogen.

$$T_{\rm rec} \approx 0.3eV \simeq 3600K.$$

Using $T_{\rm rec} = T_0(1+z_{\rm rec})$, with $T_0 = 2.7K$, gives the r**edshift of recombination**: $\boxed{z_{\rm rec} \approx 1320}$

Since matter-radiation equality is at $z_{eq} \simeq 3500$, then **recombination occurred in the matter-dominated era**. Using $a(t) = (t/t_0)^{2/3}$, the time of recombination

$$t_{\rm rec} = \frac{t_0}{(1+z_{\rm rec})^{3/2}} \sim 290\ 000 yrs$$

zeq ?

Recombination **was not an instantaneous** process but proceeded relatively quickly nevertheless, with the fractional ionisation decreasing from X = 0.9 to X = 0.1 over a time interval **Δt ~ 70 000yrs.**

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Photon Decoupling

Photons are most strongly coupled to the primordial plasma through their interactions with electrons, through Thomson scattering

$$e^- + \gamma \rightleftharpoons e^- + \gamma$$

Thomson scattering is that it **introduces polarization** along the direction of motion of the electron

The **mean free path for photons** (the mean distance travelled between scatterings) is $\lambda = \frac{1}{n_e \sigma_T}$,

and therefore the **interaction rate** at which a photon undergoes scattering $\Gamma_\gamma \approx n_e \sigma_T$,

Γ_γ decreases as the density of free electrons drops, and hence **photons and electrons decouple** when

$$\Gamma_\gamma(T_{\rm dec}) \sim H(T_{\rm dec}). \qquad X_e(T_{\rm dec}) T_{\rm dec}^{3/2} \sim \frac{\pi^2}{2\zeta(3)} \frac{H_0\sqrt{\Omega_m}}{\eta \sigma_T T_0^{3/2}}.$$

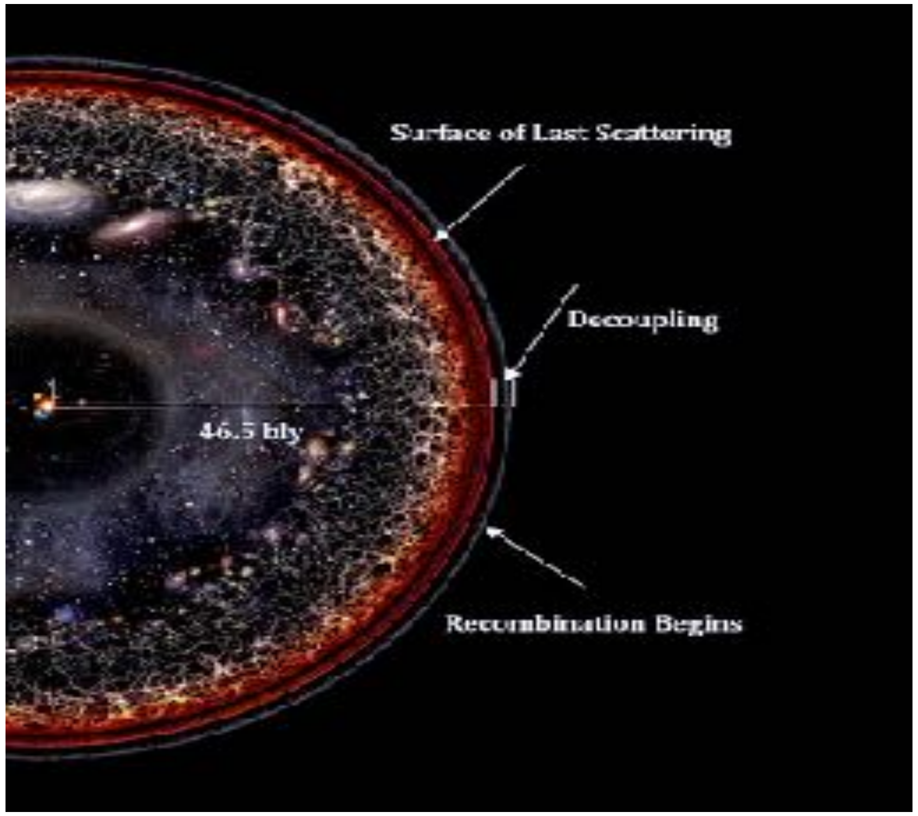
resultado?

Using the Saha equation for Xe(Tdec) $\boxed{T_{\rm dec} \sim 0.27 eV. \quad z_{\rm dec} \sim 1100, \qquad t_{\rm dec} \sim 380\ 000 \text{ yrs.}}$

Last Scattering Surface

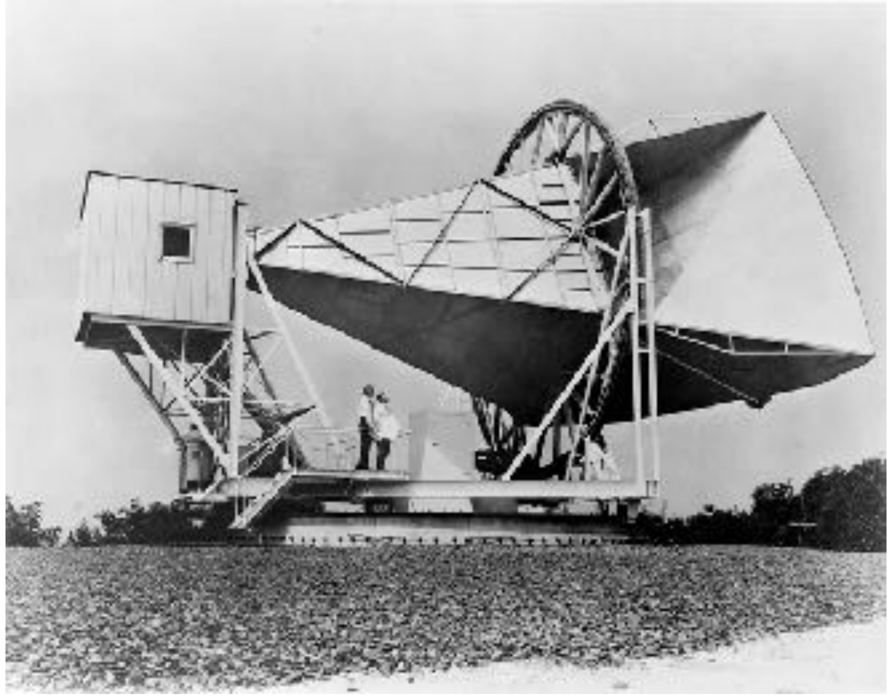
After their last scattering off an electron, **photons were able to travel unimpeded through the Universe**. These are the Cosmic Microwave Background **photons we receive today**, still with their blackbody distribution, now redshifted by a factor of 1100.

They constitute a **last scattering surface**, or more appropriately a **last scattering layer**



Isotropic CMB

- The CMB radiation was discovered in **1965** by **Arno Penzias and Robert Wilson**, while trying to identify **sources of noise in microwave** satellite communications.
- Their discovery was announced alongside the interpretation of the CMB as **relic thermal radiation** from the Big Bang by **Robert Dicke** and collaborators.
- Interestingly, the possibility of a cosmic thermal background were first entertained by **Gamow, Alpher and Herman** in **1948** as a consequence of **Big Bang nucleosynthesis**, but the idea was so beyond the experimental



nobel?

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Timeline of Observations of the CMB

1941	Andrew McKellar was attempting to measure the average temperature of the interstellar medium, and reported the observation of an average bolometric temperature of 2.3 K based on the study of interstellar absorption lines.
1946	Robert Dicke predicts ".. radiation from cosmic matter" at <20 K but did not refer to background radiation[1]
1948	George Gamow calculates a temperature of 50 K (assuming a 3-billion-year old Universe), commenting it ".. is in reasonable agreement with the actual temperature of interstellar space", but does not mention background radiation.
1948	Ralph Alpher and Robert Herman estimate "the temperature in the Universe" at 5 K. Although they do not specifically mention microwave background radiation, it may be inferred.[2]
1950	Ralph Alpher and Robert Herman re-estimate the temperature at 28 K.
1953	George Gamow estimates 7 K.
1955	Émile Le Roux of the Nançay Radio Observatory, in a sky survey at λ=33 cm, reported a near-isotropic background radiation of 3 kelvins, plus or minus 2.
1956	George Gamow estimates 6 K.
1957	Tigran Shmaonov reports that "the absolute effective temperature of the radioemission background ... is 4±3K". It is noted that the "measurements showed that radiation intensity was independent of either time or direction of observation... it is now clear that Shmaonov did observe the cosmic microwave background at a wavelength of 3.2 cm"
1960s	Robert Dicke re-estimates a MBR (microwave background radiation) temperature of 40 K
1964	A. G. Doroshkevich and Igor Novikov publish a brief paper, where they name the CMB radiation phenomenon as detectable.
1964–65	Arno Penzias and Robert Woodrow Wilson measure the temperature to be approximately 3 K. Robert Dicke, P. J. E. Peebles, P. G. Roll, and D. T. Wilkinson interpret this radiation as a signature of the big bang.
1983	RELIKT-1 Soviet CMB anisotropy experiment was launched.
1990	FIRAS on COBE measures the black body form of the CMB spectrum with exquisite precision.
1992	Scientists who analyzed data from COBE DMR announce the discovery of the primary temperature anisotropy.
1999	First measurements of acoustic oscillations in the CMB anisotropy angular power spectrum from the TOCO, BOOMERANG, and Maxima Experiments.
2002	Polarization discovered by DASI.
2004	E-mode polarization spectrum obtained by the CBI.
2005	Ralph A. Alpher is awarded the National Medal of Science for his groundbreaking work in nucleosynthesis and prediction that the universe expansion leaves behind background radiation, thus providing a model for the Big Bang theory.
2006	Two of COBE's principal investigators, George Smoot and John Mather, received the Nobel Prize in Physics in 2006 for their work on precision measurement of the CMBR.

WMAP?

PLK?

JAVazquez

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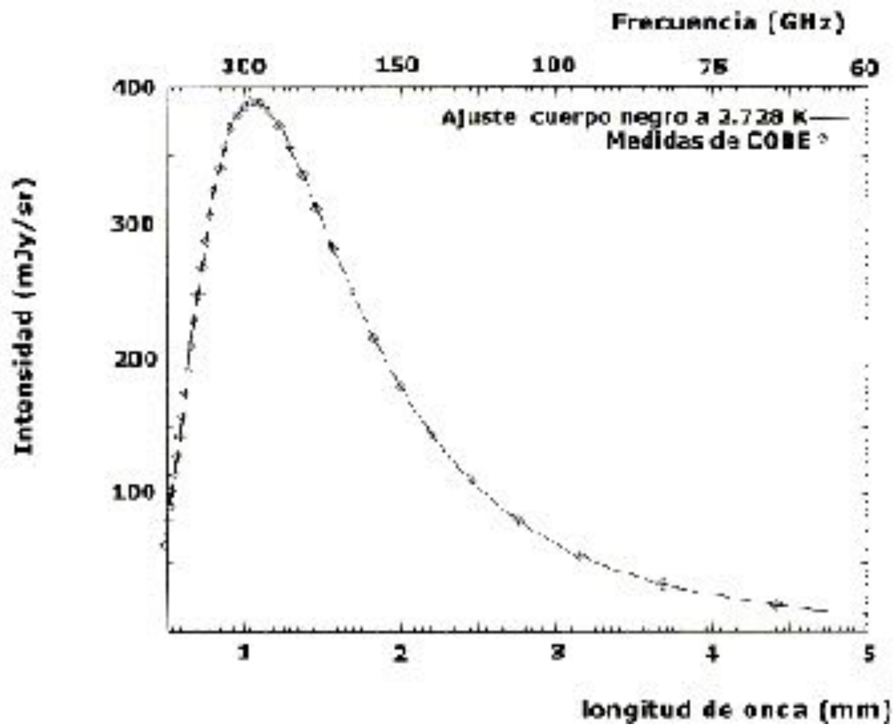


Property	Value
Temperature, $T_{\rm CMB}$	2.7255 K
Peak Wavelength, $\lambda_{\rm peak}$	0.106 cm
Number density of CMB photons, $n_{\gamma,0}$	$411\,{\rm cm}^{-3}$
Energy density of CMB photons, $u_{\gamma,0}$	$0.26\,{\rm eV\ cm}^{-3}$
Average photon energy, $\langle h\nu_{\rm CMB}\rangle$	$6.34\times10^{-4}\,{\rm eV}$
Photon/Baryon ratio, $1/\eta$	1.64×10^{9}

The **original detection** by Penzias and Wilson was at a wavelength of 73.5 mm, this being the wavelength of the telecommunication signals they were working with; this wavelength is **two orders of magnitude longer** than λpeak = 1.1mm of a T = 2.7255K blackbody.

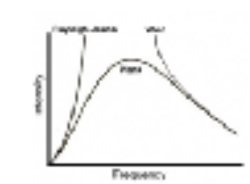
$$\langle T\rangle = \frac{1}{4\pi}\int T(\theta,\phi)\sin\theta d\theta d\phi = 2.7255 \pm 0.0006K$$

The **deviations from this mean temperature** from point to point on the sky are tiny.

$$\frac{\delta T}{T}(\theta,\phi) = \frac{T(\theta,\phi)-\langle T\rangle}{\langle T\rangle}$$

WMAP and Planck

$$\left\langle\left(\frac{\delta T}{T}\right)\right\rangle^{1/2} = 1.1\times10^{-5}$$



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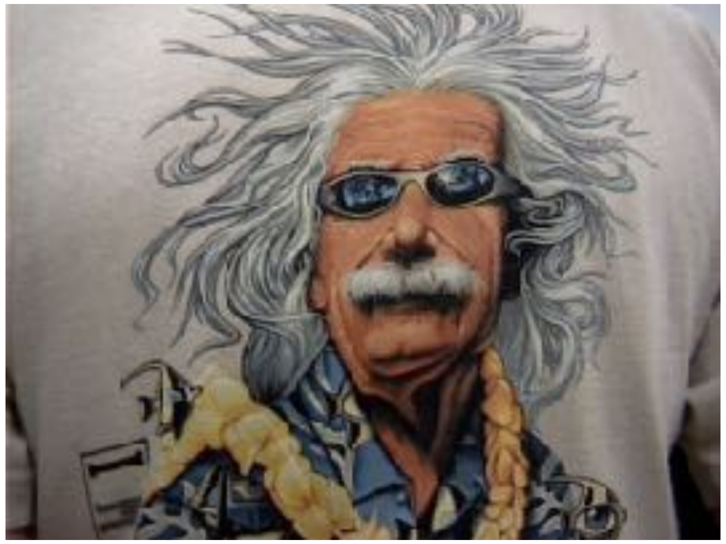
Linear Perturbations

Unperturbed



Perturbed

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Linear Perturbations

Energy-momentum perturbations

$$
\begin{aligned}
T^0_0 &= -\bar{\rho}(1+\delta), \\
T^i_0 &= (\bar{\rho}+\bar{p})v^i \equiv q^i, \\
T^0_i &= -(\bar{\rho}+\bar{p})(v_i+B_i) \\
T^i_j &= \bar{p}[(1+\pi_L)\delta^i_j + \Pi^i_j].
\end{aligned}
$$

Metric perturbations $g_{\mu\nu} \longrightarrow \bar{g}_{\mu\nu} + a^2 h_{\mu\nu},$

The most general perturbation to the background metric is given by

$$h_{\mu\nu}dx^\mu dx^\nu = -2Ad\eta^2 - 2B_i d\eta dx^i + 2H_{ij}dx^i dx^j.$$

SVT $H_{ij} = \underbrace{H_L\gamma_{ij} + \partial_{\langle i}\partial_{j\rangle}H_T}_{\text{scalar part}} + \underbrace{\partial_{(i}H^{(V)}_{j)}}_{\text{vector part}} + \underbrace{H^{(T)}_{ij}}_{\text{tensor part}},$

The Gauge Problem

change of the time coordinate can introduce a fictitious density perturbation

$\rho(\eta) \rightarrow \rho(\eta + \xi^0(\eta,\mathbf{x}))$ $\eta \rightarrow \eta + \xi^0(\eta,\mathbf{x})$

$$\begin{aligned}
A &\to A - \frac{a'}{a}T - T', \\
B &\to B + L' + kT, \\
H_L &\to H_L - \frac{a'}{a}T - \frac{k}{3}L, \\
H_T &\to H_T + kL, \\
\delta &\to \delta + 3(1+w)\frac{a'}{a}T, \\
v &\to v + L', \\
\pi_L &\to \pi_L - \frac{\bar{p}'}{\bar{p}}T = \pi_L + 3(1+w)\frac{c_s^2}{w}\frac{a'}{a}T,
\end{aligned}$$

Gauge transformations $\quad Q^{(1)} \to Q^{(1)} + \mathcal{L}_X \bar{Q},$

Where Ψ and Φ are **gauge-invariant quantities**, called Bardeen potentials

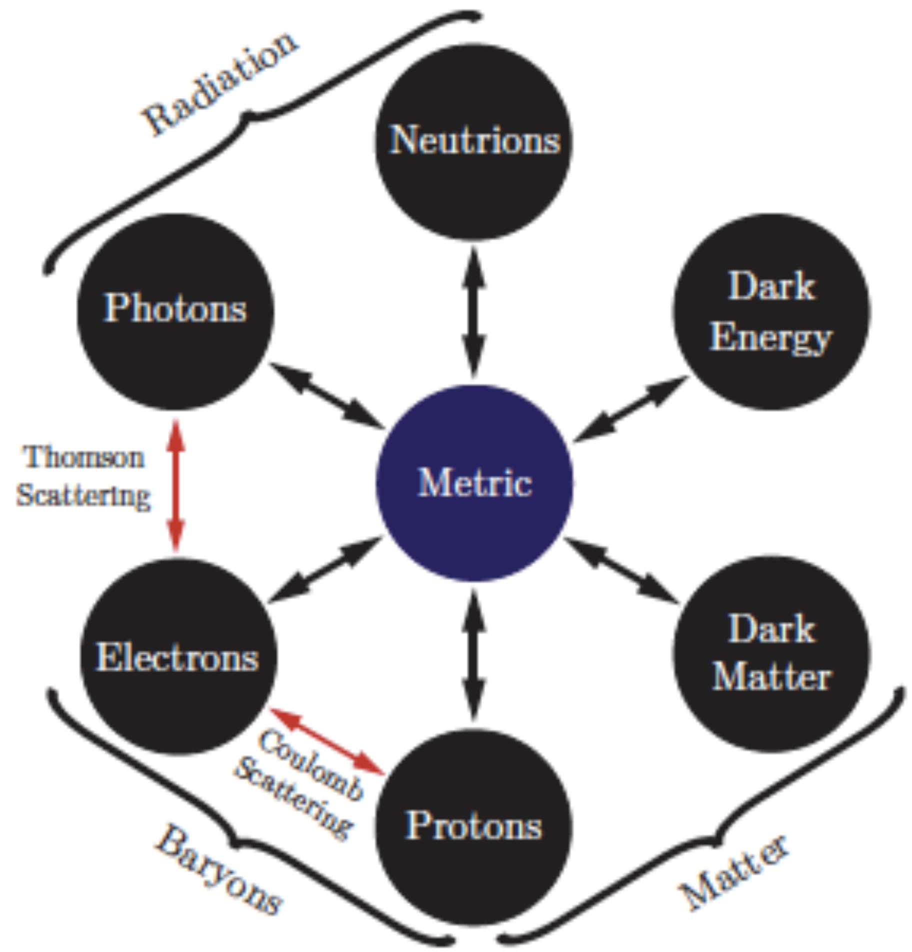
$$\Psi \equiv A - \frac{a'}{a}k^{-1}\sigma - k^{-1}\sigma', \quad \Phi \equiv H_L + \frac{1}{3}H_T - \frac{a'}{a}k^{-1}\sigma.$$

Perturbed Einstein's and conservation equation

$$\begin{aligned}
k^2\Phi + 3\frac{a'}{a}\left(\Phi' - \frac{a'}{a}\Psi\right) &= 4\pi G a^2 \bar{\rho}\delta, \\
k\left(\frac{a'}{a}\Psi - \Phi'\right) &= 4\pi G a^2 v(\bar{\rho} + \bar{p}), \\
-k^2(\Phi + \Psi) &= 8\pi G a^2 \bar{p}\Pi,
\end{aligned}$$

$$\begin{aligned}
-\delta' &= (1+w)[kv + 3\Phi'] + 3\frac{a'}{a}w\Gamma + 3\frac{a'}{a}\delta(c_s^2 - w), \\
v' &= \frac{a'}{a}(3c_s^2 - 1)v + k\Psi + \frac{kc_s^2}{1+w}\delta + \frac{kw}{1+w}\left[\Gamma - \frac{2}{3}\Pi\right],
\end{aligned}$$

The Boltzmann equation



Describes the statistical behaviour of a thermodynamic system not in a state of equilibrium

$$\frac{df}{d\eta} = C[f]$$

The Boltzmann equation

J.Santiago

$$\frac{df}{d\eta} = C[f]$$

The **distribution function** of the cosmic microwave background with temperature $\bar{T}$ is

$$\bar{f} = \left[\exp\left(\frac{E}{\bar{T}}\right) - 1\right]^{-1}.$$

We see that $\bar{f}$ depends just upon the energy E of a photon. Writing $T = T_0 a^{-1}$, we see that $\bar{f}$ is a function of aE only:

$$\bar{f}(aE) = \left[\exp\left(\frac{aE}{\bar{T}_0}\right) - 1\right]^{-1}. \tag{3.2}$$

for observers in the unperturbed background at rest $E = -a\,p$, $\bar{f}$ depends solely of $P = a^2 p$.

Let us split the spatial momentum into its **magnitude p** and the **unit vector** of photon momentum n $p^i \equiv pn^i$

$$f = f(\eta, \mathbf{x}, P, \mathbf{n})$$

The complete distribution function for each species can be split into background plus a perturbation part:

$$\boxed{f(\eta, \mathbf{x}, P, \mathbf{n}) = \bar{f}(P) + F(\eta, \mathbf{x}, P, \mathbf{n}),} \tag{3.5}$$

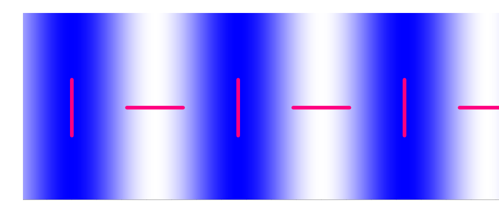
The Boltzmann equation

The **evolution of perturbations** in the universe is quantified by the Boltzmann equation:

$$\left(\frac{\partial f}{\partial \eta}\right)_P + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \eta} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial \eta} + \frac{\partial f}{\partial n^i}\frac{\partial n^i}{\partial \eta} = C[f, G],$$

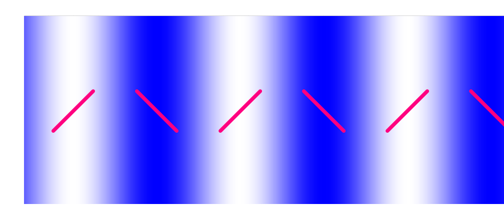
Relates the **effects of gravity** on the photon distribution function f to the **rate of interactions with other species**, given by the collision term C[f,G].

To describe the **electromagnetic wave** $\boldsymbol{E} = \left(a_1 e^{i\delta_1}\boldsymbol{\epsilon}_1 + a_2 e^{i\delta_2}\boldsymbol{\epsilon}_2\right) e^{i p \boldsymbol{n}\,\boldsymbol{x} - i\omega t}.$



The **Stokes parameters** are then defined by

$$\begin{aligned} I &\equiv \langle \boldsymbol{E}\boldsymbol{E}^\star \rangle = a_1^2 + a_2^2, \\ Q &\equiv \langle \boldsymbol{E}_1\boldsymbol{E}_1^\star - \boldsymbol{E}_2\boldsymbol{E}_2^\star \rangle = a_1^2 - a_2^2, \\ U &\equiv \left\langle \left|\frac{\boldsymbol{E}_1 + \boldsymbol{E}_2}{\sqrt{2}}\right|^2 - \left|\frac{\boldsymbol{E}_1 - \boldsymbol{E}_2}{\sqrt{2}}\right|^2 \right\rangle \\ &= 2a_1 a_2 \cos(\delta_1 - \delta_2). \end{aligned}$$



spin?

The Stokes parameters can be express as **frequency-independent** fractional thermodynamic **equivalent temperatures**.

The previous distribution applies to polarization as well by simply replacing F $\to$ G (we use G to denote the linear polarization distribution function) and $\bar{f} = \bar{f}' \to 0$

The Boltzmann equation

$$\left(\frac{\partial f}{\partial \eta}\right)_P + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \eta} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial \eta} + \frac{\partial f}{\partial n^i}\frac{\partial n^i}{\partial \eta} = C[f, G],$$

The last term vanishes, because it is of second order in perturbation theory: $\bar{f}$ does **not depend on** n^i and hence $\partial f/\partial n^i$ is a perturbation. In addition $\partial n^i/\partial \eta$, is a **change in photon direction.**

effect?

The third term can be computed from the geodesic equation

$$\frac{\partial f}{\partial P}\frac{\partial P}{\partial \eta} = -P\bar{f}_{,P}\{i\mu k[\Phi + \Psi] + 2\Phi'\},$$

$$p^0 \frac{dp^\mu}{d\eta} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\mu = 0$$

The second term

$$\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \eta} = i\mu k F(\eta, \mathbf{x}, P, \mathbf{n}).$$

Collecting the terms involving the **background only** $\left(\frac{\partial f}{\partial \eta}\right)_P = 0$

the preservation of the background black body spectrum

$$\left(\frac{\partial F}{\partial \eta}\right)_P + i\mu k F - P\bar{f}_{,P}\{i\mu k[\Phi+\Psi] + 2\Phi'\} = C[f,G]$$

Finally, making the substitution F $\rightarrow$ G, $\bar{f}' \rightarrow 0$, we get the simple evolution equation for the linear **polarization G**

$$\left(\frac{\partial G}{\partial \eta}\right)_P + i\mu k G = C_G[f,G]$$

Perturbed temperature

Writing the temperature function T in terms of the photon *brightness temperature perturbation* $\Delta \equiv \Delta T/\bar{T}$, we have

$$\boxed{T(\eta,\mathbf{x},\mathbf{n}) = \bar{T}(\eta)[1+\Delta(\eta,\mathbf{x},\mathbf{n})],} \qquad (3.13)$$

and therefore F and Δ are connected via

$$F(\eta,\mathbf{x},P,\mathbf{n}) = -P\frac{\partial \bar{f}}{\partial P}\Delta(\eta,\mathbf{x},\mathbf{n}). \qquad G(\eta,\mathbf{x},P,\mathbf{n}) = -P\frac{\partial \bar{f}}{\partial P}\mathcal{Q}(\eta,\mathbf{x},\mathbf{n}).$$

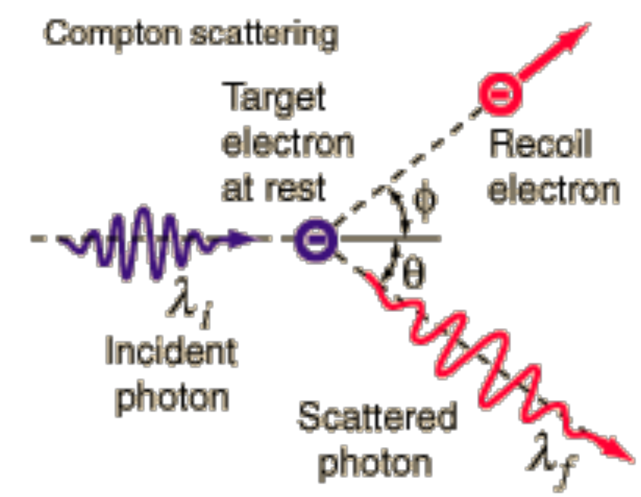
The simplify Boltzmann equation becomes

$$\boxed{\Delta' + ik\mu\Delta = -i\mu k[\Phi+\Psi] - 2\Phi' + \hat{C}[f,G]}$$

The Collision Term

The dominant term for the coupling of photons to the baryons is via inverse **Compton scattering**

$$e^-(\mathbf{q}) + \gamma(\mathbf{p}) \rightleftharpoons \mathbf{e}^-(\mathbf{q}') + \gamma(\mathbf{p}')$$



The amplitude can be calculated from the Feynman rules.

$$C[f,G] = a n_e \sigma_T \bar{f}_{,P} P \left\{ i\mu v_b + \Delta(\eta,\mathbf{x},\mathbf{n}) - \frac{1}{4}\int_{-1}^{1} \Delta(\eta,\mathbf{x},\mathbf{n}')[P_2(\lambda)P_2(\mu)+2]d\lambda \right.$$
$$\left. -\frac{1}{4}\int_{-1}^{1} \mathcal{Q}(\eta,\mathbf{x},\mathbf{n}')P_2(\mu)[-2\sqrt{6\pi}5_2 Y_2^0(\lambda)]d\lambda \right\}$$

The expansion of the temperature perturbation (Δ) and polarisations (Q and U), in terms of spherical harmonics Y ^{m}(n)

$$\Delta(\eta,\mathbf{x},\mathbf{n}) = \sum_l (-i)^l \Delta_l(k,\eta) P_l(\hat{\mathbf{k}}\cdot\mathbf{n}), \qquad (Q \pm iU)(\eta,\mathbf{x},\mathbf{n}) = \sum_{l=2} (-i)^l (E_l^0 \pm i B_l^0)\sqrt{\frac{4\pi}{2l+1}}\,{}_{\mp 2}Y_l^0(\mathbf{n}),$$

$$\boxed{C[f,G] = a n_e \sigma_T \bar{f}_{,P} P \left\{ i\mu v_b + \Delta(\eta,\mathbf{k},\mathbf{n}) + \frac{1}{10}\Delta_2 P_2(\mu) - \Delta_0 - \frac{\sqrt{6}}{10}[E_2 - \Delta_2] \right\}}$$

The Boltzmann equation thus yields to the evolution equation of temperature perturbations

$$\boxed{\Delta' + ik\mu\Delta + \kappa'\Delta = -i\mu k[\Phi + \Psi] - 2\Phi' + \kappa' \left\{ \frac{1}{4}\delta_\gamma - \Phi - i\mu v_b + \frac{1}{10}P_2(\mu)[\sqrt{6}E_2 - \Delta_2] \right\}}$$

$$Q' + ik\mu Q + \kappa' Q = \frac{\kappa'}{10}\{P_2(\mu) - 1\}\left[\sqrt{6}E_2 - \Delta_2\right].$$

$\boxed{\kappa' \equiv a n_e \sigma_T}$ is the differential optical depth

$\mu = k^{-1}\mathbf{k}\cdot\mathbf{n}$ the direction cosine.

We have use the expressions for the first few moments of the distribution function

$$T^\mu_\nu = \int \sqrt{-g}\frac{p^\mu p_\nu}{|p_0|} f(p,x) d^3p \qquad \delta = 4\Phi + \frac{1}{\pi}\int \Delta(\mathbf{n})\mathbf{d\Omega}$$

We notice that is **not manifestly gauge-invariant**,

however by defining the **gauge invariant temperature perturbation**

$$\mathcal{M} = \Delta + 2\Phi$$

$$\mathcal{M}(\eta, \mathbf{x}, \mathbf{n}) = \sum_l (-i)^l \mathcal{M}_l(\eta, \mathbf{k}) P_l(\mathbf{n}),$$

$$\boxed{\mathcal{M}' + ik\mu\mathcal{M} + \kappa'\mathcal{M} = i\mu k[\Phi - \Psi] + \kappa' \left\{ \frac{1}{4}D_g^\gamma - i\mu v_b + \frac{1}{10}P_2(\mu)\left[\sqrt{6}E_2 - \mathcal{M}_2\right] \right\}.}$$

Solving ...

$$\mathcal{M}' + ik\mu\mathcal{M} + \kappa'\mathcal{M} = i\mu k[\Phi - \Psi] + \kappa'\left\{\frac{1}{4}D_g^\gamma - i\mu v_b + \frac{1}{10}P_2(\mu)\left[\sqrt{6}E_2 - \mathcal{M}_2\right]\right\}.$$

The procedure is as follows: For each Legendre polynomials P_l

- replace $\mathcal{M}(\eta,\mu)$ by its multipole expansion
- multiply by $P_l(\mu)$
- integrate both l.h.s. and r.h.s. of the new equation over μ : $\int_{-1}^{1} d\mu$
- use the orthogonality relation $\int_{-1}^{1} d\mu P_l(\mu)P_n(\mu) = 2\delta_{ln}/(2l+1)$

HW -0 ?

$$\begin{aligned}
\mathcal{M}_0' &= -\frac{k}{3}V_\gamma, \\
\mathcal{M}_1' &= \kappa'(V_b - V_\gamma) + k(\Psi - \Phi) + k\left(\mathcal{M}_0 - \frac{2}{5}\mathcal{M}_2\right), \\
\mathcal{M}_2' &= -\kappa'(\mathcal{M}_2 - \mathcal{C}) + k\left(\frac{2}{3}V_\gamma - \frac{3}{7}\mathcal{M}_3\right), \\
\mathcal{M}_l' &= -\kappa'\mathcal{M}_l + k\left(\frac{l}{2l-1}\mathcal{M}_{l-1} - \frac{l+1}{2l+3}\mathcal{M}_{l+1}\right), \quad l > 2,
\end{aligned}$$

$$\begin{aligned}
E_2' &= -\frac{k\sqrt{5}}{7}E_3 - \kappa'(E_2 + \sqrt{6}\mathcal{C}), \\
E_l' &= k\left(\frac{2\kappa_l}{2l-1}E_{l-1} - \frac{2\kappa_{l+1}}{2l+3}E_{l+1}\right) - \kappa' E_l, \quad l > 2.
\end{aligned}$$

Massless neutrinos follow the same multipole hierarchy as M, however without polarisation

$$\begin{aligned}
\mathcal{N}_0' &= -\frac{k}{3}V_\nu, \\
\mathcal{N}_0' &= k(\Psi - \Phi) + k\left(\mathcal{N}_0 - \frac{2}{5}\mathcal{N}_2\right), \\
\mathcal{N}_l' &= k\left(\frac{l}{2l-1}\mathcal{N}_{l-1} - \frac{l+1}{2l+3}\mathcal{N}_{l+1}\right), \quad l > 1.
\end{aligned}$$

The Line of Sight Strategy

So usually, we are interested in **M(η0,μ).**

Inspecting, one notices that the **l.h.s** can be written as

$$\mathcal{M}' + ik\mu\mathcal{M} + \kappa'\mathcal{M} =$$

$$e^{-i\mu k\eta}e^{-\kappa(\eta)}\dot{L} \qquad \text{where} \qquad L \equiv e^{i\mu k\eta}e^{\kappa(\eta)}\mathcal{M}$$

Hence, the Boltzmann equation translates into

$$\dot{L} = e^{i\mu k\eta}e^{\kappa(\eta)}\left[i\mu k(\Phi - \Psi) + \kappa'\left(\frac{1}{4}D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C}\right)\right]$$

and **integrated over conformal time**

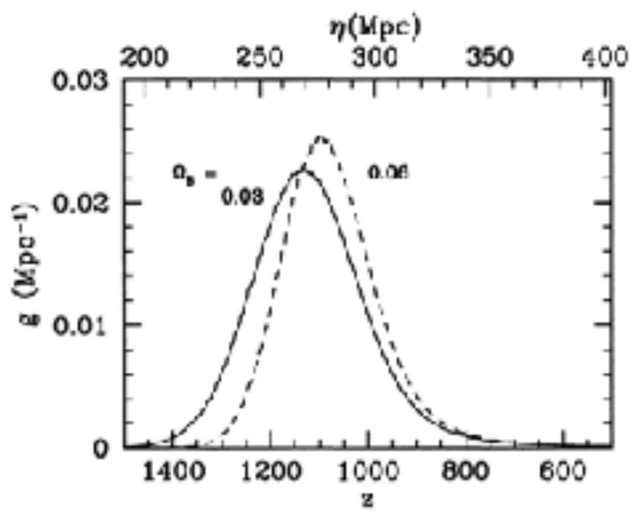
$$L(\eta_0) = \int_0^{\eta_0} d\eta e^{i\mu k\eta}e^{\kappa(\eta)}\left[i\mu k(\Phi - \Psi) + \kappa'\left(\frac{1}{4}D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C}\right)\right]$$

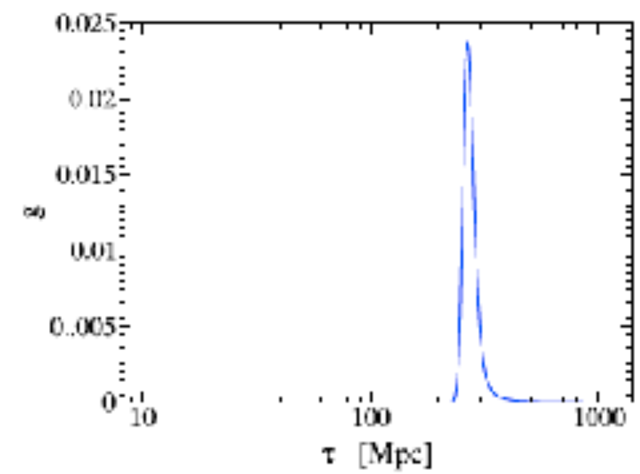
The **photon perturbation today** is given by

$$\boxed{\mathcal{M}(\mu, \eta_0) = \int_0^{\eta_0} d\eta e^{i\mu k(\eta - \eta_0)}e^{\kappa(\eta) - \kappa(\eta_0)} \times \left[i\mu k(\Phi - \Psi) + \kappa'\left(\frac{1}{4}D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C}\right)\right] \qquad (3.47)}$$

The visibility function

The product $g \equiv \kappa' \exp(\kappa(\eta) - \kappa(\eta_0))$ plays an important role and is called the visibility function. Its peak defines the epoch of recombination.





$$\mathcal{M}(\mu, \eta_0) = \int_0^{\eta_0} d\eta e^{i\mu k(\eta - \eta_0)} e^{\kappa(\eta) - \kappa(\eta_0)} \times \left[i\mu k(\Phi - \Psi) + \kappa' \left(\frac{1}{4} D_g^\gamma - i\mu V_b - \frac{1}{2}(3\mu^2 - 1)\mathcal{C} \right) \right] \tag{3.47}$$

Each term in the above Equation containing factors of μ, can be integrated by parts, order to get rid of μ

Applying this procedure to all terms involving μ yields

$$\mathcal{M}(\mu, \eta_0) = \int_0^{\eta_0} e^{\imath\mu k(\eta - \eta_0)} S_T(k, \eta) d\eta$$

$$\begin{aligned} S_T &= -e^{\kappa(\eta) - \kappa(\eta_0)}[\Phi' - \Psi'] + g' \left[\frac{V_b}{k} + \frac{3}{k^2}\mathcal{C}' \right] + g'' \frac{3}{2k^2}\mathcal{C} \\ &+ g \left[\frac{1}{4} D_g^\gamma + \frac{V_b'}{k} - (\Phi - \Psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\mathcal{C}'' \right], \end{aligned}$$

$$S_T = -e^{\kappa(\eta)-\kappa(\eta_0)}[\Phi' - \Psi'] + g'\left[\frac{V_b}{k} + \frac{3}{k^2}\mathcal{C}'\right] + g''\frac{3}{2k^2}\mathcal{C} + g\left[\frac{1}{4}D_g^\gamma + \frac{V_b'}{k} - (\Phi - \Psi) + \frac{\mathcal{C}}{2} + \frac{3}{2k^2}\mathcal{C}''\right],$$

The **density contrast D_g^γ** is the main contribution, driving the spectrum towards the **oscillatory behaviour**.

The **$(\Phi - \Psi)$ term** arises from the **gravitational redshift** when climbing out of the potential well at last scattering.

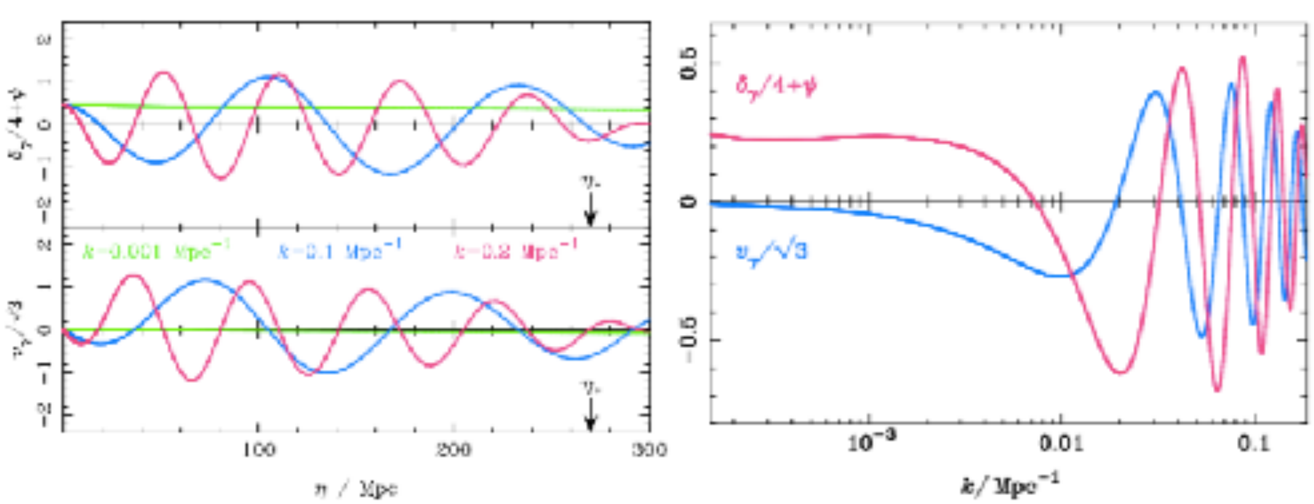
The **combination $D_g^\gamma/4 - (\Phi - \Psi)$** is known as the **ordinary Sachs-Wolfe effect (SW)**.

This gives the main contribution on scales that at decoupling were well outside the horizon

The **Doppler shift, V_b-term,** describes the blueshift caused by **last scattering electrons moving towards** the observer.

The term involving time derivatives of the potentials, **$(\Phi' - \Psi')$**, the **integrated Sachs-Wolfe effect (ISW)**.

It describes the **change of the CMB photon energy** due to the **evolution of the potentials** along the line of sight.



CMB Spectrum