(relativistic and linear) Perturbation Theory

And its observables

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Part 1. Homogeneous Universe

The unperturbed Universe

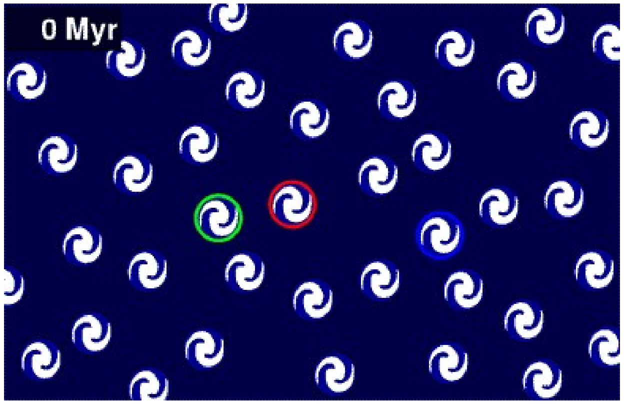
- Cosmological principle estipulates that universe is invariant:

Wherever you stand **(homogeneous)** and whichever direction you look at **(isotropic)**

- Comoving coordinates and uniform dynamics yield,

$$\vec{R} = a(t)\vec{r} \quad \Longrightarrow \quad \boxed{\vec{v} = H(t)\vec{R}(r,t)} \quad \text{where} \quad H(t) = \frac{\dot{a}}{a}$$

- Particles representing galaxies move in geodesics (worldlines) at fixed $\vec{r}$ which only intersect at the singular points far in the past (or future), these are **fundamental observers**.



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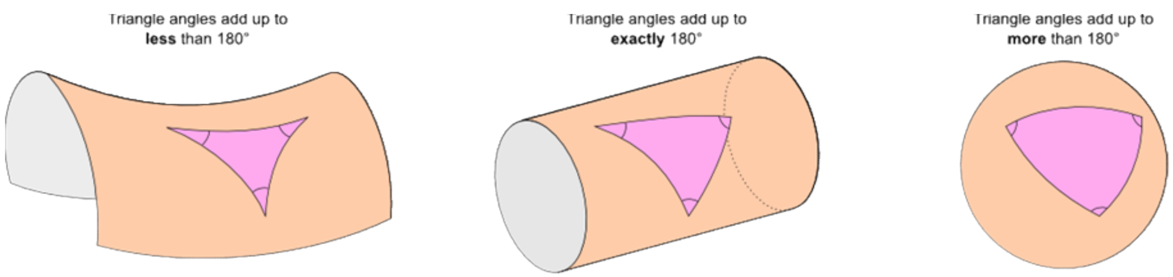
- Robertson/Walker proved that the Friedmann-Lemaitre-Robertson-Walker metric is the most general case of an homogeneous and isotropic expansion

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-k\,r^2} + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)\right]$$

Where the curvature **k** defines the geometry of space



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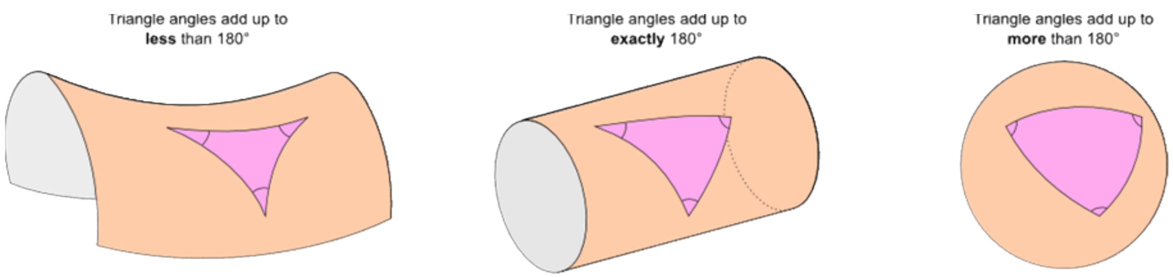

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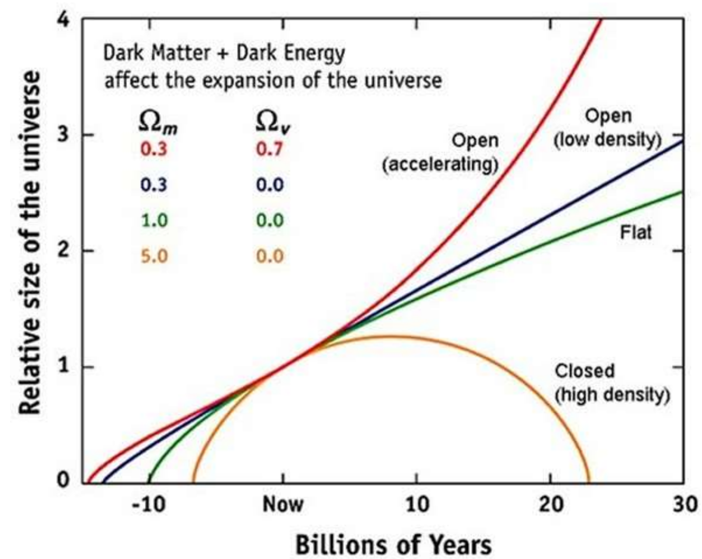


Unperturbed ingredients

- Stress Energy tensor with anisotropic stress.

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + \delta^{\mu}{}_{\nu}p + \pi^{\mu}{}_{\nu}\,, \qquad T^{\mu}{}_{\nu}u^{\nu} = -\rho u^{\mu}$$

- Proper frame of perfect fluid $T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p).$ (with $p = \omega\rho$)
- Dark matter presents no velocity dispersion, no pressure, no shear $\boxed{p_{DM} = 0}$

Unperturbed ingredients (fluid)

- Stress Energy tensor with anisotropic stress.

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + \delta^{\mu}{}_{\nu}p + \pi^{\mu}{}_{\nu} \qquad T^{\mu}{}_{\nu}u^{\nu} = -\rho u^{\mu}$$

- Proper frame of perfect fluid $T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p).$ (with $q = \pi^{\mu}{}_{\nu} = 0$)
- From distribution function

$$T^{\mu\nu} = \frac{g}{(2\pi)^3}\int f p^{\mu} p^{\nu} \frac{d^3p}{E}$$

- Energy density, momentum density and pressure

$$\rho = \frac{g}{(2\pi)^3}\int f E\, d^3p \qquad (\rho + P)\mathbf{v} = \frac{g}{(2\pi)^3}\int f \mathbf{p}\, d^3p \qquad P = \frac{g}{3(2\pi)^3}\int f p^2 \frac{d^3p}{E}$$

- For ultra-relativistic particles $p_{RAD} = E_{RAD}$, thus $\boxed{\rho_{RAD} = 3p_{RAD}}$
- If $\pi^{\mu}{}_{\nu} = 0$, this is a perfect fluid.
- Dark matter presents no pressure, no shear and no anisotropic stress. $\boxed{p_{DM} = 0}$

Unperturbed ingredients (scalar field)

- Stress Energy tensor with anisotropic stress.

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + \delta^{\mu}{}_{\nu}p + \pi^{\mu}{}_{\nu}$$

- Proper frame of perfect fluid $T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p).$ (with $p = \omega\rho$)
- Dark matter presents no pressure, no shear and no anisotropic stress.
- A scalar field has

$$T^{\mu}_{\ \nu} = g^{\mu\alpha}\varphi_{,\alpha}\varphi_{,\nu} - \delta^{\mu}_{\ \nu}\left(U(\varphi) + \frac{1}{2}g^{\kappa\lambda}\varphi_{,\kappa}\varphi_{,\lambda}\right)$$

- The homogeneous field can be described as a perfect fluid

$$u_{\mu} = \frac{\varphi_{,\mu}}{|g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa}|} \qquad \rho = -g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa} + U\,, \quad P = -g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa} - U$$

See Alma Gonzalez's Talk

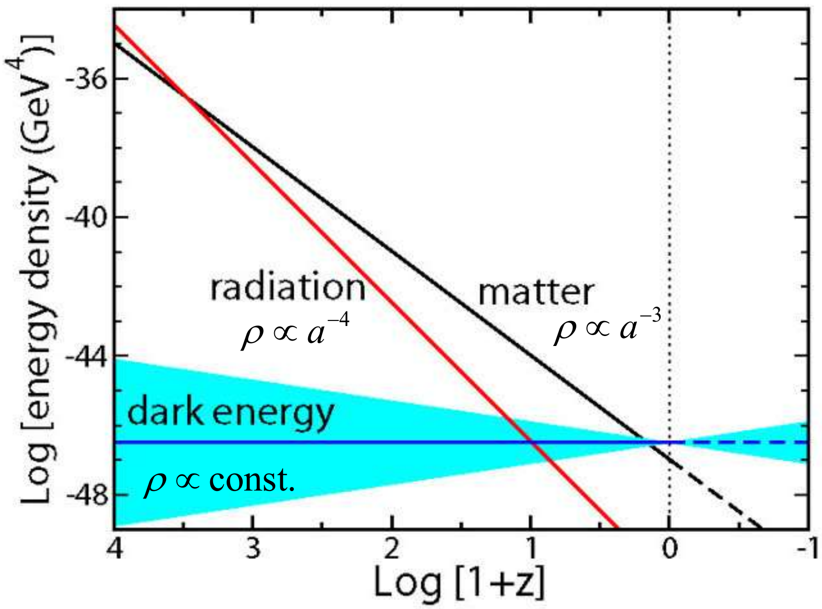
Evolution of FLRW metrics

- Einstein equations (and Conservation of $T_{\mu\nu}$) for a perfect fluid ($p = \omega\rho$)

$$\frac{d\rho}{dt} = -3H(p+\rho) \qquad \boxed{H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda} \qquad \left(\frac{H}{H_0}\right)^2 = \Omega_m + \Omega_\Lambda + \Omega_\kappa$$

- Solutions (null curvature)

$$\rho = \rho_0 a^{-3(1+\omega)} \qquad a(t) = \left(\frac{t}{t_0}\right)^{2/3(\omega+1)}$$



Evolution of FLRW metrics

- In an FLRW universe, the comoving horizon is related to the (comoving) Hubble radius

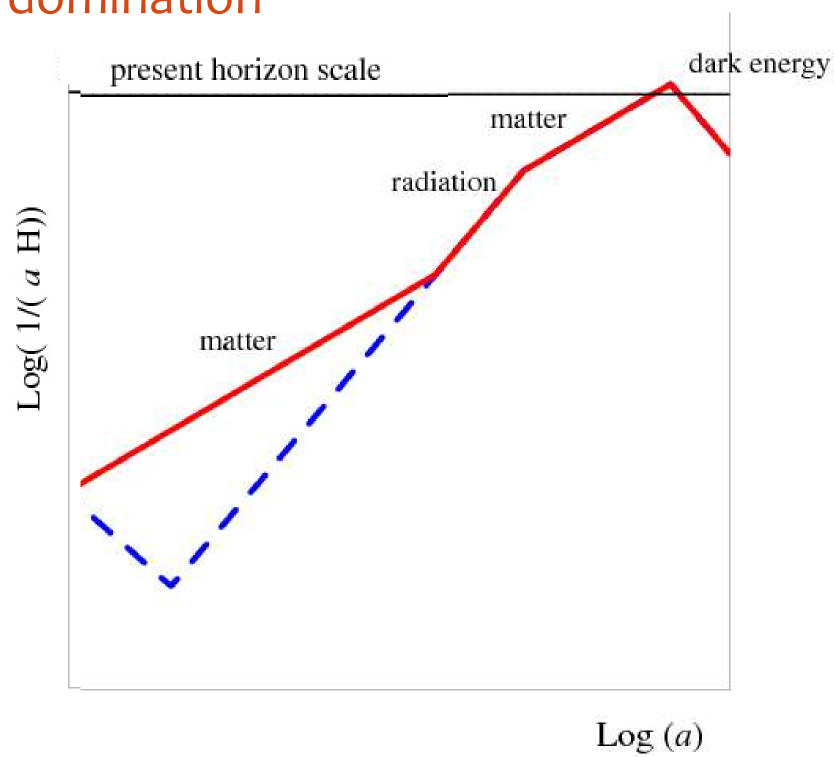
$$ds^2 = 0 \rightarrow d\eta = dr$$

$$r_c = \int_0^{\eta} d\eta' = \int_0^{a} \frac{c}{aH} d\log a = \int_0^{a} r_H d\log a = \begin{cases} r_H & \text{Radiation domination} \\ 2r_H & \text{Matter domination} \end{cases}$$

- Hubble Horizon $\quad r_H = \dfrac{c}{aH} = \dfrac{c}{\mathcal{H}}$

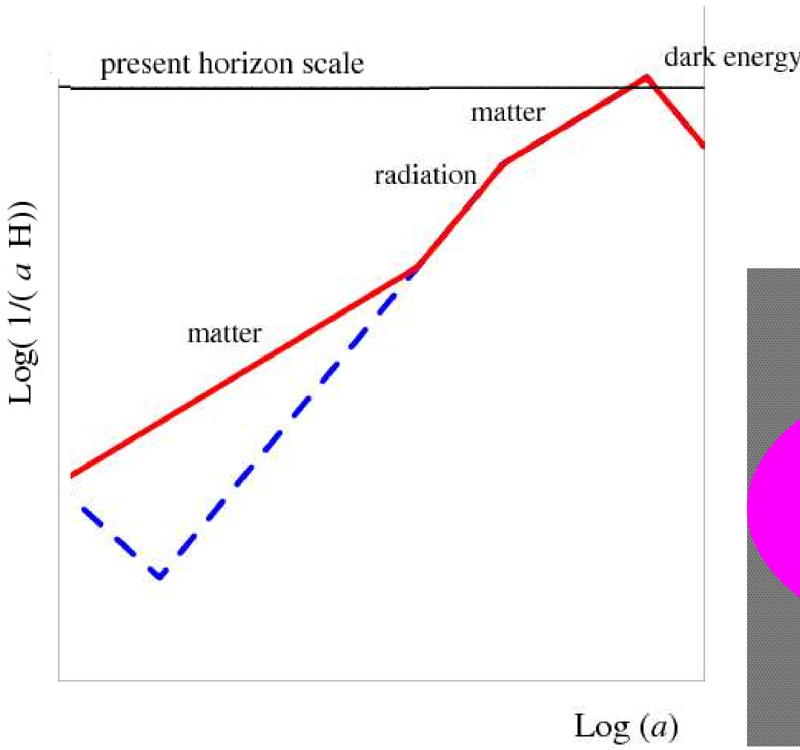
- Comoving horizon ever expanding

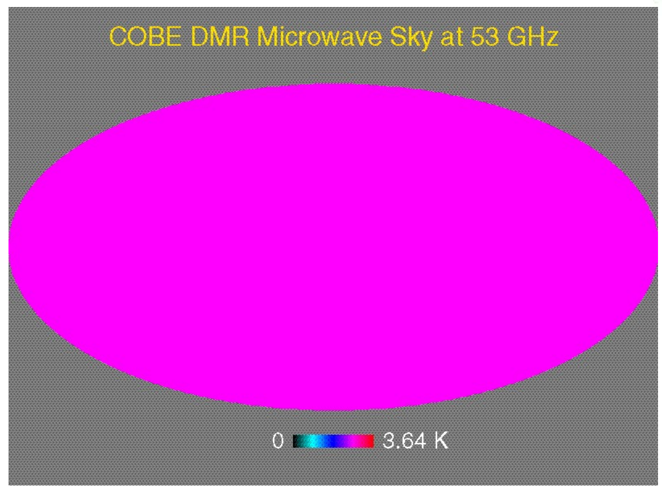
$$\frac{dr_H}{dt} = -\frac{\ddot{a}}{\dot{a}^2} = a\frac{4\pi G}{H^2}\left(p + \rho/3\right)$$

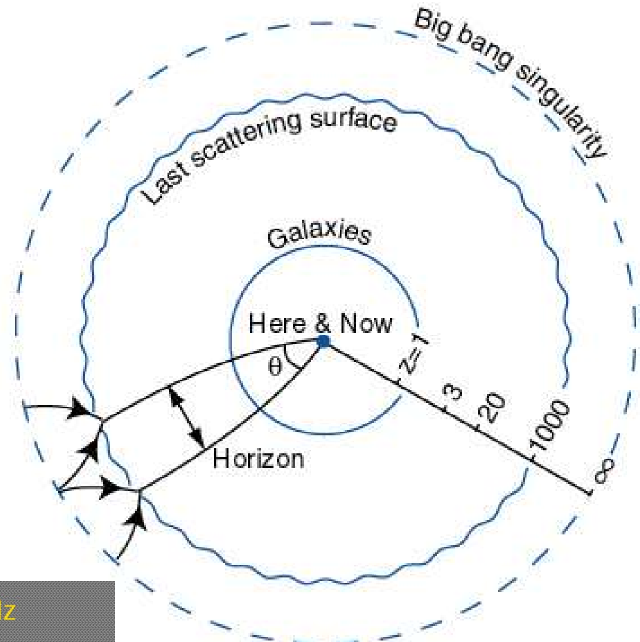


(Problem for Big Bang)

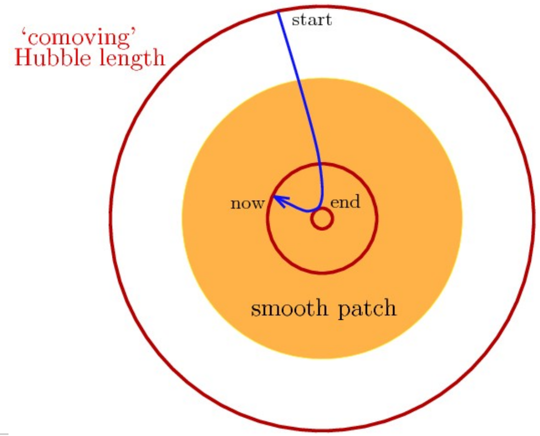
- Homogeneous Universe beyond the r_H at recombination.







Solution: Inflation



- Homogeneous Universe beyond the r_H at recombination.
- Require a shrinking r_H for early times: inflation

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\ddot{a}}{\dot{a}^2} << 0 \qquad \frac{\ddot{a}}{a} = -4\pi G(p + \frac{1}{3}\rho)$$

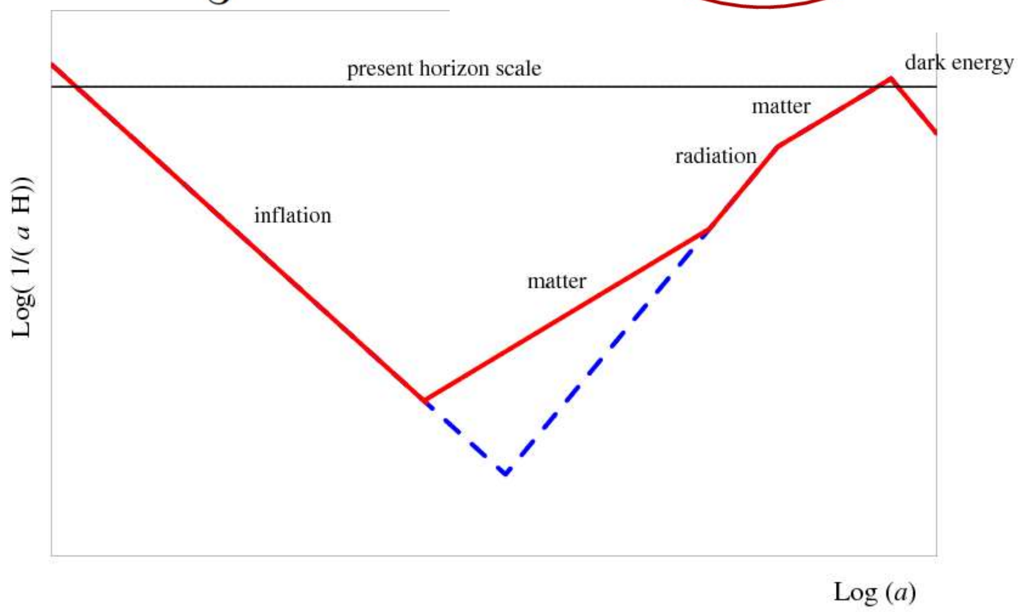
- Scalar field (inflaton)

$$\rho = \frac{1}{2}(\dot{\varphi})^2 + V(\varphi), \quad p = \frac{1}{2}(\dot{\varphi})^2 - V(\varphi)$$

- Slow roll conditions.

$$\dot{\varphi}^2 << V(\varphi) \qquad \ddot{\varphi} << 3H\dot{\varphi}$$

See Jorge Mastache's Talk

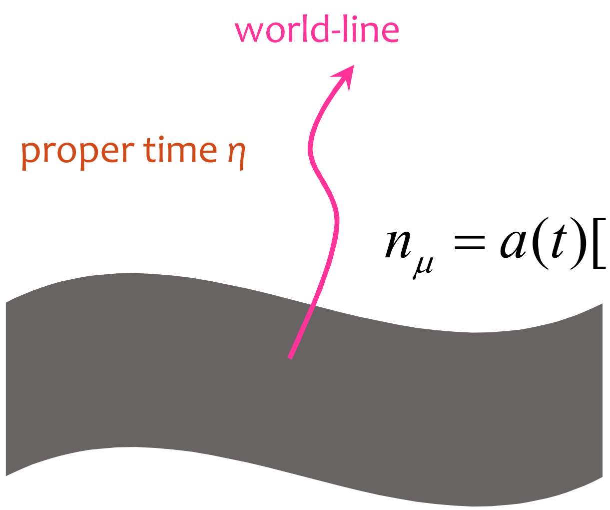


Observers and kinematics

- Comoving (orthogonal) observer,

$$\boxed{u_\mu = n_\mu = \frac{d\eta}{dx^\mu}} \qquad n_\mu n^\mu = -1$$

- Kinematic quantities defined with projection tensor : **3+1 formalism**



$$n_\mu = a(t)[-1, 0, 0, 0]$$

4-velocity of matter in FLRW

$$P_{\nu\mu} \equiv g_{\nu\mu} + n_\mu n_\nu = a(t)[\eta_{\mu\nu} + \delta_{\mu 0}\delta_{0\nu}]$$

projector on hypersurfaces of constant $\eta : \Sigma$

Observers and kinematics

- Comoving (orthogonal) observer, $n_\mu = \dfrac{d\eta}{dx^\mu}$ $\qquad n_\mu n^\mu = -1$
- Kinematic quantities defined with projection tensor , $P_{\nu\mu} \equiv g_{\nu\mu} + n_\mu n_\nu$

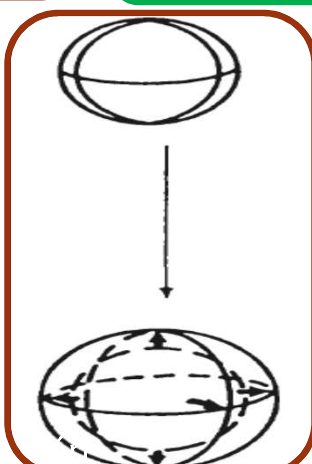
$$n_{\mu;\nu} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu n_\nu \,,$$

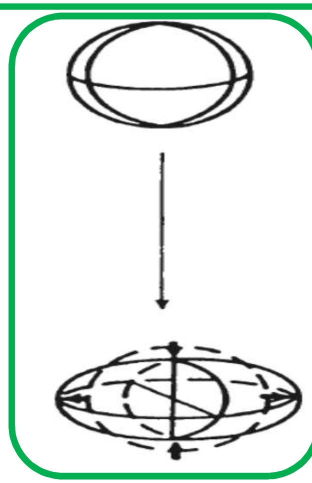
- Expansion θ, vorticity $\omega_{\mu\nu}$, shear $\sigma_{\mu\nu}$ and acceleration a_μ .

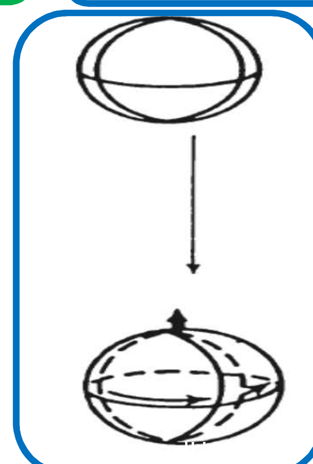
$$\theta = n^\mu{}_{;\mu} \,,$$

$$\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta \left(n_{\alpha;\beta} + n_{\beta;\alpha}\right) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu} \,,$$

$$\omega_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta \left(n_{\alpha;\beta} - n_{\beta;\alpha}\right)$$







Geometrical quantities

- The rate of change of an infinitesimally commoving volume *V* is given by $\boxed{\frac{1}{V}\frac{dV}{d\eta} = \theta = 3H}$
- The Lie derivative of the projection tensor along the velocity field is
  the extrinsic curvature of spatial hypersurfaces

$$K_{\mu\nu} \equiv \frac{1}{2}\pounds_n \mathcal{P}_{\mu\nu} = \mathcal{P}_\nu{}^\lambda n_{\mu;\lambda} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}\,. \qquad K^\mu{}_\mu = K = H$$

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- Ricci identity for the velocity field:

$$(\nabla_\mu \nabla_\alpha - \nabla_\alpha \nabla_\mu) u_\sigma = R_{\mu\alpha\sigma\lambda} u^\lambda$$

- Trace and contraction with the four velocity (assuming null acceleration):

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) = R_{\mu\lambda} u^\alpha u^\lambda$$

- Similar propagation of shear and vorticity.

Part 2. Perturbations

What are perturbations?

- **Approximating scheme** to solve problems from solutions to related simplifications. Example:

$$\sqrt{y} = \sqrt{x^2 (1+\varepsilon)} = x\sqrt{1+\varepsilon} \rightarrow \sqrt{26} = \sqrt{25+1} = 5\sqrt{1+\frac{1}{25}} \approx 5(1+1/50) \approx 5.1(=5.099)$$

For tensors $\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i)\,.$

And a Taylor expansion $\delta\mathbf{T}(t,x) = \sum_n \frac{\varepsilon^n}{n!}\delta\mathbf{T}_n(t,x)$

Why use perturbations?

- Approximating scheme to solve problems from solutions to related simplifications.

For tensors $\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i)\,.$

And a Taylor expansion
$$\delta\mathbf{T}(t,x) = \sum_n \frac{\varepsilon^n}{n!}\delta\mathbf{T}_n(t,x)$$

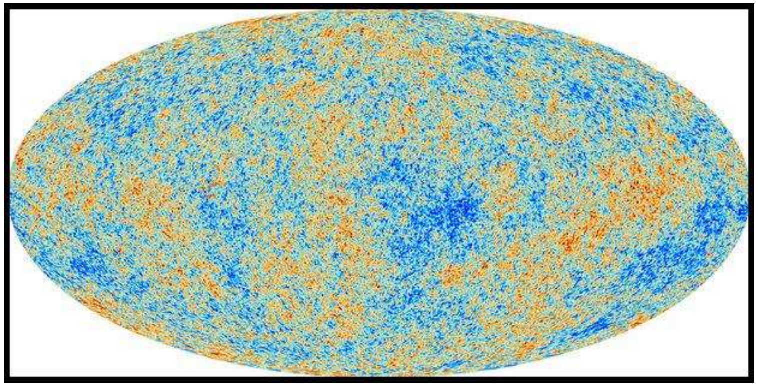
In Cosmological Perturbation Theory we deal with deviations from FLRW Universe:

- **Small**, all coordinate-dependent inhomogeneities or anisotropies.

$$\boxed{\rho(x,t) = \bar{\rho}(t) + \delta\rho(x,t) = \bar{\rho}(t)(1+\delta)}$$

- Why Perturbations? Observations from CMB:

$$\frac{\delta T}{T} \simeq 10^{-5}$$



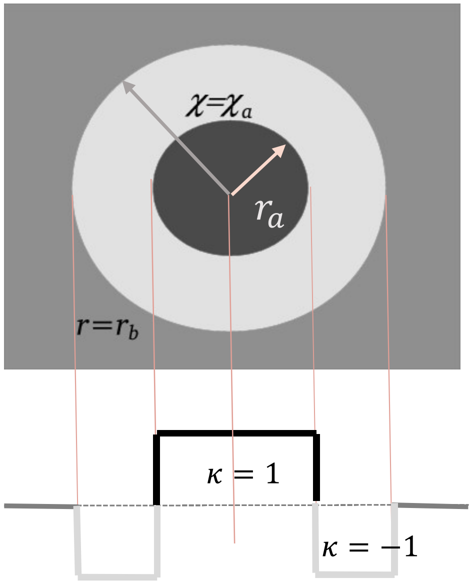
A toy model approximation

Spherical collapse model:

$$\boxed{\rho(r < r_a) = \bar{\rho}(t) + \delta\rho(t)}$$

- Top hat configuration, with an initial background expansion.

$$H_a^2 = \frac{8\pi G}{3}\rho_b = \frac{8\pi G}{3}\left[\rho_b + \delta\rho - \delta\rho\right] = \frac{8\pi G}{3}\left[\rho_a - \delta\rho\right]$$



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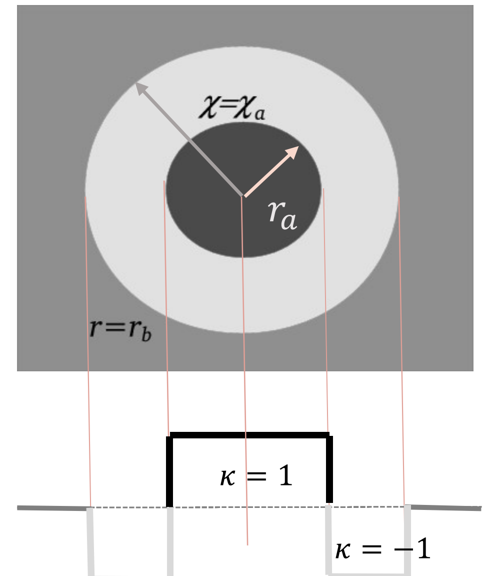

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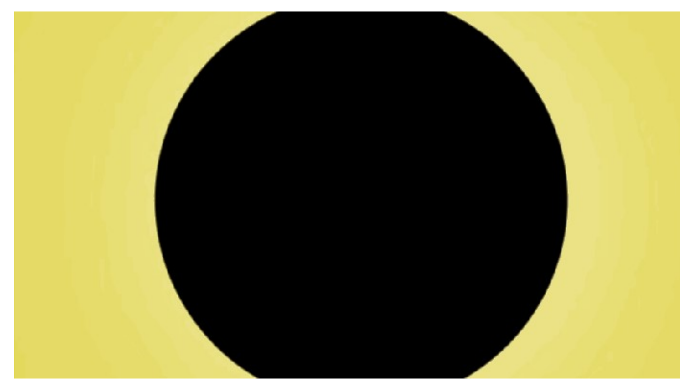
- Positive spatial curvature accounts for overdensity

$$\boxed{\frac{k\,c^2}{r^2(t_i)}=\frac{8\pi G}{3}\delta\rho(t_i)}$$



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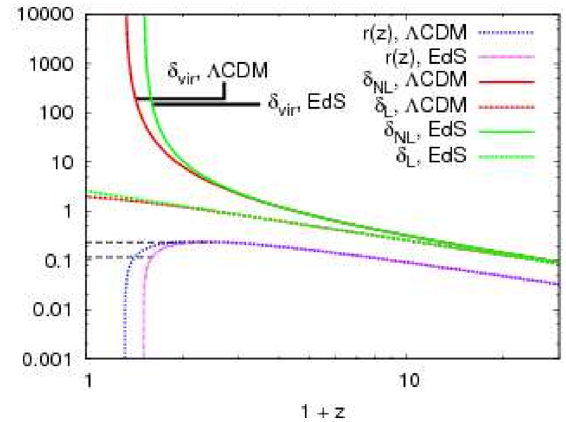

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$$\boxed{\delta^{lin}{}_{\mathrm{vir}} = \frac{3}{20}(9\pi)^{2/3}}$$

- Complete non-linear description

$$\boxed{\frac{\rho_a}{\bar{\rho}} = 1 + \frac{\delta\rho_a}{\bar{\rho}} = \frac{r_b{}^3}{r_a{}^3} = \left[\frac{\eta^2}{\frac{a_{\max}}{2}(1-\cos\eta)}\right]}$$

$$1 + \delta_{\mathrm{lin}} = 1 + \frac{3}{20}\left(6\pi\frac{t}{t_{max}}\right)^{2/3},$$

$$\boxed{\delta^{lin}{}_{\mathrm{col}} = \frac{3}{20}(12\pi)^{2/3}}$$

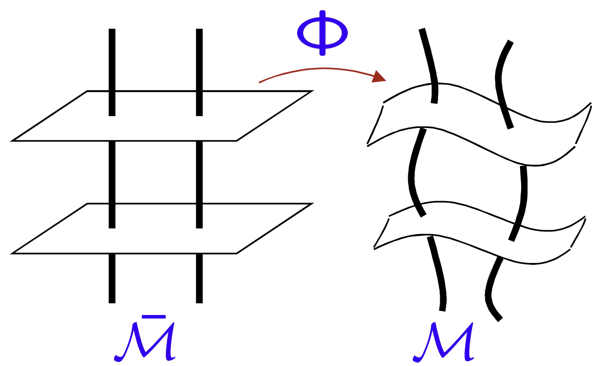
What are perturbations?

$$\boxed{\rho(x,t)=\bar{\rho}(t)+\delta\rho(x,t)=\bar{\rho}(t)(1+\delta)}$$

- Unambiguous definition of perturbations requires a map.

$$\delta\rho = \underbrace{\rho}_{\mathcal{M}} - \underbrace{\bar{\rho}}_{\bar{\mathcal{M}}}\ ?$$

$$\delta Q = Q - \Phi(\bar{Q})$$



- The map must account for a small deviation from **background.**
- No unique way of defining this map Φ (re: **gauge choice**).
- No unique way of defining perturbations (i.e. gradient expansion).

$$\nabla^2 Q/(aH)^2 \to -k^2 Q/(aH)^2 << 1 \quad \text{(at super-horizon scales)}$$

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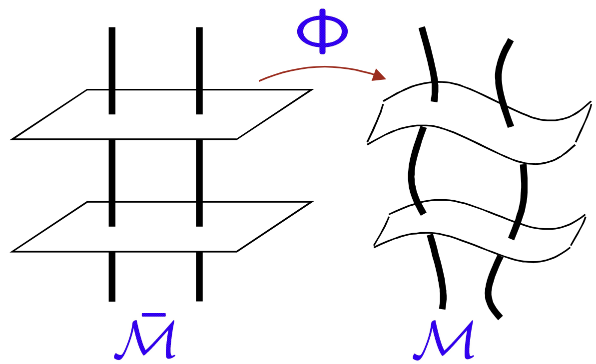

- The map must account for a small deviation from **background.**
- No unique way of defining this map (e.g. **gauge choice**).
  - What if $\bar{Q} = 0$? Then δQ is independent of mapping.
  - Then δQ is **Gauge-Independent** (Stewart-Walker Lemma)

Metric Perturbations

- The metric tensor perturbations $g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)$

$$\delta g_{00} = -2a^2\phi$$

Gravitational potential (Newtonian potential)

$$\delta g_{0i} = a^2(B_{,i} - S_i)$$

$$\delta g_{ij} = 2a^2(-2\psi + E_{,ij} + F_{i,j} + h_{ij})$$

Shift Potential

Metric Perturbations

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$$\delta g_{00} = -2a^2\phi$$ Gravitational potential (Newtonian potential)

$$\delta g_{0i} = a^2(B_{,i} - S_i)$$

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Shift Potential

$\delta g_k{}^k$ = **Local scale factor**

$(\partial_i\partial_j - \tfrac{1}{3}\nabla^2)(E' + B) =$ Shear scalar

- Split from Helmholtz Theorem (see Bardeen, 1980)
- Scalar, vector and tensors **decouple at first order**.

Metric Perturbations

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$$\delta g_{0i} = a^2(B_{,i} - S_i)$$

$$\delta g_{ij} = 2a^2(-2\psi + E_{,ij} + F_{i,j} + h_{ij})$$

Potential shift

$\delta g_k{}^k$ = Local scale factor

$(\partial_i\partial_j - \tfrac{1}{3}\nabla^2)(E + B) =$ Shear scalar σ

- Geometrical quantities:
  - Intrinsic Curvature of spatial hypersurfaces $\quad {}^{(3)}R_1 = \dfrac{4}{a^2}\nabla^2\psi_1$
  - Acceleration $\quad a_i = \phi_{,i}$
  - Expansion $\quad \boxed{\theta = \dfrac{3}{a}\left[\mathcal{H} - \mathcal{H}\phi - \psi' + \dfrac{1}{3}\nabla^2\sigma\right]}$
  - Proper time $\quad d\tau = (1+\phi)dt$

$T_{\mu\nu}$ Perturbations

- The Stress-Energy tensor split $$T_{\mu\nu} = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(x,t)$$

$$\delta T^0{}_0 = -\delta\rho_1 \,,$$
$$\delta T^0{}_i = (\rho_0 + P_0)(v_{1i} + B_{1i})$$
$$\delta T^i{}_j = \delta P_1 \delta^i{}_j + a^{-2}\pi_{(1)}{}^i{}_j \,,$$

Matter density perturbation (recall $T^\mu{}_\nu u^\nu = -\rho u^\mu$)

Velocity potential

$\delta T_k{}^k =$ Local pressure

Anisotropic stress

- Split for velocity: $$u^\mu = a^{-1}\left(1-\phi, v^i + v_{,}{}^i\right)$$

$T_{\mu\nu}$ Perturbations

- The Stress-Energy tensor split $T_{\mu\nu} = \bar{T}_{\mu\nu}(t) + \delta T_{\mu\nu}(x,t)$

$$\delta T^0{}_0 = -\delta\rho_1 \,,$$
$$\delta T^0{}_i = (\rho_0 + P_0)\,(v_{1i} + B_{1i})$$
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Matter density perturbation (recall $T^\mu{}_\nu u^\nu = -\rho u^\mu$)

Velocity potential

δT_k^k = Local pressure

Anisotropic stress

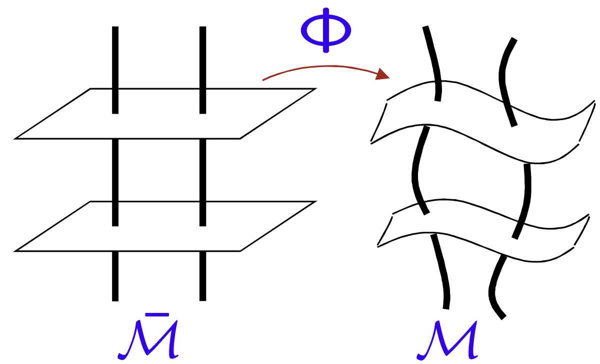
- Split for velocity: $u^\mu = a^{-1}\left(1-\phi, v^i + v_,{}^i\right)$
- Scalar field:

$$\delta T^0{}_0 = a^{-2}\bar{\varphi}'(\phi\bar{\varphi}' - \delta\varphi') - U_{,\varphi}\delta\varphi$$
$$\delta T^0{}_i = -a^{-2}\bar{\varphi}'(\delta\varphi_{,i})$$
$$\delta T^i{}_j = \left[a^{-2}\bar{\varphi}'(\delta\varphi' - \phi\bar{\varphi}') - U_{,\varphi}\delta\varphi\right]\delta^i{}_j$$

Gauges

- $\bar{Q}(t)$ depends on our choice of equal-time hypersurface at each point (x,t) :
- So $\delta Q(r,t)$ will also depend on this choice of time-slicing or **gauge choice**

$$\delta Q = Q - \Phi(\bar{Q})$$



Gauge Transformation: $x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}$

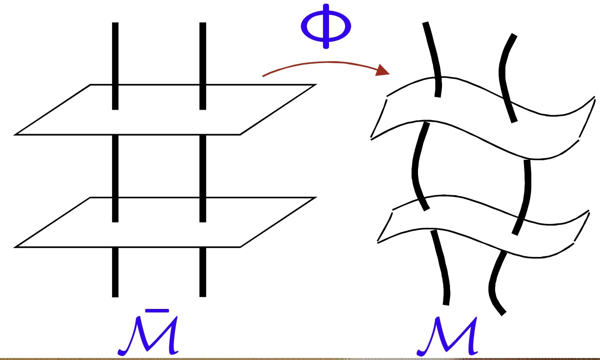
Coordinate transformation which maps points of one slicing to another

1) Must be small change
2) Helmholtz split $\xi^{\mu} = (\alpha, \beta^{i} + \beta_{,}{}^{i})$
3) Imposed to specific characteristics of $\delta Q(r,t)$

Gauges

- $\bar{Q}(t)$ depends on our choice of equal-time hypersurface at each point (x,t) :
- So $\delta Q(r,t)$ will also depend on this choice of time-slicing or **gauge choice**

$$\delta Q = Q - \Phi(\bar{Q})$$



Gauge Transformation: $x^\mu \to x^\mu + \xi^\mu$
Coordinate transformation which maps points of one slicing to another

1) Must be small change
2) Helmholtz split $\xi^\mu = (\alpha, \beta^i + \beta_{,}^{\ i})$
3) Imposed to specific characteristics of $\delta Q(r,t)$
4) **¡OJO!**: spurious quantities may appear
5) Soln: Fix the gauge completely!

Gauges

- Active approach: Map transforming perturbed quantities

$$\tilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}$$

- Vector field generating transformation $\xi^\mu = (\alpha, \beta^i + \beta_{,}^{i}) = \xi_1^{\ \mu} + \frac{1}{2}\xi_2^{\ \mu}$

- Expansion of exponential map $\exp(\pounds_\xi) = 1 + \epsilon\pounds_{\xi_1} + \frac{1}{2}\epsilon^2\pounds_{\xi_1}^2 + \frac{1}{2}\epsilon^2\pounds_{\xi_2} + \dots$

- Split of tensor transformation

$$\tilde{\overline{T}} = \overline{T}$$

$$\tilde{T}_1 = T_1 + \pounds_{\xi 1}\overline{T}$$

$$\tilde{T}_2 = T_2 + \pounds_{\xi 2}\overline{T} + (\pounds_{\xi 1})^2\overline{T} + 2\pounds_{\xi 1}T_1$$

Gauges

$$\xi^{\mu} = (\alpha, \beta^{i} + \beta_{,}^{i}) = \varepsilon \xi_1^{\mu} + \tfrac{1}{2}\varepsilon^2 \xi_2^{\mu}$$

- Passive approach: Provide relation between coordinates $\tilde{x}^{\mu}(q)$ and $x^{\mu}(q)$

$$\boxed{\widetilde{x^{\mu}}(q) = x^{\mu}(q) - \epsilon \xi_1^{\mu}(q)}$$

- Require physical (total) quantities invariant $\tilde{\rho}(\tilde{x}^{\mu}) = \rho(x^{\mu})$
- Expansion of both sides in perturbations $\rho(x^{\mu}) = \rho_0(x^0) + \epsilon \delta\rho_1(x^{\mu})$

$$\tilde{\rho}(\tilde{x}^{\mu}) = \rho_0\left(\widetilde{x^0}\right) + \epsilon \widetilde{\delta\rho_1}\left(\widetilde{x^{\mu}}\right) = \rho_0(x^0) + \epsilon\left(-\rho_0'(x^0)\xi_1^0(x^{\mu}) + \widetilde{\delta\rho_1}(x^{\mu})\right)$$

- Result: Transformation rule at first order:

$$\boxed{\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\xi_1^0}$$

- **Same applies for any other 4-scalar (c.f. active approach)** $\boxed{\tilde{Q}_1 = Q_1 + \bar{Q}'\alpha_1}$

Gauges

- Vector and tensor transformations computed through exponential map.
- Relevant results from vectors
  - Velocity transformation $\tilde{v}_1 = v_1 - \beta_1'$
  - Scalar off-diagonal metric $\tilde{B}_1 = B_1 + \beta_1' - \alpha_1$
- Results from tensor transformations
  - Scalar metric potentials
$$\widetilde{\phi_1} = \phi_1 + \mathcal{H}\alpha_1 + \alpha_1' \,,$$
$$\widetilde{\psi_1} = \psi_1 - \mathcal{H}\alpha_1 \,,$$
$$\widetilde{E_1} = E_1 + \beta_1 \,,$$
  - Gravitational Waves
$$\boxed{\widetilde{h}_{1ij} = h_{1ij}}$$

Gauges…what Gauges?

- Observers may measure different observables
- Observers that see uniform density:
  - Require: $$\delta\tilde{\rho}_1 = \delta\rho_1 - \bar{\rho}'\alpha_1 = 0 \rightarrow \alpha_{1\rho} = \frac{\delta\rho_1}{\bar{\rho}'}$$
  - Result: Curvature perturbation in **uniform density gauge**.

$$\tilde{\psi}_\rho = \psi - \mathcal{H}\frac{\delta\rho_1}{\bar{\rho}'} \equiv -\zeta$$

Gauges…what Gauges?

- Observers may measure different observables
- Observers that see uniform field:
  - Require: $$\delta\tilde{\rho}_1 = \delta\rho_1 - \bar{\rho}'\alpha_1 = 0 \rightarrow \alpha_{1\rho} = \frac{\delta\rho_1}{\bar{\rho}'}$$
  - Result: Curvature perturbation in **uniform density gauge**.

$$\tilde{\psi}_\rho = \psi - \mathcal{H}\frac{\delta\rho_1}{\bar{\rho}'} \equiv -\zeta$$

- Observers with unperturbed spatial hypersurfaces:
  - Require $\tilde{\psi} = \tilde{E} = 0$

$$\alpha_{\psi 1} = \psi_1/\mathcal{H}, \ \beta_{\psi 1} = -E_1$$

  - Result: Field perturbation in **flat gauge**.

$$\delta\tilde{\varphi}_{\rho 1} = \delta\varphi_1 + \bar{\varphi}'\frac{\psi_1}{\mathcal{H}} \equiv Q_{MS}$$

Gauges...what Gauges?

- Observers may measure different observables
- Observers that see uniform scalar field:
  - Require:

$$\alpha_{\varphi 1} = \frac{\delta\varphi_1}{\bar{\varphi}'}$$

  - Result: Curvature perturbation in **Uniform field gauge.**

$$\tilde{\psi}_{\varphi 1} = \psi_1 + \mathcal{H}\frac{\delta\varphi_1}{\bar{\varphi}'}$$

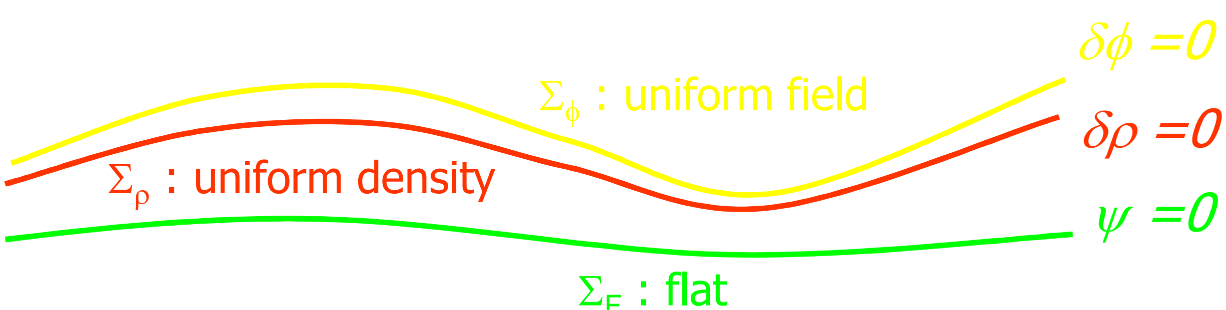
Gauges…what Gauges?

- Observers may measure different observables
- Observers that see uniform scalar field:
  - Require:
$$\alpha_{\varphi 1} = \frac{\delta\varphi_1}{\bar{\varphi}'}$$
  - Result: Curvature perturbation in **Uniform field gauge.**
$$\tilde{\psi}_{\varphi 1} = \psi_1 + \mathcal{H}\frac{\delta\varphi_1}{\bar{\varphi}'}$$
- Observers that experience no shear:
  - Require
$$\alpha_{\ell 1} = -\sigma_1 = B_1 - E_1'$$
  - Result: Metric potentials in **Longitudinal or Newtonian Gauge.**
$$\tilde{\phi}_{\ell 1} = \phi_1 + \mathcal{H}\left(B_1 - E_1'\right) + \left(B_1 - E_1'\right)' \equiv \Phi$$
$$\tilde{\psi}_{\ell 1} = \psi_1 - \mathcal{H}\left(B_1 - E_1'\right) \equiv \Psi$$

Gauge Invariants

- Combine two scalars $\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1\,,\ \ \widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1$

- Obtain a Gauge-invariant quantity from their difference $\dfrac{\psi_1}{\mathcal{H}} - \dfrac{\delta\rho_1}{\bar{\rho}\,'}$

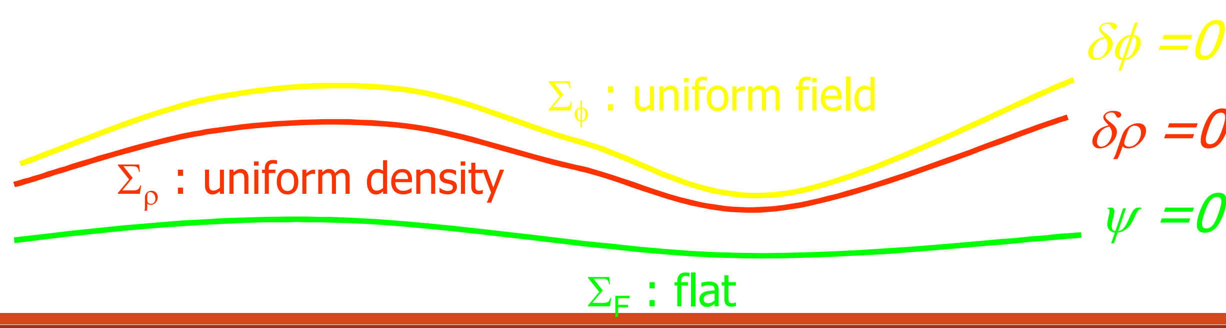


Gauge Invariants

- Combine two scalars $\tilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1\,,\ \ \widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1$

- Obtain a Gauge-invariant quantity

The **Uniform density curvature** $\frac{\psi_1}{\mathcal{H}} - \frac{\delta\rho_1}{\bar{\rho}'} = \frac{\tilde{\psi}_{\rho 1}}{\mathcal{H}} \equiv -\frac{\zeta}{\mathcal{H}}$



Gauge Invariants

- Combine two scalars $\quad \tilde{\psi}_1 = \psi_1 - \mathcal{H}\alpha_1\,, \;\; \widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1$

- Obtain a Gauge-invariant quantity
The **Uniform density curvature**

$$\frac{\psi_1}{\mathcal{H}} - \frac{\delta\rho_1}{\bar{\rho}\,'} = \frac{\tilde{\psi}_{\rho 1}}{\mathcal{H}} \equiv -\frac{\zeta}{\mathcal{H}}$$

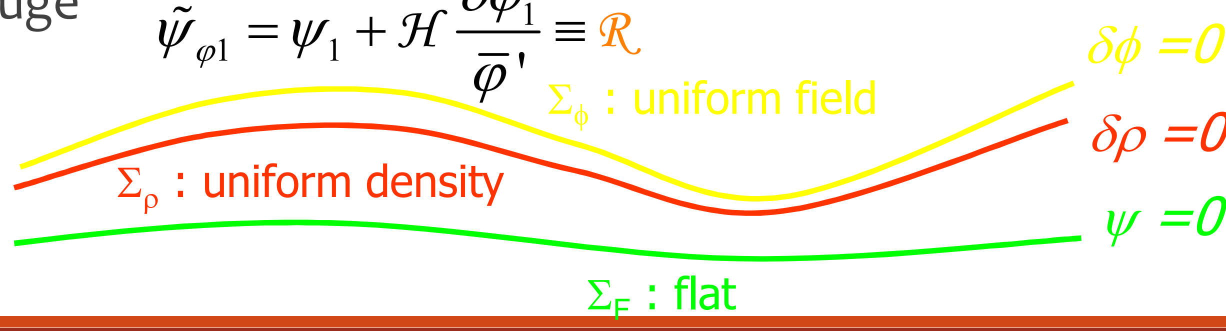
- **Bardeen Potentials** are the first
but not only gauge-invariants.

$$\tilde{\phi}_{\ell 1} = \phi_1 + \mathcal{H}\left(B_1 - E_1\,'\right) + \left(B_1 - E_1\,'\right)' \equiv \Phi$$

$$\tilde{\psi}_{\ell 1} = \psi_1 - \mathcal{H}\left(B_1 - E_1\,'\right) \equiv \Psi$$

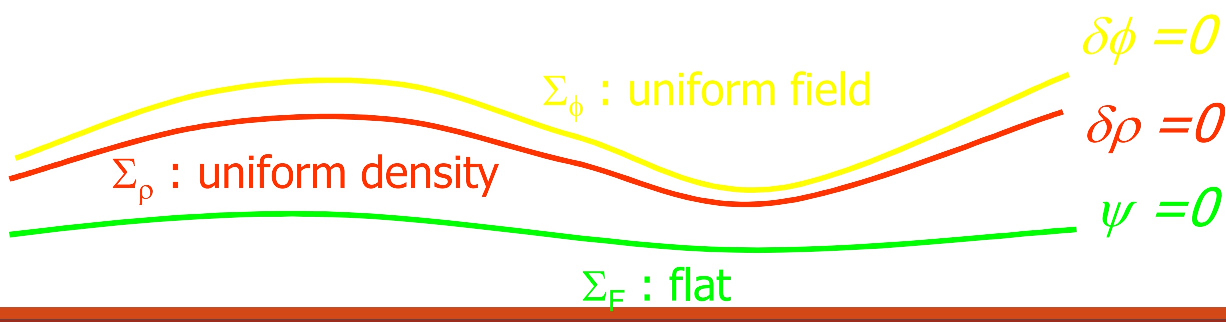
- Curvature Perturbation in
Uniform field (**comoving**) gauge

$$\tilde{\psi}_{\varphi 1} = \psi_1 + \mathcal{H}\frac{\delta\varphi_1}{\bar{\varphi}\,'} \equiv \mathcal{R}$$



Gauge lessons

- Physically meaningful $\tilde{Q}_1$ are found by fixing a gauge completely.
- Gauge-invariant $\tilde{Q}_1$ is any fixed-gauge quantity.
- Gauge transformations show only two degrees of freedom $\xi^\mu = (\alpha, \beta^i + \beta_{,}^{\ i})$
- Different problems do with specific gauges.



Part 3. Perturbation Dynamics

Conservation Equations: adiabaticity

- Perturbations can be explained (at large scales) as Q's evaluated at shifted time (cf. ϕ) :

$$\delta Q(x,\eta) = Q(\eta + \delta\eta(x)) - \bar{Q}(\eta) \approx \bar{Q}(\eta)'\delta\eta$$

- If each component of the density is separately conserved (meets a continuity equation).

$$\frac{\delta\rho_1^{(i)}}{\bar{\rho}^{(i)}\,'} = -\frac{1}{3\mathcal{H}}\frac{\delta\rho_1^{(i)}}{\rho^{(i)} + P^{(i)}} = \delta\eta \approx \phi$$

- Each component of the set of non-interacting ingredients is adiabatic

$$P^{(i)} = P^{(i)}\left(\rho^{(i)}(x^\mu)\right) \longrightarrow -3\mathcal{H}\delta\eta = \frac{\delta_1^{(\gamma)}}{4/3} = \frac{\delta_1^{(\nu)}}{4/3} = \delta_1^{(m)} = \delta_1^{(b)}$$

- **Each adiabatic component is determined by the total density perturbation**

$$\delta(r,t) = \sum_i \delta^{(i)}(r,t)$$

Conservation Equations

- Energy conservation at first order (continuity equation)

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(\rho + P)\psi' + (\rho + p)\nabla^2(v + \sigma) = 0$$

- In terms of uniform density curvature (with $c_s^2 \equiv \frac{P'}{\rho'}$) :

$$\zeta' = -\mathcal{H}\frac{\delta p_{\rm nad}}{p + \rho} - \frac{1}{3}\nabla^2 v_l \qquad \text{where} \qquad \delta p_{\rm nad} = \delta p - c_s{}^2\delta\rho$$

Conservation Equations

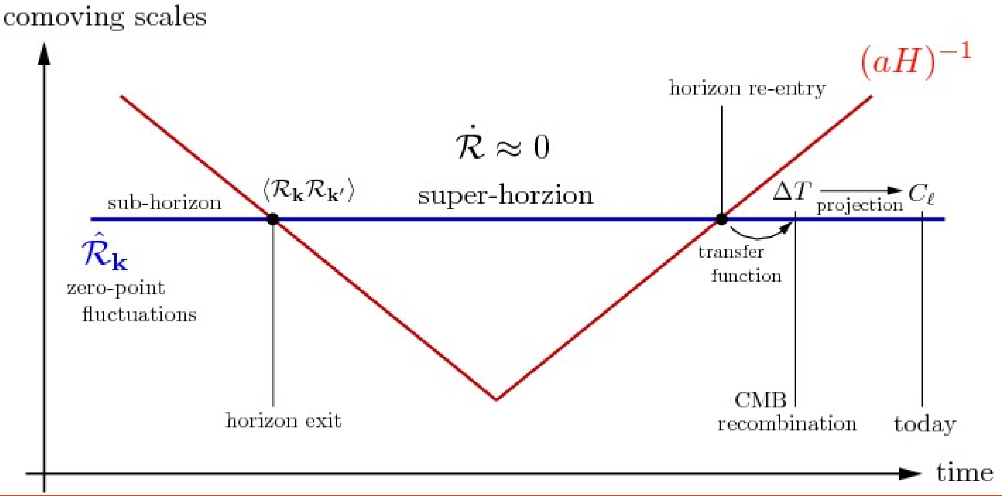
- Energy conservation at first order (continuity equation)

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(\rho + P)\psi' + (\rho + p)\nabla^2(v + \sigma) = 0$$

- In terms of uniform density curvature (with $c_s^2 \equiv \dfrac{P'}{\rho'}$):

$$\zeta' = -\mathcal{H}\frac{\delta p_{\rm nad}}{p+\rho} - \frac{1}{3}\nabla^2 v_l \longrightarrow$$ **Constant ζ for adiabatic δP and large scales: $k \ll \mathcal{H}$**

- Result valid at all orders, ζ is conserved if no entropy perturbations appear.
- Physical reasoning behind the δN formalism.



Conservation Equations

- Momentum conservation (Euler equation)

$$V' + (1 - 3c_{\rm s}^2)\mathcal{H}V + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

Conservation Equations

- Momentum conservation (Euler equation)

$$V' + (1 - 3c_{\rm s}^2)\mathcal{H}V + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

- In a comoving gauge: $V = v + B = 0$

$$(\rho + P)\phi = \delta P + (2/3)\nabla^2\Pi \longrightarrow$$

Acceleration produced by pressure gradients because: $\phi_{,i} = a_i$

Conservation Equations

- Momentum conservation (Euler equation)

$$V' + (1 - 3c_s^2)\mathcal{H}V + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

- In a comoving gauge: $V = v + B = 0$

$$(\rho + P)\phi = \delta P + (2/3)\nabla^2\Pi \longrightarrow$$ **Acceleration produced by pressure gradients**

- For pressureless dust: $(aV)' + a\phi = 0$

$\longrightarrow$ **In synchronous gauge dust velocity** V **evolves as:** $\boxed{V_{\phi 1} \approx 1/a}$

Conservation Equations

- Momentum conservation (Euler equation)

$$V' + (1 - 3c_{\rm s}^2)\mathcal{H}V + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

- In a comoving gauge: $V = v + B = 0$

$$(\rho + P)\phi = \delta P + (2/3)\nabla^2\Pi$$ ⟶ **Acceleration produced by pressure gradients**

- For pressureless dust: $(aV)' + a\phi = 0$

⟶ **In synchronous gauge dust velocity B evolves as:** $\boxed{V_{\phi 1} \approx 1/a}$

- In Longitudinal gauge, Euler + continuity:

$$\delta'_{\ell 1} + 2\mathcal{H}\delta'_{\ell 1} - \left(4\pi G\bar{\rho} + c_s{}^2\nabla^2\right)\delta_{\ell 1} = 0$$ **¡Tarea!**

Einstein Equations

- Energy and momentum constraints

$$3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) - \nabla^2\left[\psi + \mathcal{H}\sigma\right] = -4\pi G a^2 \delta\rho,$$
$$\psi' + \mathcal{H}\phi = -4\pi G a^2(\rho + P)v + B$$

- In longitudinal gauge:

$$3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi = -4\pi G a^2 \delta\rho_\ell,$$
$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2(\rho + P)v_\ell$$

$$\longrightarrow \boxed{\nabla^2\Psi = 4\pi G a^2 \delta\rho_{\mathrm{com}}}$$

Poisson Equation at all scales!

Einstein Equations

- Evolution equations:

$$\psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi = 4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2\Pi\right) ;$$

$$\sigma' + 2\mathcal{H}\sigma + \psi - \phi = 8\pi G a^2\Pi$$

- In longitudinal gauge:

$$\Psi - \Phi = 8\pi G a^2\Pi \longrightarrow$$

Equivalent potentials if no anisotropic stress

- Evolution for potentials:

$$\Psi'' + 3(1 + c_s^2)\mathcal{H}\Psi' + [2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2]\Psi = 0$$

- And for dust (cf. Friedmann eqns):

$$\Psi'' + 3\mathcal{H}\Psi' = 0 \quad \Rightarrow \quad \boxed{\Psi = \text{const.}}$$

See Josué De Santiago's Talk

Lessons so far:

- Gauges must be always specified and quantities in fully fixed gauges are always **gauge invariant quantities**.
- Uniform density curvature perturbation ζ **is conserved** on large scales.
- Bardeen potentials are conserved during pressureless matter dominated eras.
- **Spherical collapse provides intuition** of the meaning and evolution of metric perturbations.
- Adiabaticity as an indicator of evolution through growing mode.

Pending Lessons:

- Jeans scale and threshold amplitude for collapse.
- δN Formalism and the separate universe approach.
- Newtonian regime for perturbation theory.

References

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- George F. R. Ellis, Roy Maartens, Malcolm A. H. MacCallum, Relativistic Cosmology. CUP, 2012
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Chapter 3. Perturbation Solutions

Questions to answer:

- Jeans Instability.
- Velocity dispersion in dark matter.
- Map between coordinates or map between manifolds.
- Growing and decaying modes.
- Smallness of perturbations.
- Statistical treatment.