Large-scale structure formation

A very short review

Josué De Santiago

June 12, 2018

CONACYT - CINVESTAV

Table of contents

Prelude

The Vlasov equation

The Hubble law for the movement of galaxies states that

$$v_H = Hr = \mathcal{H}x$$

which means they accelerate as

$$\dot{v}_H = \frac{1}{a}\frac{d\mathcal{H}}{d\tau}x$$

where $dt = ad\tau$.

This should be originated by a gravitational potential

$$f_{exp} = -m\nabla_r\Phi_{exp} = -ma\nabla_x\Phi_{exp}$$

which can integrated to give

$$\Phi_{exp} = -\frac{d\mathcal{H}}{d\tau}\frac{x^2}{2}$$

The Vlasov equation

We are concerned with deviations from the Hubble flow

$$v = \mathcal{H}x + u$$

which means they accelerate as

$$\dot{v} = \frac{1}{a}\frac{d\mathcal{H}}{d\tau}x + \mathcal{H}\frac{dx}{d\tau} + \frac{du}{d\tau}$$

where $dt = ad\tau$.

This should be originated by a gravitational potential

$$f_{exp} = -ma\nabla_x(\Phi_{exp} + \Phi_{pec})$$

which produces an acceleration

$$\frac{d}{d\tau}(aum) = -am\nabla\Phi$$

we won't write the x or pec subindexes anymore.

The Vlasov equation

The full potential obeys the Poisson equation

$$\nabla_r^2 \phi_{tot} = 4\pi G \rho$$

which, in terms of the over density

$$\rho = \bar{\rho}(1+\delta)$$

and supplied by the Friedmann equations, can be written as

$$\nabla^2 \Phi = \frac{3}{2}\Omega_m \mathcal{H}^2 \delta$$

The Vlasov equation

The conservation of the number density of particles in phase space $f(x, p, \tau)$, is written as

$$\frac{d}{d\tau} f(x, p, \tau) = \frac{\partial f}{\partial \tau} + \frac{p}{ma} \nabla f - am \nabla \Phi \frac{\partial f}{\partial p} = 0$$

where $p = aum$.

This equation is hard to solve, so it is better to introduce the moments of the distribution

$$\begin{aligned} \rho &= \int d^3 p f \\ \rho u &= \int d^3 p \frac{p}{am} f \\ \rho u_i u_j + \sigma_{ij} &= \int d^3 p \frac{p_i p_j}{a^2 m^2} \end{aligned}$$

Using these definitions we can solve Vlasov equation!

Well, not really!

The problem is that we obtain an infinite hierarchy of equations. But at least we obtain the continuity and Euler equations

$$\begin{aligned} \frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1+\delta)u] &= 0 \\ \frac{\partial u}{\partial \tau} + \mathcal{H}u + (u \cdot \nabla)u &= -\nabla \Phi - \frac{1}{\rho}\nabla_j(\rho \sigma_{ij}) \end{aligned}$$

And we can make assumptions on the stress tensor (making it zero).

The system can be closed together with the Poisson equation

$$\nabla^2 \Phi = \frac{3}{2}\Omega_m \mathcal{H}^2 \delta$$

First order Standard Perturbation theory

Linearizing the perturbation equations

$$\begin{aligned}\frac{\partial \delta}{\partial \tau} + \nabla \cdot u &= 0 \\ \frac{\partial u}{\partial \tau} + \mathcal{H} u &= -\nabla \Phi\end{aligned}$$

Any 3D vector field is defined by its curl and divergence, which we define as

$$\theta = \nabla \cdot u \quad w = \nabla \times u$$

they obey

$$\begin{aligned}\frac{\partial \theta}{\partial \tau} + \mathcal{H} \theta &= -\nabla^2 \Phi = -\frac{3}{2} \Omega_m \mathcal{H}^2 \delta \\ \frac{\partial w}{\partial \tau} + \mathcal{H} w &= 0\end{aligned}$$

The curl decays as $w \propto a^{-1}$. We can focus on the evolution of θ and δ

First order SPT

The linearized continuity equation implies

$$\theta = -\frac{\partial \delta}{\partial \tau}$$

Substituting in the Euler equation

$$-\frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta}{\partial \tau} + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta = 0$$

This equation is separable, meaning that we can propose $\delta(x, \tau) = D_1(\tau)\delta(x, 0)$, where D_1 will always have to solutions.

For an Einstein-de Sitter universe:

$$D_1^{(+)} = a, \qquad D_1^{(-)} = a^{-3/2}, \qquad f = \frac{d \log D_1^{(+)}}{d \log a} = 1$$

For a flat universe with $\Omega_\Lambda \neq 0$ as in ΛCDM:

$$f \approx \Omega_m^{5/9}$$

First order SPT

The linearized continuity equation implies

$$\theta = -\frac{\partial \delta}{\partial \tau} = -f\mathcal{H}\delta$$

Substituting in the Euler equation

$$-\frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta}{\partial \tau} + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta = 0$$

This equation is separable, meaning that we can propose $\delta(x,\tau) = D_1(\tau)\delta(x,0)$, where D_1 will always have to solutions.

For an Einstein-de Sitter universe:

$$D_1^{(+)} = a, \qquad D_1^{(-)} = a^{-3/2}, \qquad f = \frac{d\log D_1^{(+)}}{d\log a} = 1$$

For a flat universe with $\Omega_\Lambda \neq 0$ as in ΛCDM:

$$f \approx \Omega_m^{5/9}$$

Random fields

We don't have a deterministic theory of the origin of fluctuations. But we can assume that our universe is one realization in an ensemble.

We use Gaussian random fields because:

- They are consistent with our model of primordial perturbations (inflation)
- They appear when there is a superposition of several random process (central limit theorem)
- They are easy to treat

$$p(\delta_1, \delta_2, \ldots) \propto \exp\left(-\frac{1}{2}x^T C^{-1} x\right)$$

Assuming zero mean, the fields satisfy Wick's theorem:

$$\begin{aligned}\left\langle \delta(\mathbf{k}_1)\dots\delta(\mathbf{k}_{2p+1})\right\rangle &= 0 \\ \left\langle \delta(\mathbf{k}_1)\dots\delta(\mathbf{k}_{2p})\right\rangle &= \sum_{\text{all pair associations}} \prod_{p \text{ pairs (i,j)}} \left\langle \delta(\mathbf{k}_i)\delta(\mathbf{k}_j)\right\rangle\end{aligned}$$

Statistical homogeneity and isotropy implies for the correlation function and the power spectrum

$$\langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle = \xi(r) \qquad \langle \delta(k)\delta(k')\rangle = \delta_D(k+k')P(k)$$

where

$$\xi(r) = \int d^3k P(k)\exp(ik\cdot r)$$

Halo mass function

Press-Schechter formalism

Now we are interested in the formation of virialized structures (halos)

Giving a certain initial perturbation $\delta(z_i, r)$ we can obtain the time of collapse $z_c(\delta_i)$

$$\delta_i(z_c) = \delta(z_i|z_c)$$

We will work with smoothed fields

$$\delta_s(x; R) = \int \delta_0(x')W(x - x'; R)d^3x$$

they are also Gaussian with variance

$$\sigma^2(M) =< \delta_s^2(x; R) >= \frac{1}{2\pi^2} \int P(k)\tilde{W}^2(kR)k^2 dk$$

We will see that the halo mass function depends only on the combination

$$\nu_c = \frac{\delta_i(z_c)}{\sigma(z_i, R)}$$

which can be linearly evolved to

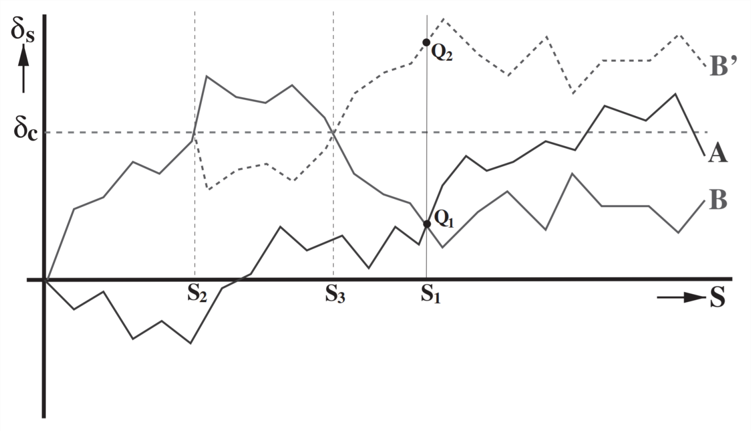
$$\nu_c = \frac{\delta_i(z_c)D(z_c, z_i)}{\sigma(z_i, R)D(z_c, z_i)} = \frac{\delta_c(z_c, R)}{\sigma(z_c, R)}$$

this is useful because $\delta_c(z_c, R) = 1.69$ in EdS and only weakly dependent on z_c for ΛCDM

Excursion set theory

If we consider a certain position x_0 we ask ourselves if we are in a halo and, if we are, which is the halo mass. In other words, whether

$$\delta_s(R, x_0) > \delta_c$$



Mo et al. 2010

The fraction of trajectories with first up crossing for $S > S_1$

$$F_{FU}(> S_1) = \int_{-\infty}^{\delta_c} [P(\delta_s, S_1) - P(2\delta_c - \delta_s, S_1)]$$

the trajectories have random jumps. If the window is a k-sharp $\delta_{k_c}(k) = \delta_0(k)\Theta(k/k_c)$

$$\langle(\Delta\delta_s)^2\rangle = \langle\left(\int_{k_c+\Delta k_c}^{k_c} \frac{e^{ik\cdot x}}{(2\pi)^3}\delta_0(k)d^3k\right)^2\rangle$$

The fraction of trajectories with first up crossing for $S > S_1$

$$F_{FU}(> S_1) = \int_{-\infty}^{\delta_c} [P(\delta_s, S_1) - P(2\delta_c - \delta_s, S_1)]$$

the trajectories have random jumps. If the window is a k-sharp $\delta_{k_c}(k) = \delta_0(k)\Theta(k/k_c)$

$$\langle(\Delta\delta_s)^2\rangle = \sigma^2(k_c + \Delta k_c) - \sigma^2(k_c)$$

The jumps are uncorrelated, meaning that they form a Markov process and

$$P(\delta_s, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta_s^2}{2\sigma^2}\right)$$

Mass function

The fraction of first up crossing is

$$F_{FU}(> \sigma_1) = \mathrm{Erf}(\frac{\nu_1}{\sqrt{2}})$$

with $\nu_1 = \delta_c/\sigma_1$.

The fraction of halos of a certain mass are

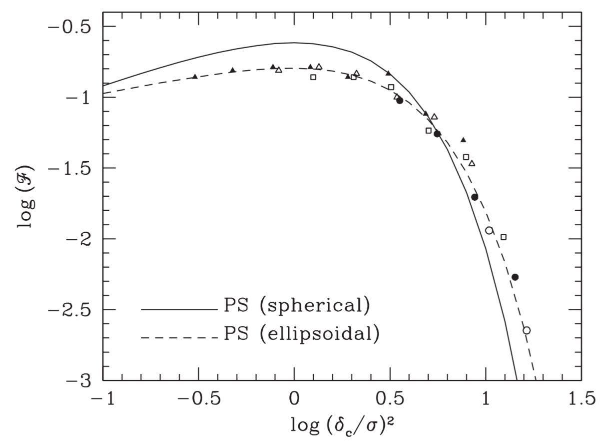
$$ndM = \bar{n}\frac{dF_{FU}(< M)}{dM}dM = \frac{\bar{\rho}}{M^2}f_{PS}(\nu)\left|\frac{d\log\nu}{d\log M}\right|$$

where

$$f_{PS} = \sqrt{\frac{2}{\pi}}\nu\exp(-\nu^2/2)$$

Mass function

Of the several adjustments to this formalism seems that including ellipsoidal collapse benefits the most



Mo et al. 2010

Higher order Standard Perturbation Theory

Fourier transform

In order to go to next orders in perturbation theory, first Fourier transform

$$\tilde{\delta}(\mathbf{k}, \tau) = \int \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^3} \exp(-i\mathbf{k} \cdot \mathbf{x})\ \delta(\mathbf{x}, \tau)$$

Assuming that the flow is irrotational:

$$\mathbf{k} \times \tilde{u} = 0 \implies \tilde{u} = C\mathbf{k}$$

where C is a scalar

$$\tilde{\theta} = i\mathbf{k} \cdot \tilde{u} = iCk^2 \implies C = -\frac{i\tilde{\theta}}{k^2}$$

meaning that

$$\tilde{u} = -\frac{i\tilde{\theta}}{k^2}\mathbf{k}$$

Fourier transform

The continuity equation

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1+\delta) u] \quad = \quad 0$$

goes to

$$\frac{\partial \tilde{\delta}(\mathbf{k},\tau)}{\partial \tau} + \tilde{\theta}(\mathbf{k},\tau) = -\int \mathrm{d}^3\mathbf{k}_1 \mathrm{d}^3\mathbf{k}_2 \delta_D(\mathbf{k}-\mathbf{k}_{12}) \alpha(\mathbf{k}_1,\mathbf{k}_2) \tilde{\theta}(\mathbf{k}_1,\tau) \tilde{\delta}(\mathbf{k}_2,\tau),$$

While the Euler equation

$$\frac{\partial u}{\partial \tau} + \mathcal{H} u + (u \cdot \nabla) u \quad = \quad -\frac{3}{2} \Omega_m \mathcal{H}^2 \delta$$

transforms to

$$\begin{aligned}\frac{\partial \tilde{\theta}(\mathbf{k},\tau)}{\partial \tau} + \mathcal{H}(\tau)\, \tilde{\theta}(\mathbf{k},\tau) + \frac{3}{2} \Omega_m \mathcal{H}^2(\tau) \tilde{\delta}(\mathbf{k},\tau) = -\int \mathrm{d}^3\mathbf{k}_1 \mathrm{d}^3\mathbf{k}_2 \delta_D(\mathbf{k}-\mathbf{k}_{12}) \\ \times \beta(\mathbf{k}_1,\mathbf{k}_2) \tilde{\theta}(\mathbf{k}_1,\tau) \tilde{\theta}(\mathbf{k}_2,\tau)\end{aligned}$$

with

$$\alpha(\mathbf{k}_1,\mathbf{k}_2) \equiv \frac{\mathbf{k}_{12} \cdot \mathbf{k}_1}{k_1^2}, \qquad \beta(\mathbf{k}_1,\mathbf{k}_2) \equiv \frac{k_{12}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2)}{2 k_1^2 k_2^2}$$

Higher order SPT

Expanding the fields

$$\tilde{\delta}(\mathbf{k},\tau) = \sum_{n=1}^{\infty} D_+^n(\tau)\delta_n(\mathbf{k}), \qquad \tilde{\theta}(\mathbf{k},\tau) = -\mathcal{H}(\tau)\sum_{n=1}^{\infty} D_+^n(\tau)\theta_n(\mathbf{k}),$$

this is valid in three special cases

- Einstein-de Sitter universe $\Omega_\Lambda = 0$ and $\Omega_m = 1$
- Spherically symmetric perturbations $\delta(\mathbf{k},\tau) = \delta(k,\tau)$
- A general cosmology when $f = \Omega_m^{1/2}$

The last one is a good approximation for ΛCDM as $f = \Omega^{5/9}$

The fields can be solved in terms of the linear fluctuations as

$$\delta_n(\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}_1 \ldots \int \mathrm{d}^3\mathbf{q}_n \, \delta_D(\mathbf{k} - \mathbf{q}_{1\ldots n}) F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) \delta_1(\mathbf{q}_1) \ldots \delta_1(\mathbf{q}_n),$$

$$\theta_n(\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}_1 \ldots \int \mathrm{d}^3\mathbf{q}_n \, \delta_D(\mathbf{k} - \mathbf{q}_{1\ldots n}) G_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) \delta_1(\mathbf{q}_1) \ldots \delta_1(\mathbf{q}_n),$$

with the kernels

$$\begin{aligned} F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) &= \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \ldots, \mathbf{q}_m)}{(2n+3)(n-1)} \Big[(2n+1)\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, \ldots, \mathbf{q}_n) \\ &\quad + 2\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, \ldots, \mathbf{q}_n) \Big], \end{aligned}$$

$$\begin{aligned} G_n(\mathbf{q}_1, \ldots, \mathbf{q}_n) &= \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \ldots, \mathbf{q}_m)}{(2n+3)(n-1)} \Big[3\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, \ldots, \mathbf{q}_n) \\ &\quad + 2n\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, \ldots, \mathbf{q}_n) \Big], \end{aligned}$$

For example, at second order

$$\tilde{\delta}(\mathbf{k},\tau) = D_+(\tau)\delta_1(\mathbf{k}) + D_+^2(\tau)\delta_2(\mathbf{k})$$

$$\tilde{\theta}(\mathbf{k},\tau) = -\mathcal{H}\left(D_+(\tau)\theta_1(\mathbf{k}) + D_+^2(\tau)\theta_2(\mathbf{k})\right)$$

where the second order perturbations are

$$\delta_2(\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\,\delta_D(\mathbf{k}-\mathbf{q}_1+\mathbf{q}_2)F_2(\mathbf{q}_1,\mathbf{q}_2)\delta_1(\mathbf{q}_1)\delta_1(\mathbf{q}_2),$$

$$\theta_2(\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}_1\mathrm{d}^3\mathbf{q}_2\,\delta_D(\mathbf{k}-\mathbf{q}_1+\mathbf{q}_2)G_2(\mathbf{q}_1,\mathbf{q}_2)\delta_1(\mathbf{q}_1)\delta_1(\mathbf{q}_2),$$

with the (symmetric) kernels

$$F_2^s(\mathbf{q}_1,\mathbf{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right) + \frac{2}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2},$$

$$G_2^s(\mathbf{q}_1,\mathbf{q}_2) = \frac{3}{7} + \frac{1}{2}\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right) + \frac{4}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2}.$$

Beyond EdS

The kernels for second order perturbation in EdS

$$F_2^s(\mathbf{q}_1,\mathbf{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{q_1 q_2}(\frac{q_1}{q_2}+\frac{q_2}{q_1}) + \frac{2}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2},$$

$$G_2^s(\mathbf{q}_1,\mathbf{q}_2) = \frac{3}{7} + \frac{1}{2}\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{q_1 q_2}(\frac{q_1}{q_2}+\frac{q_2}{q_1}) + \frac{4}{7}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2}.$$

Beyond Einstein-de Sitter: $\epsilon \simeq \frac{3}{7}\Omega_m^{-2/63}$. [Bernardeau, ApJ 433, 1 (1994)]

$$F_2(\mathbf{q}_1,\mathbf{q}_2) = \underbrace{\frac{1}{2}(1+\epsilon)}_{\simeq\frac{5}{7}+0.008} + \frac{1}{2}\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right) + \underbrace{\frac{1}{2}(1-\epsilon)}_{\frac{2}{7}-0.008}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2},$$

$$G_2(\mathbf{q}_1,\mathbf{q}_2) = \underbrace{\epsilon}_{\frac{3}{7}+0.016} + \frac{1}{2}\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right) + \underbrace{(1-\epsilon)}_{\frac{4}{7}-0.016}\frac{(\mathbf{q}_1\cdot\mathbf{q}_2)^2}{q_1^2 q_2^2}$$

The under braces are for $\Omega_m = 0.3$.

Power spectrum corrections

The first correction to the linear power spectrum has to be at fourth order in the fields

$$\begin{aligned}
\delta_D(k+k')P(k) &= \langle \delta(k)\delta(k') \rangle \\
&= \langle (\delta^{(1)}(k)+\delta^{(2)}(k)+\delta^{(3)}(k))(\delta^{(1)}(k)+\delta^{(2)}(k)+\delta^{(3)}(k)) \rangle \\
&= \delta_D(k+k')(P_L(k,\tau)+2P^{13}(k,\tau)+P^{22}(k,\tau))
\end{aligned}$$

Where

$$\begin{aligned}
P_{22}(k,\tau) &\equiv 2\int [F_2^{(s)}(\mathbf{k}-\mathbf{q},\mathbf{q})]^2 P_L(|\mathbf{k}-\mathbf{q}|,\tau)P_L(q,\tau)\mathrm{d}^3\mathbf{q} \\
P_{13}(k,\tau) &\equiv 6\int F_3^{(s)}(\mathbf{k},\mathbf{q},-\mathbf{q})P_L(k,\tau)P_L(q,\tau)\mathrm{d}^3\mathbf{q}.
\end{aligned}$$

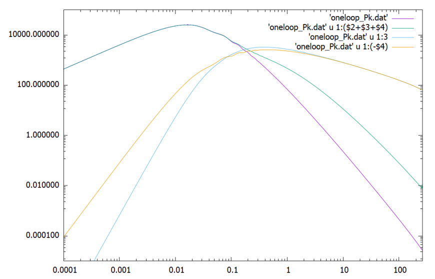
This can be "simplified" to

$$
\begin{aligned}
P_{1-loop}^{SPT} \quad = \quad & P_L(k) + \frac{1}{98}\frac{k^3}{4\pi^2}\int_0^\infty drP_L(kr)\int_{-1}^{1} dxP_L(k\sqrt{1+r^2-2rx}) \\
& \times \left(\frac{3r+7x-10rx^2}{1+r^2-2rx}\right)^2 \\
& + \frac{1}{252}\frac{k^3}{4\pi^2}P_L(k)\int_0^\infty drP_L(kr) \\
& \times \left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3}(r^2-1)^3(7r^2+2)\log\left|\frac{1+r}{1-r}\right|\right]
\end{aligned}
$$

well, actually

$$
\begin{aligned}
P_{1-loop}^{SPT} = \; & P_L(k) + \frac{2}{98}\frac{k^3}{4\pi^2}\int_0^\infty dr P_L(kr)\int_{-1}^{min(1,1/2r)} dx P_L(k\sqrt{1+r^2-2rx}) \\
& \times \left(\frac{3r+7x-10rx^2}{1+r^2-2rx}\right)^2 \\
& + \frac{1}{252}\frac{k^3}{4\pi^2} P_L(k)\int_0^\infty dr P_L(kr) \\
& \times \left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^3}(r^2-1)^3(7r^2+2)\log\left|\frac{1+r}{1-r}\right|\right]
\end{aligned}
$$

Check my code at `https://github.com/joselotl/one_loop_spt`



Lagrangian Perturbation Theory

LPT

In Lagrangian dynamics we follow the particles in its movement. Let $\mathbf{q}$ be the initial position of a particle and $\Psi(q)$ its displacement.

$$x(\tau) = \mathbf{q} + \Psi(\mathbf{q}, \tau)$$

Remember the equation for the acceleration of the peculiar velocities?

$$\frac{d}{d\tau}(aum) = -am\nabla\Phi$$

This implies that

$$\frac{d^2\mathbf{x}}{d\tau^2} + \mathcal{H}\frac{d\mathbf{x}}{d\tau} = -\nabla\Phi$$

Where ∇ is a gradient with respect to the Eulerian coordinates

LPT

In Lagrangian dynamics we follow the particles in its movement. Let $\mathbf{q}$ be the initial position of a particle and $\Psi(q)$ its displacement.

$$x(\tau) = \mathbf{q} + \Psi(\mathbf{q}, \tau)$$

Remember the equation for the acceleration of the peculiar velocities?

$$\frac{d}{d\tau}(aum) = -am\nabla\Phi$$

This implies that

$$\frac{d^2\psi}{d\tau^2} + \mathcal{H}\frac{d\psi}{d\tau} = -\nabla\Phi$$

Where ∇ is a gradient with respect to the Eulerian coordinates

LPT

A differential fraction of mass in different coordinates

$$dM = \bar{\rho}(1+\delta(\mathbf{x}))d^3x = \bar{\rho}d^3q$$

The coordinate transformation then

$$1+\delta(\mathbf{x}) = \frac{1}{\mathrm{Det}\Big(\delta_{ij}+\Psi_{i,j}\Big)} \equiv \frac{1}{J(\mathbf{q},\tau)},$$

Alternatively we can rewrite the relation between as

$$1+\delta(\mathbf{x}) = \int \delta_D(\mathbf{x}-\mathbf{q}-\Psi(\mathbf{q},\tau))d^3q = \int d^3q \frac{d^3k'}{(2\pi)^3} e^{i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{q}-\Psi(\mathbf{q},\tau))}$$

$$\delta(\mathbf{k}) = \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}}(e^{-i\mathbf{k}\cdot\Psi(\mathbf{q},\tau)}-1)$$

Solving order by order:

$$\Psi^{(n)i}(\mathbf{k},\tau) = i\frac{D_+^n}{n!}\int_{\mathbf{k}} L^{(n)i}(\mathbf{k}_1,\ldots,\mathbf{k}_n)\delta_0^{(1)}(\mathbf{k}_1)\cdots\delta_0^{(1)}(\mathbf{k}_n),$$

The first three terms are [Bouchet et al. A&A 296, 575 (1995)]

$$L^{(1)}(\mathbf{k}) = \frac{\mathbf{k}}{k^2}, \qquad L^{(2)}(\mathbf{k}_1,\mathbf{k}_2) = \frac{3}{7}\frac{\mathbf{k}}{k^2}\left[1-\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2\right],$$

$$\begin{aligned} L^{(3)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = {}& \frac{5}{7}\frac{\mathbf{k}}{k^2}\left[1-\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2\right]\left[1-\left(\frac{(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_3}{|\mathbf{k}_1+\mathbf{k}_2|k_3}\right)^2\right] \\ & -\frac{1}{3}\frac{\mathbf{k}}{k^2}\left[1-3\left(\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2}\right)^2+2\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{k}_1)}{k_1^2k_2^2k_3^2}\right], \end{aligned}$$

- Recurrence relation were unknown until recently [Matsubara, arXiv:1505.01481]

The matter power spectrum expressed in terms of displacement fields is ([Taylor & Hamilton, MNRAS 282, 767 (1996)]),

$$P_{\rm LPT}(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}} (\langle e^{i\mathbf{k}\cdot\Delta}\rangle - 1).$$

where $\mathbf{q} = \mathbf{q}_2 - \mathbf{q}_1$ is the Lagrangian coordinates separation and

$$\Delta(\mathbf{q}) \equiv \Psi(\mathbf{q}_2, \tau) - \Psi(\mathbf{q}_1, \tau).$$

Now, we can use the cumulant expansion theorem,

$$\langle e^{iX}\rangle = \exp\left(\sum_{N=1}^{\infty} \frac{i^N}{N!}\langle X^N\rangle_c\right),$$

to rewrite the PS as

$$(2\pi)^3\delta_D(\mathbf{k}) + P^{\rm LPT}(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}} \exp\left[-\frac{1}{2}k_i k_j \langle \Delta^i \Delta^j\rangle_c - \frac{i}{6} k_i k_j k_k \langle \Delta^i \Delta^j \Delta^k\rangle_c + \cdots\right]$$

Different expansions of the exponential lead to different (resummation) schemes

- iPT (Matsubara formalism): Keeps in the exponential terms evaluated at $\mathbf{q} = 0$
- CLPT: Expands terms that goes to zero as $q \to \infty$
- ...see Vlah, Seljak, and Baldauf [arXiv:1410.1617]

Redshift space distortions

Remember that for a galaxy

$$v = \mathcal{H}x + u$$

But we don't know that, we only see a recessional velocity (through its redshift)

$$v_z = \mathcal{H}s_z$$

We locate the galaxy in the incorrect position s_z

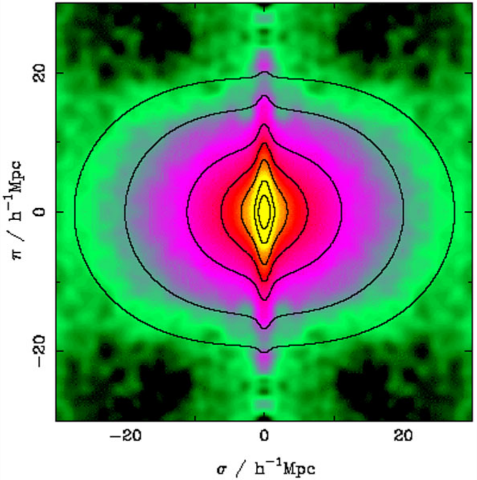
$$s = x + \frac{\hat{z} \cdot v}{\mathcal{H}} \hat{z}$$

Since $\Psi^{(n)} \propto D_+^n \implies \dot{\Psi}^{(n)} = nf\mathcal{H}\Psi^{(n)}$

$$\Psi^{s(n)} = \Psi^{(n)} + nf(\hat{z} \cdot \Psi^{(n)})\hat{z}$$

The LPT power spectrum in redshift space is

$$P_s^{\mathrm{LPT}}(\mathbf{k}) = \exp\left[-(1 + f(2+f)\mu^2)\frac{k^2}{6\pi^2}\int dp P_L(p)\right]\left[\underbrace{(1 + f\mu^2)^2 P_L(k)}_{\text{Kaiser's linear theory}} + \cdots\right]$$



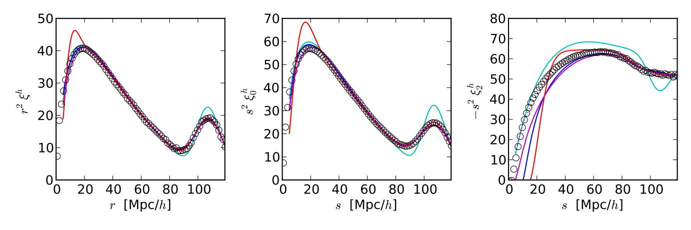
Peacock et al. 2001

Expanding in Legendre polynomials

$$\xi(r, \mu = \hat{r} \cdot \hat{z}) = \sum_l \xi_l(r) L_l(\mu)$$

Correlation function

$$\xi_s(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} P_s(\mathbf{k})$$



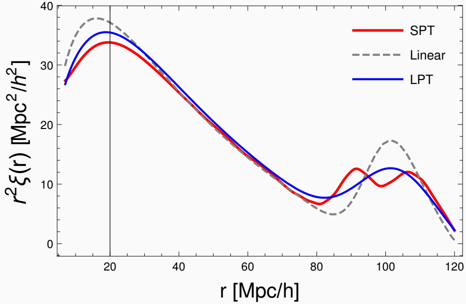
Linear(cyan), CLPT(blue), iPT(red), Zel'dovich(magenta)

White, 2014

One can insist in obtaining a correlation function from $P^{\mathrm{SPT}}_{\text{1-loop}}(k)$:

1. Cut-off the integral at some not quite large k. *e.g.* $k \simeq 100$

2. Supress the PS with a Gaussian factor $\exp(-k^2/k_{cut}^2)$



Aviles 2016

References

- Martin White, The Zel'dovich approximation (2014) Short review on LPT
- F. Bernardeau, et al., Large-scale structure of the Universe and cosmological perturbation theory (2002) Long nice review on large scale structure, a little outdated on LPT
- H. Mo, et al., Galaxy formation and evolution (2010) Pedagogical book specifically to check excursion set theory
- J. R. Bond et al., Excursion set mass functions for hierarchical Gaussian fluctuations (1999) Classic paper on excursion set, specially concerning simulations
- Michael Kopp et al., Spherical collapse and halo mass function in f(R) theories (2013)
- Jordan Carlson et al., Convolution Lagrangian perturbation theory for biased tracers (2013)
- Zvonimir Vlah et al., Lagrangian perturbation theory at one loop order: Successes, failures, and improvements (2015)

Thank you for your attention!