DS-GA 3001.008 Modeling time series data

L12. Spectral methods

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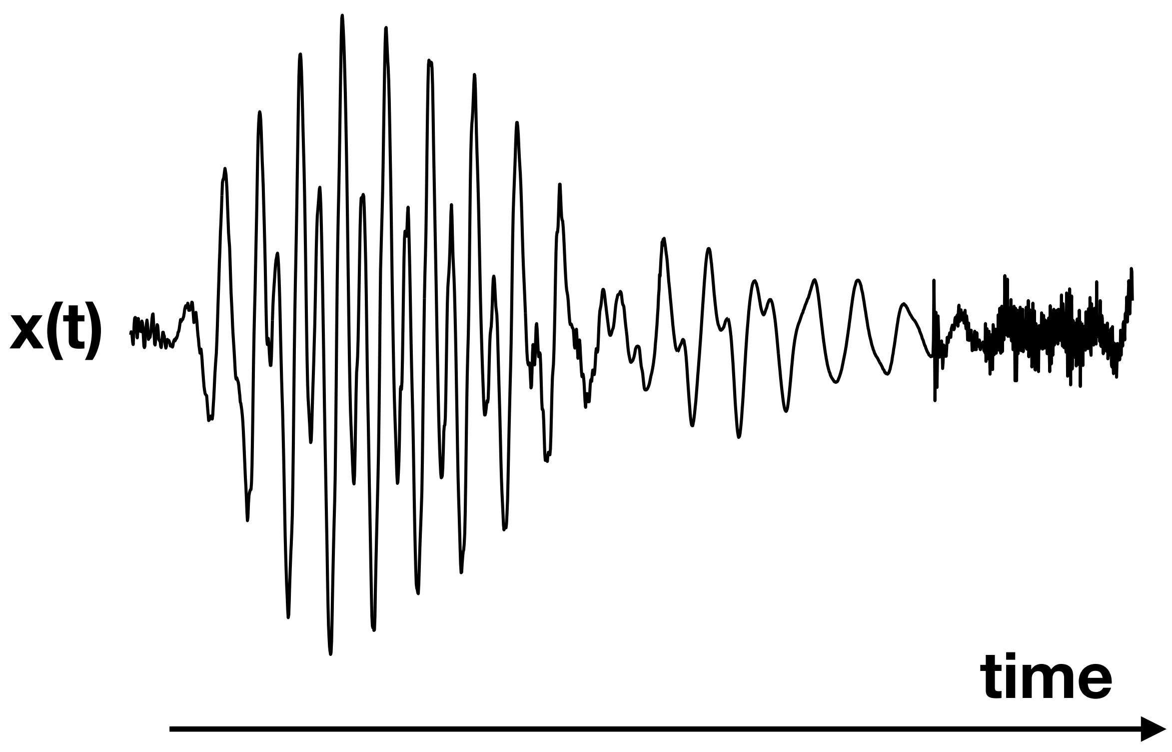
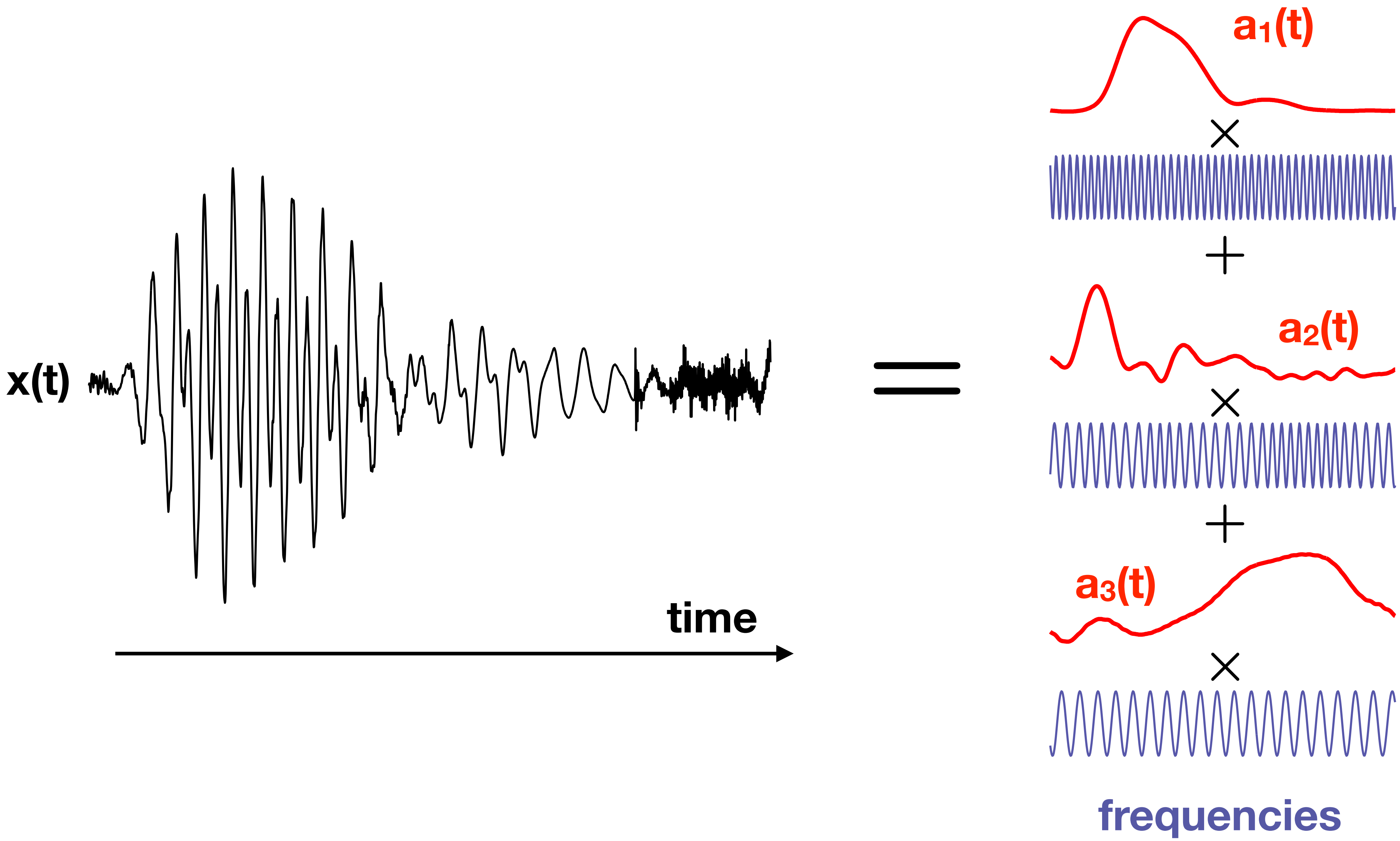
NYU, CNS & CDS

Change of representation: from time to frequencies



Change of representation: from time to frequencies



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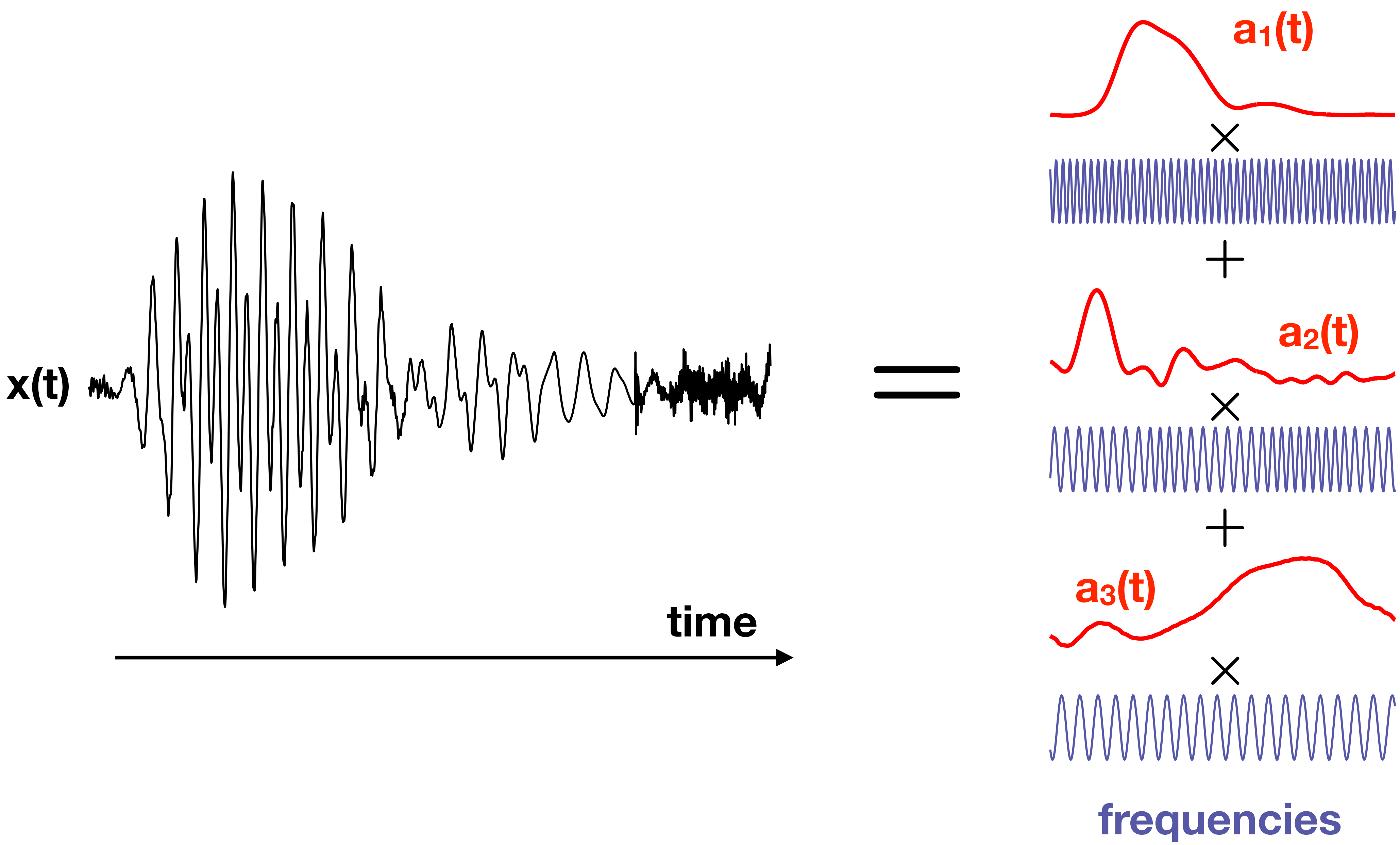
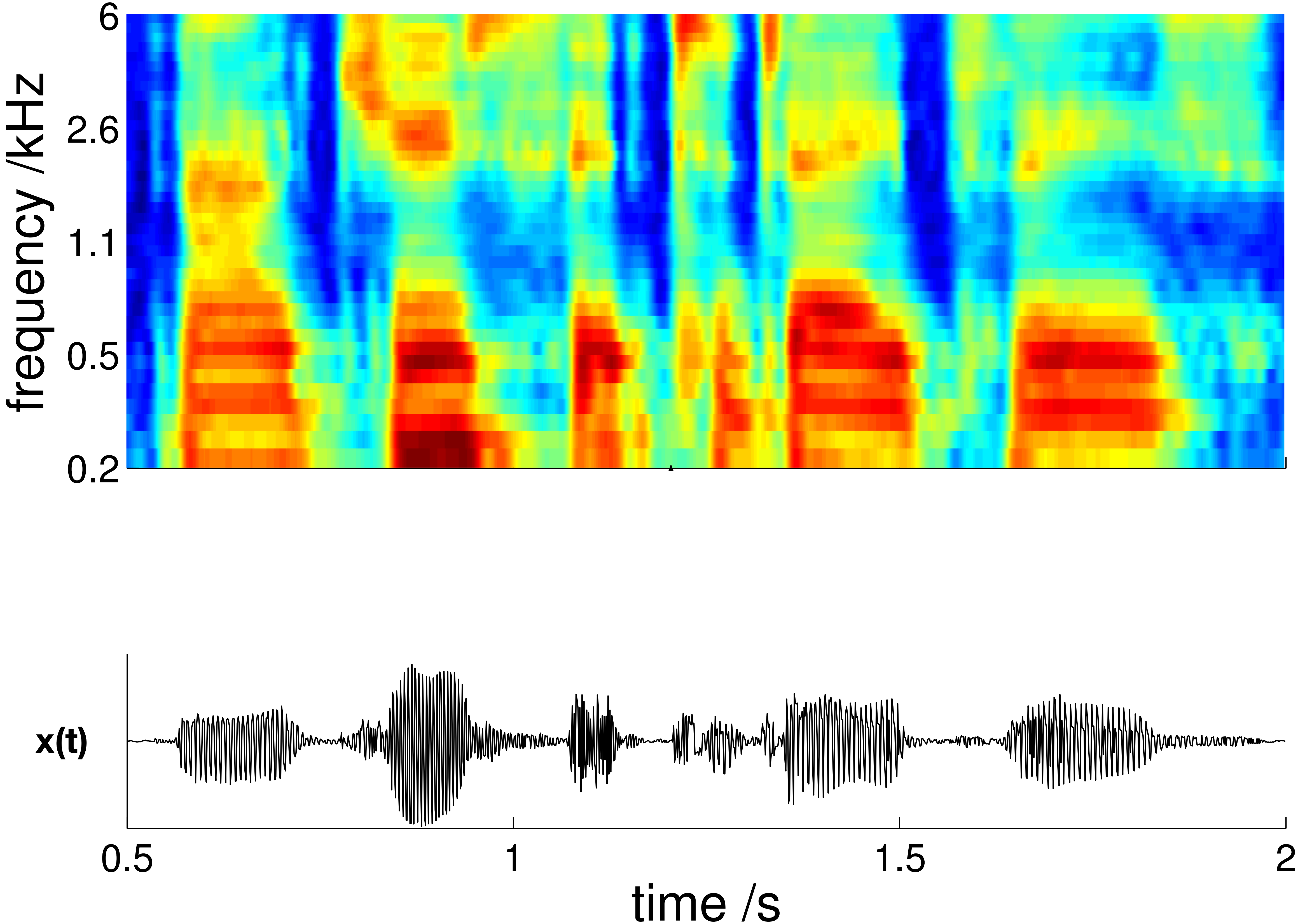


Motivation:

periodic structure is by def. predictable

if it’s there in the data then we should take advantage of it



frequency /kHz
6
2.6
1.1
0.5
0.2
x(t)
0.5
1
1.5
2
time /s

Overview:

Overview:

The traditional signal processing view: Fourier transform

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The traditional signal processing view: Fourier transform

Stats perspective (chp. 4 in tsa4.pdf)

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Probabilistic spectral analysis

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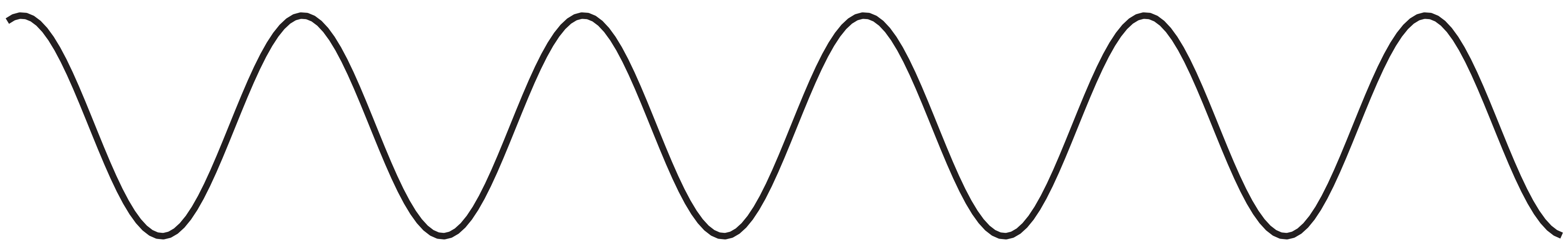
The traditional signal processing view: Fourier transform

Stats perspective (chp. 4 in tsa4.pdf)

Probabilistic spectral analysis

Modeling real-world data

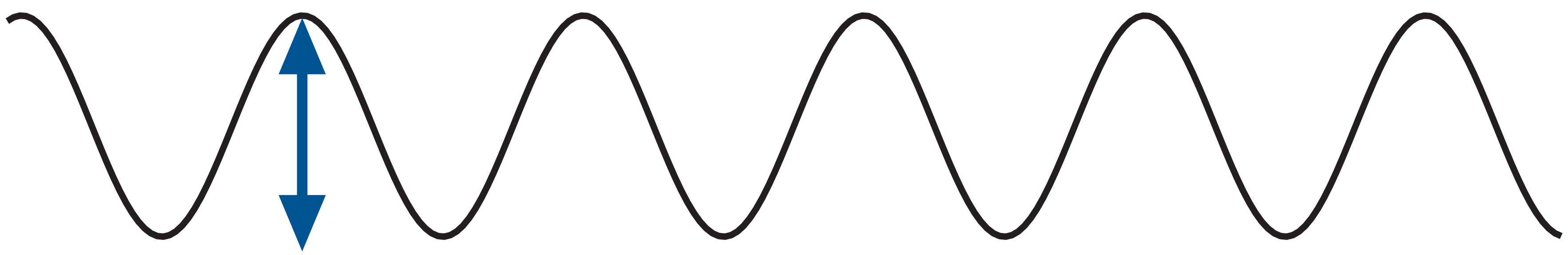
Reminder basics frequency analysis



$$x_t = A\cos(2\pi\omega t + \phi)$$

alternative forms on the board

Reminder basics frequency analysis

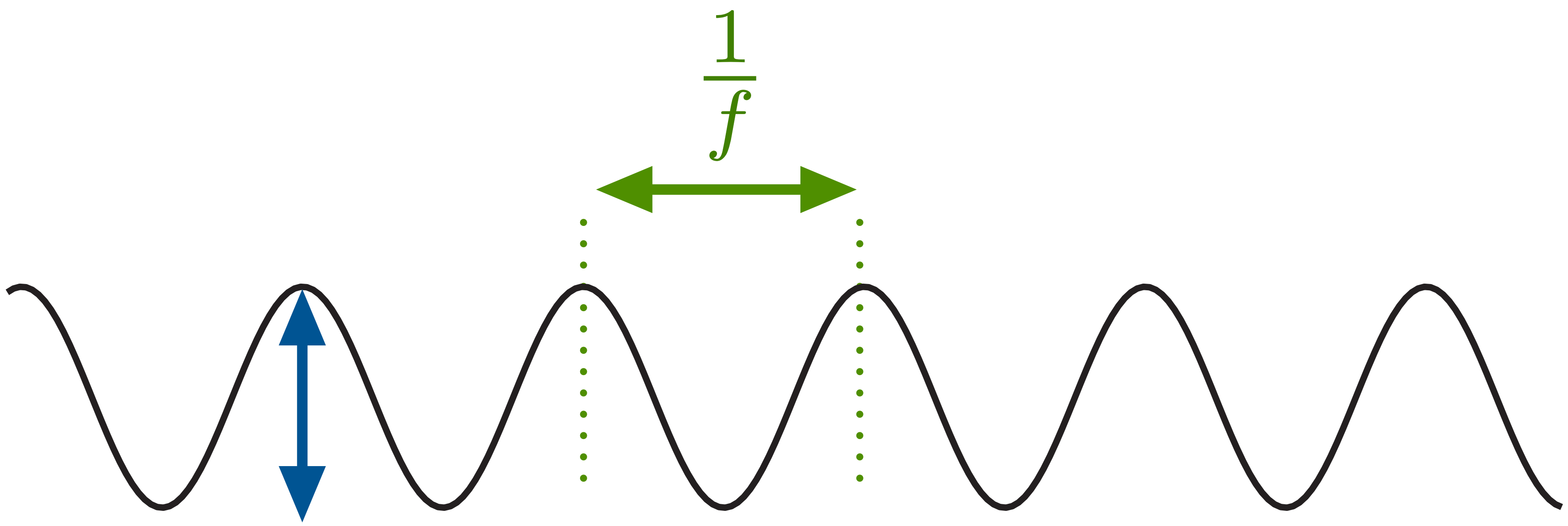


$$x_t = \boxed{A} \cos(2\pi\omega t + \phi)$$

amplitude

alternative forms on the board

Reminder basics frequency analysis



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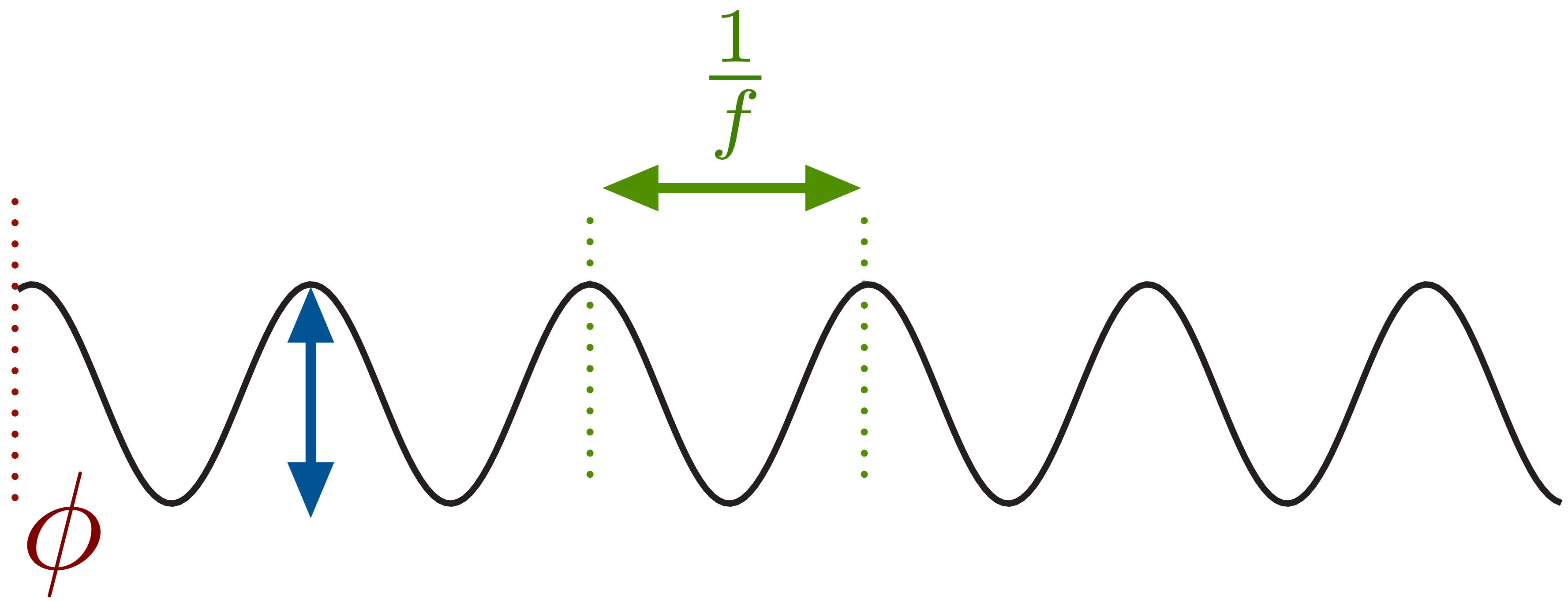
amplitude **frequency**

alternative forms on the board

Reminder basics frequency analysis

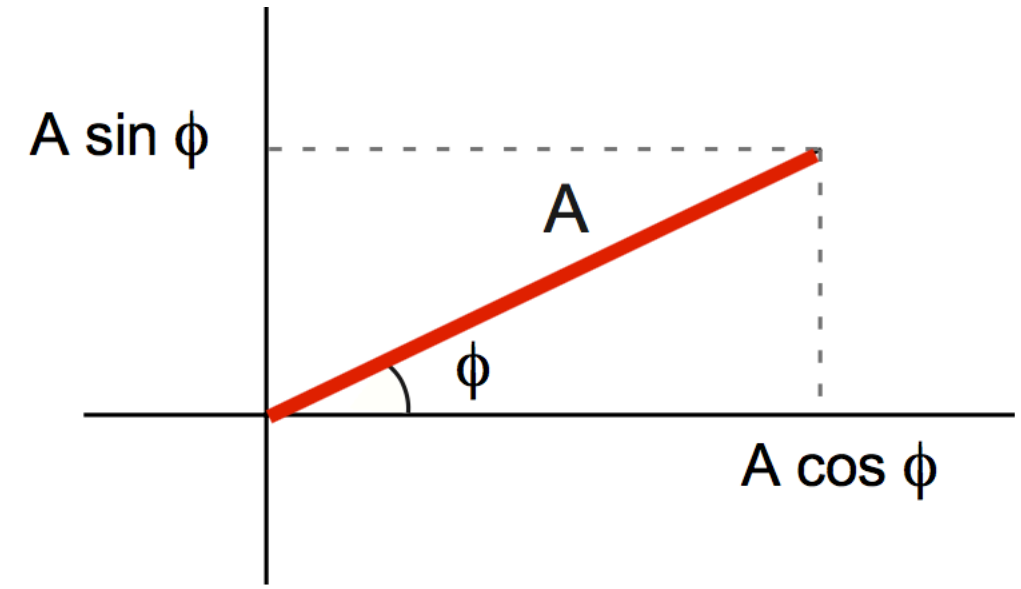


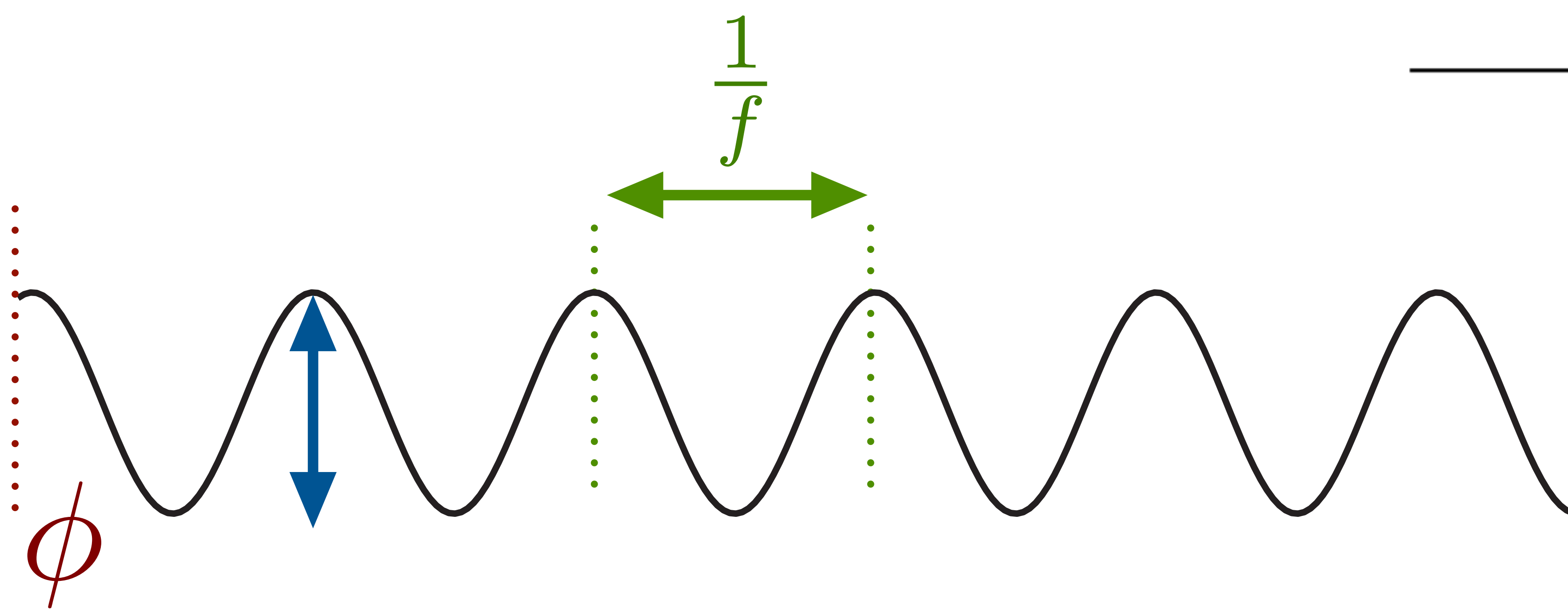
$$x_t = \boxed{A}\cos(\boxed{2\pi\omega}t + \boxed{\phi})$$

amplitude frequency phase

alternative forms on the board

Reminder basics frequency analysis





$$x_t = A\cos(2\pi\omega t + \phi)$$

amplitude frequency phase

alternative forms on the board

The Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i \omega t} dt$$

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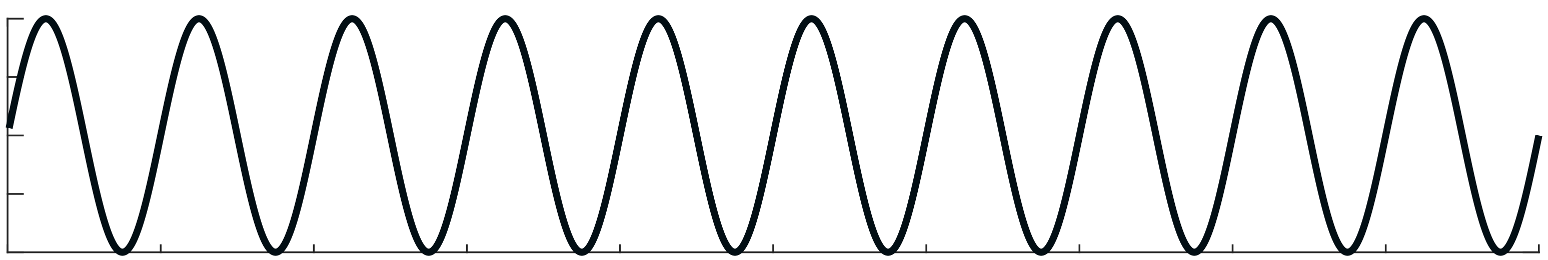
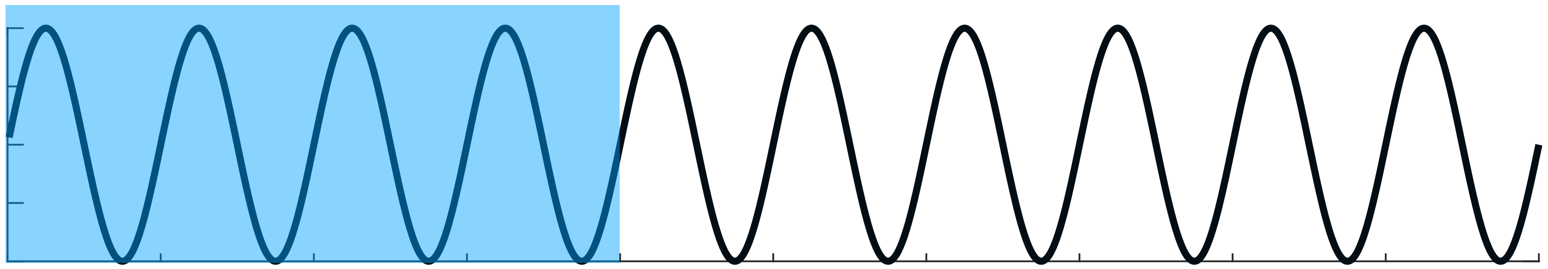


The Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i \omega t} dt$$

winding freq



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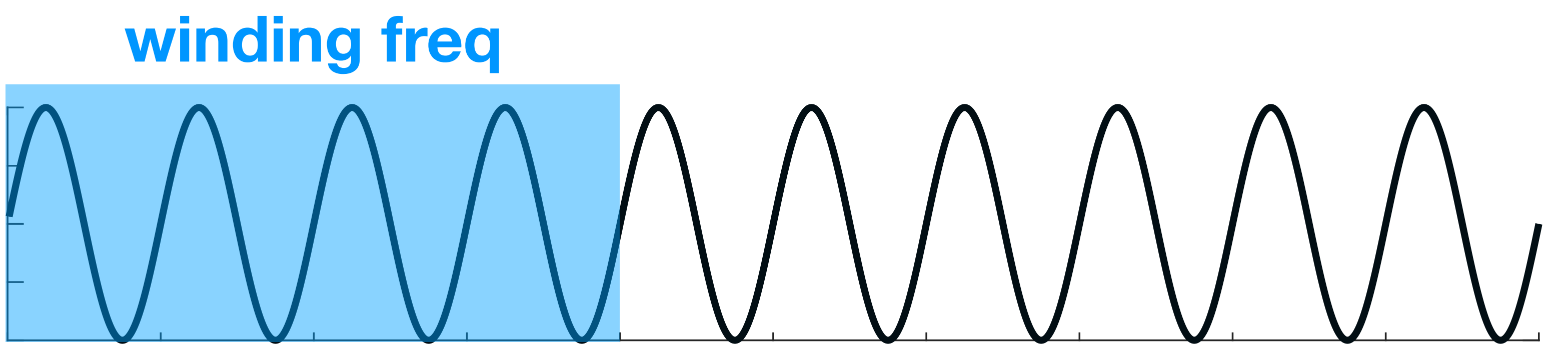

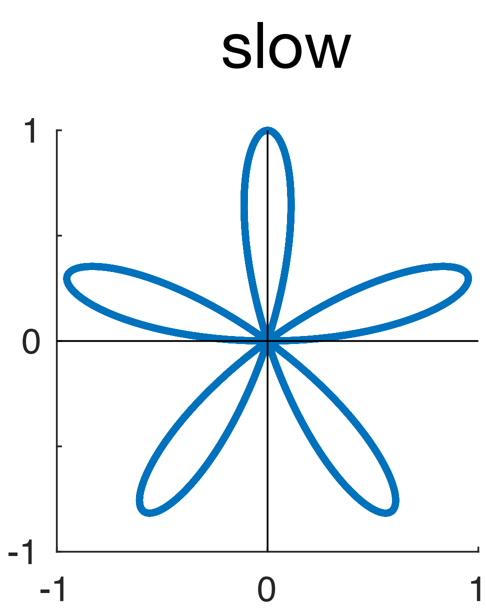

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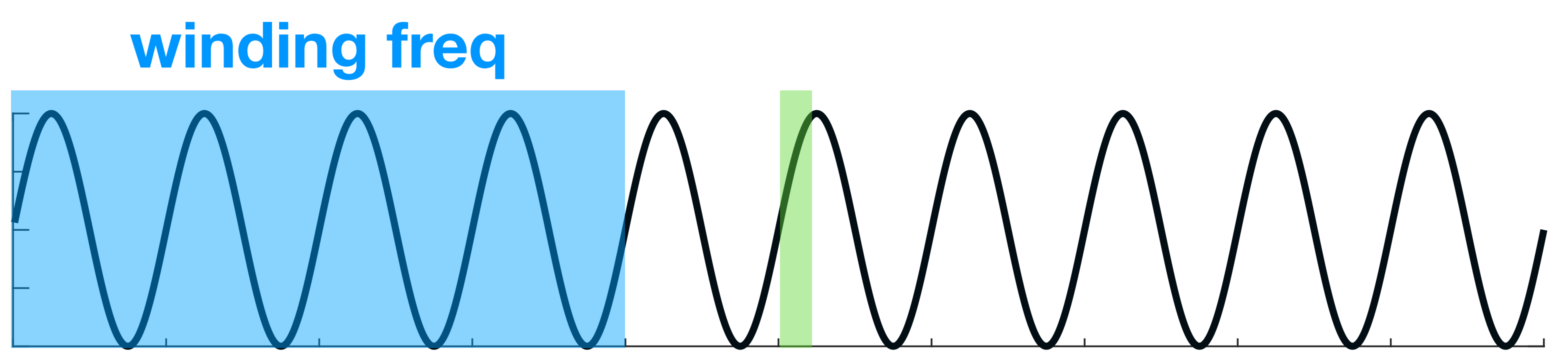

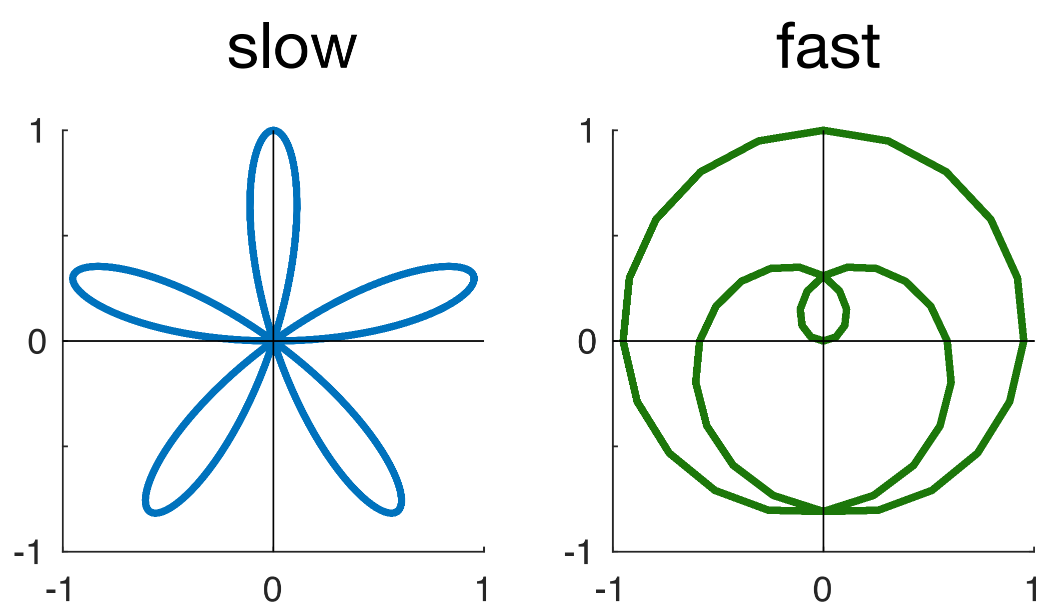

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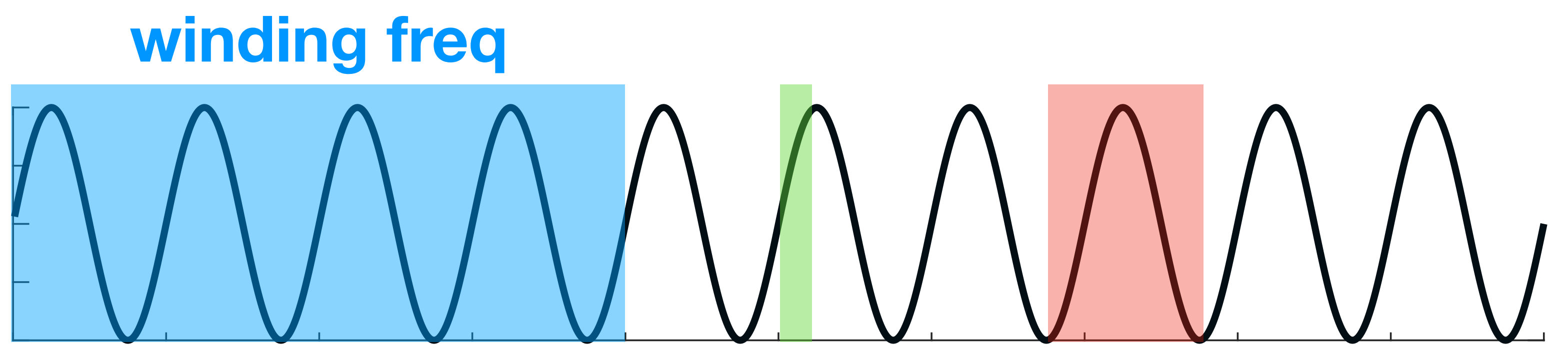

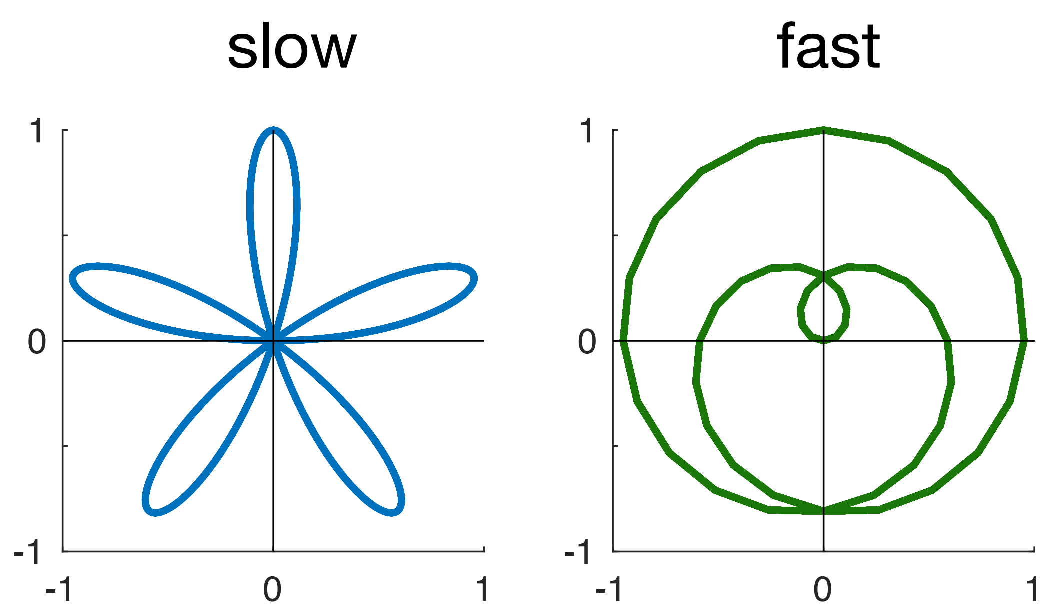

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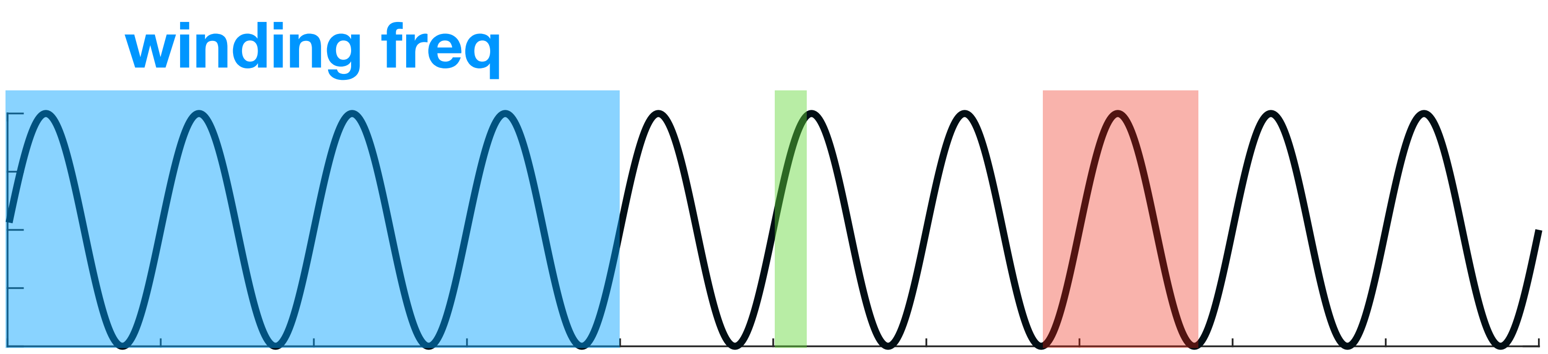

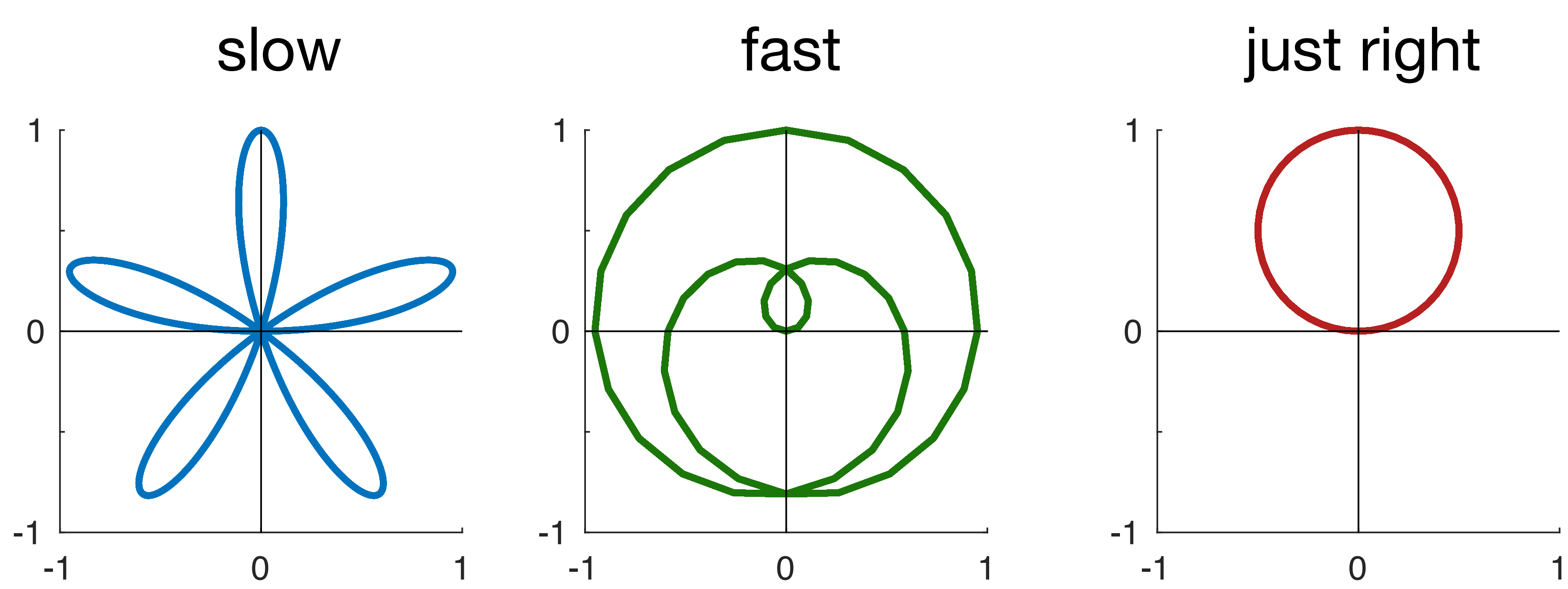

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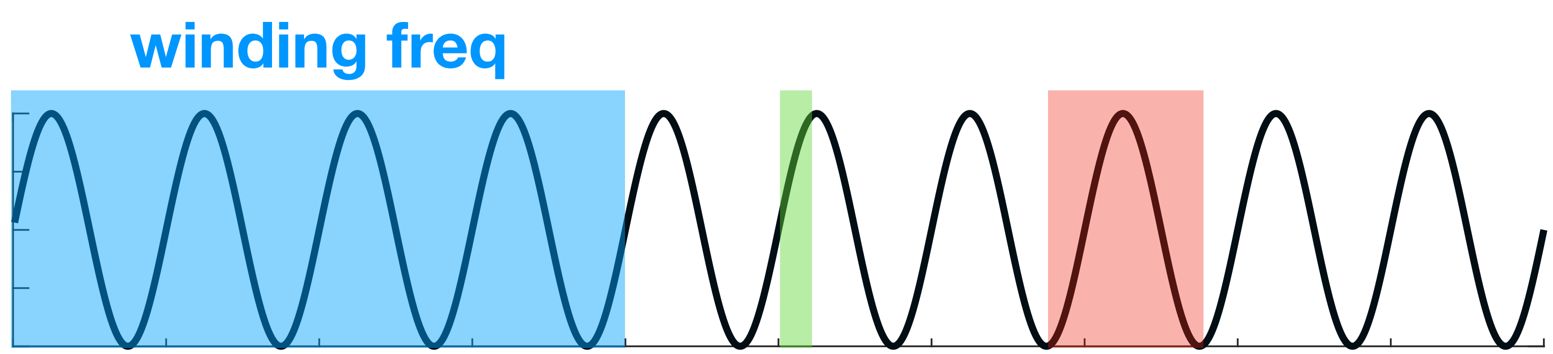
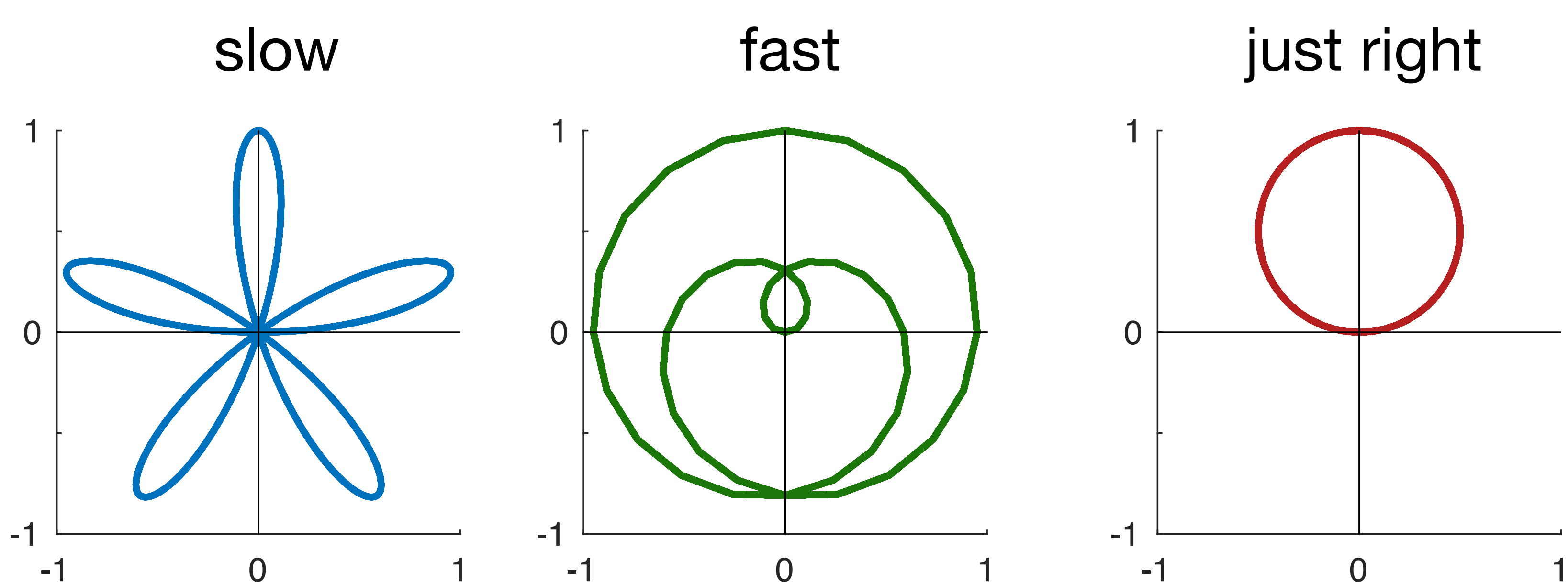
***inverse**



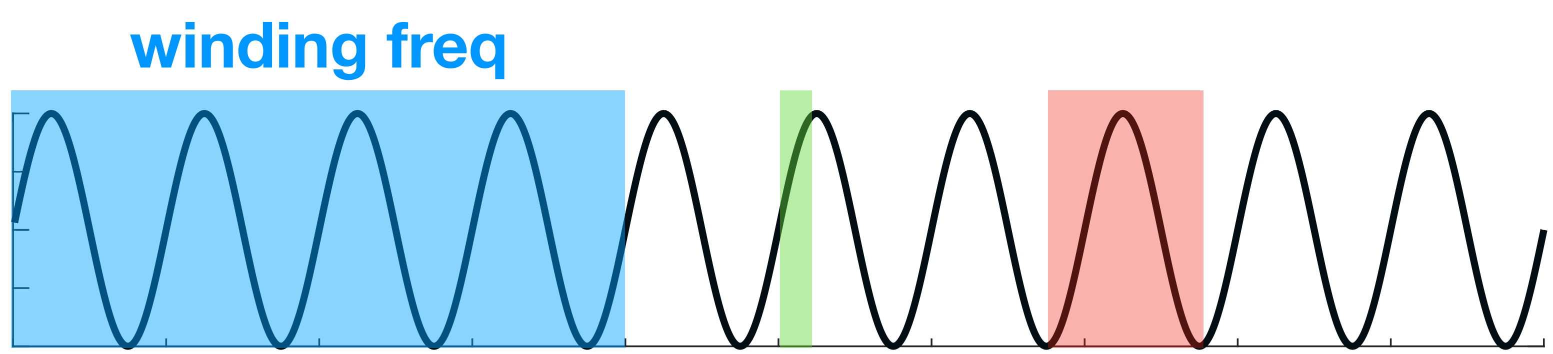


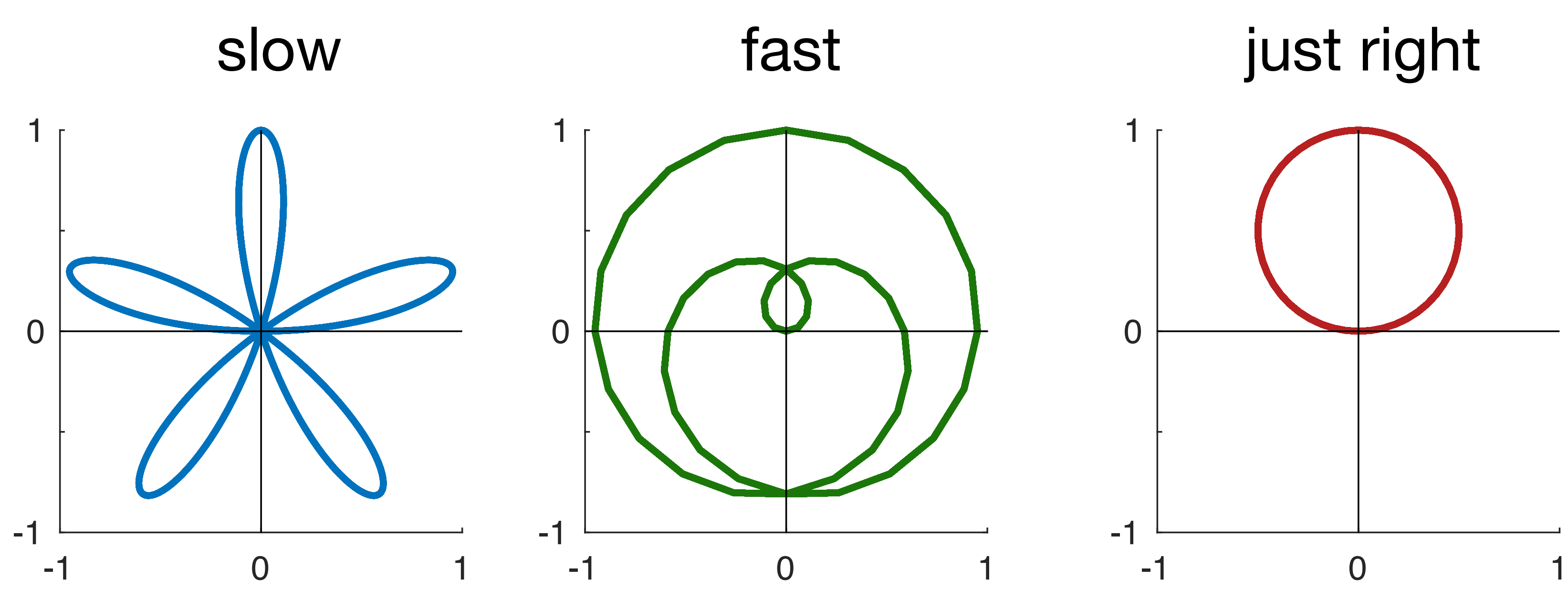
The Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i \omega t} dt$$

***inverse**

**mathematically convenient





The Fourier transform

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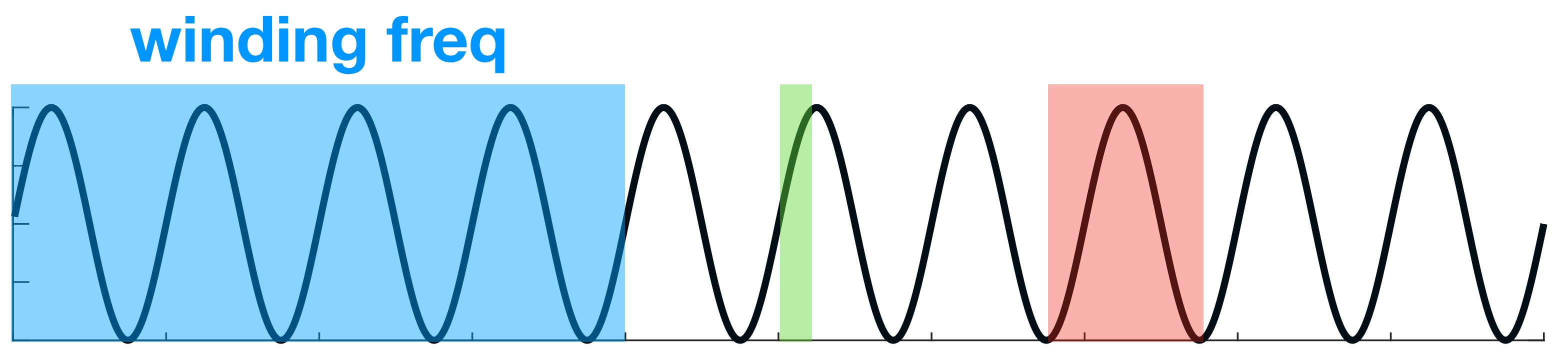
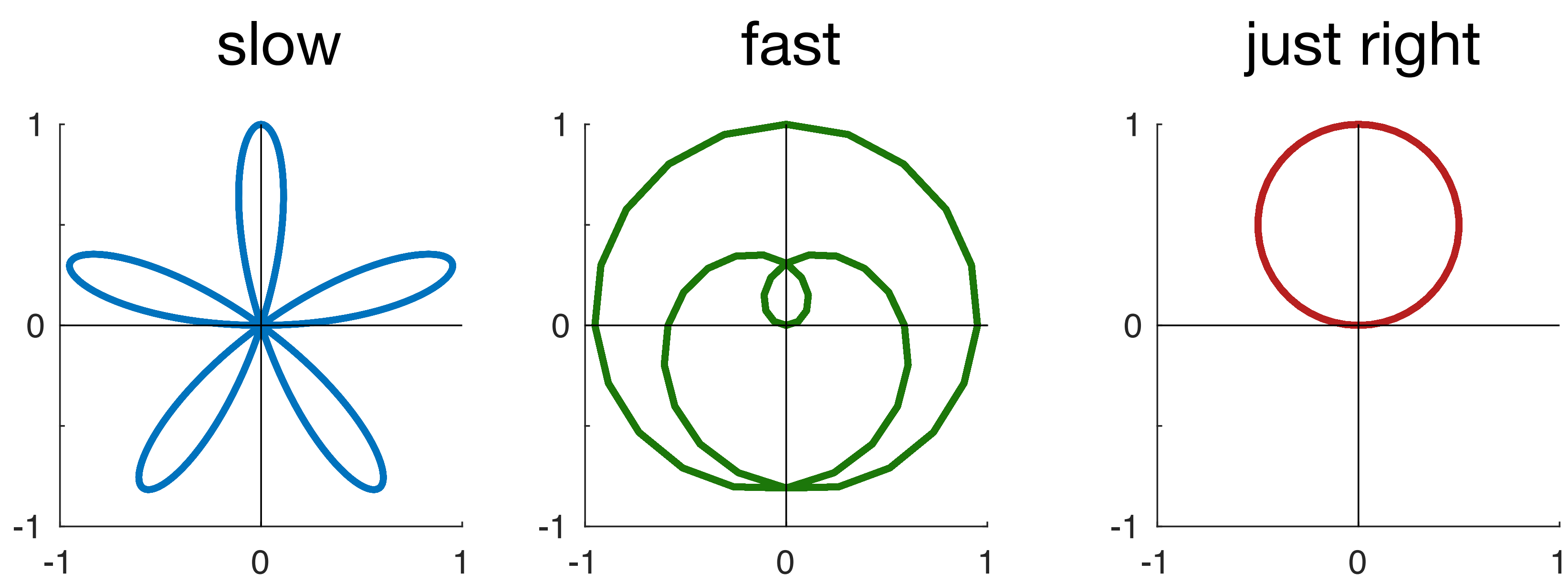
***inverse**

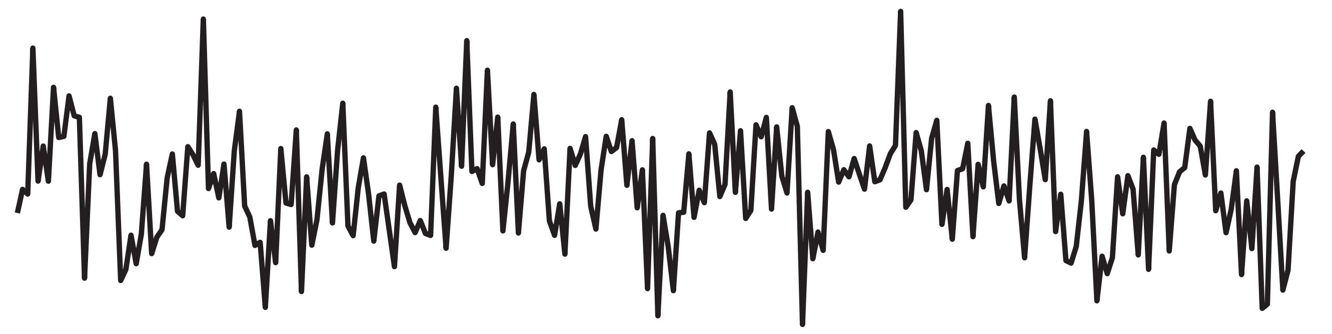
**mathematically convenient

***discrete vs continuous





Stationary time series



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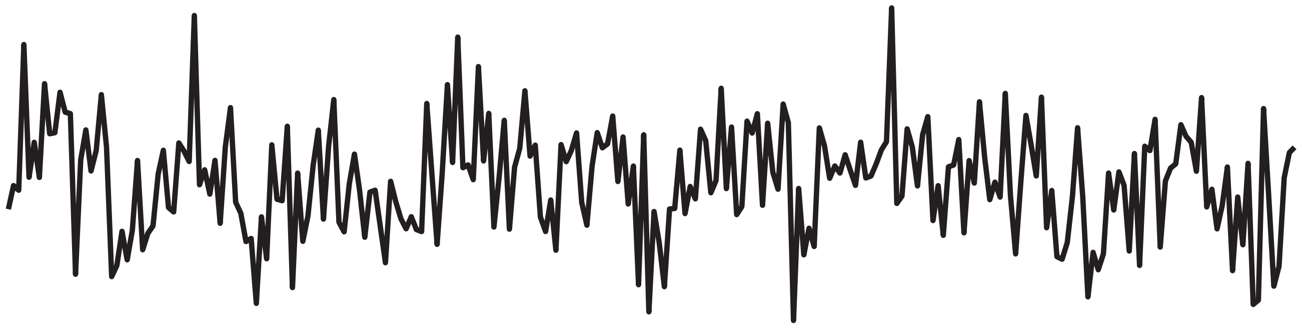
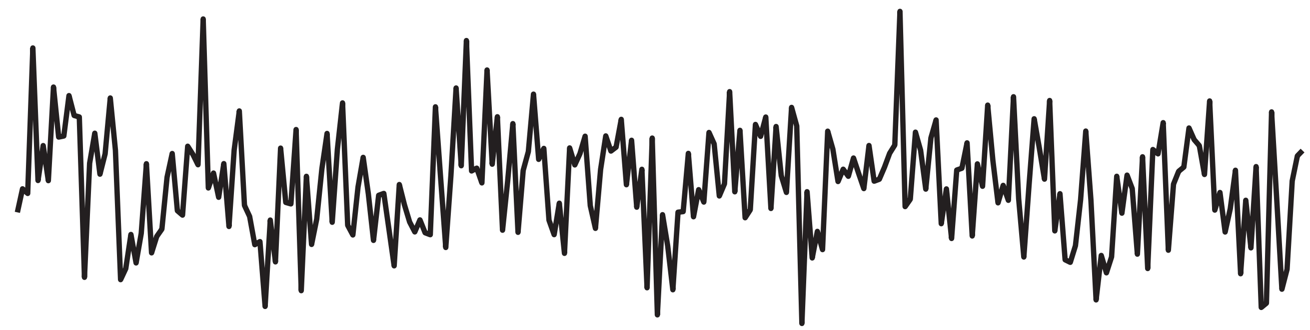


frequency domain

Stationary time series

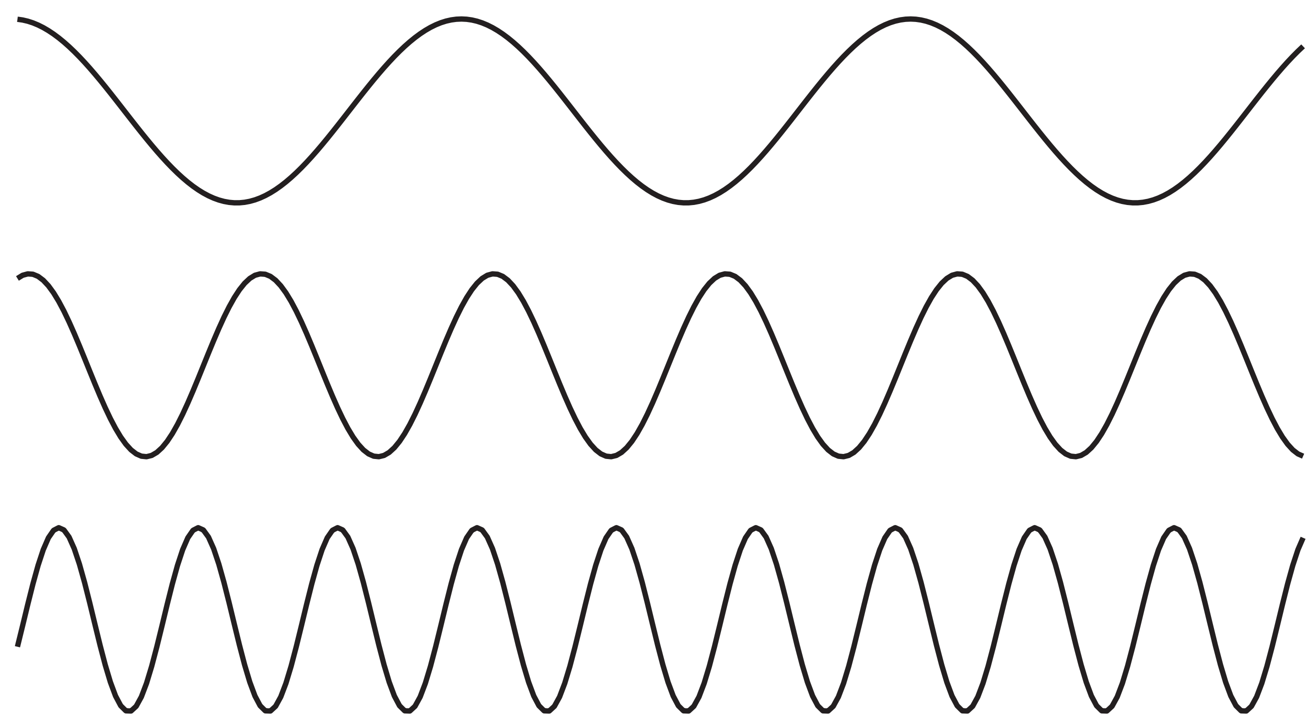


frequency domain

$a_1 \times$

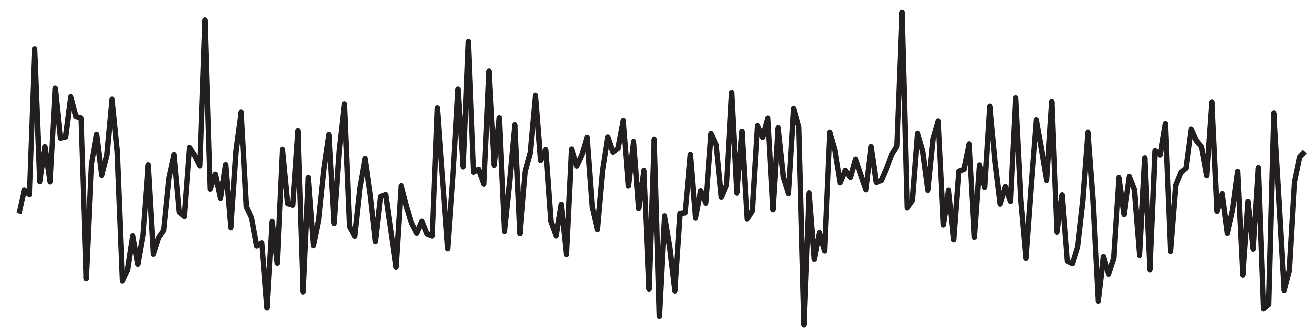
$a_2 \times$

$a_3 \times$



+ white noise

Stationary time series



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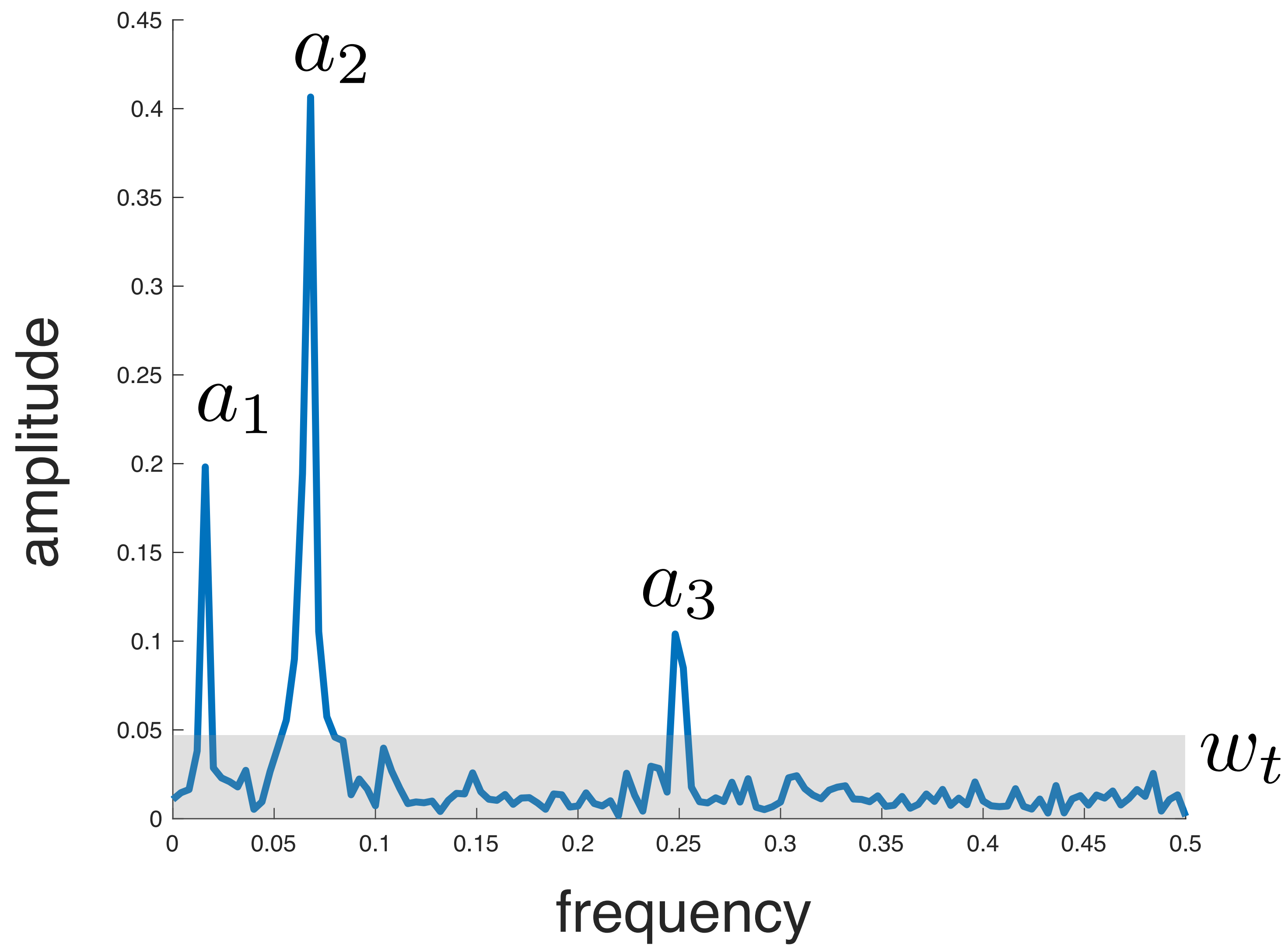

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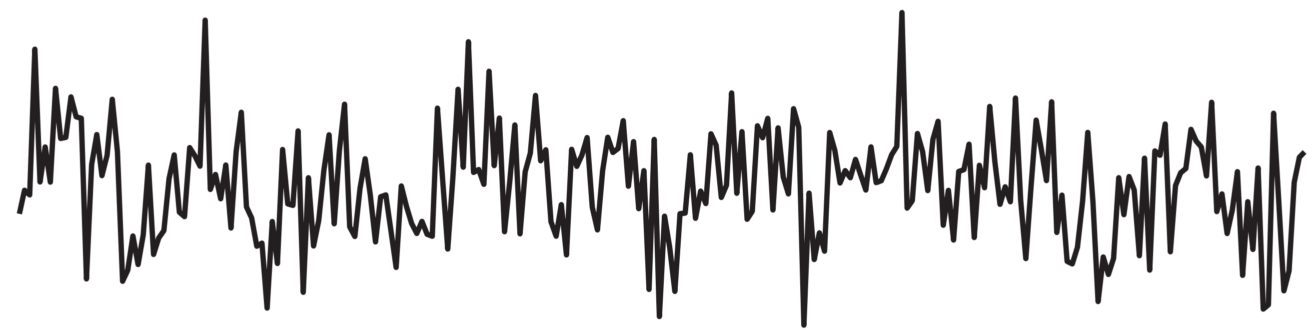
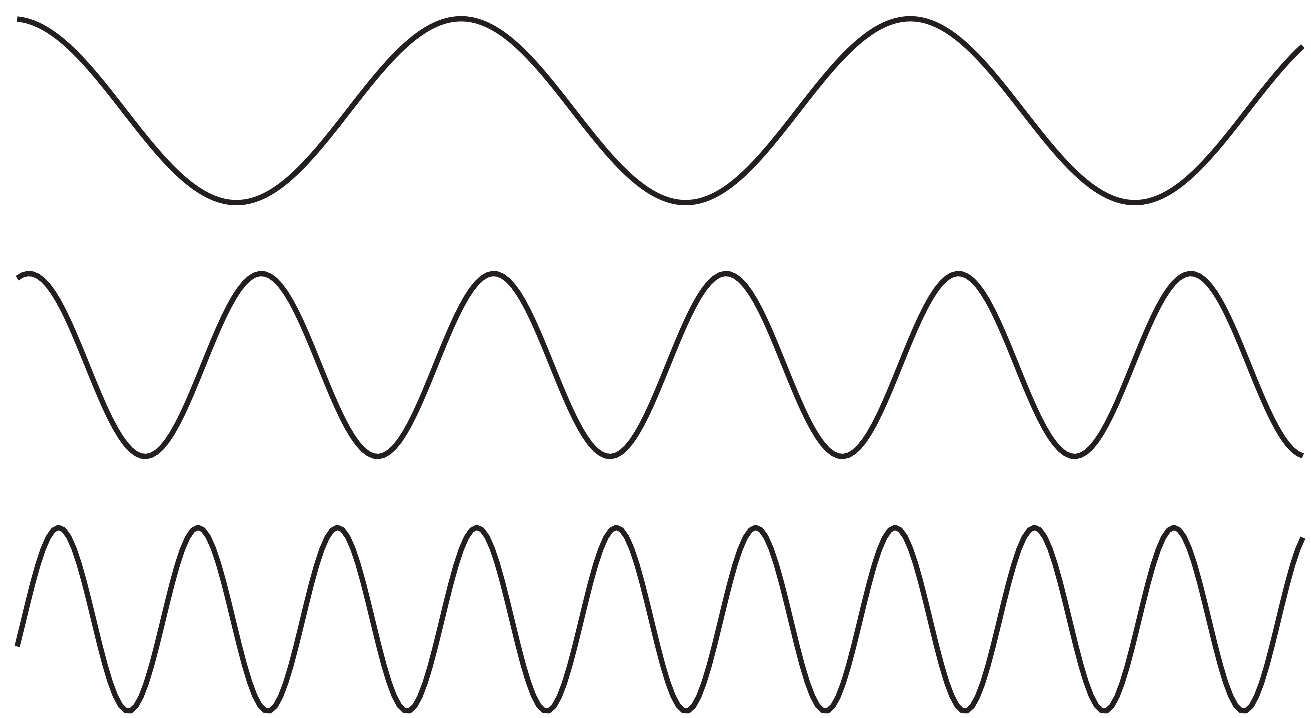


frequency domain

$a_1 \times$

$a_2 \times$

$a_3 \times$



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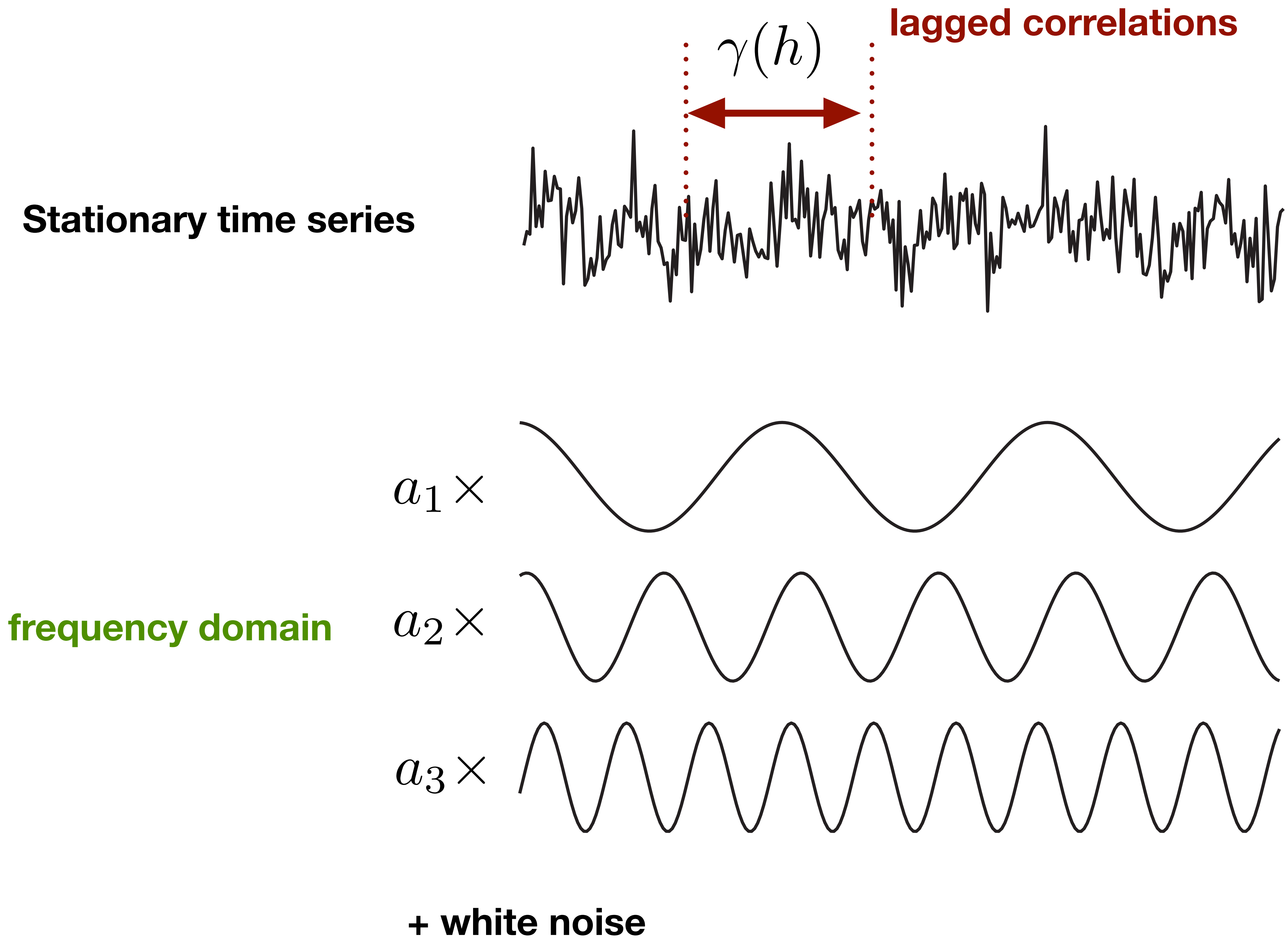
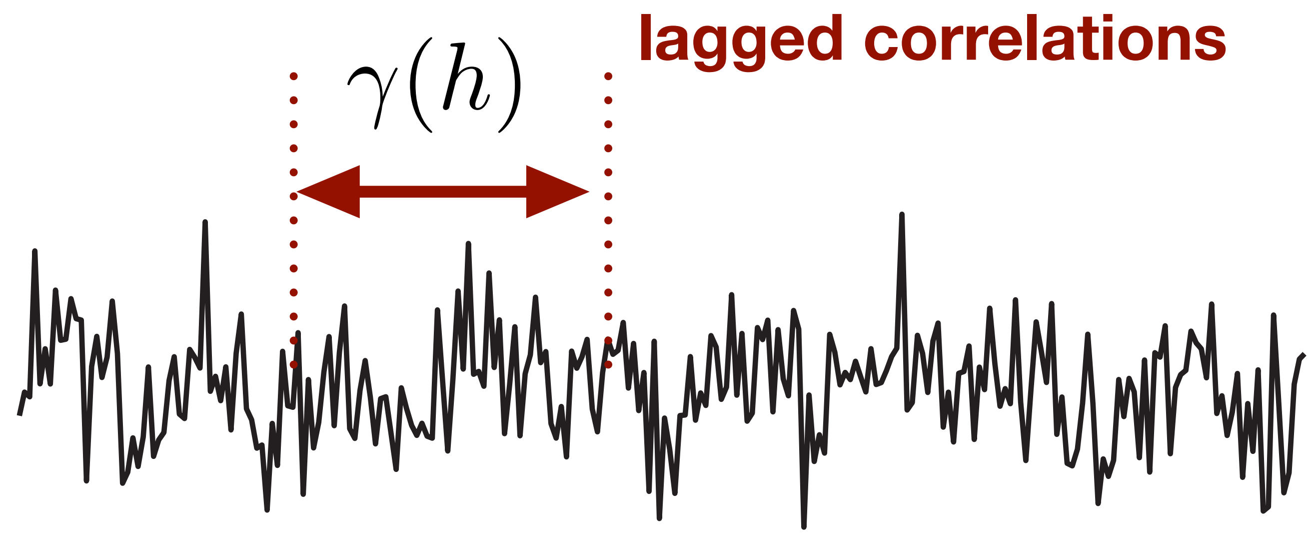

lagged correlations
$\gamma(h)$
Stationary time series
frequency domain
$a_1 \times$
$a_2 \times$
$a_3 \times$
+ white noise

Stationary time series

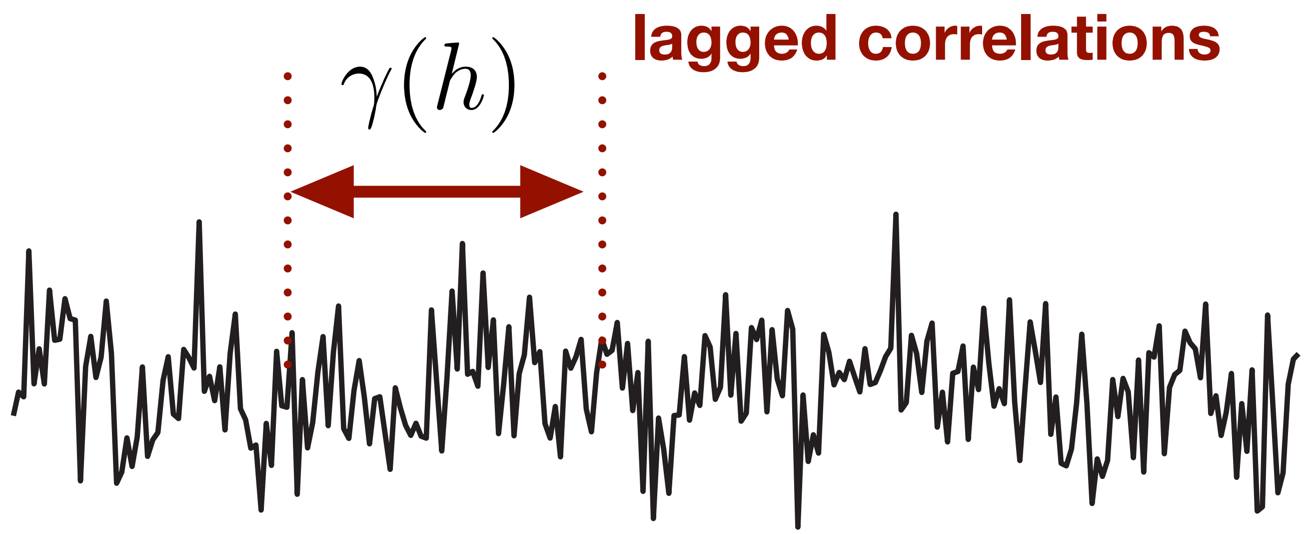


ARIMA-style

$$
\begin{aligned}
\gamma_x(h) = \operatorname{cov}(x_{t+h}, x_t) &= \operatorname{cov}(U_1 c_{t+h} + U_2 s_{t+h}, U_1 c_t + U_2 s_t) \\
&= \operatorname{cov}(U_1 c_{t+h}, U_1 c_t) + \operatorname{cov}(U_1 c_{t+h}, U_2 s_t) \\
&\quad + \operatorname{cov}(U_2 s_{t+h}, U_1 c_t) + \operatorname{cov}(U_2 s_{t+h}, U_2 s_t) \\
&= \sigma^2 c_{t+h} c_t + 0 + 0 + \sigma^2 s_{t+h} s_t = \sigma^2 \cos(2\pi\omega h),
\end{aligned}
$$

+ white noise

Stationary time series



ARIMA-style

$$x_t = \sum_{k=1}^{q} \left[U_{k1}\cos(2\pi\omega_k t) + U_{k2}\sin(2\pi\omega_k t)\right],$$

$$\gamma_x(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k h),$$

Discrete time-discrete frequencies

$$x_t = a_0 + \sum_{j=1}^{(n-1)/2} \left[a_j \cos(2\pi t\, j/n) + b_j \sin(2\pi t\, j/n) \right],$$

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Nyquist freq/ folding freq

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$$\mathcal{O}\left(n^2\right)$$

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Scaled periodogram

$$P(j/n) = a_j^2 + b_j^2$$

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Scaled periodogram

$$P(j/n) = a_j^2 + b_j^2$$

computed efficiently using **discrete Fourier transform DFT**

$$\begin{aligned} d(j/n) &= n^{-1/2} \sum_{t=1}^{n} x_t \exp(-2\pi i t j/n) \\ &= n^{-1/2} \left(\sum_{t=1}^{n} x_t \cos(2\pi t j/n) - i \sum_{t=1}^{n} x_t \sin(2\pi t j/n) \right), \end{aligned}$$

Discrete time-discrete frequencies

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$$\mathcal{O}(n \log(n))$$

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$$\mathcal{O}\left(n^2\right)$$

Scaled periodogram

$$P(j/n) = a_j^2 + b_j^2 = \frac{4}{n} |d(j/n)|^2.$$

computed efficiently using **discrete Fourier transform DFT**

$$d(j/n) = n^{-1/2} \sum_{t=1}^{n} x_t \exp(-2\pi i t j/n)$$
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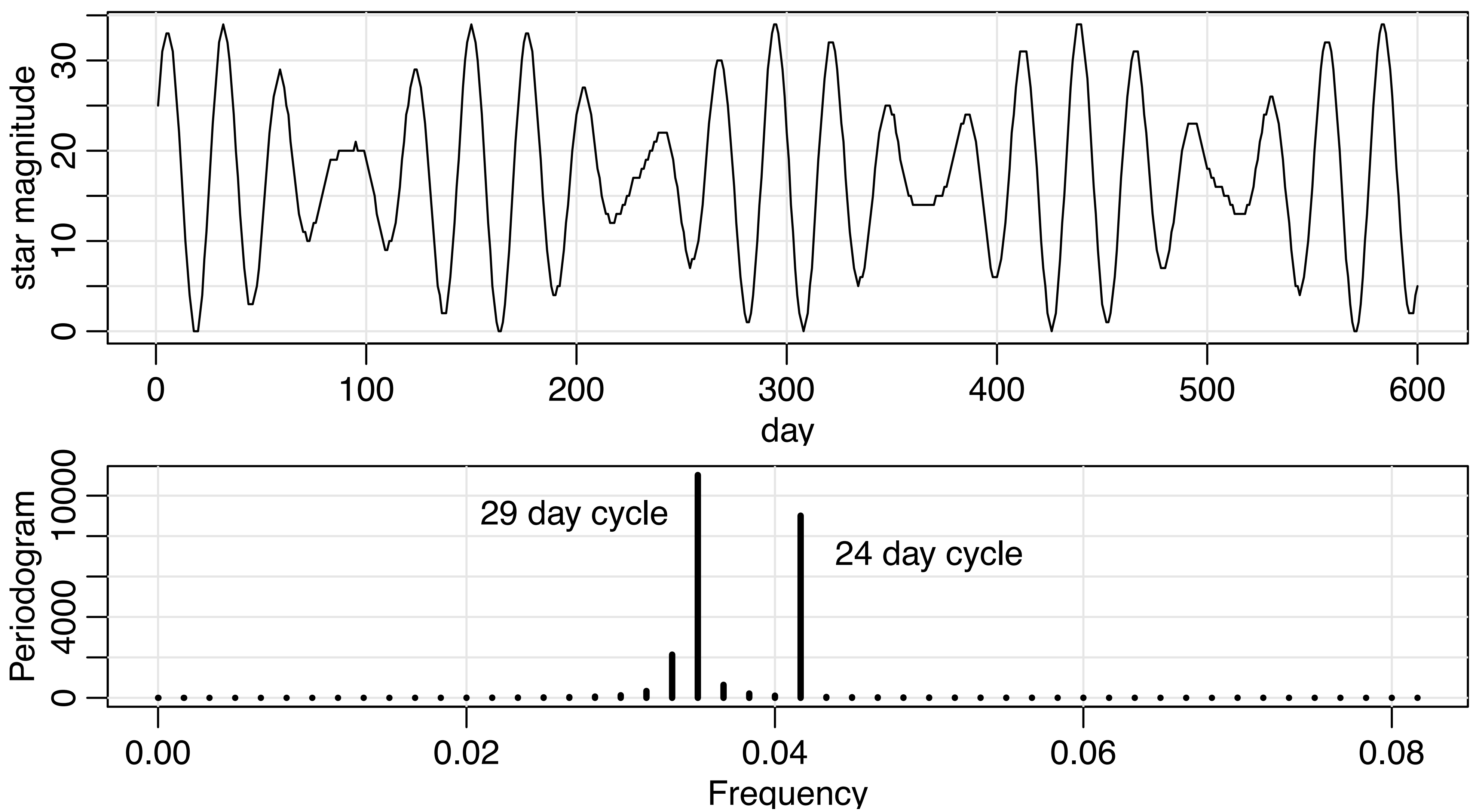
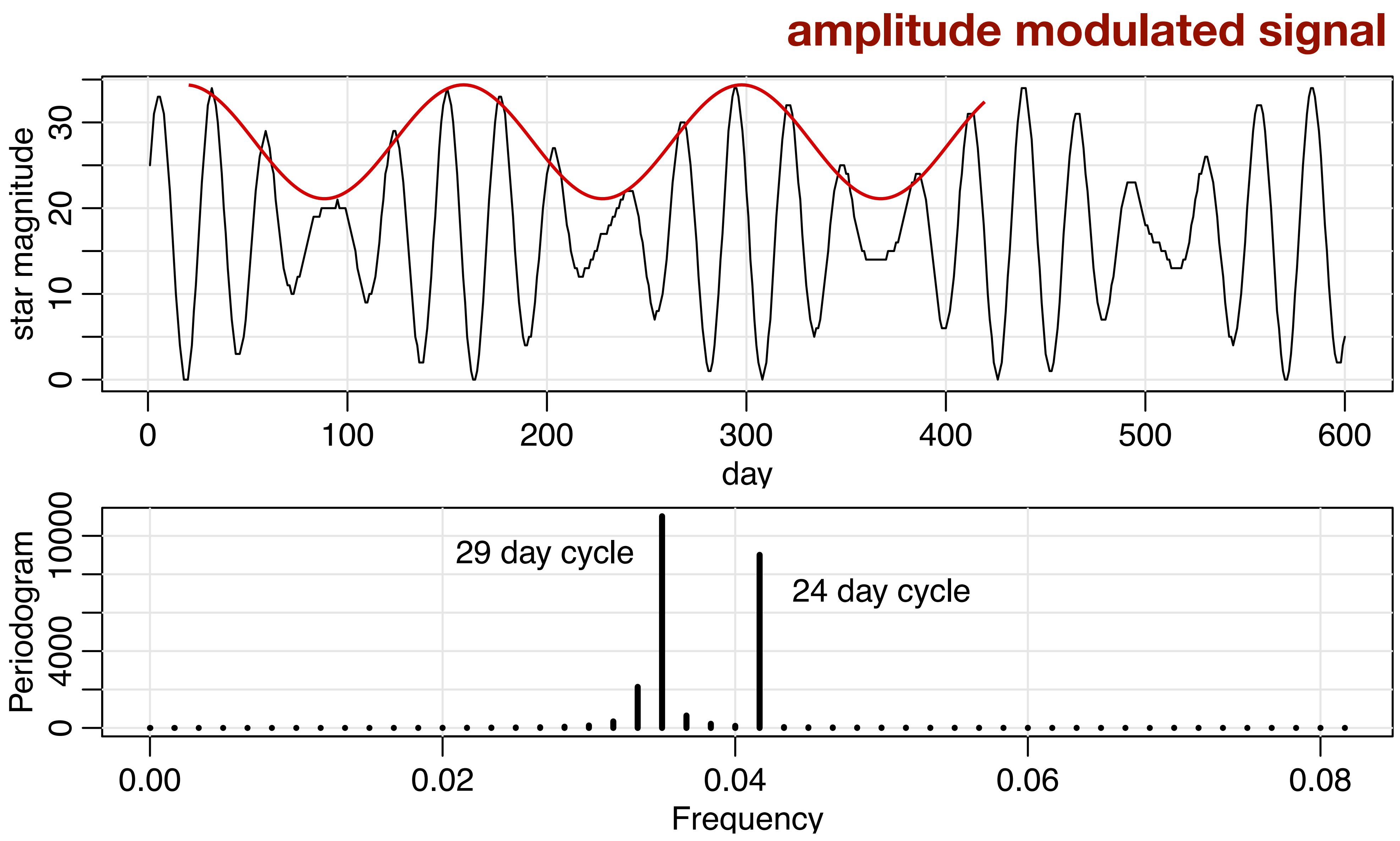


star magnitude
0
10
20
30
0
100
200
300
400
500
600
day
Periodogram
0
4000
10000
29 day cycle
24 day cycle
0.00
0.02
0.04
0.06
0.08
Frequency



amplitude modulated signal
star magnitude
0
10
20
30
0
100
200
300
400
500
600
day
Periodogram
0
4000
10000
29 day cycle
24 day cycle
0.00
0.02
0.04
0.06
0.08
Frequency

Spectral density

$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t)$$

U_1 and U_2 are uncorrelated zero-mean random variables with equal variance σ^2.

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i \omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i \omega_0 h}$$

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spectral distribution function

Spectral density

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spectral distribution function

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \leq \omega < \omega_0, \\ \sigma^2 & \omega \geq \omega_0. \end{cases}$$

Spectral density

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U_1 and U_2 are uncorrelated zero-mean random variables with equal variance σ^2.

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i \omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i \omega_0 h} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \omega h} dF(\omega)$$

spectral distribution function

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2 & \omega \ge \omega_0. \end{cases}$$

~cumulative variance

Spectral density

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~cumulative variance

always exists, unique

$$dF(\omega) = f(\omega)\, d\omega.$$

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty, \tag{4.15}$$

then it has the representation

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h}\, f(\omega)\, d\omega \quad h = 0, \pm1, \pm2, \ldots \tag{4.16}$$

as the inverse transform of the spectral density,

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h} \quad -1/2 \leq \omega \leq 1/2. \tag{4.17}$$

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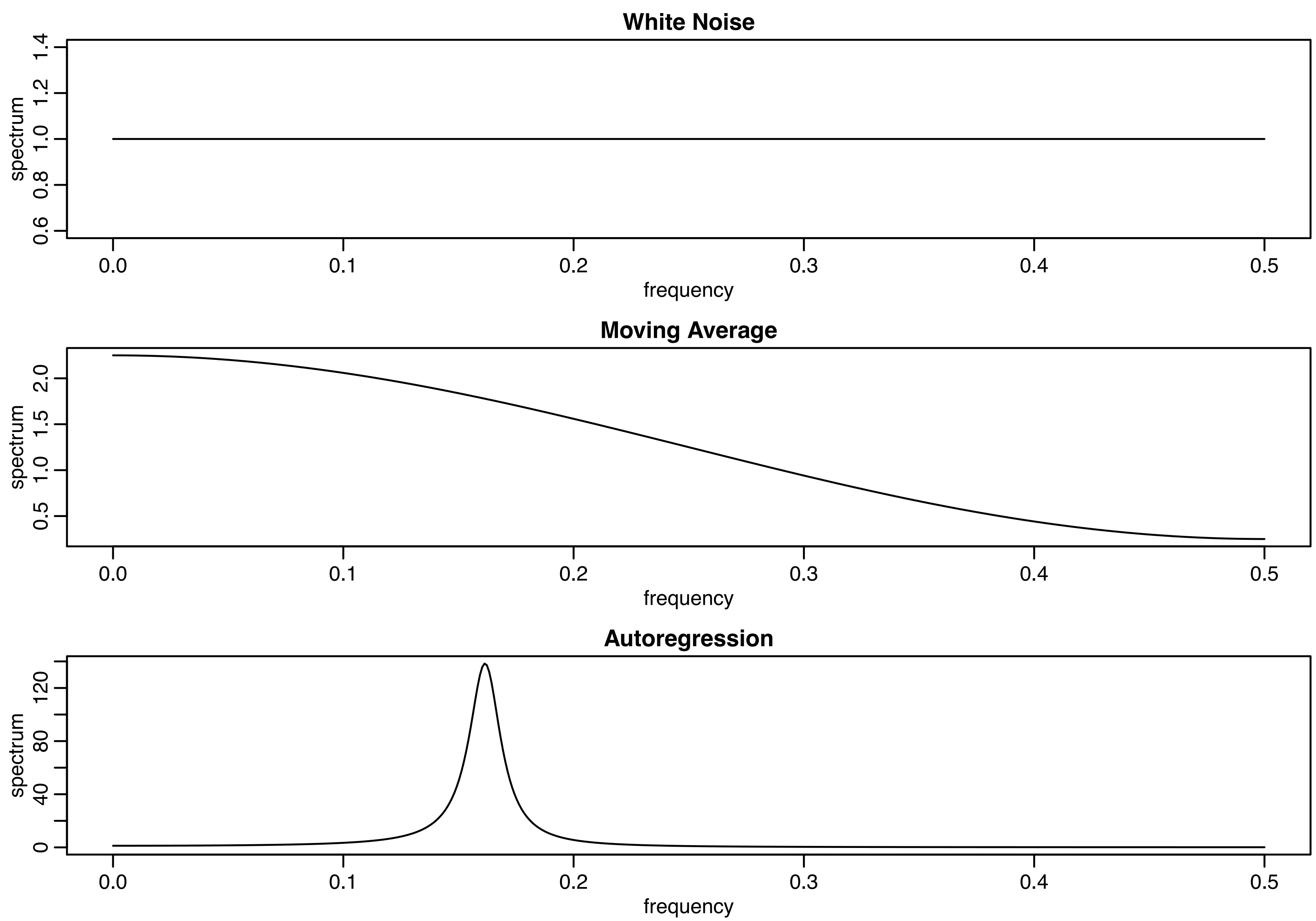
One-to-one map covariance function and f, invertible

Spectral analogue of a probability density

Example: white noise $f_w(\omega) = \sigma_w^2$ **flat spectrum**



check at home for AR(1), MA(1)

Linear combinations, filters

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

Impulse response

proof on the board

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Impulse response

Fourier Transform

frequency response

$$A(\omega) = \sum_{j=-\infty}^{\infty} a_j \, \mathrm{e}^{-2\pi i \omega j},$$

proof on the board

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Impulse response

Fourier Transform

frequency response

$$A(\omega) = \sum_{j=-\infty}^{\infty} a_j \, \mathrm{e}^{-2\pi i \omega j},$$

spectral density

$$f_y(\omega) = |A(\omega)|^2 \; f_x(\omega),$$

proof on the board

Spectral analysis as PCA

$$\mathrm{cov}(X) = \Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix}.$$

Spectral analysis as PCA

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For n sufficiently large, the eigenvalues of Γ_n are

$$\lambda_j \approx f(\omega_j) = \sum_{h=-\infty}^{\infty} \gamma(h)\, \mathrm{e}^{-2\pi i h j/n},$$

A Bayesian perspective

A Bayesian perspective

Why bayesian:

A Bayesian perspective

Why bayesian:

No explicit notion of measurement noise

A Bayesian perspective

Why bayesian:

No explicit notion of measurement noise

No way to deal with missing data

A Bayesian perspective

Why bayesian:

No explicit notion of measurement noise

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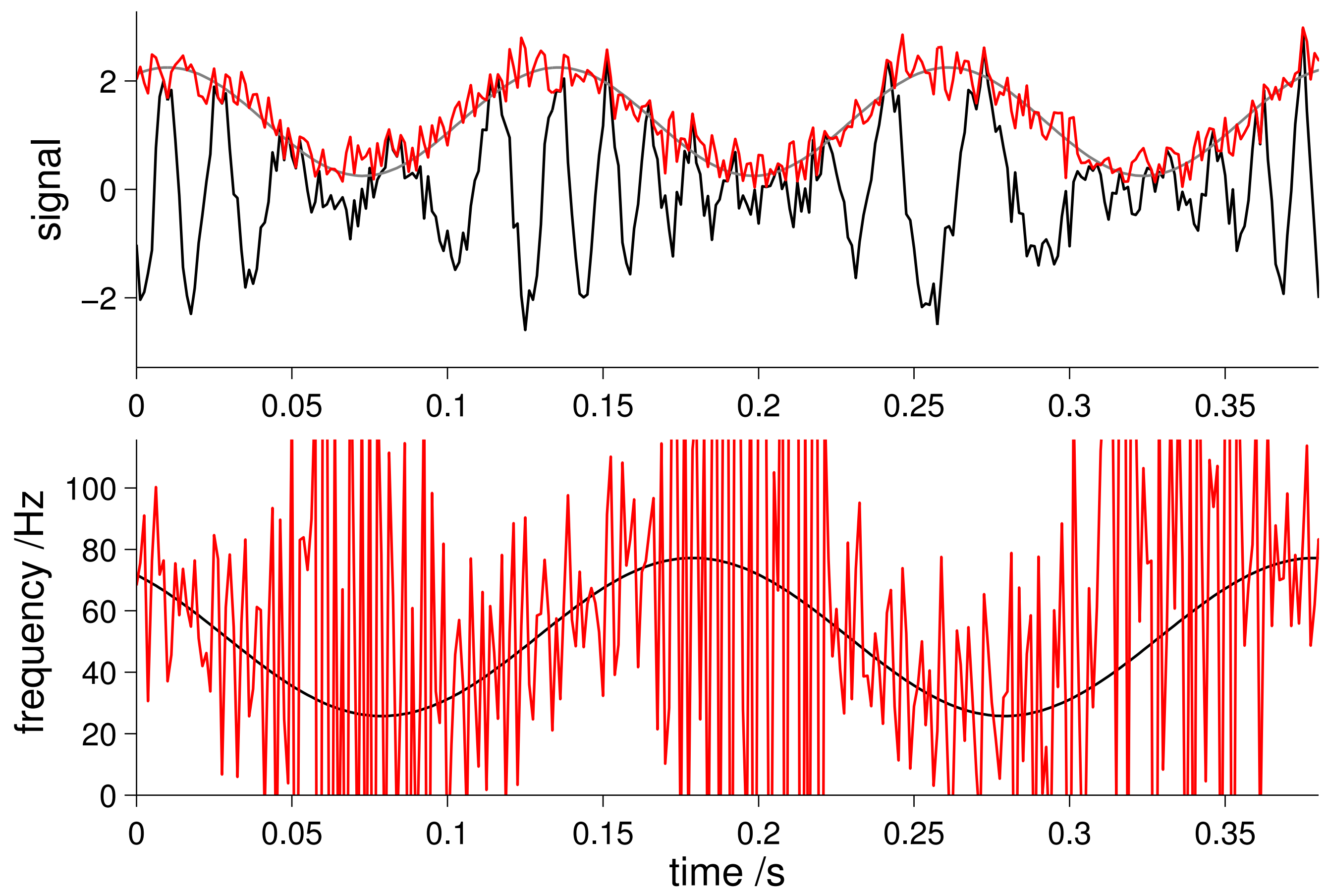
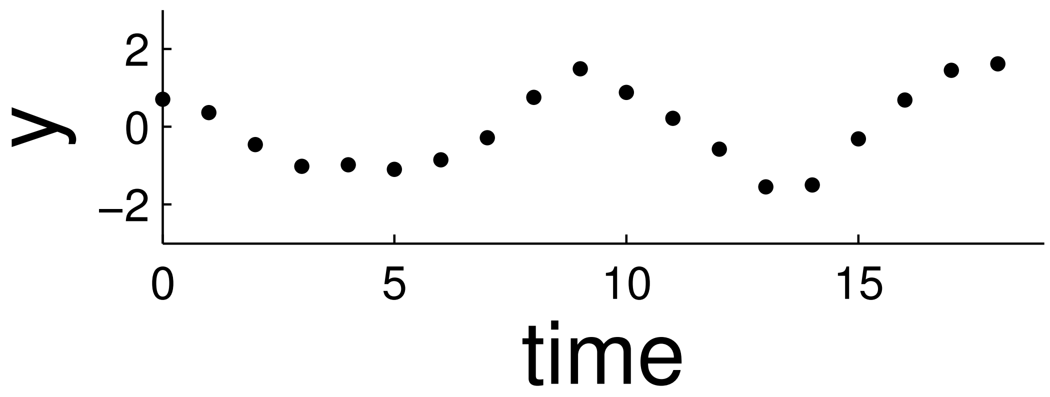


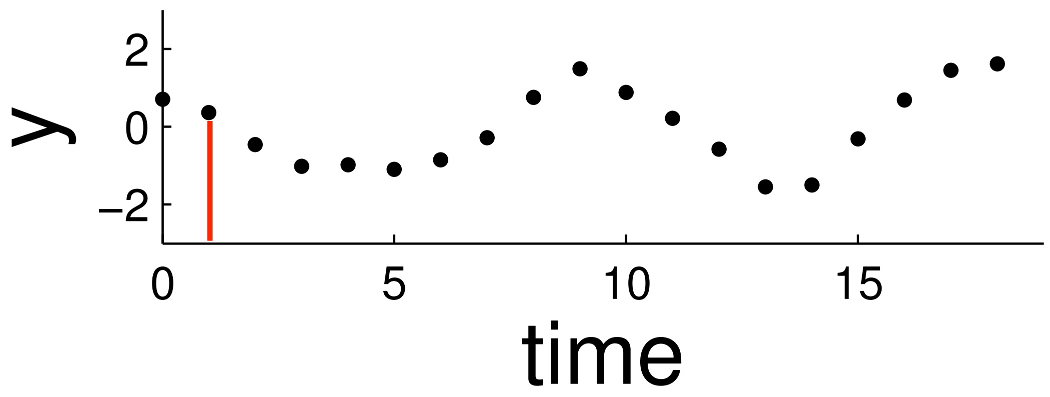
A Bayesian perspective

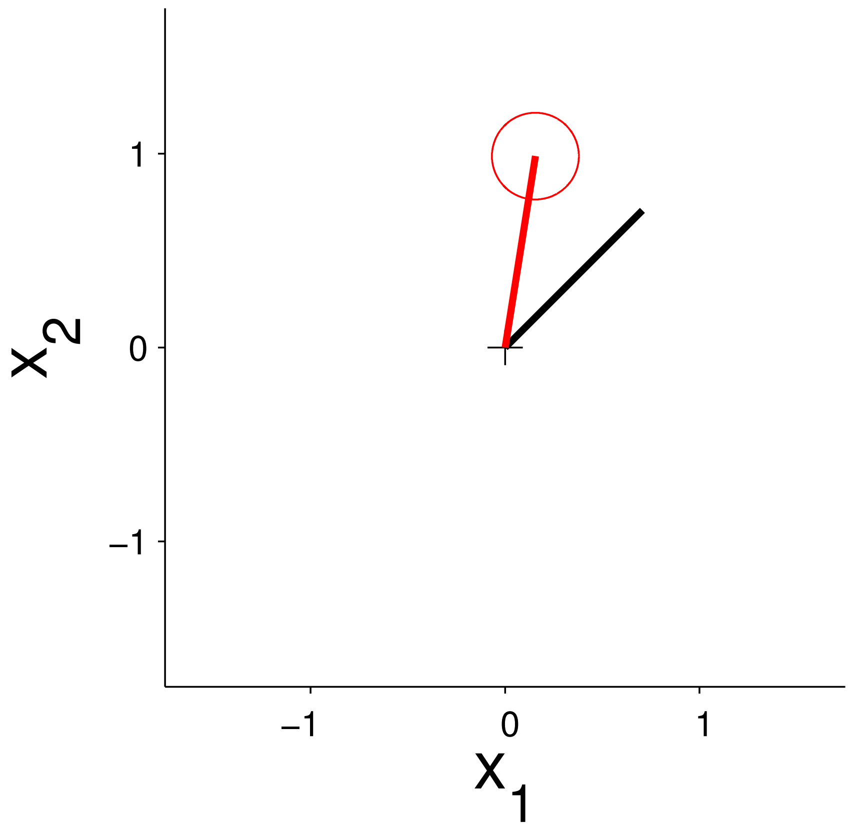


$$\mathrm{y}(t) = \Re\left(\mathrm{a}(t)\exp(i\phi(t))\right)$$

Qi and Minka 2002, Cemgil and Godshill 2005.

A Bayesian perspective





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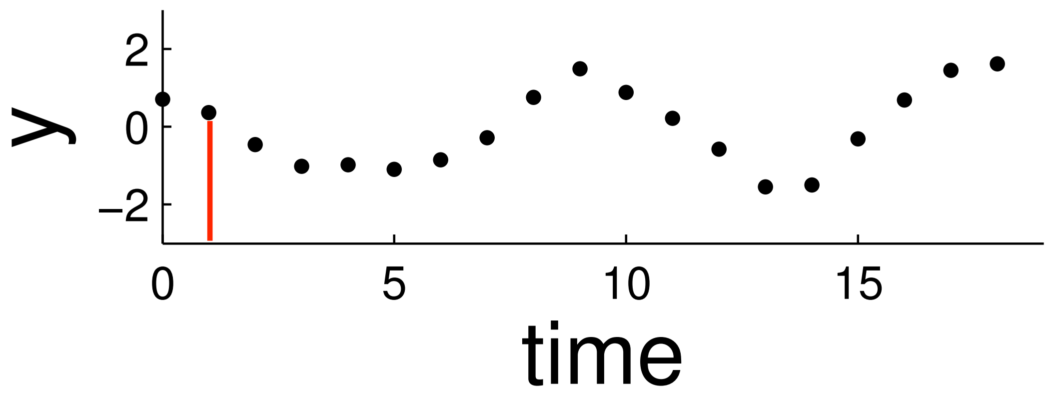

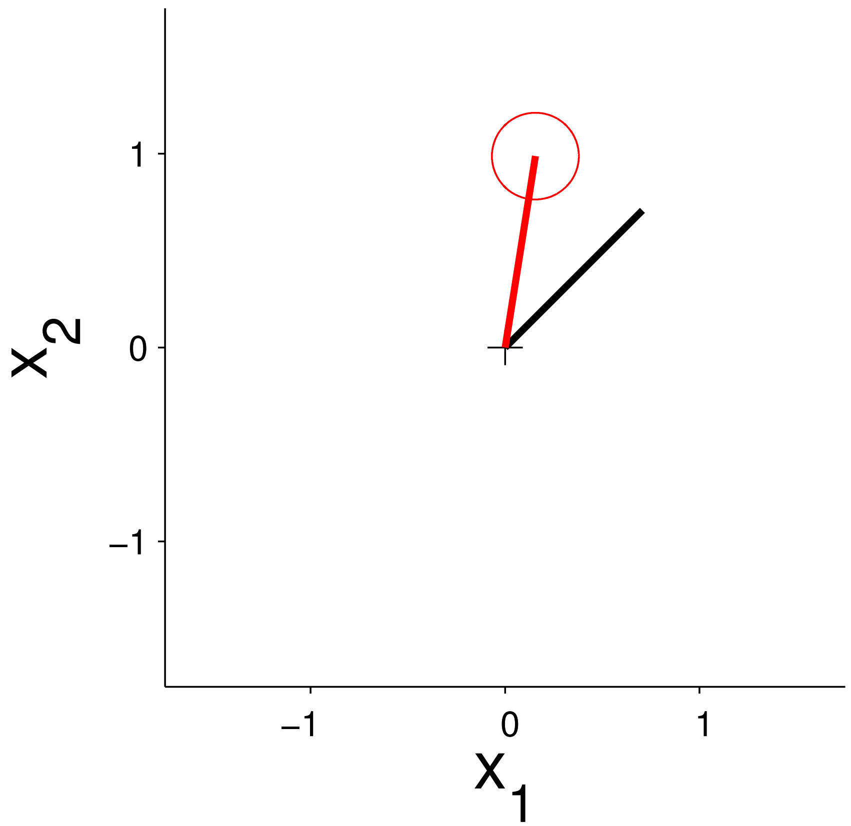

$$y(t) = \Re\left(a(t)\exp(i\phi(t))\right)$$

$$y_t = [1, 0]x_t + \sigma_y \eta_t$$

$$x_t = \lambda R(\bar{\omega})x_{t-1} + \sigma_x^2 \boldsymbol{\epsilon}_t$$

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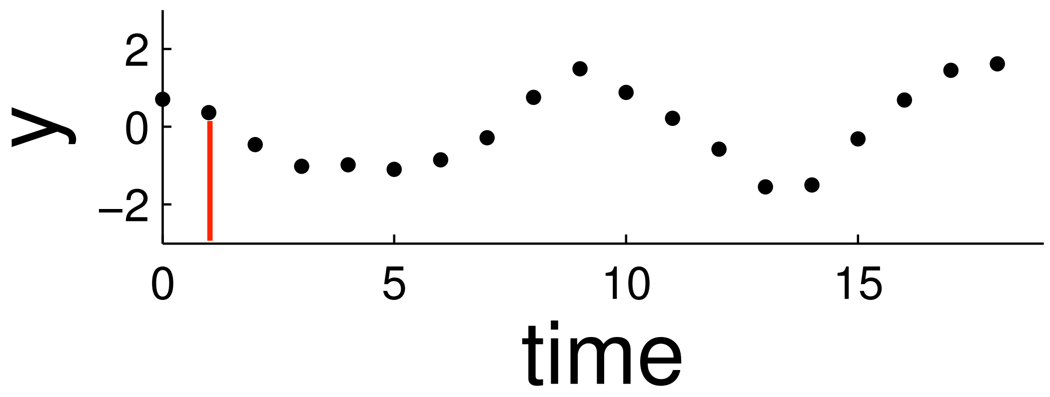
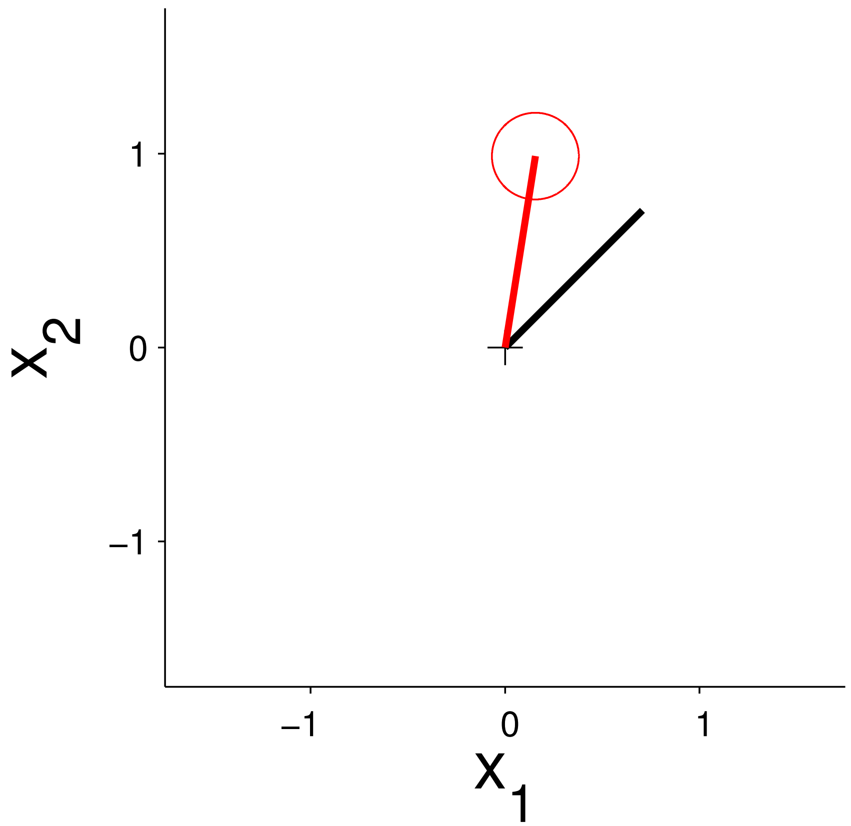
A Bayesian perspective





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Rotation matrix: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

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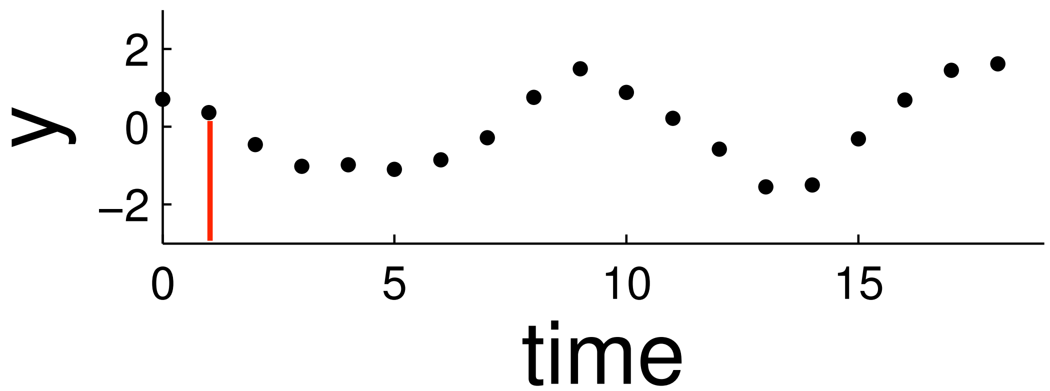
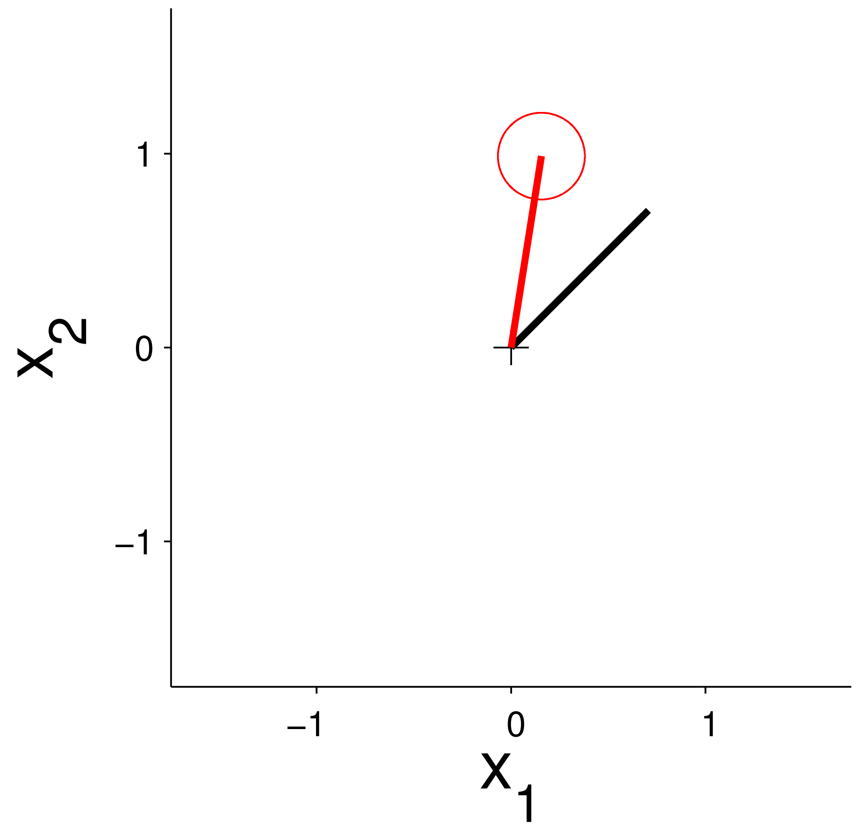
A Bayesian perspective





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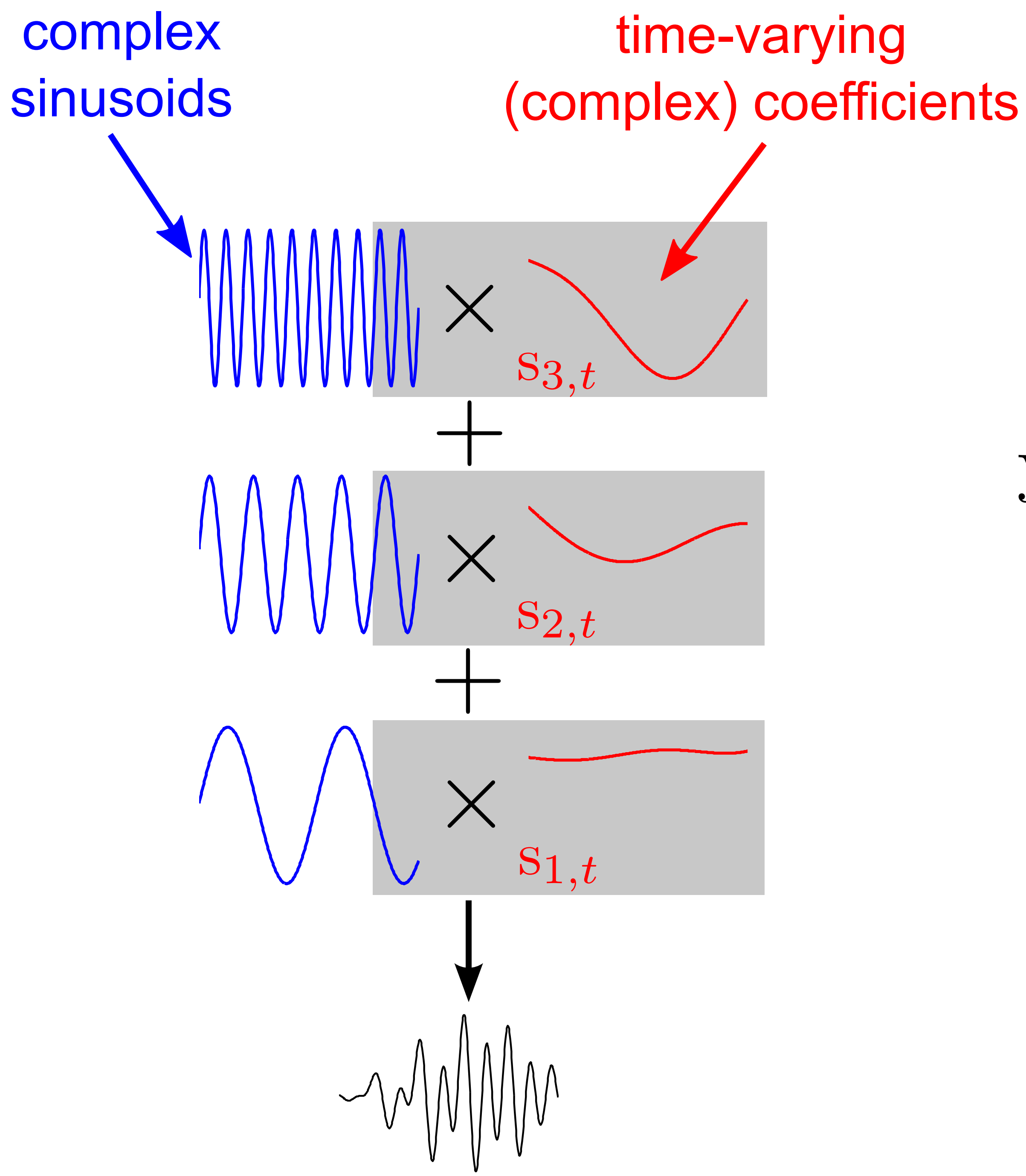
Just an extended Kalman filter

Rotation matrix: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Qi and Minka 2002, Cemgil and Godshill 2005.

Generative model in complex domain

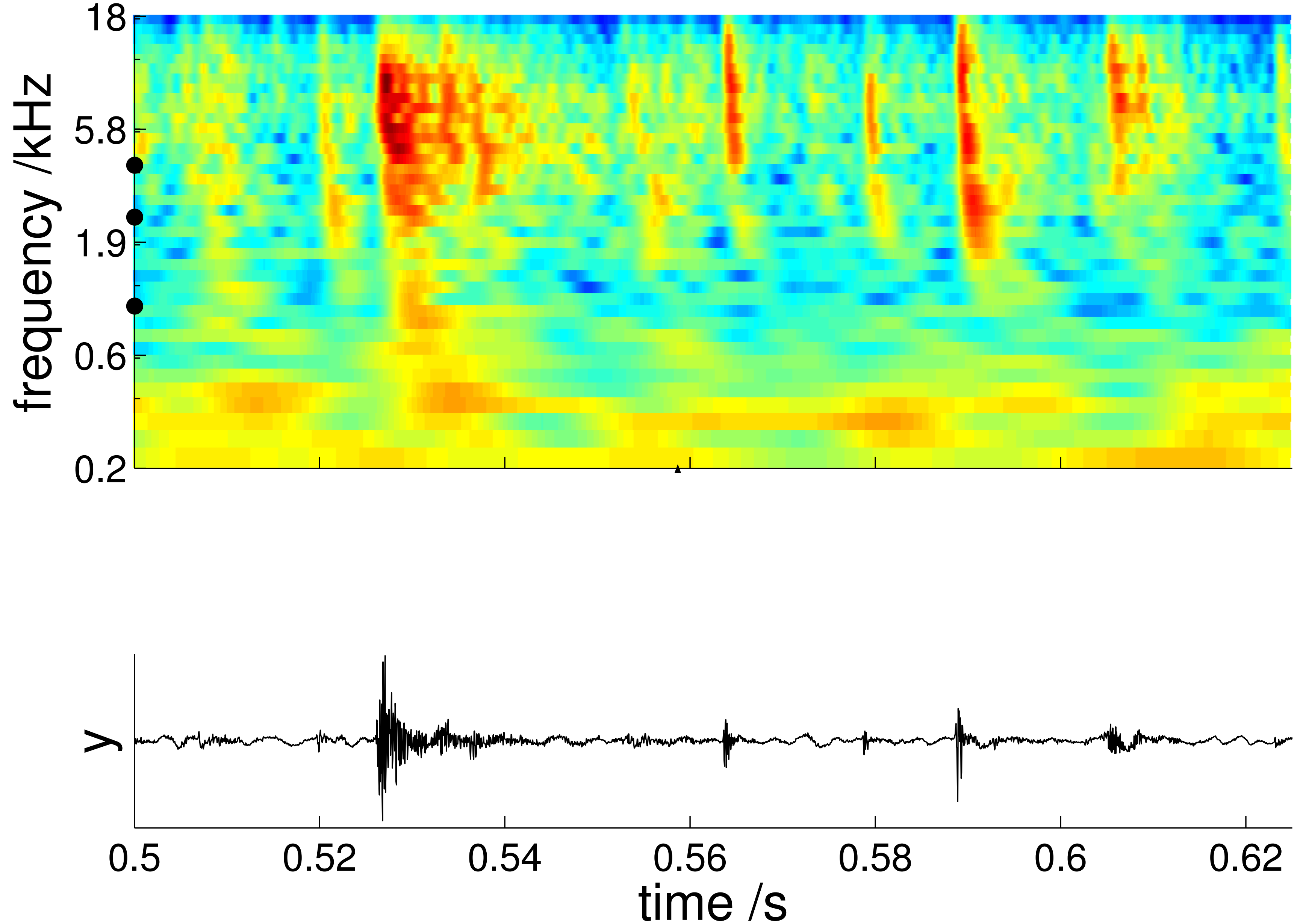


$$y_t = \sum_d \Re\left(e^{i\omega_d t} s_{d,t}\right) + \sigma_y \eta_t$$

$$\Re\left(s_{d,t}\right) \sim \mathcal{GP}(0, \Gamma)$$

$$\Im\left(s_{d,t}\right) \sim \mathcal{GP}(0, \Gamma)$$

The interesting cases are not stationary



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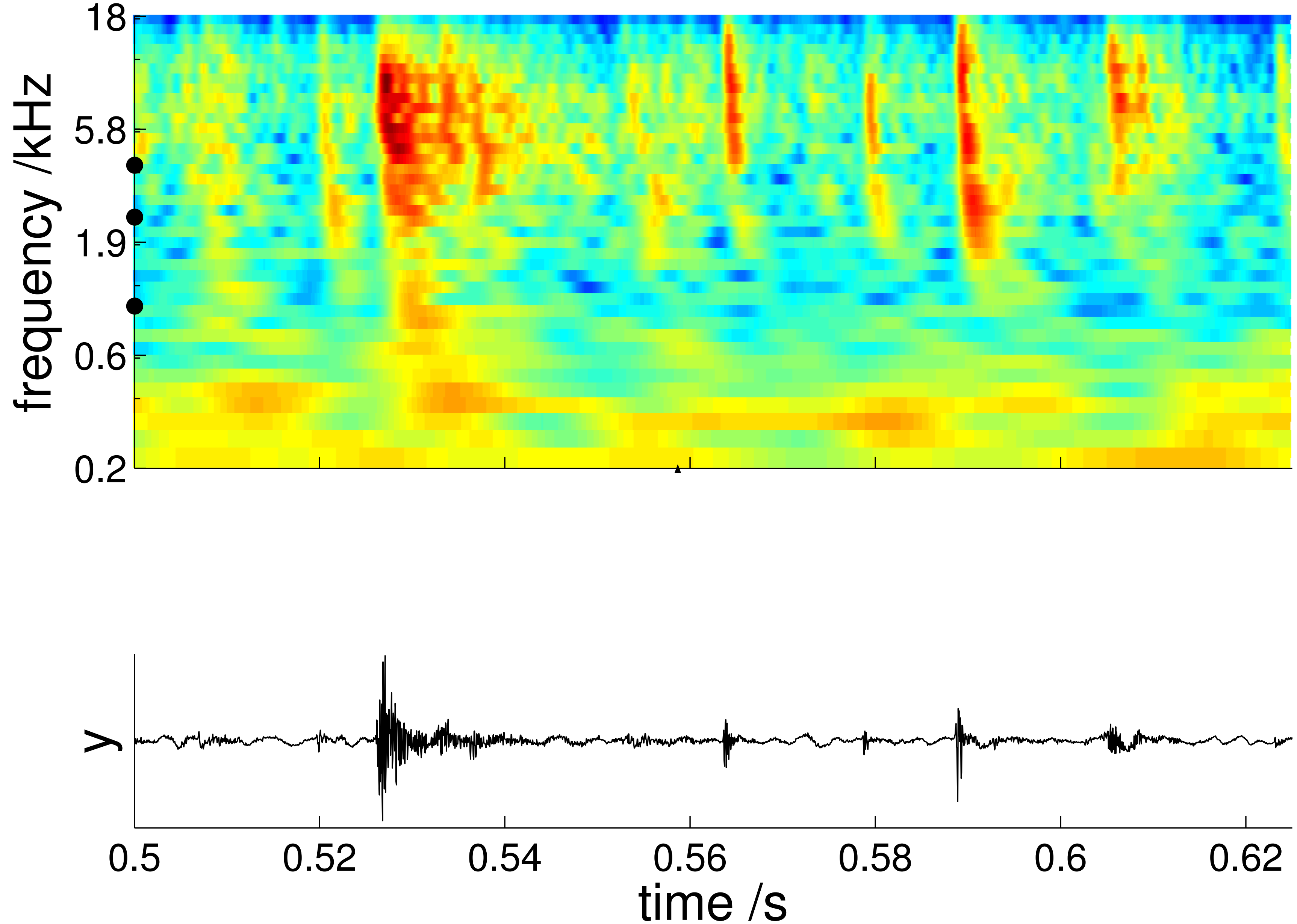
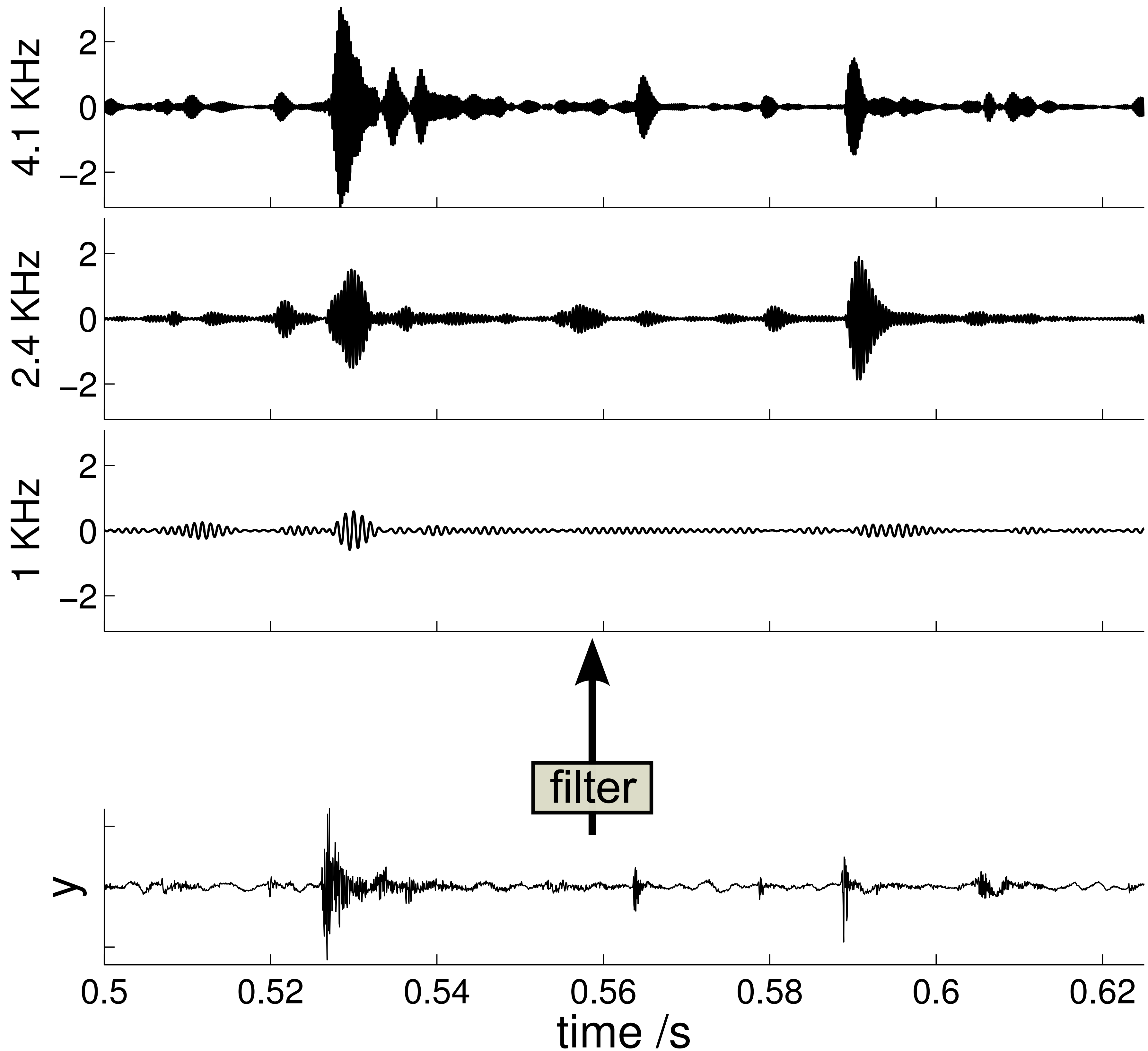
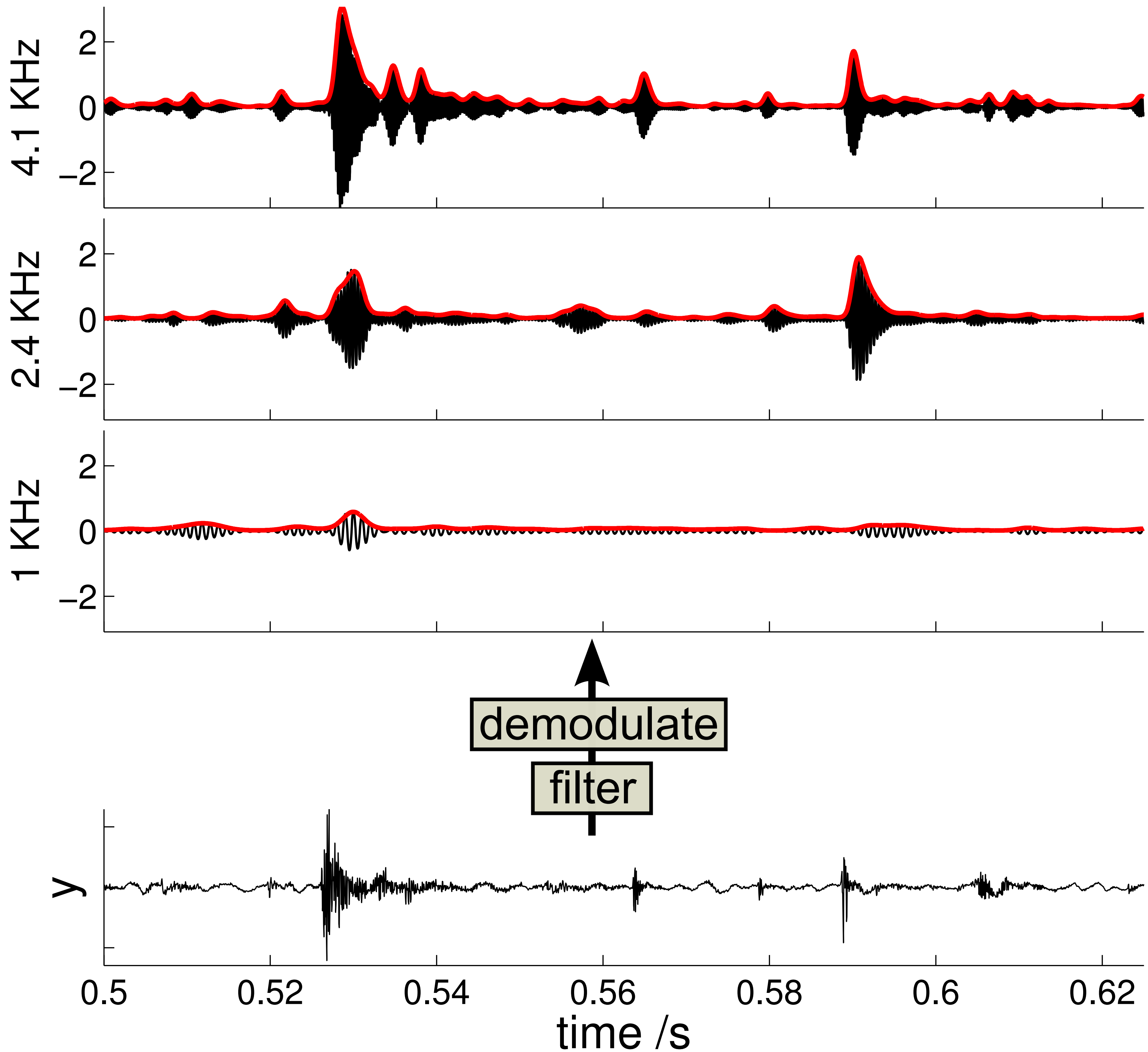
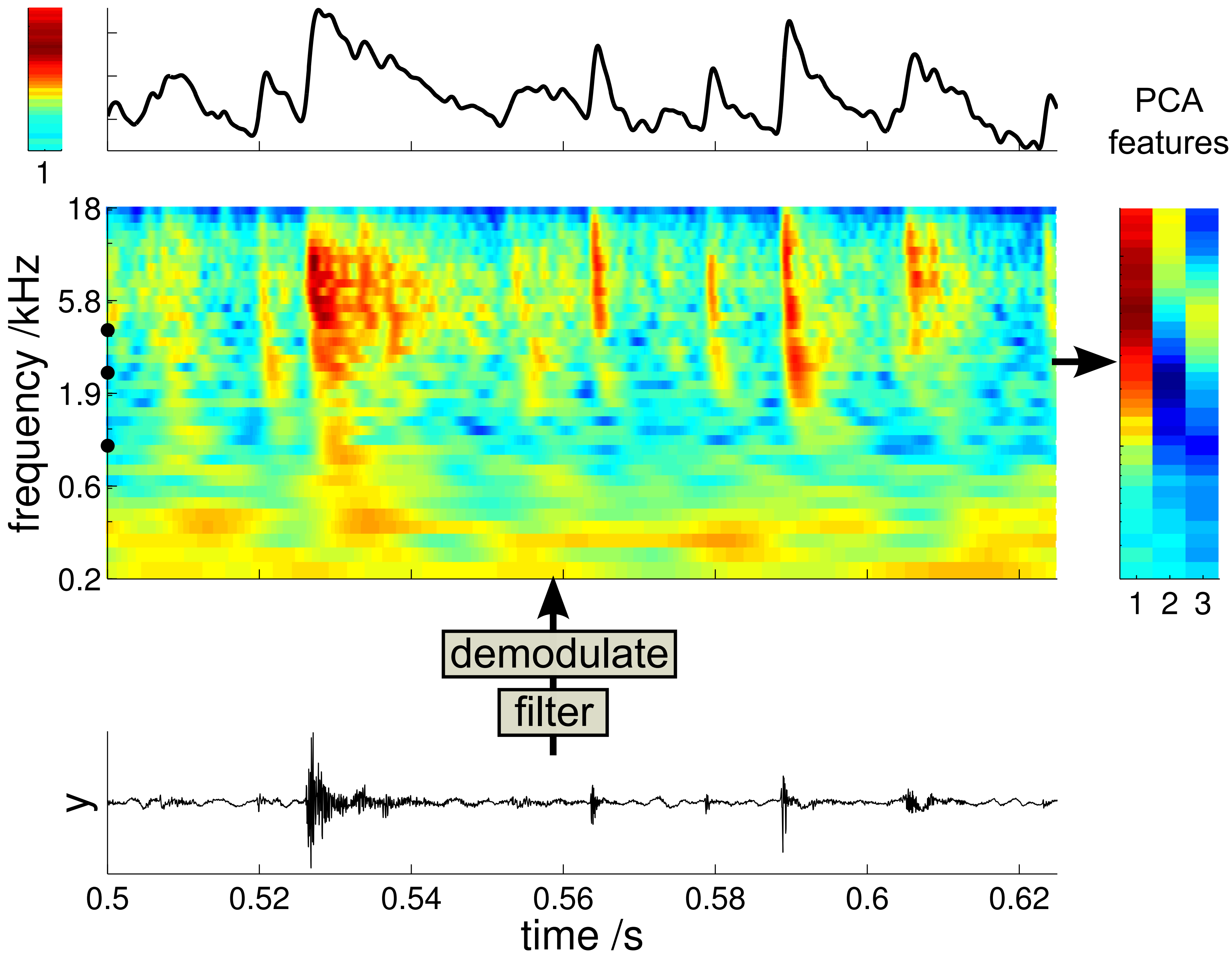



4.1 KHz
2.4 KHz
1 KHz
2
0
−2
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

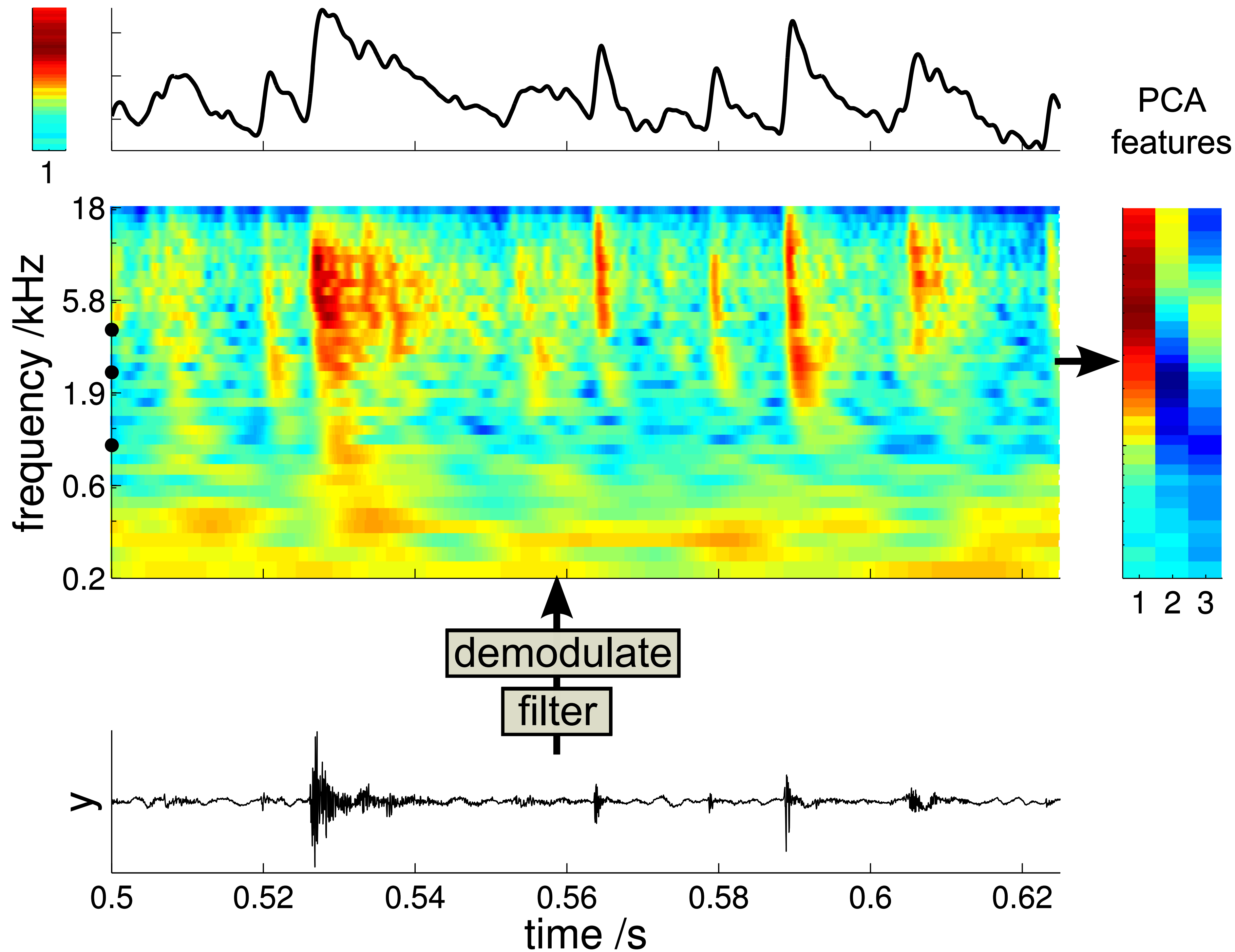

4.1 KHz
2.4 KHz
1 KHz
2
0
−2
demodulate
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

Fire

Fire

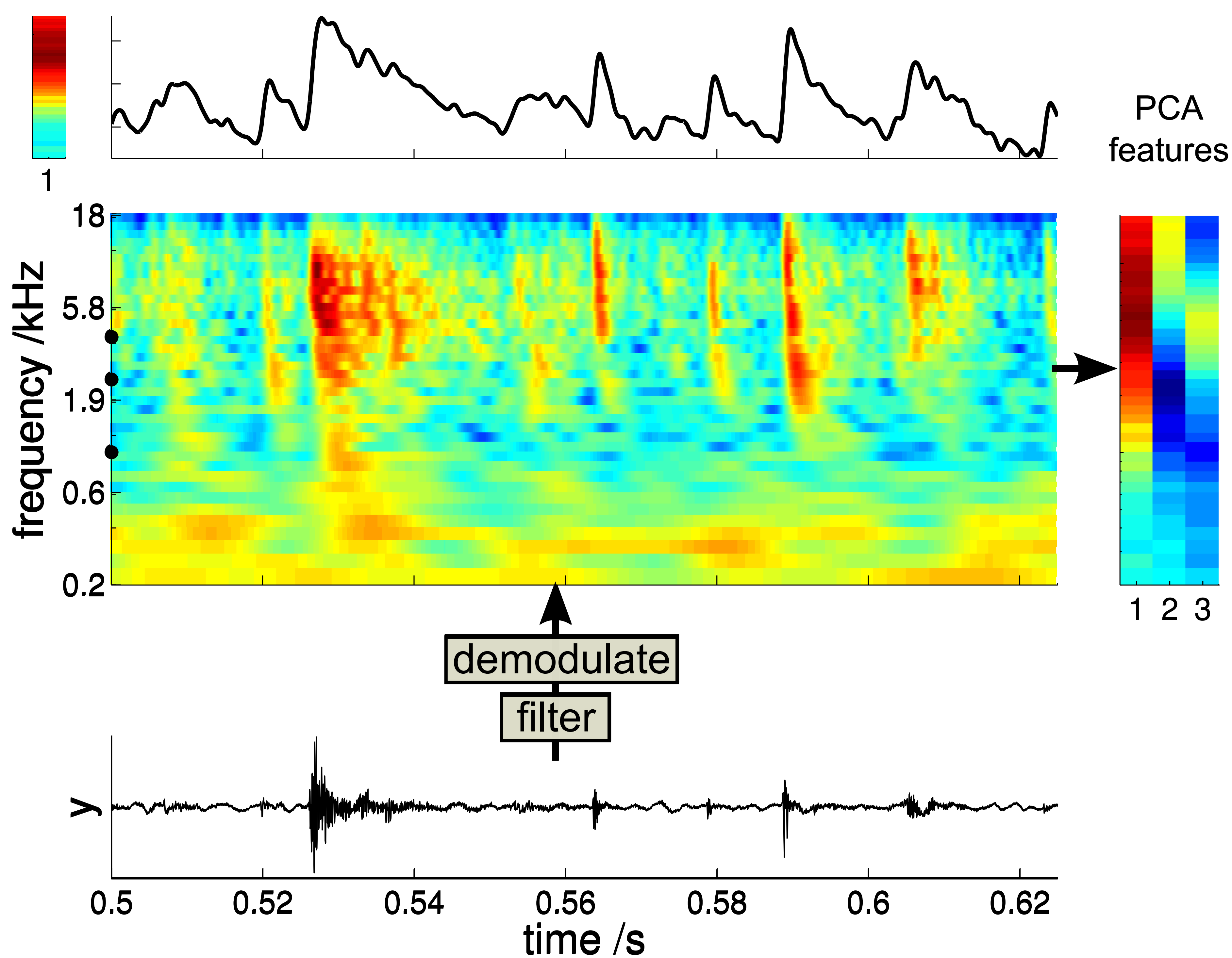

1
18
5.8
1.9
0.6
0.2
frequency /kHz
PCA
features
1 2 3
demodulate
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

Fire



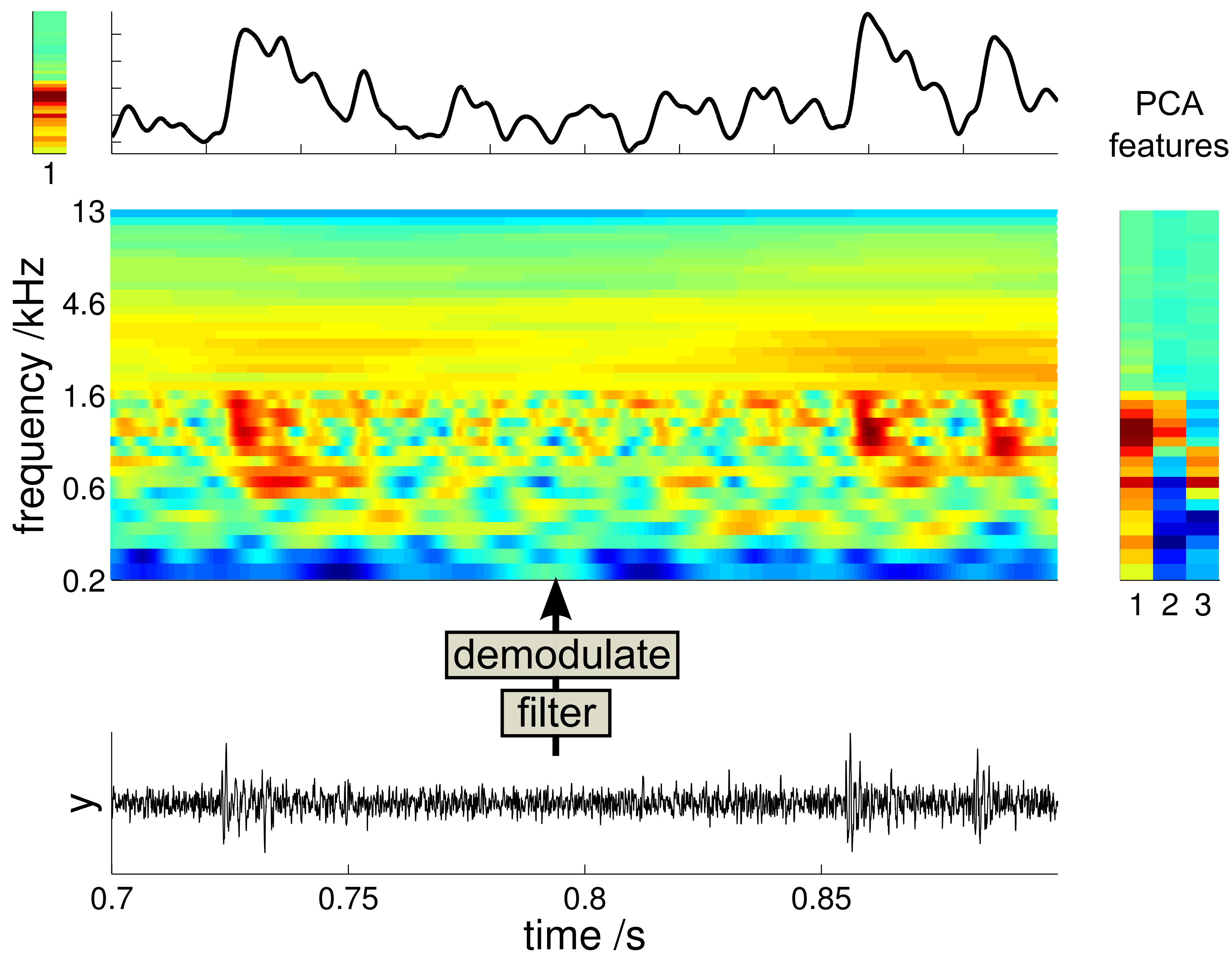
PCA
features
1
18
5.8
1.9
0.6
0.2
frequency /kHz
1 2 3
demodulate
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

Fire



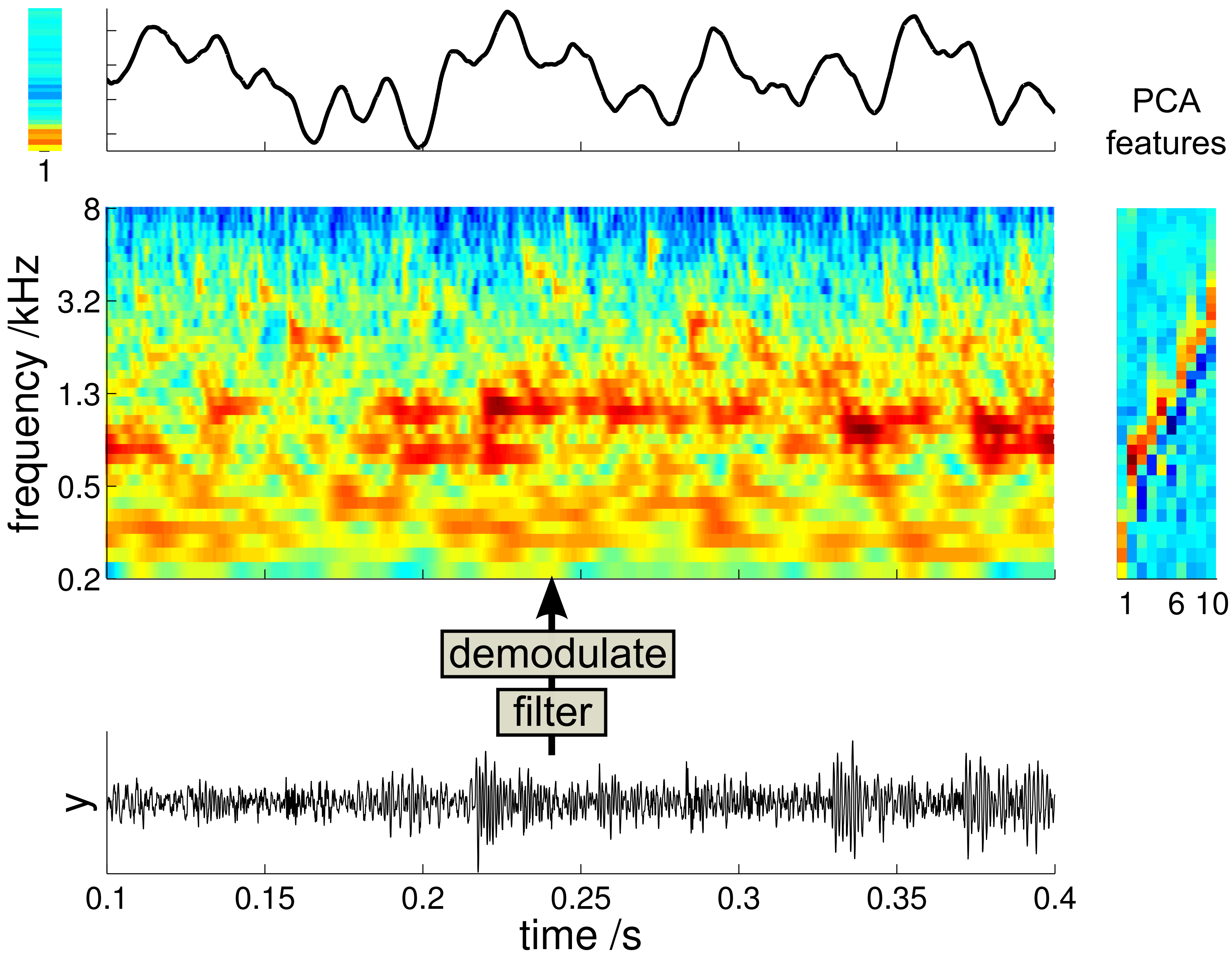
PCA
features
1
18
5.8
1.9
0.6
0.2
frequency /kHz
1
2
3
demodulate
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

Wind



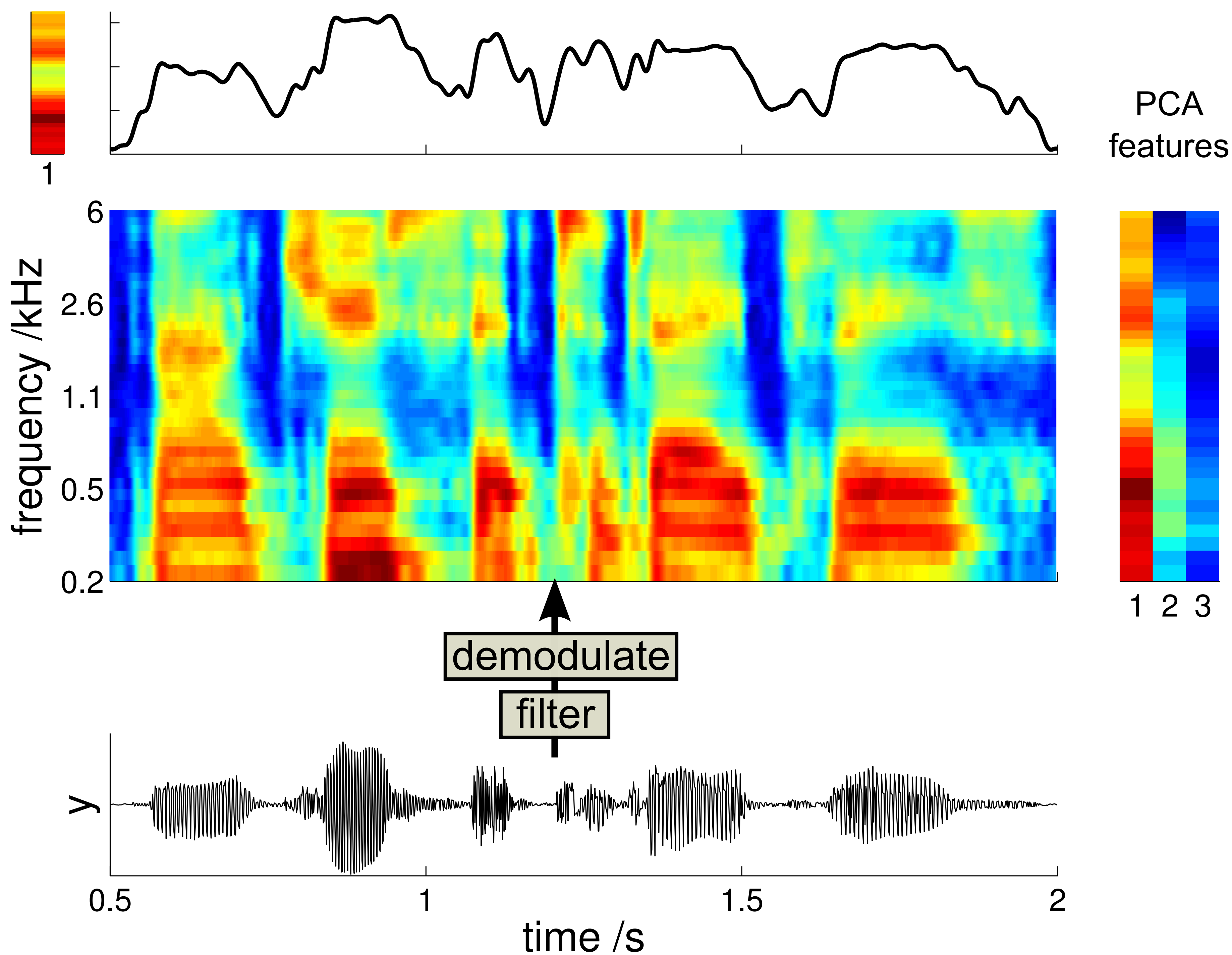
1
PCA
features
13
4.6
1.6
0.6
0.2
frequency /kHz
1 2 3
demodulate
filter
y
0.7
0.75
0.8
0.85
time /s

Water



1
PCA features
8
3.2
1.3
0.5
0.2
frequency /kHz
1
6
10
demodulate
filter
y
0.1
0.15
0.2
0.25
0.3
0.35
0.4
time /s

Speech



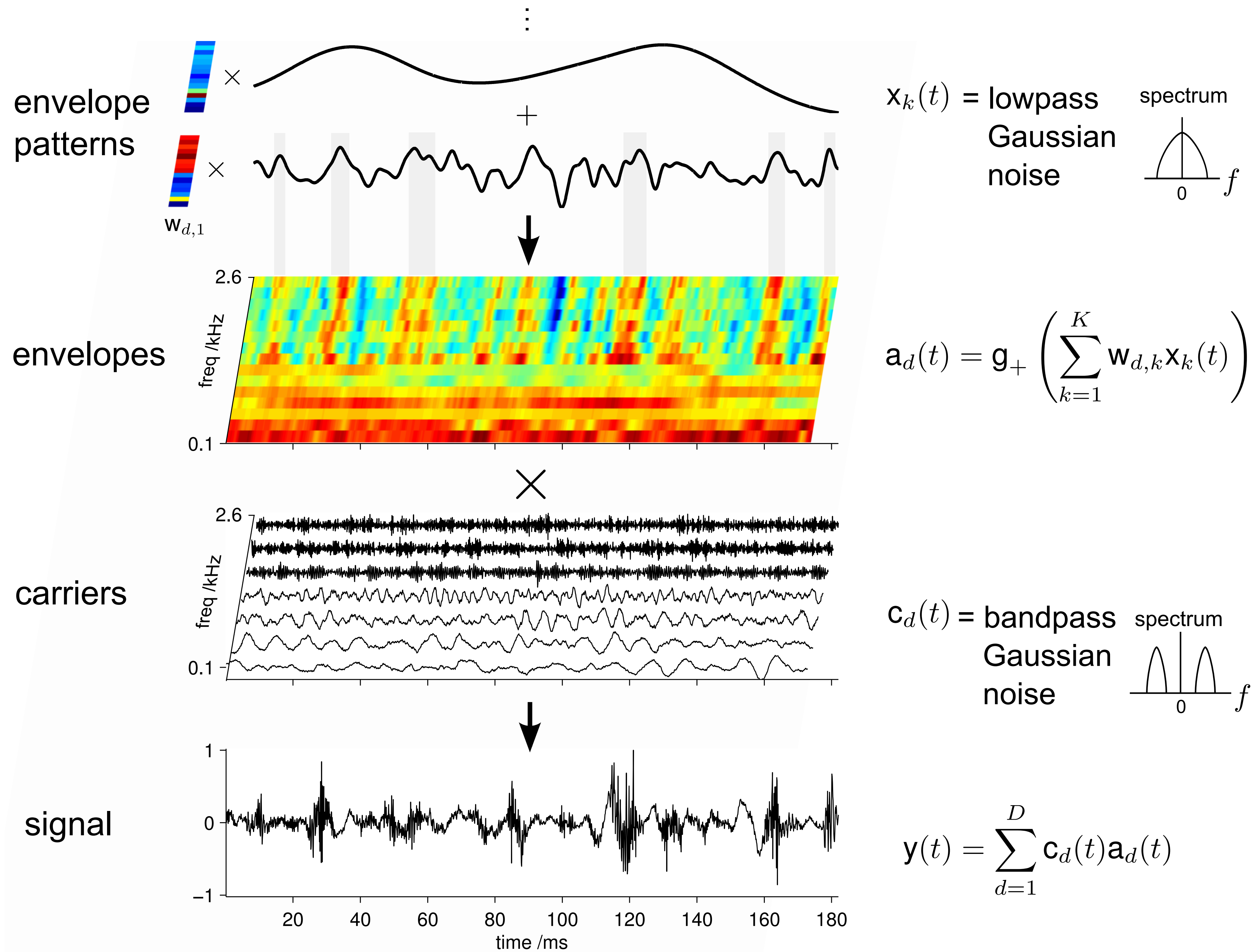
1
PCA
features
6
2.6
1.1
0.5
0.2
frequency /kHz
1 2 3
demodulate
filter
y
0.5
1
1.5
2
time /s

Spectral analysis reveals a lot of potentially useful statistical regularities in real-world data

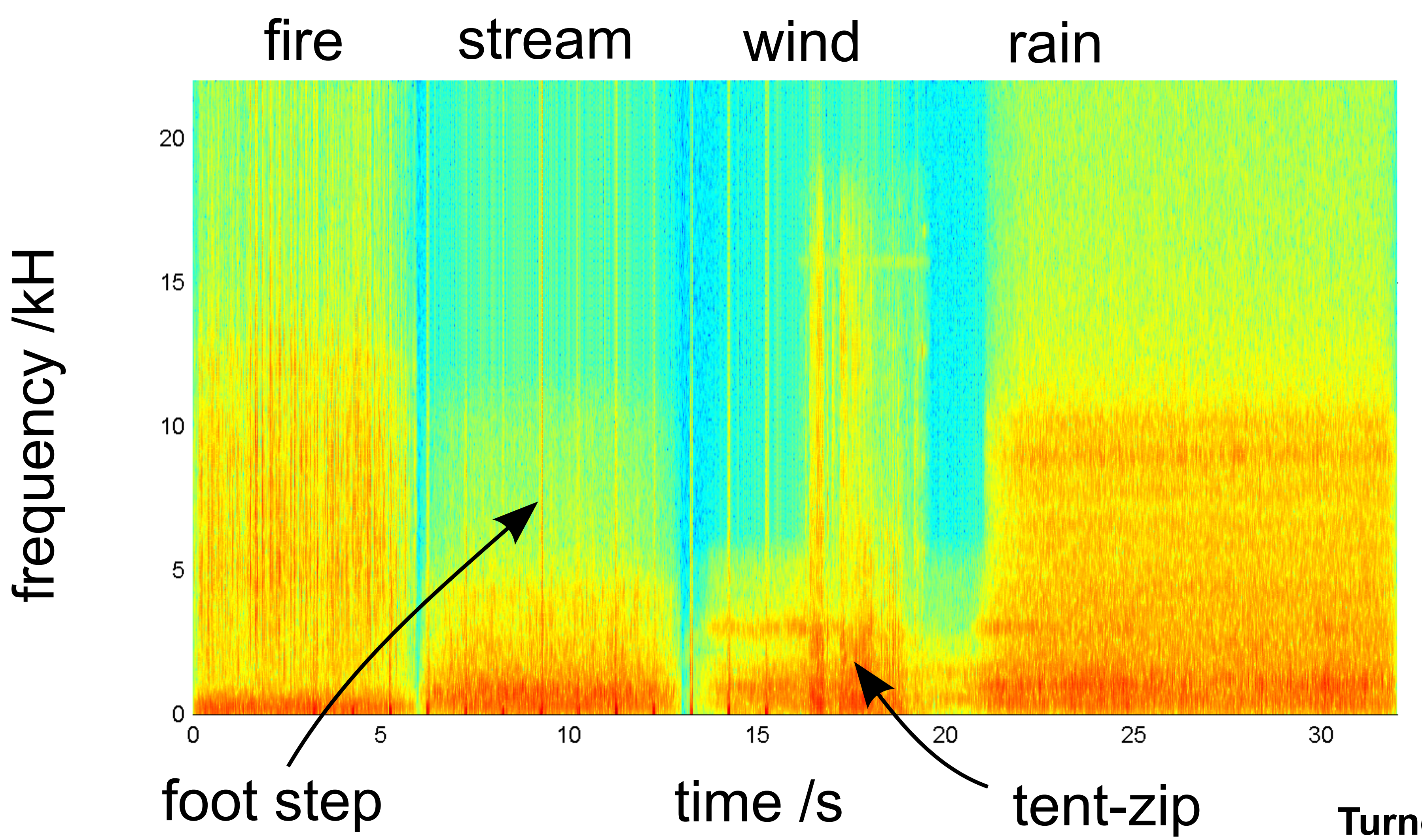
energy in sub-bands (power-spectrum)
patterns of co-modulation
time-scale of the modulation
depth of the modulation (sparsity)

We can build probabilistic models that capture such phenomena and fit them to data

Statistical Model



Sound Generation Demo



Turner, 2010

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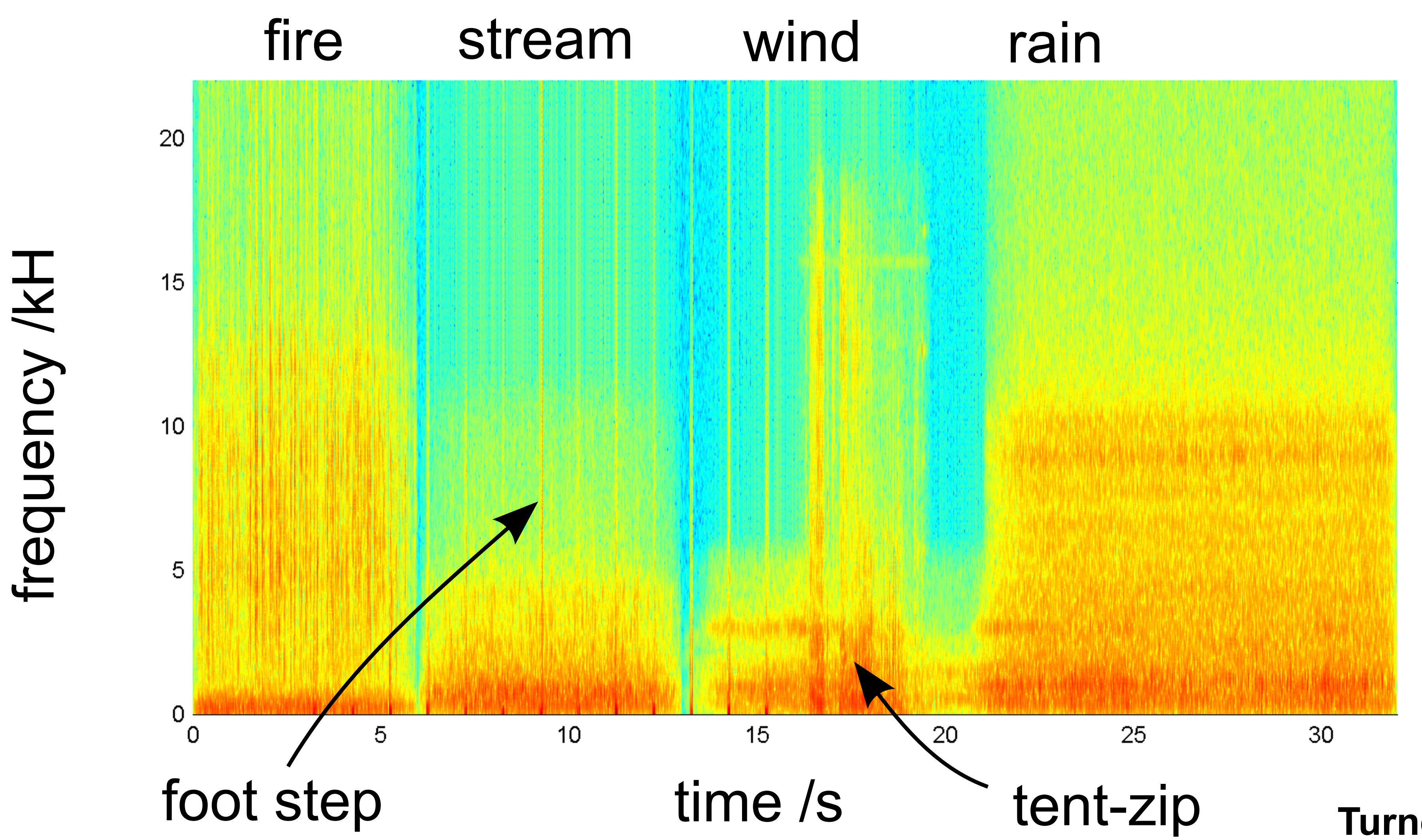
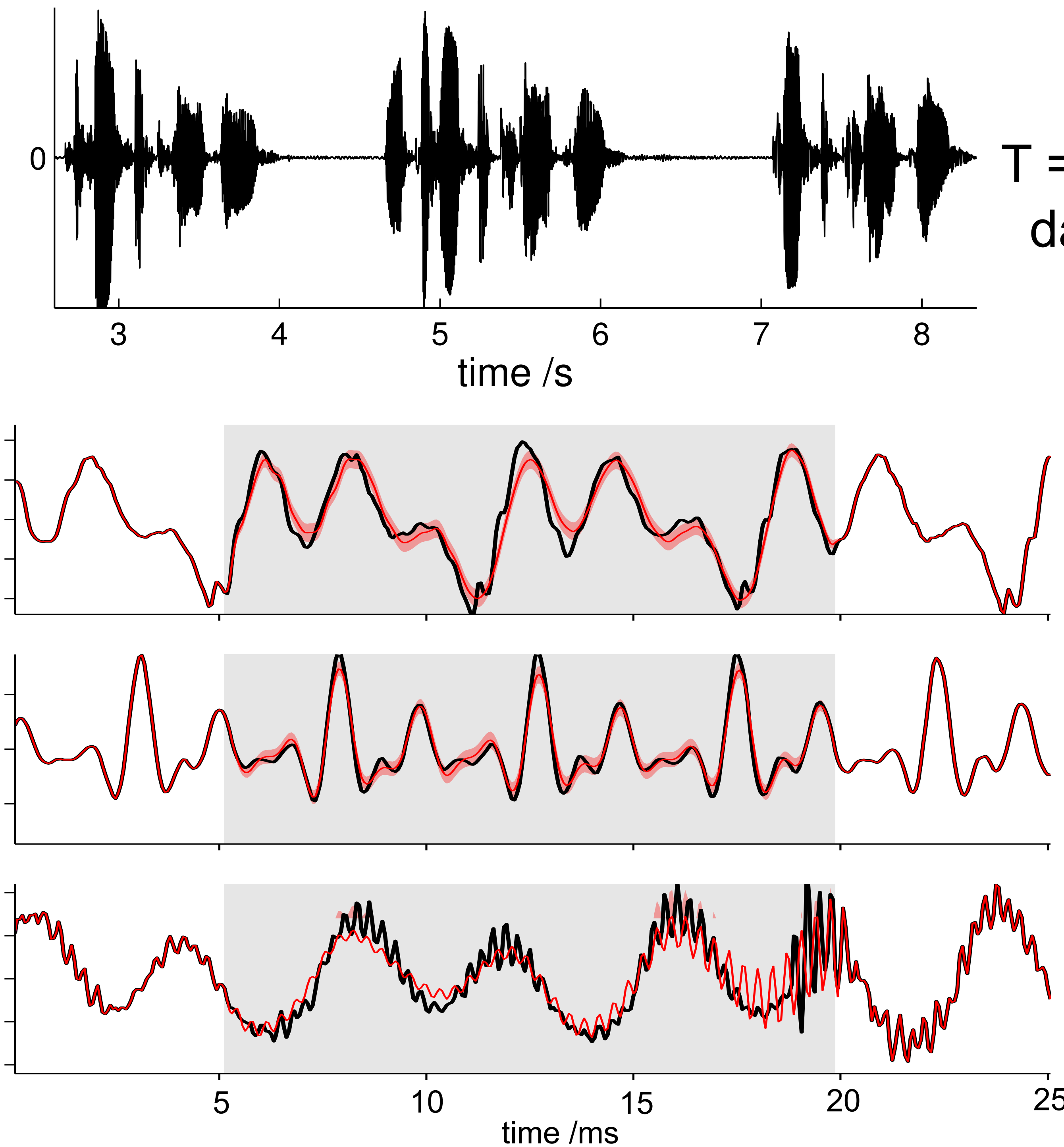


Turner, 2010

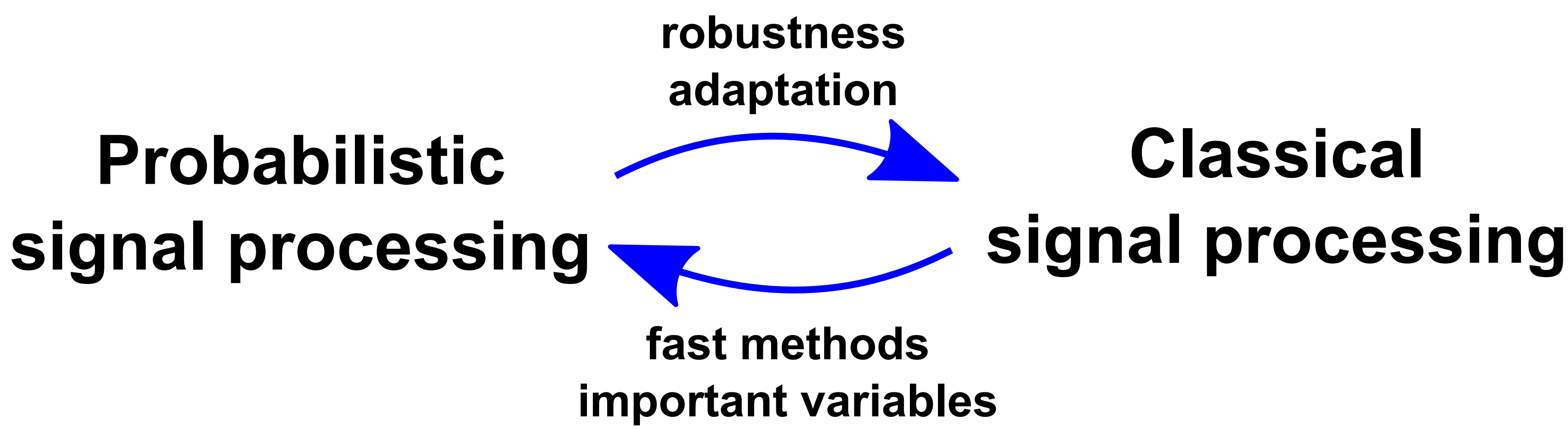
Application: speech imputation

audio
time-series
data



$y(t)$
$T = 10^5\text{-}10^7$
datapoints

reconstruction
using a GP
model



extra reading:

Turner, R. E. and Sahani, M.
"Time-frequency analysis as probabilistic inference"
IEEE Transactions on Signal Processing. 2014

Turner, R. and Sahani M. "Probabilistic amplitude and frequency demodulation."
Advances in Neural Information Processing Systems. 2011.

Turner, R. E., and Sahani, M.
"Demodulation as probabilistic inference."
IEEE Transactions on Audio, Speech, and Language Processing 19.8 (2011): 2398-2411.