Homework 3

Due: Wednesday, February 24, 2016

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but *not your name or your CS login*) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

a. Let A be a set with n elements. Let T be the set of ordered pairs (X, Y) where X and Y are subsets of A. Let S be the set of 0/1/2/3 strings of length n. That is, elements of S are strings of length n where each character is 0, 1, 2, or 3. Give (and prove) a bijection between T and S.

   Let f be a function that takes subsets of T and converts them to a 0/1/2/3 string. 0/1/2/3 represent 4 cases by which $t \in T$ is converted to a 0, 1, 2, or 3 in S.

   The first case is that the subset of T $t = \{\emptyset\}$, and that length = 1 0/1/2/3 string is a 0 in S.

   The second case is that in which the subset of T has only X. In that case, the length = 1 0/1/2/3 string is a 1 in S.

   The third case is that in which the subset of T has only Y. In that case, the length = 1 0/1/2/3 string is a 2 in S.

   The fourth and final case is that in which the subset is equal to T, and contains both X and Y. In this case, the length = 1 0/1/2/3 string is a 3 in S.

   To prove a bijection between T and S, I will first prove that T and S are surjective. By the above function, all cases of T are converted uniquely to a value of S. Only one permutation of T can produce either a 0, 1, 2, or 3. Thus, T and S are surjective.

   To prove injectivity, it must follow that 0, 1, 2, or 3 correspond to a single input from T.

b. If there exists a bijection between two finite sets M and N, what can you conclude about the sizes of M and N?

   A surjective relationship between sets A and B means that the $|A|$ is $\geqslant |B|$. An injective relationship betwee sets A and B means that the $|B| \geqslant |A|$.

Since $|A| \geqslant$ the $|B|$ and $|B| \geqslant$ the $|A|$, it can be said that $|A| = |B|$, and thus the same size.

Problem 2

a. Prove by induction that for all positive integers n, there exists a positive integer m such that:

$$m^2 \leq n < (m+1)^2$$

Proof. by induction Base case: let n = 1, $\exists\ m = 1 s.t. 1^2 \leq 1 < 2^2$.

Inductive Hypothesis: assume $m^2 \leq k < (m+1)^2$.
Inductive Step: $m^2 \leq k+1 < (m+1)^2$.
$m^2 < k+1 \leq (m+1)^2$.
Because $m^2 \leq k$, $m^2 < k+1$.
$m^2 \leq k+1$. Now, let's focus on $k+1 < (m+1)^2$. Case 1: $k+1 = (m+1)^2$: Let $x = m+1$.
$x^2 <= k+1 < (x+1)^2$. We're done.
Case 2: $k+1 < (m+1)^2$. Let $x = m, x^2 \leq k+1 < (x+1)^2$. We're done.
In every case, $\exists m s.t. m^2 \leq n < (m+1)^2$.

b. Prove by contradiction that there exists a **unique** such m.
Given $\forall n \in \mathbb{Z}, \exists m$ s.t. $m^2 \leq n < (m+1)^2$.
Assume that there exist two values x and y that fulfill $x^2 \leq n < (x+1)^2$ and $y^2 \leq n < (y+1)^2$ s.t. $x \neq y$.
Since $x \neq y$, $x > y$ or $y > x$.
Assume here that $x > y$.
Since $x > y$, $x^2 > y^2$.
$x^2 \geq (y+1)^2$.
Now, we are left with two equations:
1) $x^2 \leq n < (x+1)^2$.
2) $y^2 \leq n < (y+1)^2$.

By the above, we see that $x^2 \leq n$.
We also see that $n < (y+1)^2$.
However, as illustrated above, $x^2 \geq (y+1)^2$.
This is a contradiction, so the assumption that $x^2 \leq n < (x+1)^2$ and $y^2 \leq n < (y+1)^2$ s.t. $x \neq y$ is false.
Thus, there exists a unique such m.

Problem 3

a. Prove by contradiction that for any integer n, $n^2 - 2$ is not divisible by 4.

Assume $\exists n$ s.t. $(n^2 - 2)/4 \in \mathbb{Z}$.
Thus, $(n^2 - 2)$ is divisible by 4.
n^2 is not divisible by 4, and is even.
n^2 takes the form $(2k)^2$, pursuant to the form of even numbers.
This is expanded as $4k^2$, or $4(k^2)$.
$4(k^2)$ is divisible by 4, so the claim is false that $\exists n$ s.t. $(n^2 - 2)/4 \in \mathbb{Z}$.

b. Prove that for any integer n, n^3 is odd if and only if n is odd.

Case 1: n is odd Consider n = (2k+1) for some k.
$n^3 = (2k+1)^3$, or $8k^3 + 12k^2 + 6k + 1$.
This can be rewritten as $2(4k^3 + 6k^2 + 3k) + 1$.
This refactored number takes the form 2k + 1 for some k.
Thus, when n is of the form (2k+1), or odd, n^3 is odd as well.
Case 2: n is even Consider n = (2k) for some k.
$n^3 = (2k)^3$, or $8k^3$.
$8k^3$ is even, so when n is even, n^3 is also even. In case 1, when n is odd, n^3 is odd.
In case 2, when n is even, n^3 is even.
Thus, n^3 is odd iff n is odd.

c. Prove that for any integer n, $n^3 - n$ is divisible by 3.

Claim: $\forall n, (n^3 - n)/3 \in \mathbb{Z}$.

Proof. $(n^3 - n) = n(n^2 - 1) = n(n+1)(n-1)$.
$n(n+1)(n-1)$ represents 3 consecutive values of n.
To prove the original claim, we must show that either $(n-1)$, (n), or $(n+1)$ is divisible by 3.
Claim: from $(n-1)(n)(n+1)$, either $(n-1)$, (n), or $(n+1)$ is divisible by 3.
Case 1: $(n-1)$ is divisible by 3.
$n - 1 = 3k$, so $n = 3k + 1$.
$(3k+1)^3 - (3k+1) = (27k^2 + 6k + 1)$
$27k^2 + 6k + 1 - 1 = n - 1 = 3k$.
$3(9k^2 + 2k) = 3u$. $3u = n - 1$.
$n - 1$ is divisible by 3, so we're done.
Case 2: n is divisible by 3.
$n = 3k$.
$(3k)^3 - 3k = n$. $(3k^3 - 3k) = (27k^3) - 3k = 3(9k - 1)$
$3(9k - 1) = 3u$.
$3u = n$.
n is divisible by 3, so we're done.

Case 3: $n+1$ is divisible by 3. $n+1=3k$, so $n=3k-1$.
$(3k-1)^3-(3k-1)=n$. $27k^3-27k^2+6k=n$. $3(9k^3-9k^2+2k)=n$. $3(u)=n$.
n is divisible by 3, so we're done.

In all 3 cases, either $(n-1)$, (n), or $(n+1)$ are divisible by 3, so the multiplication is also divisible by 3.

Thus, for any integer n, n^3-n is divisible by 3.

Problem 4

Consider the following relation on the set of integers:

$\forall a, b \in \mathbb{Z},\ (a, b) \in R$ if and only if the remainder when a is divided by 3 is the same as the remainder when b is divided by 3.

a. Prove that R is an equivalence relation. For R to be an equivalence relation, R must be reflexive, symmetric, and transitive.

Reflexivity: For R to be reflexive, $\forall x \in R$, xRx.
For xRx to be true, $x\%3 = x\%3$. Since we know this to be true, R is reflexive.

Symmetry: For R to be symmetric, if aRb, then bRa.
For aRb to be true, WLOG, if $a\%3 = 1$, then $b\%3 = 1$.
If $b\%3 = a\%3$, then bRa. Thus, if aRb, then bRa.
This is the definition of symmetry, so R is symmetric.

Transitivity: For R to be transitive, for $(a, b, c) \in R$, if aRb and bRc, then aRc.

In regards to aRb, WLOG, if $a\%3 = 1$, then $b\%3 = 1$.
Similarly, if $b\%3 = 1$, then $c\%3 = 1$.

R is reflexive, symmetric, and transitive. Thus, R is an equivalence relation.

Since $a\%3 = b\%3 = c\%3$, aRc is also $\in R$.

b. How many distinct equivalence classes are in this equivalence relation? What are they?

In this example, equivalence classes can be thought of as the possible results when $x \in \mathbb{Z}$ is divided by 3.
These classes are: $x\%3 = 0$, $x\%3 = 1$, and $x\%3 = 2$.
These classes can be represented, in order, as $3k \in \mathbb{Z}$, $3k+1 \in \mathbb{Z}$, and $3k+2 \in \mathbb{Z}$.

Note: An equivalence class is defined for an equivalence relation, R on set A, as follows: $[a]_R = \{x \in A \mid (a, x) \in R\}$

Problem 5

This is an optional problem. It will not affect your grade.

Let $C(n)$ be the number of 0/1 strings of length n that do not contain consecutive 1s. For example, $C(4) = 8$ because there are 8 0/1 strings of length 4 without consecutive 1s: 0000, 0001, 0010, 0100, 1000, 0101, 1010, and 1001.

Prove that $\forall n \in \mathbb{Z}^+$, $C(n) = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+2}\right)$.

Hint: Show that for any $n \geq 3$, $C(n) = C(n-1) + C(n-2)$. Also, you may use the following without proof:

- $1 + \frac{1+\sqrt{5}}{2} = (\frac{1+\sqrt{5}}{2})^2$
- $1 + \frac{1-\sqrt{5}}{2} = (\frac{1-\sqrt{5}}{2})^2$