# Project 4 – Questions 1–5

Aaron Toomer and Milena Silva

## Question 1 – Definitions

Let $P(i, j, k)$ represent the probability that the current player wins given:

- $i$: the current player's score,
- $j$: the opponent's score,
- $k$: the current turn total.

If $i + k \geq 100$, then the player can hold and win:

$$P(i, j, k) = 1$$

Otherwise, the value iteration update is:

$$P(i, j, k) = \max(P(i, j, k, \text{roll}), P(i, j, k, \text{hold}))$$

Where:
**Roll Option:**

$$P(i, j, k, \text{roll}) = \frac{1}{6}(1 - P(j, i, 0)) + \sum_{d=2}^{6} \frac{1}{6} P(i, j, k + d)$$

**Hold Option:**

$$P(i, j, k, \text{hold}) = 1 - P(j, i + k, 0)$$

## Question 2 – Probabilities by Hand

### (a) $P(99, 99, 0)$

The roll option is:

$$P(99, 99, 0, \text{roll}) = \frac{1}{6}(1 - P(99, 99, 0)) + \frac{5}{6}(1)$$

Let $x = P(99, 99, 0)$.

Solving:

$$x = \frac{6 - x}{6} \Rightarrow 6x = 6 - x \Rightarrow 7x = 6 \Rightarrow x = \frac{6}{7} \approx 0.857$$

Thus:

$$P(99, 99, 0) \approx 0.857$$

### (b) $P(98, 99, 0)$ and $P(99, 98, 0)$

Let $x = P(98, 99, 0)$ and $y = P(99, 98, 0)$.

$$x = \frac{6 - y}{6}, \quad y = \frac{6 - x}{6}$$

Solving the system:

$$x = \frac{6 - \frac{6-x}{6}}{6} \Rightarrow 36x = 30 + x \Rightarrow 35x = 30 \Rightarrow x = \frac{6}{7} \approx 0.857$$

$$y = \frac{6 - x}{6} = \frac{6 - \frac{6}{7}}{6} = \frac{36}{42} = \frac{6}{7} \approx 0.857$$

### (c) $P(97, 99, 0)$, $P(97, 99, 2)$, and $P(99, 97, 0)$

Given:

$$P(97, 99, 2) \approx 0.855$$

Rolling from $(97, 99, 0)$:

$$P(97, 99, 0, \text{roll}) = \frac{1}{6}(1 - P(99, 97, 0)) + \frac{1}{6}(0.855) + \frac{4}{6}(1) \Rightarrow \frac{5.855 - P(99, 97, 0)}{6}$$

Holding:

$$P(97, 99, 0, \text{hold}) = 1 - P(99, 97, 0)$$

Assuming $P(99, 97, 0) \approx 0.861$:

$$P(97, 99, 0, \text{roll}) \approx \frac{5.855 - 0.861}{6} = \frac{4.994}{6} \approx 0.833$$

So:

$$P(97, 99, 0) \approx 0.833$$

Rolling from $(99, 97, 0)$:

$$P(99, 97, 0, \text{roll}) = \frac{1}{6}(1 - 0.833) + \frac{5}{6}(1) = 0.0278 + 0.8333 = 0.861$$

Holding:

$$1 - 0.833 = 0.167$$

Final:

$$P(99, 97, 0) \approx 0.861$$

## Final Q2(c) Answers

$$P(97, 99, 0) \approx 0.833, \quad P(97, 99, 2) \approx 0.855, \quad P(99, 97, 0) \approx 0.861$$

## Question 4 – Probability of Winning at the Start

(a) The probability that the first player wins assuming both play optimally is:

$$P(0, 0, 0) \approx 0.5306$$

(b) The lowest turn total on the first turn where the player should hold is:

$$21$$

(c) If Player 1 scores no points on their first turn:

- Probability that Player 2 wins is: 0.4694
- Player 2 should hold on their first turn when turn total reaches: 21

## Question 5 – Policy Output Summary

The full 100x100 hold-threshold table is large, but analysis reveals the following pattern:

- As the player's score increases, the turn total required to hold generally decreases.
- When the opponent is close to 100, the optimal strategy becomes more aggressive — players tend to hold earlier to avoid the risk of losing.
- When the player is behind, they are more likely to roll longer, especially if the opponent is near 100.

This reflects a dynamic policy that adapts not only to the player's current score but also to the opponent's threat level.