Chebyshev Polynomials

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Chebyshev Polynomial Definition

The kth Chebyshev polynomial can be defined as

$$T_k(x) = \cos(k\theta) \tag{1}$$

where $\theta = \arccos(x)$.

Orthogonality (based on ATAP exercise 3.7)

Theorem 3.1 in ATAP states if f is Lipschitz continuous on [-1, 1], it has a unique representation as a Chebyshev series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \tag{2}$$

which is absolutely and uniformly convergent. The coefficients are given for $k \geq 1$ by the formula

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx \tag{3}$$

and for $k = 0$ by the same formula with the factor $2/\pi$ changed to $1/\pi$. Here, we derive the weighted orthogonality conditions for Chebyshev polynomials, and show how these can be used to derive eq 3.

The **orthogonality condition for Chebyshev polynomials** for integers j and k to be derived is

$$\int_{-1}^{1} \frac{T_j(x)\, T_k(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0, & j \neq k \\ \pi, & j = k = 0 \\ \frac{\pi}{2}, & j = k \geq 1 \end{cases} \tag{4}$$

Inserting the definition of Chebyshev polynomials

$$\int_{-1}^{1} \frac{T_j(x) T_k(x)}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{\cos(j \arccos(x)) \cos(k \arccos(x))}{\sqrt{1-x^2}} dx \tag{5}$$

Making the change of variables $x = \cos(\theta)$, $dx = -\sin(\theta) d\theta$, the definition becomes

$$-\int_{-1}^{1} \cos(j\theta) \cos(k\theta) d\theta \tag{6}$$

As x ranges from -1 to 1 in the integral, θ ranges from π to 0. Allowing θ to instead range from 0 to π accounts for the minus sign in the integral above. To summarize, we have so far found

$$\int_{-1}^{1} \frac{T_j(x) T_k(x)}{\sqrt{1-x^2}} dx = \int_{0}^{\pi} \cos(j\theta) \cos(k\theta) d\theta \tag{7}$$

Using trigonometric identities

$$\int_{0}^{\pi} \cos(j\theta) \cos(k\theta) d\theta = \frac{1}{2} \int_{0}^{\pi} \cos((j-k)\theta) d\theta + \frac{1}{2} \int_{0}^{\pi} \cos((j+k)\theta) d\theta \tag{8}$$

Since j and k are integers, both $(j-k)$ and $(j+k)$ are integers, say m. The integral of $\int_0^\pi \cos(mx)dx = 0$ for all integers $m \neq 0$. In other words, eq 8 is only non zero when $j = k$. Considering this case

$$\int_0^\pi \cos(j\theta)\cos(k\theta)d\theta = \frac{1}{2}\int_0^\pi \cos(0)d\theta + \frac{1}{2}\int_0^\pi \cos(2k\theta)d\theta \tag{9}$$

Now, if $j = k = 0$

$$\frac{1}{2}\int_0^\pi \cos(0)d\theta + \frac{1}{2}\int_0^\pi \cos(0)d\theta = \int_0^\pi \cos(0)d\theta = \pi \tag{10}$$

If $j = k \geq 1$, then $2k$ is an integer, and its cosine integral vanishes

$$\frac{1}{2}\int_0^\pi \cos(0)d\theta + \frac{1}{2}\int_0^\pi \cos(2k\theta)d\theta = \frac{1}{2}\int_0^\pi \cos(0)d\theta = \frac{\pi}{2} \tag{11}$$

all three cases have been considered, leading to the orthogonality conditions in eq 4.

The coefficients can now be derived beginning with eq 2. Multiplying by $T_j(x)/\sqrt{1-x^2}$

$$\frac{T_j(x)f(x)}{\sqrt{1-x^2}} = \sum_{k=0}^{n} \frac{a_k T_j(x)T_k(x)}{\sqrt{1-x^2}} \tag{12}$$

where on the RHS, $T_j(x)/\sqrt{1-x^2}$ may be moved inside the sum because both terms are independent of k. Integrating both sides over the domain $[-1, 1]$

$$\int_{-1}^{1} \frac{T_j(x)f(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{a_k T_j(x)T_k(x)}{\sqrt{1-x^2}}dx = \sum_{k=0}^{\infty} a_k \int_{-1}^{1} \frac{T_j(x)T_k(x)}{\sqrt{1-x^2}}dx \tag{13}$$

where again the integral may be pulled inside the sum, as the integral depends on x, the sum on k. By the orthogonality conditions (eq 4), the last integral is only nonzero when $j = k$, causing the infinite sum to collapse, leaving

$$\int_{-1}^{1} \frac{T_j(x)f(x)}{\sqrt{1-x^2}}dx = a_j \int_{-1}^{1} \frac{T_j(x)^2}{\sqrt{1-x^2}}dx = \begin{cases} a_0 \cdot \pi, & j = 0 \\ a_j \cdot \frac{\pi}{2}, & j \geq 1 \end{cases} \tag{14}$$

leading directly to eq 3.

Extrema and Roots (based on ATAP exercise 3.12)

The roots of Chebyshev polynomials in [-1, 1] are found by setting

$$T_k(x) = \cos(k\theta) = 0 \tag{15}$$

Which occurs when $k\theta = \frac{\pi}{2} + m\pi$ for some integer m. Solving gives

$$\theta_{roots} = \frac{\pi}{2k} + \frac{m\pi}{k} \tag{16}$$

The extrema are found by setting

$$T_k'(x) = \frac{k\sin(k\theta)}{\sin(\theta)} = 0 \tag{17}$$

This occurs when $\sin(k\theta) = 0$, or $k\theta = m\pi$. Solving gives

$$\theta_{ext} = \frac{m\pi}{k} \tag{18}$$