

Chebyshev Polynomials

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Chebyshev Polynomial Definition

The k th Chebyshev polynomial can be defined as

$$T_k(x) = \cos(k\theta) \quad (1)$$

where $\theta = \arccos(x)$.

Orthogonality (based on ATAP exercise 3.7)

Theorem 3.1 in ATAP states if f is Lipschitz continuous on $[-1, 1]$, it has a unique representation as a Chebyshev series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x) \quad (2)$$

which is absolutely and uniformly convergent. The coefficients are given for $k \geq 1$ by the formula

$$a_k = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx \quad (3)$$

and for $k = 0$ by the same formula with the factor $2/\pi$ changed to $1/\pi$. Here, we derive the weighted orthogonality conditions for Chebyshev polynomials, and show how these can be used to derive eq 3.

The **orthogonality condition for Chebyshev polynomials** for integers j and k to be derived is

$$\int_{-1}^1 \frac{T_j(x) T_k(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & j \neq k \\ \pi, & j = k = 0 \\ \frac{\pi}{2}, & j = k \geq 1 \end{cases} \quad (4)$$

Inserting the definition of Chebyshev polynomials

$$\int_{-1}^1 \frac{T_j(x) T_k(x)}{\sqrt{1-x^2}} dx = \int_{-1}^1 \frac{\cos(j \arccos(x)) \cos(k \arccos(x))}{\sqrt{1-x^2}} dx \quad (5)$$

Making the change of variables $x = \cos(\theta)$, $dx = -\sin(\theta)d\theta$, the definition becomes

$$- \int_{-1}^1 \cos(j\theta) \cos(k\theta) d\theta \quad (6)$$

As x ranges from -1 to 1 in the integral, θ ranges from π to 0 . Allowing θ to instead range from 0 to π accounts for the minus sign in the integral above. To summarize, we have so far found

$$\int_{-1}^1 \frac{T_j(x) T_k(x)}{\sqrt{1-x^2}} dx = \int_0^\pi \cos(j\theta) \cos(k\theta) d\theta \quad (7)$$

Using trigonometric identities

$$\int_0^\pi \cos(j\theta) \cos(k\theta) d\theta = \frac{1}{2} \int_0^\pi \cos((j-k)\theta) d\theta + \frac{1}{2} \int_0^\pi \cos((j+k)\theta) d\theta \quad (8)$$

Since j and k are integers, both $(j - k)$ and $(j + k)$ are integers, say m . The integral of $\int_0^\pi \cos(mx)dx = 0$ for all integers $m \neq 0$. In other words, eq 8 is only non zero when $j = k$. Considering this case

$$\int_0^\pi \cos(j\theta) \cos(k\theta)d\theta = \frac{1}{2} \int_0^\pi \cos(0)d\theta + \frac{1}{2} \int_0^\pi \cos(2k\theta)d\theta \quad (9)$$

Now, if $j = k = 0$

$$\frac{1}{2} \int_0^\pi \cos(0)d\theta + \frac{1}{2} \int_0^\pi \cos(0)d\theta = \int_0^\pi \cos(0)d\theta = \pi \quad (10)$$

If $j = k \geq 1$, then $2k$ is an integer, and its cosine integral vanishes

$$\frac{1}{2} \int_0^\pi \cos(0)d\theta + \frac{1}{2} \int_0^\pi \cos(2k\theta)d\theta = \frac{1}{2} \int_0^\pi \cos(0)d\theta = \frac{\pi}{2} \quad (11)$$

all three cases have been considered, leading to the orthogonality conditions in eq 4.

The coefficients can now be derived beginning with eq 2. Multiplying by $T_j(x)/\sqrt{1-x^2}$

$$\frac{T_j(x)f(x)}{\sqrt{1-x^2}} = \sum_{k=0}^n \frac{a_k T_j(x)T_k(x)}{\sqrt{1-x^2}} \quad (12)$$

where on the RHS, $T_j(x)/\sqrt{1-x^2}$ may be moved inside the sum because both terms are independent of k . Integrating both sides over the domain $[-1, 1]$

$$\int_{-1}^1 \frac{T_j(x)f(x)}{\sqrt{1-x^2}}dx = \int_{-1}^1 \sum_{k=0}^n \frac{a_k T_j(x)T_k(x)}{\sqrt{1-x^2}}dx = \sum_{k=0}^n a_k \int_{-1}^1 \frac{T_j(x)T_k(x)}{\sqrt{1-x^2}}dx \quad (13)$$

where again the integral may be pulled inside the sum, as the integral depends on x , the sum on k . By the orthogonality conditions (eq 4), the last integral is only nonzero when $j = k$, causing the infinite sum to collapse, leaving

$$\int_{-1}^1 \frac{T_j(x)f(x)}{\sqrt{1-x^2}}dx = a_j \int_{-1}^1 \frac{T_j(x)^2}{\sqrt{1-x^2}}dx = \begin{cases} a_0 \cdot \pi, & j = 0 \\ a_j \cdot \frac{\pi}{2}, & j \geq 1 \end{cases} \quad (14)$$

leading directly to eq 3.