Exam 2 Warm Up

Calculus 1 Spring 2025

I. MASONS NOTE

A lot of this is completely copy-pasted from my previous quiz warm ups, is to be used as a study resource, and **is not an indication of the questions on the exam**.

II. THE DERIVATIVE

$$\frac{d}{dx}[\quad] \tag{1}$$

means take the derivative with respect to x of whatever lies in $[\quad]$. If it is $f(x)$, then

$$\frac{d}{dx}f(x) = f'(x) \tag{2}$$

and $f'(x)$ is called the derivative of $f(x)$. Geometrically, **the derivative is the slope of the function** (as shown later). It is useful: The derivative of position is velocity, the derivative of charge flow is current, many computer algorithms rely on taking derivatives to navigate towards minimum values, etc.

III. 2.7 DERIVATIVES AND RATES OF CHANGE

A. Slopes of Secant Lines and Average Velocities are the Same

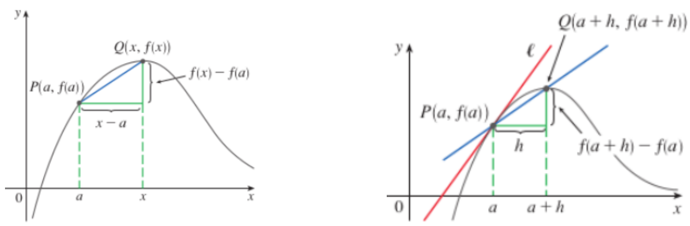
In section 2.1, the conceptual jump was learning the average velocity between two points is the same thing as finding the slope of the secant line between them. This section formalizes this idea. The definition of the derivative at point a is

$$f'(a) = \lim_{x\to a}\frac{f(x) - f(x)}{x - a} \tag{3}$$

The definition of the instantaneous velocity of an object with position $f(t)$ at time $t = a$ is

$$f'(a) = v(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} \tag{4}$$

To show they are the same, define $x = a + h$, which gives $h = x - a$. Based on this definition, the $\lim_{x\to a}$ is the same thing as the $\lim_{h\to 0}$, so the above equations become equivalent. The following figures show geometric representations of both definitions, and that both compute the same thing, the rise over run of a (an infinitesimal) triangle.



(a) (b)

FIG. 1

IV. 2.8 THE DERIVATIVE AS A FUNCTION

In the previous section, the derivative existed at a point a, i.e. $f'(a)$. Here, we consider the derivative as a function that takes values at many points. Use the same definition we saw in section 2.7, but replace a with x so that the derivative is general.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{5}$$

A. Worked Example

(Exam 2 Sample, Question 8) Find $f'(x)$ using the definition of the derivative if $f(x) = x^2 + x$. By the power rule, we expect to end up with $f'(x) = 2x + 1$. Using the definition gives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \tag{6}$$

Simplifying, and evaluating the limit gives

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{h(2x + h + 1)}{h} = 2x + 1 \tag{7}$$

Which agrees with the power rule as expected.

V. 3.1 DERIVATIVES OF POLYNOMIALS AND EXPONENTIAL FUNCTIONS

A. Derivative of a Constant Function

If $f(x) = c$, where c is some constant, then

$$\frac{d}{dx} f(x) = \frac{d}{dx} c = 0 \tag{8}$$

The graph of $f(x) = c$ is the line $y = c$ shown below. Asking "what is the derivative of $f(x)$?" is the same question as "what is the slope of $f(x)$?". The **slope of a constant function is 0**, so $f'(x) = 0$ for any constant function.

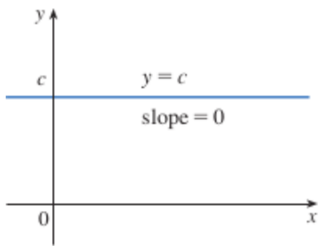


FIG. 2

B. Power Rule

If $f(x) = x^n$, where n is a real number, then

$$\frac{d}{dx} x^n = n x^{n-1} \tag{9}$$

Using the power rule, if $f(x) = x$, then $f'(x) = 1$. From a geometric perspective, $f(x) = x$ is the line with slope 1 everywhere, therefore its derivative is 1, as shown in the figure below.

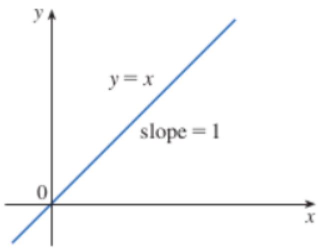


FIG. 3

C. Constant Multiple Rule

If c is some constant and f is differentiable, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \tag{10}$$

D. Worked Example

Find $g'(x)$ if $g(x) = 3x^{13}$.

$$g'(x) = 3\frac{d}{dx}x^{13} = 39x^{12} \tag{11}$$

E. The Sum and Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \tag{12}$$

F. Worked Example

Find $g'(x)$ if $g(x) = 2x^3 + 14x + 99999999$.

$$g'(x) = \frac{d}{dx}[2x^3 + 14x + 99999999] = \frac{d}{dx}2x^3 + \frac{d}{dx}14x + \frac{d}{dx}99999999 = 6x^2 + 14 \tag{13}$$

G. Exponential Functions

The exponential function, e^x, is the function who is its own derivative. If $f(x) = e^x$, then $f'(x) = e^x$. However, if $f(x) = e^{cx}$, where c is some constant, then

$$\frac{d}{dx}e^{cx} = \frac{d}{dx}[cx]e^{cx} = ce^{cx} \tag{14}$$

I give a spiel about the use of e^x on pages 2 and 14 of my exam 2 review in the Canvas files.

H. Worked Example

If $g(x) = 3e^{-x}$, then by the constant multiple rule and the derivative of exponential functions,

$$\frac{d}{dx}3e^{-x} = 3\frac{d}{dx}[-x]e^{-x} = -3e^{-x} \tag{15}$$

VI. 3.2 THE PRODUCT AND QUOTIENT RULES

A. Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \tag{16}$$

B. Worked Example

Find $f'(x)$ if $f(x) = x^2 g(x)$.

$$f'(x) = \frac{d}{dx}x^2 g(x) = 2xg(x) + x^2 g'(x) \tag{17}$$

C. Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \tag{18}$$

D. Worked Example

Find $f'(x)$ if $f(x) = 4x/e^x$.

$$f'(x) = \frac{d}{dx}\frac{4x}{e^x} = \frac{4e^x - 4xe^x}{e^{2x}} = \frac{4e^x(1-x)}{e^{2x}} = 4e^{-x}(1-x) \tag{19}$$

VII. 3.3 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$\frac{d}{dx}\sin(x) = \cos(x)$,

$\frac{d}{dx}\cos(x) = -\sin(x)$,

$\frac{d}{dx}\tan(x) = \sec^2(x)$,

$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$,

$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$,

and last but not least $\frac{d}{dx}\cot(x) = -\csc^2(x)$. Heres a visual representation of $\frac{d}{dx}\sin(x) = \cos(x)$ for fun.

VIII. 3.4 THE CHAIN RULE

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f(g(x))$ is differentiable, and

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) \tag{20}$$

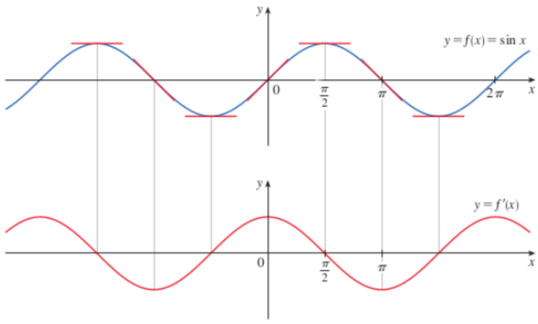


FIG. 4

A. Worked Example

Find $h'(x)$ if $h(x) = e^{\sin(\sqrt{x})}$. Let $h(x) = f(g(x))$ where $f(x) = e^x$ and $g(x) = \sin(\sqrt{x})$.

$$h'(x) = \frac{d}{dx} e^{\sin(\sqrt{x})} = e^{\sin(\sqrt{x})} \frac{d}{dx} \sin\left(\sqrt{x}\right) \tag{21}$$

We need the chain rule again. Then $\sin(\sqrt{x}) = r(t(x))$, where $r(x) = \sin(x)$ and $t(x) = \sqrt{x} = x^{1/2}$. So

$$\frac{d}{dx} \sin\left(\sqrt{x}\right) = \cos\left(\sqrt{x}\right) \frac{1}{2} x^{-1/2} \tag{22}$$

and the final answer is

$$h'(x) = \frac{d}{dx} e^{\sin(\sqrt{x})} = e^{\sin(\sqrt{x})} \frac{d}{dx} \sin\left(\sqrt{x}\right) = e^{\sin(\sqrt{x})} \cos\left(\sqrt{x}\right) \frac{1}{2} x^{-1/2} = \frac{e^{\sin(\sqrt{x})} \cos\left(\sqrt{x}\right)}{2\sqrt{x}} \tag{23}$$

IX. 3.5 IMPLICIT DIFFERENTIATION

A. Implicit Equations and Differentiation

Have so far dealt with "explicit" equations, where one variable is written explicitly in terms of another. Examples are $y = x\sin(x)$ and $y = x^2 + 14x$. There are also "implicit" equations, which only have a relation between two variables. Examples are $x^2 + y^2 = 25$ and $x^3 + y^3 = 6xy$. To do **implicit differentiation, differentiate both sides of the equation with respect to x, then algebraically solve for dy/dx.**

B. Worked Example

Find the derivative of $x^3 + y^3 = 6xy$ using implicit differentiation

$$\begin{aligned} 3x^2 + 3y^2 \frac{dy}{dx} &= 6x\frac{dy}{dx} + 6y \\ 3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned} \tag{24}$$

C. Why?

When taking the derivative of y, multiplying by $\frac{dy}{dx}$ comes from the chain rule. We are assuming there may be some hidden dependence of y on x. The idea is to look at the variable we are taking the derivative with respect to, if they "match" (i.e. $\frac{d}{dx}x$), do normal differentiation, if they don't "match" (i.e. $\frac{d}{dx}y$), do implicit differentiation. This is not formal, but hopefully helps with intuition.

X. 3.6 DERIVATIVE OF LOGARITHMIC AND INVERSE TRIGONOMETRIC FUNCTIONS

There are two derivative rules from this subsection

A. Derivative of Logarithmic Functions

$$\frac{d}{dx}(\log_b x) = \frac{1}{x\ln(b)} \tag{25}$$

and

$$\frac{d}{dx}b^x = b^x \ln(b) \tag{26}$$

using the first rule with $b = e$, we see

$$\frac{d}{dx}(\log_e x) = \frac{d}{dx}\ln(x) = \frac{1}{x\ln(e)} = \frac{1}{x} \tag{27}$$

B. Worked Example

Find the derivative of $y = \sqrt{\ln(x)}$

$$y' = \frac{1}{2}\ln(x)^{-1/2}(\frac{1}{x}) = \frac{1}{2x\sqrt{\ln(x)}} \tag{28}$$

C. Derivatives of Inverse Trigonometric Functions

The derivatives of trigonometric functions were introduced in section 3.3, here we visit the inverse trigonometric functions. To see how they work, if $y = \tan^{-1}(x)$, then $\tan(y) = x$. This idea is the starting point for taking the derivative. The trig identities $\sin^2(x) + \cos^2(x) = 1$ and $\tan^2(x) = \sec^2(x) + 1$ are used to simplify the resulting expression.

D. Worked Example

Find the derivative of $y = \cos^{-1}(x)$. We know $\cos(y) = x$, doing implicit differentiation on this gives

$$-\sin(y)\frac{dy}{dx} = 1 \quad \rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin(y)} \tag{29}$$

Using trig identities, $\sin(y) = \sqrt{1-\cos^2(y)}$. From the original expression, $cos(y) = x$. Making both of these substitutions gives

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2(y)}} \quad \rightarrow \quad \frac{-1}{\sqrt{1-x^2}} \tag{30}$$

XI. 3.7 RATES OF CHANGE IN NATURAL AND SOCIAL SCIENCES

There is no "new math" in this section, but applications of the derivative to scientific fields are explored. The book gives uses in physics, chemistry, biology, economics, as well as others.

A. Worked Example

(Exam 2 Sample, Question 7) Suppose a particle has position function $s = f(t) = t^2 - 6t + 10$, for $t \geq 0$.

1) Determine where the particle is at rest:

The particle will be at rest when the velocity is 0. (If I throw a ball up, it has positive velocity. On its way down, it has negative velocity. At the point the ball turns around, it must have 0 velocity.) Using the fact that velocity is the time derivative of position,

$$v(t) = f'(t) = 2t - 6 \tag{31}$$

Then

$$v(t) = 0 \quad \rightarrow \quad 2t - 6 = 0 \quad \rightarrow \quad t = 3 \tag{32}$$

At time $t = 3$, the particle has 0 velocity.

2) On what interval(s) is the particle moving in the positive direction?:

The particle is moving in the positive direction when $v(t) > 0$.

$$v(t) > 0 \quad \rightarrow \quad 2t - 6 > 0 \quad \rightarrow \quad t > 3 \tag{33}$$

When $t > 3$, the particle is moving in the positive direction.

3) Determine the total distance traveled by the particle in the first 5 seconds:

The total distance traveled is the sum of the paths where the direction was unchanged. Here, the particle is moving negatively from $t = 0$ to $t = 3$, and positively from $t = 3$ to $t = 5$. The distance from $t = 0$ to $t = 3$ is

$$|f(0) - f(3)| = |10 - 1| = 9 \tag{34}$$

and the distance from $t = 3$ to $t = 5$ is

$$|f(3) - f(3)| = |1 - 5| = 4 \tag{35}$$

Then the total distance is $9 + 4 = 13$.

XII. 3.9 RELATED RATES

Idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity, normally one quantity is easier to measure than the other. Next, find an equation to relate the two quantities of interest, then use implicit differentiation with respect to time. Lab 10 is a good study resource for these questions.

A. Worked Example

(Exam 2 Sample, Question 6) A ladder 12 feet in length is propped up against a vertical wall. The foot of the ladder slides away from the wall along the floor, while the top of the ladder maintains contact with the wall. When the foot of the ladder is 8 feet from the wall it is moving away from the wall at 4 feet per second. At that point how fast is the top of the ladder moving down the wall?

These questions should always start with drawing a picture, but I can't (easily) do that here. Set up a triangle where the hypotenuse is the ladder (12ft), the bottom of the triangle is 8ft (given from the question), then using the Pythagorean theorem, the height of the triangle is ≈ 8.9 ft. Now, start with the Pythagorean theorem in general

$$x^2 + y^2 = z^2 \quad \rightarrow \quad x^2 + y^2 = 12^2 \tag{36}$$

Since the length of the ladder, z, is fixed. Perform implicit differentiation with respect to time

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \tag{37}$$

Using the given information, and what we deduced about x and y, the only unknown in the above equation is dy/dt. Using what we know, solve for dy/dt

$$2(8)(4) + 2(8.9)\frac{dy}{dt} = 0 \quad \rightarrow \quad \frac{dy}{dt} = -3.6 \tag{38}$$

XIII. PRACTICE PROBLEMS

Exam 2 Sample Questions posted by Dr. Tran in an announcement on 3/12. My solutions are posted in files, some of the problems are repeated here.