Calculus 1 Exam 1 Notes

Calculus 1 Spring 2025

I. 2.1 THE TANGENT AND VELOCITY PROBLEMS

A. Tangent line as the limit of secant lines

The slope of the secant line between two points $P = (p_x, p_y), Q = (q_x, q_y)$ is

$$m_{PQ} = \frac{p_y - q_y}{p_x - q_x} \tag{1}$$

and represents the "rise over run" of a given function. As the distance between p_x and q_x shrinks, the secant line approaches the tangent line. In other words, the tangent line is the limit of the secant line as the distance between p_x and q_x approaches 0. The following figure demonstrates the idea, ℓ being the tangent line.

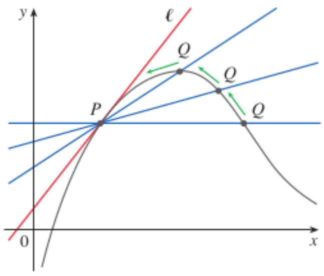

FIG. 1

B. Building the Tangent Line

To estimate the slope of the tangent line at a point P, take multiple values of q_x and approach p_x from both sides, keeping track of the values of q_x and m_{PQ} in a table. What value is m_{PQ} approaching as $q_x \to p_x$ from the left and right? If the two sides approach the same finite value, this will be the estimation of the slope at P, call it m_P.

$$m_P \approx \lim_{q_x \to p_x^-} \frac{p_y - q_y}{p_x - q_x} = \lim_{q_x \to p_x^+} \frac{p_y - q_y}{p_x - q_x} \tag{2}$$

With m_P in hand, use the point slope formula

$$y - p_y = m_P(x - p_x) \tag{3}$$

and we have an estimation of the tangent line at P. This line is (at least an estimate for) ℓ in fig (1).

C. Velocity

Finding the average velocity between two points is the same thing as finding the slope of the secant line between them, as done in eq (1). The instantaneous velocity at a point P is analogous to estimating the slope at a point as done in eq (2).

II. 2.2 THE LIMIT OF A FUNCTION

A. Definition

A limit refers to the behavior that a function ($f(x)$) approaches as the input (x) approaches a certain value (c).

$$\lim_{x\to c} f(x) = L \qquad \iff \qquad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \tag{4}$$

L representing the limit, or value that $f(x)$ approaches (the arrow reading "if and only if"). Note that x never gets to the point c, just arbitrarily close.

B. Understanding the Definition

In the following figure, $\lim_{x\to 4} f(x) = 4$, even though $f(4)$ is undefined, because x never gets to 4 when the limit is taken, and the function values approach 4 from both sides. Also, $\lim_{x\to 2} f(x) = \text{DNE}$, since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$.

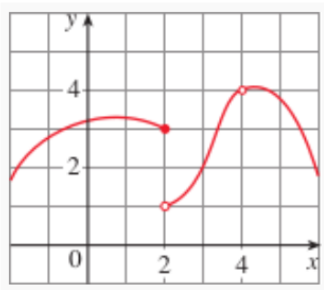

FIG. 2

Limits may also take the values $\pm\infty$. The following figure shows an example where $\lim_{x\to a} f(x) = \infty$.

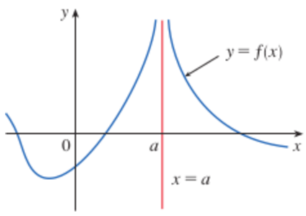

FIG. 3

III. TABLE METHOD

To find

$$\lim_{x\to c} f(x) = L \tag{5}$$

using the table method, build a column of x values approaching c from the left right, and build a corresponding column with $f(x)$ values. The limit L is determined based on $f(x)$ as x is closer and closer to c. Again, this is estimating the slope at a point, as done in eq (2).

IV. 2.3 CALCULATING LIMITS USING LIMIT LAWS

A. Limit Laws

There are 5 limit laws, know them as well as how to apply them. As an example of two, the "constant multiple law" and "sum law", if $\lim_{x\to 4} f(x) = 3$ and $\lim_{x\to 4} g(x) = -1$, then

$$\lim_{x\to 4}[2f(x) + 3g(x)] = 2\lim_{x\to 4} f(x) + 3\lim_{x\to 4} g(x) = 2(3) + 3(-1) = 3 \tag{6}$$

B. Direct Substitution

When evaluating $\lim_{x\to c} f(x)$, if plugging in $x = c$ does not give an indeterminate form, this is "direct substitution". Always try this first, as it is the easiest method, but most likely to fail.

C. Algebraic Manipulation

When direct substitution fails, the limit may be rearranged with algebra to be put into a form which can be evaluated by direct substitution. Lab 2 is a good study guide for this topic. In it, methods include multiplying by the conjugate, factoring, getting a common denominator to add fractions, etc.

D. Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, except possibly at a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \tag{7}$$

then $\lim_{x\to a} g(x) = L$. In words, if $f(x)$ is both bigger than or equal to something, and less than or equal to something, then $f(x)$ is also equal to something. It's use comes in deducing answers to more complex questions through a simpler approach.

As an example, find

$$\lim_{x\to 0} x\cos\left(\frac{27\pi}{x}\right) \tag{8}$$

$cos(ax)$ for any number a is bounded between -1 and 1. The only effect a has is on the frequency, so

$$-1 \leq \cos(27\pi x) \leq 1 \tag{9}$$

Multiplying by the same thing everywhere

$$-x \leq x\cos(27\pi x) \leq x \tag{10}$$

Taking the limit everywhere

$$\lim_{x\to 0} -x \leq \lim_{x\to 0} x\cos(27\pi x) \leq \lim_{x\to 0} x \tag{11}$$

Evaluate the outside limits

$$0 \leq \lim_{x\to 0} x\cos(27\pi x) \leq 0 \tag{12}$$

Then by the squeeze theorem,

$$\lim_{x\to 0} x\cos\left(\frac{27\pi}{x}\right) = 0 \tag{13}$$

V. 2.4 PRECISE DEFINITION OF A LIMIT

A. Definition

The limit of $f(x)$ as x approaches a is L if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \textit{ then } |f(x) - L| < \epsilon \tag{14}$$

The inequalities in eq 14 can be expanded to

$$a - \delta < x < a + \delta \textit{ and } L - \epsilon < f(x) < L + \epsilon \tag{15}$$

Graphically, the following figure shows how the definition "traps" the limit from all sides

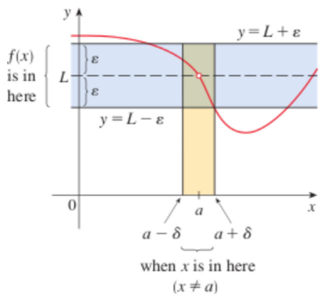

FIG. 4

B. Solving Problems

To use the definition to prove a limit, begin your "scratch" work with $|f(x) - L| < \epsilon$. Plug in the $f(x)$ and L from the specific problem, and work until you have something of the form $|x - a| <$ some(ϵ), where the last term just means some number in terms of epsilon, for example $\epsilon/5$. Then, choose $\delta =$ some(ϵ). Begin the proof with $|x - a| < \delta =$ some(ϵ), then work until you have $|f(x) - L| < \epsilon$. I have worked examples from the solutions to lab 3.

VI. 2.5 CONTINUITY

A. Definition

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a) \tag{16}$$

in words, there are no holes in the graph, and the limit can be evaluated by direct substitution.

B. Types

The book gives 3 types of discontinuities. Removable, jump, and infinite. Lab 3 has good practice for the different types.

A function can also be continuous from one side, and discontinuous from the other. In the following figure, there are discontinuities at $x = 1$ (removable), $x = 3$ (jump), and $x = 5$ (removable). Note $x = 1$ and $x = 5$ are discontinuous from either side, but $x = 3$ is continuous from the left, since $\lim_{x \to 3^-} f(x) = f(3)$.

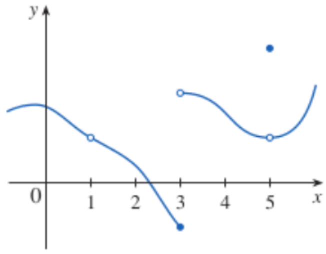

FIG. 5

C. Intermediate Value Theorem

Suppose f is continuous on $[a, b]$, let N be any real number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

To use the IVT to prove solutions exist (that is, values of x that give $f(x) = 0$), find two inputs x_1 and x_2 that give $f(x_1)f(x_2) < 0$. Because one function value was negative, and the other positive, the IVT guarantees there is some value of c between x_1 and x_2 that gives $f(x) = 0$, proving c is a solution.

D. Piecewise Functions

VII. 2.6 LIMITS AT INFINITY; HORIZONTAL ASYMPTOTES

Horizontal asymptotes describe the functions behavior as x goes to $\pm\infty$. I'll leave this as an example to be worked, show $\lim_{x\to\infty} \sqrt{x^2 - 1} + x = 0$. The function is shown in the figure below, with a horizontal asymptote at $x = 0$ as expected.

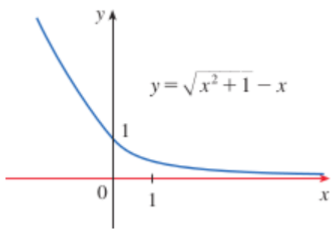

FIG. 6