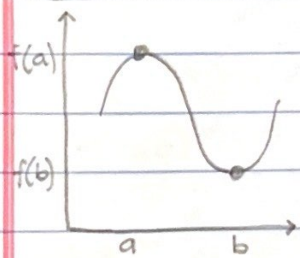
Mason Mault

1

Chapter 4 & 5 Notes

Sec 4.1

Maximum and Minimum Values

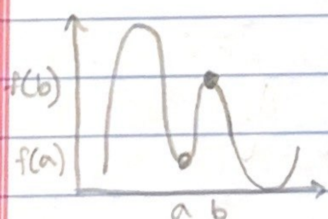


$f(a)$ a maximum
$f(b)$ a minimum

D

Let $c \in \mathbb{R}$ be in the domain of f. Then $f(c)$ is the

global/absolute maximum if $f(c) \geq f(x)$ $\forall x \in D$
global/absolute minimum if $f(c) \leq f(x)$ $\forall x \in D$



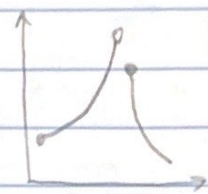
$f(b)$ a local maximum
$f(a)$ a local minimum

$f(c)$ is the
local maximum if $f(c) \geq f(x)$ if $x \approx c$
local minimum if $f(c) \leq f(x)$ if $x \approx c$

The Extreme Value Theorem

If f continuous on $[a,b]$, then f attains an absolute maximum $f(c)$ and absolute minimum $f(d)$ at some $c, d \in [a,b]$

Break hypothesis,
function has no
max value.



Fermats Theorem

If f has a local min/max at c, if $f'(c)$ exists,
the $f'(c) = 0$.

A critical number of f is a number c in the domain of f st either $f'(c) = 0$ or $f'(c)$ DNE

Closed Interval Method

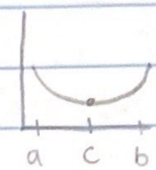
To find the absolute max/min values of a continuous function f on closed interval $[a,b]$:

1) Find values of f at its critical numbers in (a,b)
2) Find values of f at endpoints $[a,b]$
3) Largest of 1, 2 = absolute max
Smallest of 1, 2 = absolute min

③

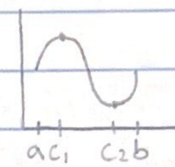
Sec 4.2

Rolle's Theorem



Let f satisfy the following

1) f continuous on [a,b]
2) f differentiable on (a,b)
3) f(a) = f(b)



Then $\exists c \in (a,b)$ st $f'(c) = 0$

ex) Throw a ball up at time a, catch it at time b. There must be a time $c \in (a,b)$ st $f'(c)=0$, that is when the ball turns around

Mean Value Theorem

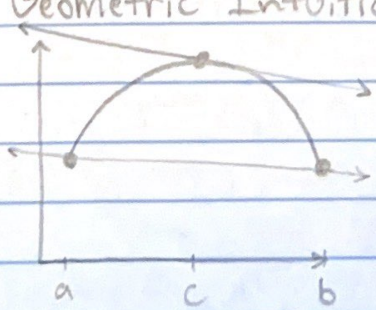
Let f satisfy the following

1) f continuous on [a,b]
2) f differentiable on (a,b)

Then $\exists c \in (a,b)$ st

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \text{ equiv, } f'(c)(b-a) = f(b)-f(a)$$

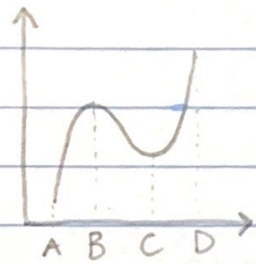
Geometric Intuition



The secant line from a to b has same slope as tangent line at point c.

sec 4.3

What does f' say about f?



Between AB, CD, tangent lines have positive slope & $f'(x) > 0$.
BC, tangent has negative slope & $f'(x) < 0$

Seems that f increases when $f'(x) > 0$ & decreases when $f'(x) < 0$. Now to prove

Proof Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Assume $f'(x) > 0$ on the interval, want to show $f(x_1) < f(x_2)$. f must be differentiable (since $f'(x)$ defined as >0 on (x_1, x_2).
By mean value theorem, garunteed a $c \in (x_1, x_2)$ st

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

By assumption $f'(c) > 0$, and $x_1 < x_2 \rightarrow (x_2 - x_1) < 0$.
If RHS > 0, so must the LHS. $f(x_2) - f(x_1) > 0$ gives

$$f(x_1) < f(x_2) \quad \blacksquare$$

The other way would work the same way. ($f(x_1) > f(x_2)$ if $f'(x) < 0$)

Main point: if $f'(x) > (<)\ 0$ on some interval, $f(x)$ increasing (decreasing) on that interval.

First Derivative Test

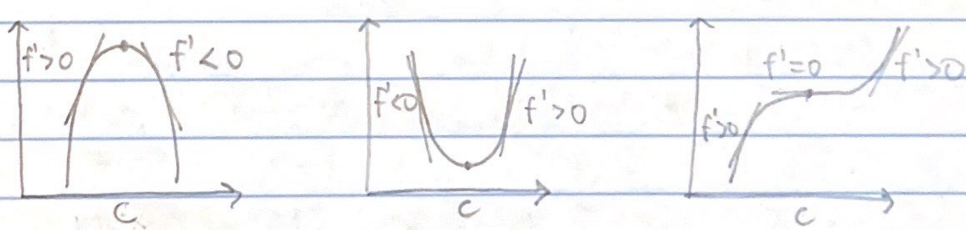
Saw that max/min occur at critical numbers c, but not all c's give a max/min. How to tell whats happening at c?

Suppose c is a critical number of continuous f.

a) If f' changes from + → − at c, f has a local max at c.

b) If f' changes from − → + at c, f has a local min at c.

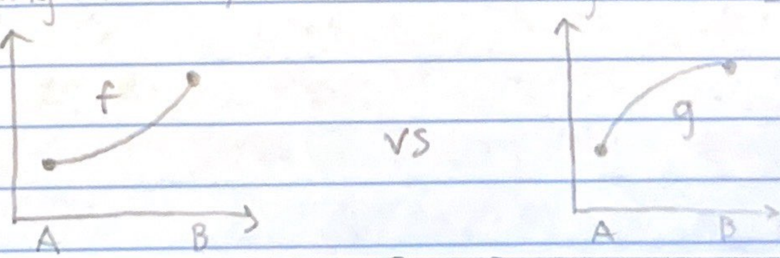
c) If f' does not change sign at c, there is no local min/max. at c.



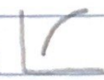
max   min   nothing

Second Derivative Test

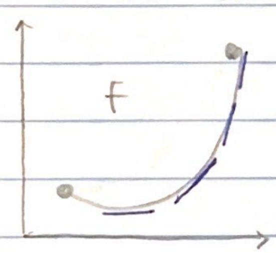
How can we distinguish between the two curves joing A to B, both increasing? Their concavity

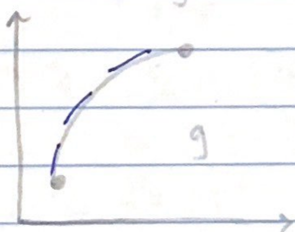


f would be concave up 

g would be concave down 

Look at each curves tangent lines





If the graph of a function lies above all its tangents on an interval (f), it is concave up. If it lies below its tangents on an interval (g), it is concave down.

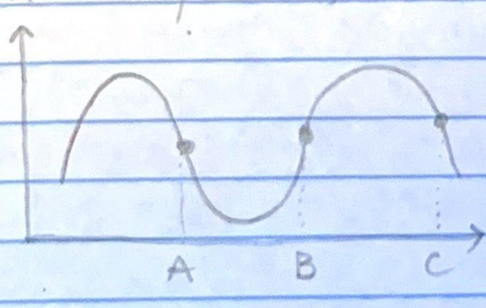
Concavity Test

(a) If $f''(x) > 0$ on interval, I, f concave up on I

(b) If $f''(x) < 0$ on I, concave down on I.

Inflection Point

If f is continuous at point P, and f changes concavity at P, P is an inflection point of f.



A,B,C inflection points

Second Derivative Test Pt 2

Suppose f'' is continuous near critical point c

a) If $f'(c) = 0$ and $f''(c) > 0$, f has local min at c

b) If $f'(c) = 0$ and $f''(c) < 0$, f has local max at c

Curve Sketching

Use info about f' & f'' to graph f.

Process: Find some f(x) you want to graph, find f', f''. Find critical numbers, use first and second derivative tests at these critical numbers, plot info gained.

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Worked example for Sec 4.3

53) $f(x) = x\sqrt{6-x}$

a) Find increasing/decreasing intervals

b) Find local min/max

c) Find intervals of concavity and inflection points

d) Sketch the curve

Get y' and y''

$$y' = \sqrt{6-x} - \frac{x}{2}(6-x)^{-\frac{1}{2}} = \boxed{\sqrt{6-x} - \frac{x}{2\sqrt{6-x}}} \rightarrow \boxed{\frac{3(4-x)}{2\sqrt{6-x}} = y'}$$

after algebra, $\boxed{y'' = \frac{3(x-8)}{4(6-x)^{3/2}}}$

a) Need critical points, points at which $y'=0$.

$$\frac{3(4-x)}{2\sqrt{6-x}} = 0$$ goes to 0 @ $x=4$, und past $x=6$

need to check $(-\infty, 4), (4,6)$

f increasing when $f' > 0$, check $x=3$ and $x=5$

$x=3 \rightarrow \frac{3(1)}{2\sqrt{3}} > 0 \rightarrow$ $\boxed{f \text{ increasing } (-\infty, 4)}$

$x=5 \rightarrow \frac{3(-1)}{2\sqrt{1}} < 0 \rightarrow$ $\boxed{f \text{ decreasing } (4,6)}$

$\boxed{f \text{ undefined } (6, \infty)}$

(a)

b) If f increasing until $x=4$, then decreasing after $x=4$, . Another perspective is f' changed from $+ \to -$, by first derivative test this point is a local max.

$f(4) = 4\sqrt{6-4} = \boxed{4\sqrt{2}\text{, max @ } x=4}$

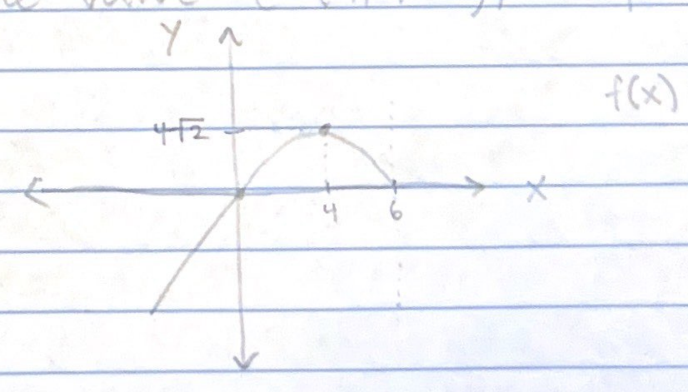
c) Use concavity test. down $f''<0$
Concave up $f''>0$, Check $x=3,5$

$f''(3) = \frac{3(-5)}{4(3)^{3/2}} < 0$, concave down $(-\infty,4)$

$f''(5) = \frac{3(-3)}{4(1)^{3/2}} < 0$, concave down $(4,6)$

Concave down on $(-\infty,6)$, no inflection points since concavity never changed.

d) Concave down everywhere, increasing $(-\infty,4)$, dec $(4,6)$, only extreme value @ $(4, 4\sqrt{2})$, und past 6, $f(0)=0$



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Sec 4.4

Indeterminate Forms & L'Hospitals Rule

If the limit has form

$\lim_{x \to a} \frac{f(x)}{g(x)}$, where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$,

called an indeterminate form of type $\frac{0}{0}$.
Same can occur if $f(x) \to g(x) \to \infty$ as $x \to a$.

L'Hospitals Rule

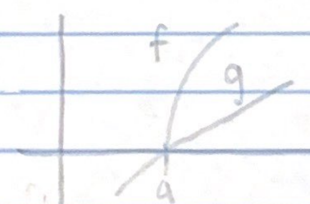
Suppose f and g differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). Suppose

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, or that

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ if the RHS exists.

Intuition help: Consider



Zooming in on a, the graphs look ≈ linear.

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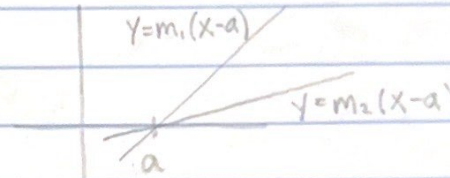
If the graphs were linear



ratio of graphs

$$\frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2} \approx \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Notes

- Rule says limit of quotient of functions is equal to limit of quotient of derivatives, if conditions met.

- Also works for one sided limits, infinite limits.

ex) $\displaystyle \lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{\left(\frac{1}{x}\right)}{1}$

$\displaystyle = \lim_{x \to 1} \frac{1}{x} = \boxed{1}$

ex) $\displaystyle \lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} \boxed{= \infty}$

Indeterminant Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, can be difficult to compute $\lim_{x \to a} [f(x) g(x)]$.

These limits called indeterminant form of type $0 \cdot \infty$.

Deal with these by rewriting fg as a quotient

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

This converts the problem to one of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and we can use L'Hospitals rule.

ex) $\lim\limits_{x \to 0^+} x\ln x = \lim \dfrac{\ln x}{1/x} = \lim \dfrac{(1/x)}{(-1/x^2)} = \lim\limits_{x \to 0^+} (-x) \boxed{= 0}$

could also do

$\lim x\ln x = \lim \dfrac{x}{1/\ln(x)}$, but top is easier, so always choose the easier version.

Indeterminant Differences

If $\lim f(x) = \lim g(x) = \infty$, then

$\lim [f(x) - g(x)]$ may be hard to compute. Deal with these by converting difference to a quotient.

ex) $\lim\limits_{x \to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1} \right) \overset{\lim}{=} \dfrac{x-1-\ln x}{(x-1)\ln x} \overset{\lim}{=} \dfrac{1 - \frac{1}{x}}{(x-1)\frac{1}{x} + x\ln x}$

$= \lim \dfrac{x-1}{1 - \frac{1}{x} + x\ln x} = \lim \dfrac{x-1}{x-1+x\ln x} = \lim \dfrac{1}{1 + x\left(\frac{1}{x}\right) + \ln x}$

$= \lim\limits_{x \to 1^+} \dfrac{1}{2 + \ln x} = \boxed{\dfrac{1}{2}}$

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Indeterminant Powers

Limits of type $\lim_{x \to a} = [f(x)]^{g(x)}$

Can result in 3 indeterminant form, 0^0, ∞^0, 1^∞.

Treat these in one of two ways,

let $y = [f(x)]^{g(x)} \to \ln(y) = g(x)\ln(f(x))$, or

$[f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$

Either method leads to indeterminant form $0 \cdot \infty$

ex) $\lim_{x \to 0^+} (1+\sin 4x)^{\cot x}$

Let $y = (1+\sin 4x)^{\cot x}$, then take natural log

$\ln(y) = \cot(x)\ln[1+\sin 4x] = \dfrac{\ln[1+\sin 4x]}{\tan(x)}$

$\lim_{x \to 0^+} \ln(y) = \lim_{x \to 0^+} \dfrac{\ln[1+\sin 4x]}{\tan x} = \dfrac{\left[\dfrac{4\cos x}{1+\sin 4x}\right]}{\sec^2 x} = 4$

computed lim of $\ln(y)$

$\lim \ln(y) = 4$

$\lim \boxed{y = e^4}$

(14)

Sec 4.5

Summary of Curve Sketching

Make a "checklist" of info we want

A) Domain, set of x's for which $f(x)$ defined

B) Intercepts, y int $= f(0)$. x int, set $f(x)=0$, solve x.

C) Symmetry, if applicable, use even/oddness/periodicity

D) Asymptotes, find them

E) Intervals of increase/decrease, check if $f'(x) >, < 0$

F) Local Min/Max, 1st derivative test, check if $f'(c)$ changes sign at critical point c.

G) Concavity and Inflection Points, check if $f''(x) >, < 0$

H) Sketch

Slant Asymptotes

If $\lim_{x\to\pm\infty} \left[f(x) - (mx+b) \right] = 0$, for $m \neq 0$, the line $y = mx + b$ is a slant asymptote.

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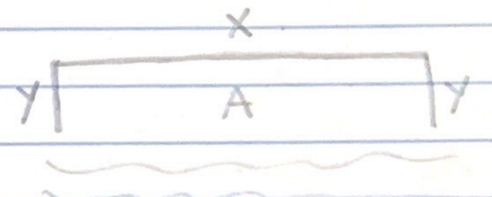
Sec 4.7

Optimization Problems

Apply the methods for finding extreme values to solve real world problems. In business, you want to minimize cost and maximize profit.

ex) Farmer has 2400ft of fence, wants to build fence around rectangular field bordering a straight river, no fence along river. What dimensions of fence give largest area?

Possible configuration:



Area $= xy$

Total fence $= 2y + x = 2400\text{ft} \rightarrow x = 2400\text{ft} - 2y$

Plug into area

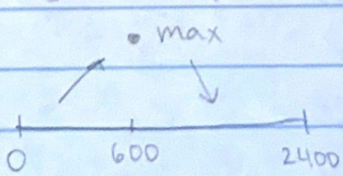
$A = (2400 - 2y)y = \boxed{2400y - 2y^2 = A}$

Constraints: If 100% of fence went to sides y, each would get 1200ft $\rightarrow 0 \le y \le 1200$

Want to maximize $A = 2400y - 2y^2$ on $0 \le y \le 1200$

$A' = 2400 - 4y$

$A' > 0 \rightarrow 2400 > 4y \rightarrow 600 > y$



$\boxed{A(600) = 720{,}000}$

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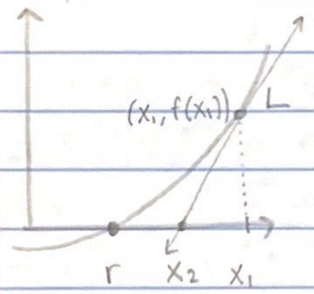
Sec 4.8

Newtons Method

If the equation we want to solve is hard, or unsolvable analytically, approximate the solution.

Want to solve $f(x)=0$, solutions are x intercepts.

Say we have initial guess x_1, solution is r



Compute tangent line L at point $(x_1, f(x_1))$. The x intercept, x_2, is the new starting point. Repeat until x_n close enough to r, $|x_n - r| < \varepsilon$.

Want a formula for x_2 in terms of x_1 (x_n in terms x_{n-1}), by definition of derivative

$$\lim_{x \to x_1} f'(x) = \frac{f(x_1) - f(x)}{x_1 - x} \quad , \text{ if } x_1 \text{ near } x$$

$$f'(x_1) \approx \frac{f(x_1) - f(x)}{x_1 - x} \rightarrow (x_1 - x) f'(x_1) = f(x_1) - f(x)$$

let $f(x) = y$, mult both sides by -1

$$(x - x_1) f'(x_1) = y - f(x_1)$$

x intercept of L is x_2, so $(x_2, 0)$ is on the line L, so

$$0 - f(x_1) = f'(x_1)(x_2 - x_1) \rightarrow \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

So in general, $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ where $f'(x_n) \neq 0$

where $\lim_{n \to \infty} x_n = r$.

ex) With $x_1 = 2$, find x_3 for $x^3 - 2x - 5 = 0$

$f' = 3x^2 - 2$

$f(x_1) = f(2) = 8 - 4 - 5 = -1 \rightarrow x_2 = 2 + \frac{1}{10} = \boxed{\frac{21}{10} = x_2}$

$f'(x_1) = 12 - 2 = 10$

$f(x_2) = f(2.1) = 0.061 \rightarrow x_3 = 2.1 - \frac{0.061}{11.23} = \boxed{2.09457 = x_3}$

$f'(2.1) = 11.23$

Sec 4.9

Antiderivatives

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

ex) $f(x) = x^2 \rightarrow F(x) = \frac{1}{3}x^3$

Taking the derivative of F(x) gets back to f(x).

The same is true for $F(x) = \frac{1}{3}x^3 + 4$, or $\frac{1}{3}x^3 + 694{,}713$

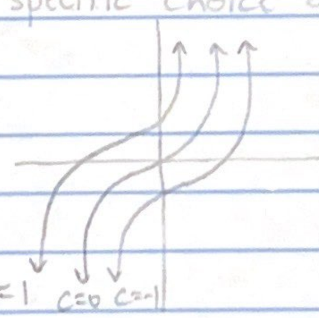
Theorem

If F an antiderivative of f on I, the most general antiderivative of f on I is

$$F(x) + C$$

where C is any constant.

Back to the example above, the general antiderivative is $F(x) = \frac{1}{3}x^3 + C$. The specific choice of C vertically shifts the graph



ex) Find all functions g such that

$x^{-\frac{1}{2}}$

$$g'(x) = 4\sin(x) + 2x^4 - \frac{1}{\sqrt{x}}$$

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antiderivative then

$$\boxed{-4\cos(x) + \frac{2}{5}x^5 - 2\sqrt{x} + C = g(x)}$$

ex) Find f if $f' = e^x + 20(1+x^2)^{-1}$, $f(0) = -2$

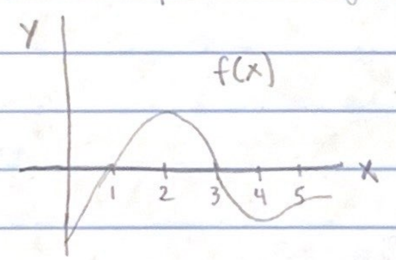
$f = e^x + 20\tan^{-1}x + C$

$f(0) = -2 = e^0 + 20\cancel{\tan^{-1}(0)} + C \rightarrow -2 = 1 + C \rightarrow C = -3$

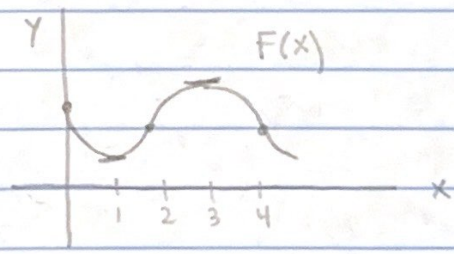
$$\boxed{f(x) = e^x + 20\tan^{-1}x - 3}$$

Graphing Antiderivatives

ex) Graph of f given, Know F(0)=2, graph F



Note, slope of F is f. Know we'll have horizontal tangents at x=1,3. Decreasing from (0,1) ∪ (3,∞), increasing (1,3). f changes sign at x=2,4, indicating inflection points



Linear Motion

Position $\xrightarrow{\frac{d}{dt}}$ Velocity $\xrightarrow{\frac{d}{dt}}$ Acceleration

Suppose we know acceleration, and the initial values for position ($s(0)$) and velocity ($v(0)$). By taking two antiderivatives, we can obtain position.

ex) Particle moving in straight line with $a(t) = 6t + 4$.
$v(0) = -6\,cm/s$, $s(0) = 9\,cm$. Find $s(t)$

$v(t) = 3t^2 + 4t + C$
$v(0) = -6\,cm/s = 3(0)^2 + 4(0) + C \rightarrow C = -6\,cm/s$
$v(t) = 3t^2 + 4t - 6\,cm/s$

$s(t) = t^3 + 2t^2 - 6t + D$
$s(0) = 9 = D$
$\boxed{s(t) = t^3 + 2t^2 - 6t + 9}$

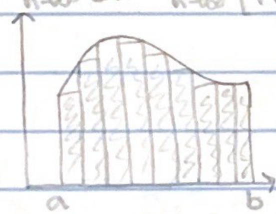
Sec 5.1

Area

The area A of region S that lies under the graph of continuous function f is the limit of sums of areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + \ldots + f(x_n)\Delta x \right] \quad \text{(right endpoint)}$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \left[f(x_0)\Delta x + \ldots + f(x_{n-1})\Delta x \right] \quad \text{(left)}$$



Subdivide S into n strips of width $\Delta x = \frac{b-a}{n}$

where $x_i = a + i\Delta x$

Distance

Given velocity and initial condition, compute distance

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t$$

which is basically the same as A. "The distance travelled is equal to the area under the curve of the velocity function".

Sec 5.2

The definite integral

If f defined on $a \leq x \leq b$, divide $[a,b]$ into n subintervals of equal width $\Delta x = b-a/n$. Let $x_0(=a), x_1, \ldots, x_n(=b)$ be the endpoints of the subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. The definite integral of f from a to b

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided the limit exists and gives the same result for all choices of sample points. If it does, f integrable on $[a,b]$.

where the limit is defined

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon, \quad \forall n \in \mathbb{Z} \ni \forall x_i^* \in [x_{i-1}, x_i]$$

Reimann Sum

Thrm: If f continuous on $[a,b]$, or if f has finite jump discontinuities, then f is integrable on $[a,b]$. $\int_a^b f(x)\,dx$ exists

Thrm: If f integrable on $[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x, \quad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

Sums of Powers ($\sum_{i=1}^{n} i^2$, etc) & Properties of Sums ($\sum_{i=1}^{n} ca_i$) Im not writing

Midpoint Rule

Choose x_i^* to be the midpoint of interval, usually better when approximating integrals.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\left[f(\bar{x}_1)+\dots+f(\bar{x}_n)\right]$$

$\Delta x = \frac{b-a}{n}$, $\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$

Sec 5.3

Fundamental Theorem of Calculus

Gives inverse relation between derivatives & integrals.

Part 1) Deal with equations of form: $g(x) = \int_a^x f(t)\,dt$

if x constant, so is the integral, If not, it will vary in x.

FTC1: If f continuous on $[a,b]$, then g, where

$$g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

"The derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit"

FTC2: If f continuous on $[a,b]$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is antiderivative of f, i.e. $F' = f$

using FTC1:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

using FTC2:

$$\int_a^x F'(t)\,dt = F(x) - F(a)$$