Exam 3 Warm Up

Calculus 1 Spring 2025

I. MASONS NOTE

A lot of this is completely copy-pasted from my previous quiz warm ups, or from my chapter 4 notes in files. It is to be used as a study resource, and **is not an indication of the questions on the exam**.

II. 3.8 EXPONENTIAL GROWTH AND DECAY

The function e^x represents many real world quantities, as it is some function whose derivative is proportional to itself (e^{ix} is even cooler, but is not introduced in this course). e^x is the solution to the following differential equation

$$\frac{dy}{dt} = ky \tag{1}$$

where k is some constant. If $k > 0$, the equation is called the "Law of Natural Growth". If $k < 0$, it is called the "Law of Natural Decay". To see how this differential equation and its solution work, let $y = ce^{kt}$, where c is some constant. Then $\frac{dy}{dt} = kce^{kt}$, plugging this into eq 1 gives

$$\frac{dy}{dt} = ky \;\rightarrow\; kce^{kt} = kce^{kt} \tag{2}$$

which is true, showing the differential equation is satisfied. A fair question is what *is* c? Differential equations are always paired with initial conditions (or boundary values), which tell us c. For example, if we care about how a population grows as a function of time, we need to know the population at time $t = 0$, this is one example of c. To demonstrate,

$$y(t) = ce^{kt} \;\rightarrow\; y(0) = ce^{k0} = c \;\rightarrow\; c = y(0) \tag{3}$$

Another fair question is what is k? Continuing the example, if the function represents some population (so $y(t) = P(t)$), then the differential equation is $\frac{dP}{dt} = kP$, solving for k gives

$$k = \frac{dP/dt}{P} \tag{4}$$

so k is the rate of change of population (or whatever quantity) with respect to the current population (or whatever quantity), and is called the "relative growth rate".

A. Worked Example (problem 3 in section 3.8)

A culture of the bacterium Salmonella enteritidis initially contains 50 cells. When introduced into a nutrient broth, the culture grows at a rate proportional to its size. After 1.5 hours the population has increased to 975.

a) Find an expression for the number of bacteria after t hours.

We know the solution of the differential equation is $y(t) = ce^{kt} = y(0)e^{kt}$. If the initial population is 50, then $y(0) = 50$, and $y(t) = 50e^{kt}$. To solve for k, use the fact that at $t = 1.5$, $y(t) = 975$.

$$y(1.5) = 50e^{k(1.5)} = 975 \;\rightarrow\; e^{k(1.5)} = \frac{975}{50} \tag{5}$$

take the natural log of both sides

$$k(1.5) = \ln\left(\frac{975}{50}\right) \;\rightarrow\; k = 1.9803 \tag{6}$$

and $y(t) = 50e^{(1.9803)t}$.

b) Find the number of bacteria after 3 hours.

$$y(3) = 50e^{(1.9803)(3)} = 19,014 \tag{7}$$

c) Find the rate of growth after 3 hours.

If $y(t)$ represents population, $y'(t)$ represents population growth. Evaluating $y'(3)$ gives the rate of growth after 3 hours. First, notice $y'(t) = ky(t)$, so

$$y'(3) = ky(3) = (1.9803)(19,014) = 37,653 \tag{8}$$

d) After how many hours will the population reach 250,000?

In other words, this is asking what value of t gives $y(t) = 250,000$.

$$y(t) = 50e^{(1.9803)t} = 250,000 \ \rightarrow \ \ln\left(\frac{250,000}{50}\right) = 1.9803t \ \rightarrow \ t = 4.3 \tag{9}$$

III. 3.10 LINEAR APPROXIMATION

At some point $x = a$, zooming in on the tangent line gives a decent approximation of the function $f(x)$. This is (incredibly) useful when $f(x)$ is computationally expensive to evaluate or is not analytical. The point we care about is $(a, f(a))$, and its slope is $f'(a)$. Using point slope formula, the equation of the tangent line is

$$f(x) = f(a) + f'(a)(x - a) \tag{10}$$

which approximates $f(x)$ near the point $x = a$. The book refers to this as $L(x)$, the linearization of $f(x)$ at a, to distinguish it from $f(x)$. So we could write

$$L(x) = f(a) + f'(a)(x - a) \tag{11}$$

The image below demonstrates the relationship between $f(x)$ and $L(x)$.

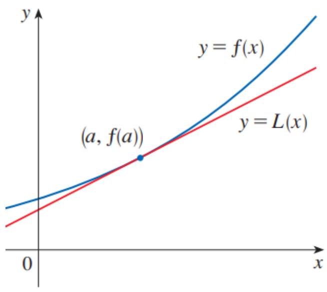

FIG. 1: For fun, $L(x)$ is also known as the first order Taylor expansion of $f(x)$ at $x = a$. Including more terms (of higher order) gives better and better approximations - you'll see them in cal 2.

A. Worked Example (problem 32 in section 3.10)

Use a linear approximation to estimate $1/4.002$.

Let $f(x) = 1/x$, then $f'(x) = -1/x^2$. We'll need $f(4) = 1/4$ and $f'(4) = -1/16$. Using the linearization of $f(x)$ at $x = 4$

$$L(x) = f(a) + f'(a)(x - a) \ \rightarrow \ L(x) = f(4) + f'(4)(x - 4) \tag{12}$$

plugging in values

$$L(x) = \frac{1}{4} - \frac{1}{16}(x-4) = \frac{4}{16} - \frac{(x-4)}{16} = \frac{4-x+4}{16} = \frac{8-x}{16} \tag{13}$$

so $L(x) = (8-x)/16$ is an approximation built around $a = 4$, and is reasonable for values of x near this, specifically $x = 4.002$. Then

$$L(x) = \frac{8-x}{16} \quad \rightarrow \quad L(4.002) = \frac{8-4.002}{16} \approx 0.2495875 \tag{14}$$

IV. 4.1 ABSOLUTE MIN AND MAX

A. Critical Numbers/Points

A **critical number/point** of $f(x)$ is a number c in the domain of $f(x)$ such that either $f'(c) = 0$ or $f'(c)$ DNE. Critical points represent the x value where the function switches from increasing to decreasing, or vice versa. To find critical points of a function $f(x)$, take the derivative and set it equal to 0, and solve for x.

B. Worked Example (from Lab 12)

Find the critical points of $f(x) = x^4 - 2x^2 + 4$.

First, take the derivative: $f'(x) = 4x^3 - 4x$. Next, set this equal to 0 and solve for x.

$$4x^3 - 4x = 0 \quad \rightarrow \quad 4x(x^2 - 1) = 0 \tag{15}$$

From this, we get

$$4x = 0 \quad \text{and} \quad x^2 - 1 = 0 \tag{16}$$

From $4x = 0$, we learn $x = 0$ is a critical point. From $x^2 - 1 = 0$, we learn $x = \pm 1$ are two more critical points. So, $x = -1, 0, 1$ are the critical points for the given $f(x)$.

C. Min/Max on a Closed Interval

To find the absolute max/min of a continuous function $f(x)$ on the closed interval $[a, b]$:

1) Find the critical points of $f(x)$ in (a, b)
2) Find the values of $f(x)$ at its critical points
3) Find the values of $f(x)$ at its endpoints (at $f(a)$ and $f(b)$)
4) Largest of (1) and (2) is the absolute max
5) Smallest of (1) and (2) is the absolute min

The local min/max will be the values from (1) and (2) where the function changed from increasing to decreasing. Note that local min/max cannot occur at the endpoints of an interval. Any absolute min/max that lies inside the interval will also count as a local min/max.

D. Worked Example from Lab 12

Find the absolute min/max, as wall as local min/max of $f(x) = x^4 - 2x^2 + 4$ on the closed interval $[0, 3]$.

This is the same function as above, which has critical points $x = -1, 0, 1$. However, the critical point $x = -1$ lies outside of the closed interval, so we do not include it in the following steps. Next, evaluate $f(x)$ at all critical points in the closed interval and endpoints.
$f(0) = 4$, $f(1) = 3$, and $f(3) = 67$.

Absolute Max: $f(3) = 67$
Absolute Min: $f(1) = 3$
Local Max: $f(0) = 4$
Local Min: $f(1) = 3$ also counts as a local min because there are function values on the left and right of $x = 1$. If the interval were from $[1, 3]$, then $f(1)$ would not count as a local min, since there would be no values to check on the left (ask for clarification if needed).

V. 4.3 DERIVATIVES AND GRAPH SHAPE

Use the first and/or second derivatives to learn information about a function. The sign (aka $\pm$) of the first derivative tells us whether the function is increasing or decreasing, the sign of the second derivative tells us whether the function is concave up or down.

A. Intervals of Increasing and Decreasing

To find the intervals a function is increasing or decreasing, find its critical points. One approach is to use the table method that has been introduced in lecture. A second approach is to place the critical points on a number line, and test x values in the intervals between critical points. The tested x values will be plugged into $f'(x)$. Only one value is tested per interval, which is sufficient because a function can only change direction at critical points. The sign of the interval indicates its direction.

B. Worked Example

Find the increasing/decreasing intervals of f, and its local min/max values, where $f(x) = x^4 - 2x^2 + 4$.

This is the same function as above, which has critical points $x = -1, 0, 1$. Method 1 is using the table method (which gets you the same result). Method 2 is placing the critical points on a number line, and testing values in each interval created by the critical points.
For the interval $(-\infty, -1)$, choose $x = -2$, For $(-1, 0)$, choose $x = -1/2$. For $(0, 1)$, choose $x = 1/2$. For $(1, \infty)$, choose $x = 2$. Evaluate at these 4 points: $f(-2) = -24$, $f(-1/2) = 3/2$, $f(1/2) = -3/2$, and $f(2) = 24$.
Using this information, the number line would look like Which tells us that $f(x)$ is

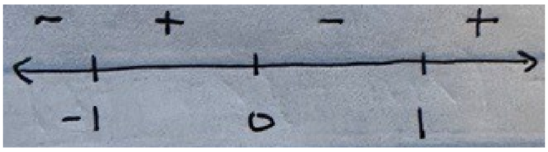

FIG. 2

Increasing: $(-1, 0) \bigcup (1, \infty)$
Decreasing: $(-\infty, -1) \bigcup (0, 1)$

To find local min/max, evaluate $f(x)$ at the critical points, and look at which values are biggest/smallest.
$f(-1) = 3$, $f(0) = 4$, and $f(1) = 3$. These values tell us
Local Min: $x = -1, 1$
Local Max: $x = 0$

C. Inflection Points

An inflection points is a point where a function $f(x)$ changes concavity, which is another word for the curvature of the function. Finding inflection points is analogous to finding critical points, except $f''(x)$ is set equal to zero and solved for x.

D. Intervals of Concavity

To find the intervals of concavity, find the functions inflections points, and repeat the same process used to find increasing and decreasing from the first derivative. This time, the only change will be when testing x values in the intervals, they are plugged in to the second derivative. Examples shown in lab 12 solutions.

VI. 4.4 INDETERMINATE FORMS AND L'HOPITAL'S RULE

If direct substitution of a limit problem fails (gives some indeterminate form) L'Hopitals Rule can help solve the problem. The rules is

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{17}$$

A. Worked Example (page 11 chapter 4 notes)

Find $\lim_{x\to 1}\frac{\ln(x)}{x-1}$.

Direct substitution gives

$$\lim_{x\to 1}\frac{\ln(x)}{x-1} = \frac{\ln(1)}{1-1} = \frac{0}{0} \tag{18}$$

an indeterminate form. Apply L'Hopitals rule

$$\lim_{x\to 1}\frac{\ln(x)}{x-1} \quad\to\quad \lim_{x\to 1}\frac{(1/x)}{1} \quad\to\quad \lim_{x\to 1}\frac{1}{x} = 1 \tag{19}$$

VII. 4.5 CURVE SKETCHING

Use the ideas learned so far to sketch the graph of some given function. The early ideas of the course, such as domains and asymptotes, play a role, as well as more recent ideas, like intervals of increasing/decreasing, and intervals of concavity. The only "new" part of this section is actually drawing the graph.

A. Worked Example (problem 8 sample exam 3)

Sketch the graph of a function $y = f(x)$ that satisfies $f(0) = 0$, is decreasing on $(-\infty, 2)\bigcup(2,\infty)$, is concave up on $(2,\infty)$, concave down on $(-\infty, 2)$, and satisfies $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -\infty$.

Turn the intervals into sign charts for visualization in the image below

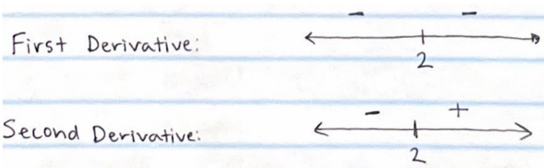

FIG. 3

Graphing the function, indicating all of the requirements where they are met below.

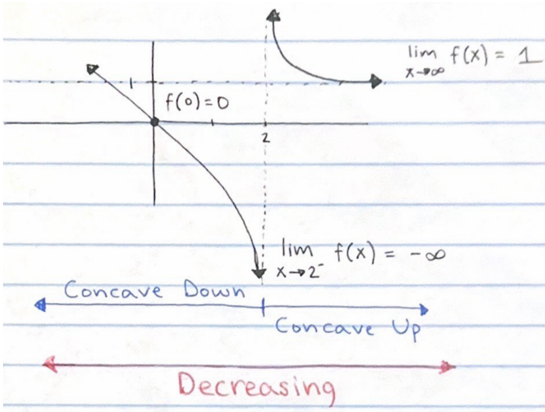

FIG. 4

VIII. 4.7 OPTIMIZATION PROBLEMS

This section applies the methods of finding extreme values (minimums and maximums) to real world problems. If you're running a business, it's beneficial to maximize profit and minimizing cost. Most of these problems work best with pictures, so I'll skip a worked example here, and list places to find them.

A. Where to Find Worked Examples

Exam 3 Solutions, lab 13 solutions (files soon if not already).

IX. 4.9 ANTIDERIVATIVES

So far, we've been given the function $f(x)$ and asked "What is the rate of change of $f(x)$?". The derivative, $f'(x)$, gives the answer. But what if we are given the rate of change, and want to find the original function? The antiderivative, $F(x)$, answers this question.

(Definition) A function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$. (In other words, the antiderivative $F(x)$ is correct if taking its derivative returns $f(x)$.)

Pretending we know no rules for antiderivates, how could we find the antiderivative of $f(x) = 4x$? We know the antiderivative is correct if taking the derivative gives $4x$. This suggests the antiderivative should be something of order x^2, because its derivative is $2x$. Fix the missing coefficient with a 2, so the antiderivative $F_1(x) = 2x^2$. Test $F_1(x)$:

$$F_1'(x) = f(x) \quad \rightarrow \quad 4x = 4x \tag{20}$$

However, $F_2(x) = 2x^2 + 14$ also passes this test, and $F_3(x) = 2x^2 + 93248624095642$, and so on, since the derivative of any constant is 0. For this reason, antiderivatives are only unique up to a constant value. So, the final answer to the problem is $F(x) = 2x^2 + C$.

The following image demonstrates the nonuniqeness of antiderivatives for $f(x) = x^2$, whose anditerivative is $F(x) = \frac{1}{3}x^3 + C$. The value of C is specific to the problem at hand. For example, when finding position from velocity, the value of C is determined by the initial velocity, as shown below.

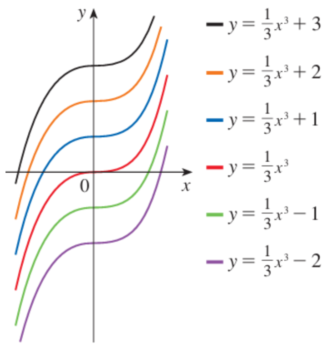

FIG. 5

A. Antiderivative of X^n

The general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C \tag{21}$$

and is like the power rule in reverse.

B. Worked Example (problem 9 sample exam 3)

Find $F(x)$ if $f''(x) = x^2 + e^x + 4\sin(x)$, given $f'(0) = 3$ and $f(0) = 4$.

Taking the antiderivative of $f''(x)$ gives $f'(x)$ up to a constant, to be determined by the initial conditions.

$$f'(x) = \frac{x^3}{3} + e^x - 4\cos(x) + C_1 \;\rightarrow\; f'(0) = \frac{0^3}{3} + e^0 - 4\cos(0) + C_1 = 3 \;\rightarrow\; 1 - 4 + C_1 = 3 \tag{22}$$

solving gives $C_1 = 6$, and $f'(x) = \frac{x^3}{3} + e^x - 4\cos(x) + 6$. Repeat to find $f(x)$

$$f(x) = \frac{x^4}{12} + e^x - 4\sin(x) + 6x + C_2 \;\rightarrow\; f(0) = 1 + C_2 = 4 \tag{23}$$

solving gives $C_2 = 3$ and the final answer is $f(x) = \frac{x^4}{12} + e^x - 4\sin(x) + 6x + 3$

X. PRACTICE PROBLEMS

Exam 3 Sample Questions posted by Dr. Tran in an announcement on 4/11. Solutions posted in an announcement by me on 4/12.