Quiz 3 Warm Up

Calculus 1 Spring 2025

I. 3.5 IMPLICIT DIFFERENTIATION

A. Implicit Equations and Differentiation

Have so far dealt with "explicit" equations, where one variable is written explicitly in terms of another. Examples are $y = x\sin(x)$ and $y = x^2 + 14x$. There are also "implicit" equations, which only have a relation between two variables. Examples are $x^2 + y^2 = 25$ and $x^3 + y^3 = 6xy$. To do implicit differentiation, differentiate both sides of the equation with respect to x, then algebraically solve for dy/dx.

B. Example Problem

Find the derivative of $x^3 + y^3 = 6xy$ using implicit differentiation

$$\begin{aligned} 3x^2 + 3y^2\frac{dy}{dx} &= 6x\frac{dy}{dx} + 6y \qquad (1) \\ 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

C. Why?

When taking the derivative of y, multiplying by $\frac{dy}{dx}$ comes from the chain rule. We are assuming there may be some hidden dependence of y on x. The idea is to look at the variable we are taking the derivative with respect to, if they "match" (i.e. $\frac{d}{d{\color{red}x}}{\color{red}x}$), do normal differentiation, if they don't "match" (i.e. $\frac{d}{d{\color{red}x}}{\color{blue}y}$), do implicit differentiation. This is not formal, but hopefully helps with intuition.

II. 3.6 DERIVATIVE OF LOGARITHMIC AND INVERSE TRIGONOMETRIC FUNCTIONS

There are two derivative rules from this subsection

A. Derivative of Logarithmic Functions

$$\frac{d}{dx}(\log_b x) = \frac{1}{x\ln(b)} \qquad (2)$$

and

$$\frac{d}{dx}b^x = b^x\ln(b) \qquad (3)$$

using the first rule with $b = e$, we see

$$\frac{d}{dx}(\log_e x) = \frac{d}{dx}\ln(x) = \frac{1}{x\ln(e)} = \frac{1}{x} \qquad (4)$$

B. Example Problem

Find the derivative of $y = \sqrt{\ln(x)}$

$$y' = \frac{1}{2}\ln(x)^{-1/2}(\frac{1}{x}) = \frac{1}{2x\sqrt{\ln(x)}} \tag{5}$$

C. Derivatives of Inverse Trigonometric Functions

The derivatives of trigonometric functions were introduced in section 3.3, here we visit the inverse trigonometric functions. To see how they work, if $y = \tan^{-1}(x)$, then $\tan(y) = x$. This idea is the starting point for taking the derivative. The trig identities $\sin^2(x) + \cos^2(x) = 1$ and $\tan^2(x) = \sec^2(x) + 1$ are used to simplify the resulting expression.

D. Example Problem

Find the derivative of $y = \cos^{-1}(x)$. We know $\cos(y) = x$, doing implicit differentiation on this gives

$$-\sin(y)\frac{dy}{dx} = 1 \quad \rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin(y)} \tag{6}$$

Using trig identities, $\sin(y) = \sqrt{1-\cos^2(y)}$. From the original expression, $cos(y) = x$. Making both of these substitutions gives

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2(y)}} \quad \rightarrow \quad \frac{-1}{\sqrt{1-x^2}} \tag{7}$$

III. PRACTICE PROBLEMS

The solution to most of the problems are worked out in my exam 2 review (files->unfiled->exam material->exam 2), the page number indicated in the question if so. The two problems not worked in the exam review are worked in this document on the final page. Try them before looking at the solutions for the most benefit.

Implicit Differentiation:

1) $x^2 + y^2 = 25$ page 8

2) $xy = \sqrt{x^2+y^2}$ page 9

Derivatives of Logarithmic Functions:

3) $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$ worked below, try it yourself first!!!!!

Derivatives of Inverse Trigonometric Functions:

4) $y = \cos^{-1}(2x^3)$ worked below, try it yourself first!!!!!

5) $y = x\tan^{-1}(x)$ page 11

IV. WORKED PRACTICE PROBLEMS

A. Solution to 3

(Example 7 in chapter 3.6 of the textbook)
To take the derivative of $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$ using logarithmic differentiation, take the natural log of both sides, and use log rules to simplify the right hand side

$$\ln(y) = \frac{3}{4}\ln(x) + \frac{1}{2}\ln\left(x^2+1\right) - 5\ln(3x+2) \tag{8}$$

Differentiate implicitly with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{1}{2(x^2+1)}(2x) - \frac{5}{(3x+2)}(3) = \frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{(3x+2)} \tag{9}$$

Multiply by y to get $\frac{dy}{dx}$ by itself

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{(3x+2)}\right) \tag{10}$$

We are almost done, $\frac{dy}{dx}$ is by itself, but we want it in terms of x. Replacing y with the original expression gives the final answer

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{(3x+2)}\right) \tag{11}$$

You could do this with a combination of the product and quotient rules as well (like I did in one of the problems from lab 10), but this method is more efficient.

B. Solution to 4

From $y = \cos^{-1}(2x^3)$ we get $\cos(y) = 2x^3$. Differentiate implicitly with respect to x

$$-\sin(y)\frac{dy}{dx} = 6x^2 \quad \rightarrow \quad \frac{dy}{dx} = \frac{-6x^2}{\sin(y)} \tag{12}$$

Using our favorite trigonometric identity, $\sin(y) = \sqrt{1-\cos^2(y)}$, and substituting the original expression

$$\frac{dy}{dx} = \frac{-6x^2}{\sqrt{1-\cos^2(y)}} = \frac{-6x^2}{\sqrt{1-(2x^3)^2}} = \frac{-6x^2}{\sqrt{1-4x^6}} \tag{13}$$