Gravitational Wave Notes

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I. CHAPTER 30: GAUGE FREEDOM

A. Introduction

Gravitational waves are required by any relativistic theory of gravity. Information about changes in a dynamic sources gravitational field can move through space away from the source no faster than speed c. When observed far from the source, the gravitational waves will be tiny deviations from flat space, justifying the weak field approximation to simplify the mathematics.

B. Weak Field Approximation (Chapter 22)

If the gravitational field in a region of spacetime is sufficiently weak, the metric used to describe it should fit the following metric to a good approximation. This coordinate system may be thought of as quasi-cartesian.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

Where $h_{\mu\nu} = h_{\nu\mu}$ (the metric is symmetric) and $|h_{\mu\nu}| << 1$. $h_{\mu\nu}$ is the **metric perturbation**, the deviation from the flat space metric $\eta_{\mu\nu}$. The weak field approximation drops terms that are second order and higher in the perturbation ($>= |h_{\mu\nu}|^2$). This approximation allows the raising and lowering of the perturbation indices with the flat space metric instead of the full metric as shown. The definition of the inverse metric is

$$g^{\alpha\mu} g_{\mu\nu} = \delta^\alpha_\nu \tag{2}$$

Note $g^{\alpha\mu}$ must consist of a flat space metric and a perturbation of order $h_{\mu\nu}$, as does $g_{\mu\nu}$ in eq (1). Denote the perturbation component of the inverse metric as $m^{\alpha\mu}$. We can insert eq (1) into eq (2) with $g^{\alpha\mu} = \eta^{\alpha\mu} + m^{\alpha\mu}$.

$$\begin{aligned} (\eta^{\alpha\mu} + m^{\alpha\mu})(\eta_{\mu\nu} + h_{\mu\nu}) &= \delta^\alpha_\nu \\ \delta^\alpha_\nu + h^\alpha_\nu + m^\alpha_\nu + m^{\alpha\mu} h_{\mu\nu} &= \delta^\alpha_\nu \end{aligned} \tag{3}$$

The term $m^{\alpha\mu} h_{\mu\nu}$ is neglected as it is second order in the perturbation, leaving

$$h^\alpha_\nu = -m^\alpha_\nu \tag{4}$$

Raising the final index would give $h^{\alpha\nu} = -m^{\alpha\nu}$. This shows raising all indices of eq (1) has the effect of a minus sign on the perturbation

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{5}$$

With $h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$. The gradients of $h_{\mu\nu}$ may also be raised and lowered with the flat space metric, for example

$$\partial^\alpha h_{\mu\nu} = g^{\alpha\beta} \partial_\beta h_{\mu\nu} \approx (\eta^{\alpha\beta} - h^{\alpha\beta}) \partial_\beta h_{\mu\nu} \approx \eta^{\alpha\beta} \partial_\beta h_{\mu\nu} \tag{6}$$

The last approximation as $\partial_\beta h^{\alpha\beta} h_{\mu\nu}$ is again second order in the perturbation. After defining more tools, this weak field approximation will allow us to simplify the weak field Einstein equation to two non-coupled linear differential equations when investigating gravitational waves.

(Notes: Searching for "linearized gravity" has been a more helpful search than "weak field approximation". The name linear as all equations are linear in the perturbation.)

C. The Trace-Reversed Perturbation

When working with gravitational waves, the form of the Einstein equation that is often most convenient is

$$G^{\gamma\sigma} = R^{\gamma\sigma} - \frac{1}{2}g^{\gamma\sigma}R = 8\pi G T^{\gamma\sigma} \tag{7}$$

To first order in the perturbation, the Einstein equation becomes (exercise 30.1.1 NEED TO DO)

$$\frac{1}{2}\left(\partial^\gamma\partial_\mu h^{\mu\sigma} + \partial^\sigma\partial_\mu h^{\mu\gamma} - \partial^\gamma\partial^\sigma h - \partial^\mu\partial_\mu h^{\gamma\sigma} - \eta^{\gamma\sigma}\partial_\beta\partial_\mu h^{\mu\beta} + \eta^{\gamma\sigma}\partial^\mu\partial_\mu h\right) = 8\pi G T^{\gamma\sigma} \tag{8}$$

Where $h = \eta^{\mu\nu}h_{\mu\nu}$. This can be moderately simplified by defining the **trace-reversed metric perturbation** $H_{\mu\nu}$.

$$H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{9}$$

We can obtain the scalar H by raising both sides with the inverse metric (exercise 30.2.1)

$$\begin{aligned} \eta^{\mu\nu}H_{\mu\nu} &= \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h\eta^{\mu\nu}\eta_{\mu\nu} \\ H &= h - 2h \\ H &= -h \end{aligned} \tag{10}$$

Note $\eta^{\mu\nu}\eta_{\mu\nu} = 4$. Giving the reason the name is trace-reversed. This holds the same information as the original perturbation, rearranged only for the purpose of simplification. The Einstein tensor is itself the trace reversed Ricci tensor (minute 48, Lecture 14 Linearized Gravity 1, Scott Hughes). With this knowledge, we know using a trace reversed perturbation will clean up a few terms in the Einstein equation. Using eq (9) and the substitution given by eq (10) we get (exercise 30.2.2)

$$h_{\mu\nu} = H_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}H \tag{11}$$

Raising both indices gives $h^{\mu\nu} = H^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}H$. This is substituted into the Einstein equation (eq (8)) to begin simplification (exercise 30.3.1)

$$\begin{aligned} \frac{1}{2}(\partial^\gamma\partial_\mu(H^{\mu\sigma} - \frac{1}{2}\eta^{\mu\sigma}H) + \partial^\sigma\partial_\mu(H^{\mu\gamma} - \frac{1}{2}\eta^{\mu\gamma}H) + \partial^\gamma\partial^\sigma H - \partial^\mu\partial_\mu(H^{\gamma\sigma} - \frac{1}{2}\eta^{\gamma\sigma}H) \\ -\eta^{\gamma\sigma}\partial_\beta\partial_\mu(H^{\mu\beta} - \frac{1}{2}\eta^{\mu\beta}H) - \eta^{\gamma\sigma}\partial^\mu\partial_\mu H) = 8\pi G T^{\gamma\sigma} \end{aligned} \tag{12}$$

The partial derivative of any term involving the scalar H will be 0. The weak-field Einstein equation then reduces to

$$-\partial^\gamma\partial_\mu H^{\mu\sigma} - \partial^\sigma\partial_\mu H^{\mu\gamma} + \partial^\mu\partial_\mu H^{\gamma\sigma} + \eta^{\gamma\sigma}\partial_\beta\partial_\mu H^{\mu\beta} = -16\pi G T^{\gamma\sigma} \tag{13}$$

After multiplying by -2 and killing any $\partial_\alpha H$ or $\partial^\alpha H$, where $\partial^\mu\partial_\mu = -(\partial/\partial t^2) + \nabla^2$. This equation is getting more approachable to solve. The last step is to make use of our freedom to choose coordinates. As described in chapter 23, even though the Einstein equation determines the geometry of spacetime surrounding gravitating objects, we are completely free to choose the coordinates defining that geometry. The next section gives an introduction to gauge transformations, which will be the final step in getting the weak field Einstein equation in a nicer form.

D. Gauge Transformation Basics

Consider a coordinate transformation from the quasi-cartesian coordinates x^α to new quasi-cartesian coordinates $x^{\alpha'}$.

$$x^{\alpha'} = x^\alpha + \epsilon^\alpha \tag{14}$$

Where $|\epsilon^\alpha| << 1$. The coordinate transformation partials are

$$\frac{\partial x^{\alpha'}}{\partial x^\beta} = \delta^\alpha_\beta + \partial_\beta\epsilon^\alpha \tag{15}$$

$$\frac{\partial x^\beta}{\partial x^{\alpha'}} = \delta^\beta_\alpha - \partial_\alpha\epsilon^\beta \tag{16}$$

To see how the perturbation transforms, use the tensor transformation rule for the full metric (exercise 30.4.1)

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \tag{17}$$

Inserting eq (15) and eq (16) into eq (17), and substituting eq (1) as we are in the weak field approximation gives

$$\begin{aligned} g'_{\mu\nu} &= (\delta^{\alpha}_{\mu} - \partial_{\mu}\epsilon^{\alpha})(\delta^{\beta}_{\nu} - \partial_{\nu}\epsilon^{\beta})(\eta_{\alpha\beta} + h_{\alpha\beta}) \\ g'_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} + O(h^2) \end{aligned} \tag{18}$$

Where $O(h^2)$ represents terms of second order or greater in the perturbation, which are dropped. We want to cast this in the form of

$$g'_{\mu\nu} = \eta'_{\mu\nu} + h'_{\mu\nu} \tag{19}$$

From this, we deduce

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} \tag{20}$$

Similarly (exercise 30.4.2 NEED TO DO)

$$H'_{\mu\nu} = H_{\mu\nu} - \partial_{\mu}\epsilon_{nu} - \partial_{\nu}\epsilon_{\mu} + \eta_{\mu\nu}\partial_{\alpha}\eta^{\alpha} \tag{21}$$

We now know how the metric perturbation transforms under coordinate transformations. We can then ask, "How does the Riemann tensor change under this transformation?". The Reimann tensor takes form

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_{\beta}\partial_{\mu}h_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}h_{\beta\mu} - \partial_{\alpha}\partial_{\mu}h_{\beta\nu} - \partial_{\beta}\partial_{\nu}h_{\alpha\mu}\right) \tag{22}$$

Inserting eq (20) into the above, every term involving ϵ^{α} cancels, and we are left with

$$R'_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_{\beta}\partial_{\mu}h'_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}h'_{\beta\mu} - \partial_{\alpha}\partial_{\mu}h'_{\beta\nu} - \partial_{\beta}\partial_{\nu}h'_{\alpha\mu}\right) \tag{23}$$

To first order in the perturbation. That is, transforming the metric perturbation with eq (14) has no effect on the Riemann tensor, and in turn no effect on the Einstein equation in the weak field approximation. A coordinate transformation of the kind given in eq (14) is called a **gauge transformation**, a specific choice of quasi-cartesian coordinates is then called a **gauge**. If small transformations in the perturbation have no effect on the outcome on the Einstein equation, we can cleverly choose the gauge to make solutions easier to find. This is the final tool which will unlock the desired form of the Einstein equation. To summarize and peak ahead, if $h_{\mu\nu}$ satisfies the Einstein equation, so does $h'_{\mu\nu}$. Given any solution to the weak field Einstein equation, we can generate an infinite family of solutions by applying different gauge transformations of form eq (14).

E. The Lorenz Gauge

Suppose a given trace-reversed perturbation $H^{\mu\nu}$ solves the weak field Einstein equations (eq (13)). Raising both indices of (eq (21)) gives

$$H'^{\mu\nu} = H^{\mu\nu} - \partial^{\mu}\epsilon^{\nu} - \partial^{\nu}\epsilon^{\mu} + \eta^{\mu\nu}\partial_{\alpha}\epsilon^{\alpha} \tag{24}$$

Given any initial $H^{\mu\nu}$, we can make $\partial_{\mu}H'^{\mu\nu} = 0$, if we are careful to choose our transformation functions ϵ^{α} such that (exercise 30.6.1)

$$\partial^{\mu}\partial_{\mu}\epsilon^{\nu} = \partial_{\mu}H^{\mu\nu} \tag{25}$$

To demonstrate, hit all terms in eq (24) with ∂_{μ} and use eq (25)

$$\begin{aligned} \partial_{\mu}H'^{\mu\nu} &= \partial_{\mu}H^{\mu\nu} - \partial^{\mu}\partial_{\mu}\epsilon^{\nu} - \partial^{\nu}\partial_{\mu}\epsilon^{\mu} + \eta^{\mu\nu}\partial_{\mu}\partial_{\alpha}\eta^{\alpha} \\ \partial_{\mu}H'^{\mu\nu} &= \partial_{\mu}H^{\mu\nu} - \partial_{\mu}H^{\mu\nu} - \partial^{\nu}\partial_{\mu}\epsilon^{\mu} + \partial^{\nu}\partial_{\alpha}\epsilon^{\alpha} \\ \partial_{\mu}H'^{\mu\nu} &= -\partial^{\nu}\partial_{\mu}\epsilon^{\mu} + \partial^{\nu}\partial_{\mu}\epsilon^{\mu} \\ \partial_{\mu}H'^{\mu\nu} &= 0 \end{aligned} \tag{26}$$

Where again $\partial^\mu \partial_\mu = -(\partial/\partial t^2) + \nabla^2$. This differential operator is well studied, and it is proven that solutions to the equation of form

$$\partial^\mu \partial_\mu f = g \tag{27}$$

Always exist for well defined driving functions g, making it a valid condition to require. Therefore, we can always choose nice transformation functions ϵ^α such that $\partial_\mu H^{'\mu\nu} = 0$ is true. This is the final tool needed to simplify the weak field Einstein equation, repeated here

$$-\partial^\gamma \partial_\mu H^{\mu\sigma} - \partial^\sigma \partial_\mu H^{\mu\gamma} + \partial^\mu \partial_\mu H^{\gamma\sigma} + \eta^{\gamma\sigma} \partial_\beta \partial_\mu H^{\mu\beta} = -16\pi G T^{\gamma\sigma} \tag{28}$$

As **gauge freedom** (our ability to choose a gauge without changing any physically observable quantities) allows us to choose a gauge such that $\partial_\mu H^{'\mu\nu} = 0$, which three terms in the above contain, the weak field Einstein equation finally takes the form

$$\partial^\mu \partial_\mu H^{\gamma\sigma} = -16\pi G T^{\gamma\sigma} \tag{29}$$

$$\partial_\mu H^{'\mu\nu} = 0 \tag{30}$$

Two non-coupled linear differential equations, as was the goal.

F. Additional Gauge Freedom

Requiring the condition $\partial_\mu H^{'\mu\nu} = 0$ does not exhaust our gauge freedom. Suppose we have a solution to the weak field Einstein equation $H^{\mu\nu}$ that also already satisfies $\partial_\mu H^{\mu\nu} = 0$. We can apply a gauge transformation to obtain an $\partial_\mu H^{'\mu\nu}$ that still solves the Einstein equation, as that equation is invariant under gauge transformations. This is analogous to how the Riemann tensor was unchanged by a substitution of the transformed perturbation from eq (22) to eq (23). We can continuously apply gauge transformations to generate new solutions to the Einstein equation.