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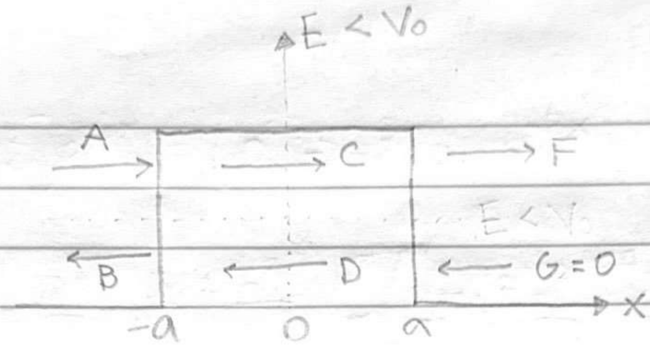
4) 

Potential defined

$$V(x) = \begin{cases} 0, & x < -a \\ V_0, & -a < x < a \\ 0, & x > a \end{cases}$$

The Schrodinger Eigenvalue Equation has 2 forms

$$\left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V_0 \right) \psi(x) = E\psi(x), \quad -a < x < a$$

$$\left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) \psi(x) = E\psi(x), \quad x < -a \text{ or } x > a$$

Define wave vector K and decay factor q

$$K = \sqrt{\frac{2mE}{\hbar^2}}, \quad q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

equations become

$$\frac{d^2\psi}{dx^2} = q^2\psi \qquad -a < x < a$$

$$\frac{d^2\psi}{dx^2} = -K^2\psi \qquad x < -a \text{ or } x > a$$

the general solutions (with $G=0$)

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x<-a \\ Ce^{qx} + De^{-qx}, & -a<x<a \\ Fe^{ikx}, & x>a \end{cases}$$

Will use BCs to determine T & R.
Need to ensure continuity in the wave functions and their derivatives at $x=-a$, $x=a$.

(1) $\psi(-a)$: $Ae^{-ika} + Be^{ika} = Ce^{-qa} + De^{qa}$

(2) $\frac{d\psi}{dx}(-a)$: $ikAe^{-ika} - ikBe^{ika} = qCe^{-qa} - qDe^{qa}$

(3) $\psi(a)$: $Ce^{qa} + De^{-qa} = Fe^{ika}$

(4) $\frac{d\psi}{dx}(a)$: $qCe^{qa} - qDe^{-qa} = ikFe^{ika}$

Get C & D in terms of F
From 4
$\frac{ik}{q}Fe^{ika} = Ce^{qa} - De^{-qa} \rightarrow Ce^{qa} = De^{-qa} + \frac{ik}{q}Fe^{ika}$

$C = De^{-2qa} + \frac{ik}{q}Fe^{ika}e^{-qa} = \boxed{De^{-2qa} + \frac{ik}{q}Fe^{a(ik-q)} = C}$

(*) $Ce^{qa} = Fe^{ika} - De^{-qa}$

$\left[De^{-2qa} + \frac{ik}{q}Fe^{a(ik-q)}\right]e^{qa} = Fe^{ika} - De^{-qa}$

$$De^{-qa} + \frac{iK}{q} Fe^{iaK} = Fe^{iKa} - De^{-qa}$$

(**)

$$2De^{-qa} = Fe^{iKa}\left(1 - \frac{iK}{q}\right) \rightarrow \boxed{De^{-qa} = \frac{1}{2} Fe^{iKa}\left(1 - \frac{iK}{q}\right)}$$

$$2D = Fe^{i(aK+a)}\left(1 - \frac{iK}{q}\right)$$

$$\boxed{D = \frac{1}{2} Fe^{i(aK+a)}\left(1 - \frac{iK}{q}\right)}$$, plug D into (*)

$$Ce^{qa} = Fe^{iKa} - \frac{1}{2} Fe^{iKa}\left(1 - \frac{iK}{q}\right) = Fe^{iKa} - \frac{1}{2}Fe^{iKa} + \frac{1}{2}Fe^{iKa}\left(\frac{iK}{q}\right)$$

$$Ce^{qa} = \frac{1}{2}Fe^{iKa} + \frac{1}{2}Fe^{iKa}\left(\frac{iK}{q}\right) = \boxed{\frac{1}{2}Fe^{iKa}\left(1 + \frac{iK}{q}\right) = Ce^{qa}} \quad (***)$$

So far found De^{-qa} and Ce^{qa}, work with A, B

Multiply (1) by iK to use (2)

$$iKAe^{-iKa} + iKBe^{iKa} = iKCe^{-qa} + iKDe^{qa}$$

$$iKBe^{iKa} = iKCe^{-qa} + iKDe^{qa} - iKAe^{-iKa}$$, into (2)

$$-iKAe^{-iKa} + \left[iKCe^{-qa} + iKDe^{qa} - iKAe^{-iKa}\right] = -qCe^{-qa} + qDe^{qa}$$

$$-2iKAe^{-iKa} = -iKCe^{-qa} - iKDe^{qa} - qCe^{-qa} + qDe^{qa}$$

$$2iKAe^{-iKa} = Ce^{-qa}(iK + q) + De^{qa}(iK - q)$$

$$\boxed{2Ae^{-iKa} = Ce^{-qa}\left(1 + \frac{q}{iK}\right) + De^{qa}\left(1 - \frac{q}{iK}\right)} \quad (\$)$$

Need to swap sign on exponents in (**) and (***) to plug into ($), From (**), mult by e^{2qa}

$$De^{-qa}\left(e^{2qa}\right) = \frac{1}{2}Fe^{iKa}\left(1-\frac{iK}{q}\right)e^{2qa}$$

$$\boxed{De^{qa} = \frac{1}{2}Fe^{a(iK+2q)}\left(1-\frac{iK}{q}\right)}$$, now (***)

$$Ce^{qa}\left(e^{-2qa}\right) = \frac{1}{2}Fe^{iKa}\left(1+\frac{iK}{q}\right)e^{-2qa}$$

$$\boxed{Ce^{-qa} = \frac{1}{2}Fe^{a(iK-2q)}\left(1+\frac{iK}{q}\right)}$$

Plug 2 above boxes back into ($)

$$2Ae^{-iKa} = \left[\frac{1}{2}Fe^{aiK}e^{-2aq}\left(1+\frac{iK}{q}\right)\right]\left(1+\frac{q}{iK}\right) = \left(1-\frac{iq}{K}\right)$$

to get all i's on top

$$+\left[\frac{1}{2}Fe^{a(iK+2q)}\left(1-\frac{iK}{q}\right)\right]\left(1-\frac{q}{iK}\right) = \left(1+\frac{iq}{K}\right)$$

$$= \frac{Fe^{iaK}}{2}\left[e^{-2aq}\left(1+\frac{iK}{q}\right)\left(1-\frac{iq}{K}\right) + e^{2aq}\left(1-\frac{iK}{q}\right)\left(1+\frac{iq}{K}\right)\right]$$

$$= \frac{Fe^{iaK}}{2}\left[e^{-2aq}\left(1-\frac{iq}{K}+\frac{iK}{q}+\frac{Kq}{qK}\right) + e^{2aq}\left(1+\frac{iq}{K}-\frac{iK}{q}+\frac{Kq}{qK}\right)\right]$$

$$= \frac{Fe^{iaK}}{2}\left[e^{-2aq}\left(2-\frac{iq}{K}+\frac{iK}{q}\right) + e^{2aq}\left(2+\frac{iq}{K}-\frac{iK}{q}\right)\right]$$

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$$= \frac{Fe^{iaK}}{2}\left[2e^{-2qa} - \frac{iq}{K}e^{-2qa} + \frac{iK}{q}e^{-2qa} + 2e^{2qa} + \frac{iq}{K}e^{2qa} - \frac{iK}{q}e^{2qa}\right]$$

multiply by forms of 1 $\left(\frac{q}{q} \text{ or } \frac{K}{K}\right)$ to get terms with same denominator

$$= \frac{Fe^{iaK}}{2}\left[2(e^{-2qa} + e^{2qa}) - \frac{iq^2}{Kq}e^{-2qa} + \frac{iK^2}{Kq}e^{-2qa}\right.$$

$$\left. + \frac{iq^2}{Kq}e^{2qa} - \frac{iK^2}{Kq}e^{2qa}\right]$$

can now group

$$= \frac{Fe^{iaK}}{2}\left[2(e^{-2aq} + e^{2qa}) + \frac{iq^2}{Kq}\left(e^{2qa} - e^{-2qa}\right) - \frac{iK^2}{Kq}\left(e^{-2qa} + e^{-2qa}\right)\right]$$

$$= \frac{Fe^{iaK}}{2}\left[2(e^{-2aq} + e^{2qa}) + \frac{i(q^2 - K^2)}{Kq}\left(e^{2qa} - e^{-2qa}\right)\right]$$

$$= Fe^{iKa}\left[2\left(\frac{e^{-2aq} + e^{2qa}}{2}\right) + \frac{i(q^2 - K^2)}{Kq}\left(\frac{e^{2qa} - e^{-2qa}}{2}\right)\right]$$

$$= Fe^{iKa}\left[2\cosh(2aq) + \frac{i(q^2 - K^2)}{Kq}\sinh(2aq)\right]$$

finally

$$2Ae^{-iKa} = Fe^{iKa}\left[2\cosh(2aq) + \frac{i(q^2 - K^2)}{Kq}\sinh(2aq)\right]$$

giving

$$Ae^{-iKa} = Fe^{iKa}\left[\cosh(2aq) + \frac{i(q^2-K^2)}{2Kq}\sinh(2aq)\right]$$

Now have A strictly in terms of F. The transmission coefficient is the inverse ratio of the intensity of the wave sent into the potential (A^2) over the wave tunneling through the potential (F^2). $T \to \left(\frac{|F|}{|A|}\right)^2$

$\left(\frac{|Ae^{-iKa}|}{|Fe^{iKa}|}\right)^2$ should be the same as $\left(\frac{|A|}{|F|}\right)^2$ since the phase wont affect the magnitude.

$$\left(\frac{|A|}{|F|}\right)^2 = \left[\cosh(2aq) + \frac{i(q^2-K^2)}{2Kq}\sinh(2qa)\right]^2$$

$$= \cosh^2(2aq) + \left(\frac{q^2-K^2}{2Kq}\right)^2\sinh^2(2qa)$$

can reduce with $\cosh^2\alpha = 1+\sinh^2\alpha$

$$= 1+\sinh^2(2aq) + \left(\frac{q^2-K^2}{2Kq}\right)^2\sinh^2(2qa)$$

first expand

$$\left(\frac{q^2-K^2}{2Kq}\right)\left(\frac{q^2-K^2}{2Kq}\right) = \frac{q^4-2K^2q^2+K^4}{4K^2q^2}$$

(13).

$$= 1 + \sinh^2(2qa) + \left(\frac{q^4 - 2K^2q^2 + K^4}{4K^2q^2}\right)\sinh^2(2qa)$$

$$= 1 + \left(1 + \frac{q^4 - 2K^2q^2 + K^4}{4K^2q^2}\right)\sinh^2(2qa)$$

$$= 1 + \left(\frac{4K^2q^2 + q^4 - 2K^2q^2 + K^4}{4K^2q^2}\right)\sinh^2(2qa)$$

$$= 1 + \left(\frac{q^4 + K^4 + 2K^2q^2}{4K^2q^2}\right)\sinh^2(2qa)$$

$$= 1 + \left(\frac{(q^2 + K^2)^2}{4K^2q^2}\right)\sinh^2(2qa)$$

sub q & K from first page

$$= 1 + \frac{\left(\frac{2m(V-E)}{\hbar^2} + \frac{2mE}{\hbar^2}\right)^2}{4\left(\frac{2m(V-E)}{\hbar^2}\right)\left(\frac{2mE}{\hbar^2}\right)}\sinh^2\left(2a\sqrt{\frac{2m(V-E)}{\hbar^2}}\right)$$

$$= 1 + \frac{\left(\frac{2mV}{\hbar^2}\right)^2}{\left(\frac{16m^2E(V-E)}{\hbar^4}\right)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V-E)}\right)$$

$$= 1 + \frac{\left(\frac{4m^2V^2}{\hbar^4}\right)}{\left(\frac{16m^2E(V-E)}{\hbar^4}\right)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V-E)}\right)$$

$$= 1 + \frac{V^2}{4E(V-E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V-E)}\right)$$

making the transmission coefficient

$$\boxed{T = \left(1 + \frac{V^2}{4E(V-E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V-E)}\right)\right)^{-1}}$$

with reflection coefficient

$$\boxed{R = 1 - T}$$