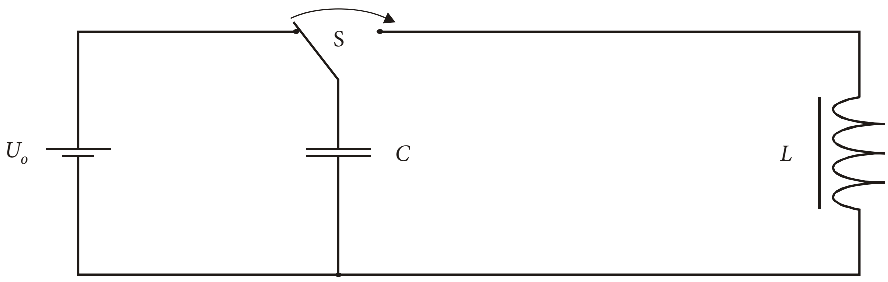
LC Oscillator

We investigate a circuit with a coil and a capacitor which is charged to a voltage V_0 (see diagram). At time $t = 0$ the switch S is changed to the other state, connecting the capacitor directly to the coil.



According to the loop rule, the voltage induced in the coil equals the voltage across the capacitor:

$$v_{ind}(t) = v_C(t) \tag{1}$$

The current flowing through the coil is determined by the self-inductance of the coil:

$$v_{ind}(t) = -L \cdot \dot{i}(t) \tag{2}$$

The voltage across the capacitor depends on the charge stored on the plates of the capacitor and its capacity:

$$v_C(t) = \frac{q(t)}{C} \tag{3}$$

The current through the capacitor is determined by the rate of change of the charge:

$$i(t) = \dot{q}(t) \tag{4}$$

Equations (1) to (4) can be combined to yield the differential equation

$$\ddot{q}(t) = -\frac{1}{L \cdot C} \cdot q(t) \,. \tag{5}$$

This is the characteristic differential equation of a simple harmonic motion. With the replacements

$$\begin{aligned} q(t) &\to y(t) \\ \frac{1}{L \cdot C} &\to \omega^2 \end{aligned} \tag{6}$$

it is formally equivalent to the differential equation of a spring pendulum. Its solution can be written as

$$q(t) = q_0 \cdot \cos(\omega \cdot t) \tag{7}$$

with the angular frequency

$$\omega = \sqrt{1/L \cdot C} \tag{8}$$

and the period $T = 2\pi \cdot \sqrt{L \cdot C}$, respectively.

Equations (3) and (4) lead to expressions for the voltage across the capacitor and the current:

$$v_C(t) = \frac{q(t)}{C} = \frac{q_0}{C} \cdot \cos(\omega \cdot t) \text{ and } i(t) = \dot{q}(t) = -q_0 \cdot \sin(\omega \cdot t) \cdot \omega \tag{9}$$

or, using $v_{C0} = q_0 / C = V_0$ and $i_0 = \omega \cdot q_0$ $\hat{i} = \omega \cdot \hat{q}$ simply

$$v_C(t) = v_0 \cdot \cos(\omega \cdot t) \text{ and } i(t) = -i_0 \cdot \sin(\omega \cdot t) \,. \tag{10}$$

Conservation of Energy

One of the assumptions in the derivation of equation (1) was that we can neglect the resistance of wires. Therefore, there is no energy loss due to heat. The energy of the oscillation is in the electric and the magnetic field.

The energy in the electric field is

$$W_e(t) = \frac{1}{2} \cdot \frac{q(t)^2}{C} = \frac{1}{2} \cdot \frac{q_0^{\,2}}{C} \cdot \cos^2(\omega \cdot t) \tag{11}$$

And the energy in the magnetic field

$$W_m(t) = \frac{1}{2} \cdot L \cdot i(t)^2 = \frac{1}{2} \cdot L \cdot i_0^{\,2} \cdot \sin^2(\omega \cdot t)\,. \tag{12}$$

The formal analogy with the oscillation is again obvious: The energy in the electric field corresponds to the elastic energy of the spring, where the reciprocal value of the capacity takes the role of the spring constant. The energy in the magnetic field can be interpreted as kinetic energy, the current corresponding to the pendulum mass‘s velocity and the inductance of the coil acting as its inertia. The latter seems reasonable in the view of Lenz’s law.

With equation (9) fort he current and expression (8) for the angular frequency we find for the energy in the magnetic field

$$\begin{aligned} W_m(t) &= \frac{1}{2} \cdot L \cdot i(t)^2 = \frac{1}{2} \cdot L \cdot \omega^2 \cdot q_0^{\,2} \cdot \sin^2(\omega \cdot t) \\ &= \frac{1}{2} \cdot \frac{\cancel{L}}{\cancel{L} \cdot C} \cdot q_0^{\,2} \cdot \sin^2(\omega \cdot t) = \frac{1}{2} \cdot \frac{q_0^{\,2}}{C} \cdot \sin^2(\omega \cdot t) \end{aligned} \tag{13}$$

Obviously, the conservation of energy also holds true for the undamped LC oscillator:

$$W = W_e(t) + W_m(t) = \frac{1}{2} \cdot \frac{q_0^{\,2}}{C} = \frac{1}{2} \cdot L \cdot i_0^{\,2} \tag{14}$$

The energy oscillates between the electric and the magnetic field at twice the frequency of voltage and current.

Conclusions

We saw that we find the same “motion” (i.e. the simple harmonic motion) in two completely different physical topics. Every equation for the spring pendulum can be transformed into the corresponding equation for the LC oscillator with the following substitutions:

$$\begin{aligned} &y(t) \leftrightarrow q(t) \\ &v_y(t) \leftrightarrow i(t) \\ &k \leftrightarrow C^{-1} \\ &m \leftrightarrow L \end{aligned} \tag{15}$$