Conservation of Energy for a Mass on a Spring

The conservation of the total energy for a mass on a spring is often proved ignoring the gravitational potential energy, e.g. by assuming the mass is oscillating horizontally, or by just mentioning that the latter is somehow "compensated" for by the already stretched spring in the equilibrium position. It is an instructive exercise to show that this simplification is justified.

A spring with spring constant *k* is stretched when a mass is attached to it. In the equilibrium position, the elongation is (see figure 1)

$$y_0 = \frac{m \cdot g}{k} .$$

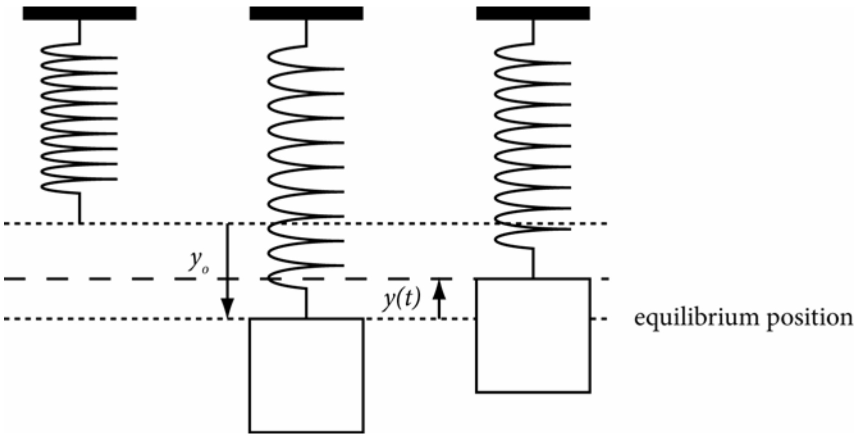

Figure 1: Mass on a spring

For a displacement *y(t)* from the equilibrium position, the elastic energy E_{el} of the spring and the gravitational potential energy E_{pot} are:

$$E_{el} = \frac{1}{2} \cdot k \cdot \left(y_0 - y(t)\right)^2 \text{ and } E_{pot} = m \cdot g \cdot y(t) + E_0$$

with an arbitrary constant E_0. Using the expression for the equilibrium elongation of the spring, we find

$$\begin{aligned} E_{el} + E_{pot} &= \frac{1}{2} \cdot k \cdot y_0^{\,2} - k \cdot y_0 \cdot y(t) + \frac{1}{2} \cdot k \cdot y^2(t) + m \cdot g \cdot y(t) + E_0 \\ &= \frac{1}{2} \cdot k \cdot \frac{m \cdot g}{k} \cdot y_0 - k \cdot \frac{m \cdot g}{k} \cdot y(t) + \frac{1}{2} \cdot k \cdot y^2(t) + m \cdot g \cdot y(t) + E_0 \\ &= \frac{1}{2} \cdot m \cdot g \cdot y_0 + \frac{1}{2} \cdot k \cdot y^2(t) + E_0 . \end{aligned}$$

By choosing an appropriate reference level for the gravitational potential energy such that

$$E_0 = -\frac{1}{2} \cdot m \cdot g \cdot y_0 ,$$

we have found the familiar term used in most textbooks.

Graphing the elastic energy of the spring E_{el}, the gravitational potential energy E_{pot} and the kinetic energy of the mass E_{kin} for a simple harmonic motion $y(t) = A \cdot \cos(\omega \cdot t)$ reveals the conservation of energy, as expected (see figure 2).

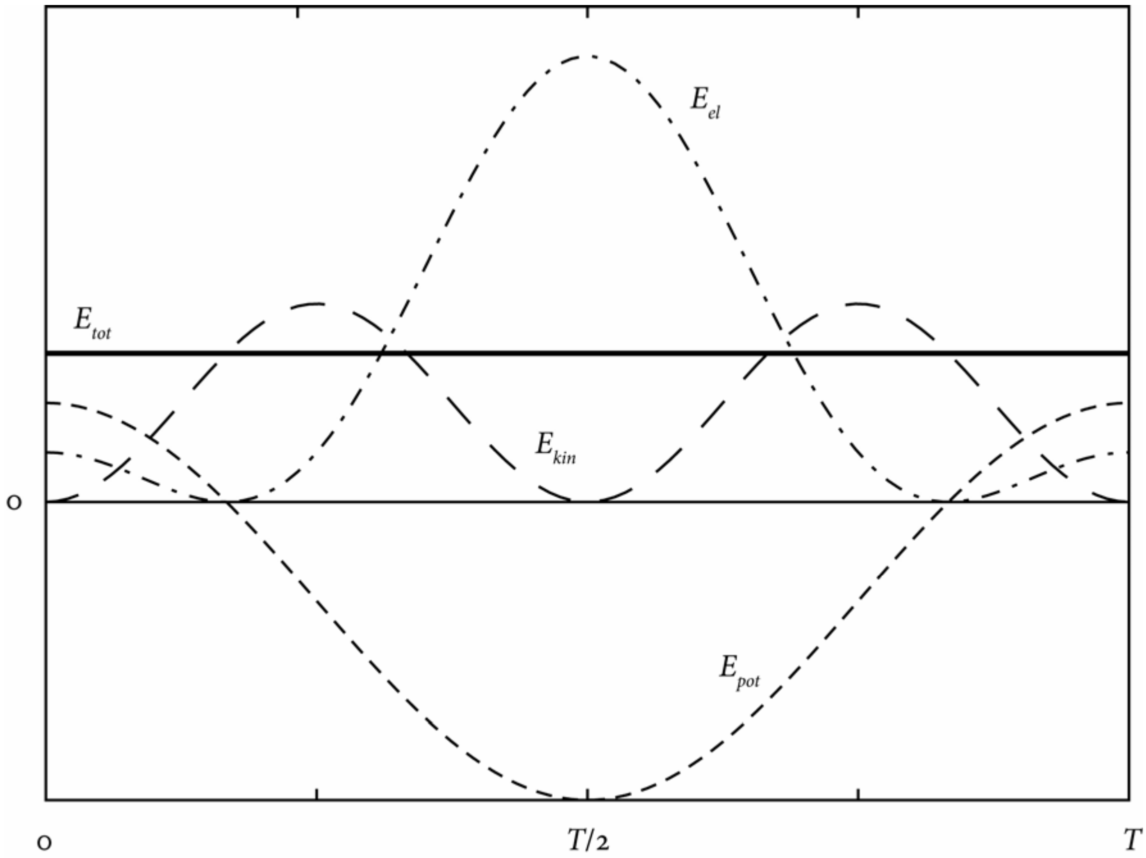

Figure 2: Conservation of energy for a mass on a spring