Error Calculation

1. Types of Errors

We discern three types of errors:

- *Gross errors:* Making a wrong reading of a value on a meter stick is a typical gross error. Gross errors can be avoided by careful experimentation.
- *Systematic errors:* Systematic errors are caused by imprecise measuring meters or methods, e.g. a meter stick which is slightly too long. Systematic errors bias all measured values in the same direction and cannot be eliminated by repeated measurements. They are usually very difficult to detect.
- *Random errors:* There is no absolutely precise measurement. Even the most sophisticated meters have only a limited precision. In electronic meters, the electronic "noise" caused by the thermal motion of the electrons sets a lower limit on precision. Since random errors do not change the "correct" result in any preferred direction, their influence can be reduced by repeated measurements.

2. Rule of Thumb

The result of a calculation can have at most as many significant figures as the least precise quantity used in the calculation.

The rule of thumb only gives a rough idea of the precision of the result.

3. Absolute and Relative Errors

The *absolute* error of a measured value a is Δa, if a is in the interval $[a - \Delta a, a + \Delta a]$ with absolute certainty. We write the result as $a \pm \Delta a$, e.g. $N_A = (6.02 \pm 0.02) \cdot 10^{23}\ \mathrm{mol}^{-1}$. In this example, there is no point in writing the quantity as $N_A = 6.0221367 \cdot 10^{23}\ \mathrm{mol}^{-1}$ since the value of the last five figures is smaller than the error.

We always round the error to one significant figure* and the result to the same decimal place. Without explicitly giving an error, we imply it to be of the order of the last figure of the number. For a physicist, a value of 1.5 is *not* the same as 1.500!

Example: The result of a calculation is 153.173 ms and the calculated error $3.21 \cdot 10^{-4}$ s. The standard form of the result is therefore (153.2 ± 0.3) ms.

The *relative* error of a measured value a is the ratio of the absolute error and the value itself:

$$r_a = \frac{\Delta a}{a} .$$

Relative errors are often given as a percentage, e.g. $N_A = 6.0221367 \cdot 10^{23}\ \mathrm{mol}^{-1} \pm 0.3\ \%$.

4. Calculating Errors

Only rarely does an experiment give us directly the quantities we are interested in. Instead, we calculate them from measured values. The error of the result can be calculated from the measurement errors. For "small" relative errors we can apply the following rules for *error propagation* (see "Formeln, Tabellen, Begriffe", p 184):

- Addition and subtraction: Absolute errors are added.
- Multiplication and division: Relative errors are added.
- Powers: Relative errors are multiplied by the power.

For more complicated operations (e.g. trigonometric functions), the error of the result can be determined by a worst case analysis (see examples).

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* As an exception to the rule a value starting with the (significant) figure 1 is often rounded to two significant figures.

5. Diagrams

In diagrams, values are represented by dots or small boxes and – whenever possible - by *error bars* (see fig. 1).

If we expect a linear relation between two quantities, we can fit a straight line to the data points, either by hand, with the calculator or a spreadsheet program (*linear regression*). The hypothesis is verified if the straight line crosses all error bars.

If we assume the relation to be non-linear (e.g. $y \propto x^2$), we can either fit a non-linear function (e.g. quadratic regression) with the calculator or a spreadsheet, or find a graphical representation in which a linear relation can be expected (e.g. y vs. x^2).

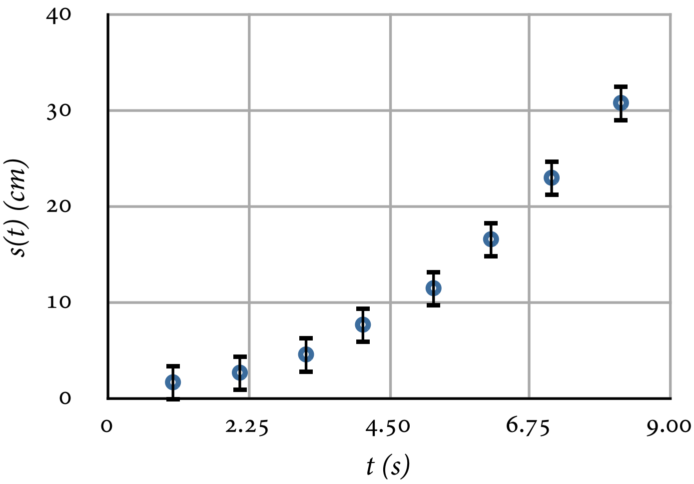

Figure 1: Diagram with error bars

6. Examples

a) Finding absolute errors

- Measuring lengths using a meter stick with centimetre division: Write down the division (1 cm) in your journal. Readings can be made to a precision of at least 0.5 cm.
- Measuring time using a stop watch with division 0.01 s: The response time of a human and therefore the absolute error is of the order of 0.1 s. Do not be fooled by digital displays! The precision can be improved by repeated measurements of the same event.

b) The speed of a vehicle can be determined by measuring the time needed for it to cover a certain distance. The measured values are $t \pm \Delta t$ and $s \pm \Delta s$. The speed is $v = s/t$. According to the quotient rule, the relative errors of distance and time have to be added: $r_v = r_s + r_t = \Delta s/s + \Delta t/t$.

The absolute error of the speed is $\Delta v = v \cdot r_v = v \cdot \left(\frac{\Delta s}{s} + \frac{\Delta t}{t} \right)$.

c) Determining the error by worst case analysis

For complicated expressions or functions to which the rules for error propagation do not apply, proceed as follows: Within their respective error bounds, increase or decrease every value in order to make the result as great as possible. This is the upper bound for the result. The lower bound can be determined in exactly the same way.