Kalman Model

1 8 State Position, Velocity, and Time Filtering Model

1.1 Physical Model

In GNSS Estimation, the primary estimated parameters are position, velocity, receiver clock offset, and receiver clock drift. These parameters can be estimated as an 8 state discrete extended Kalman filter.

One such model has velocity states and the receiver clock drift states that are treated as first order Gauss-Markov processes.

The following is a modified excerpt from (Grewel and Andrews, 2001), (p. 81-82).

> Let x_k be a zero-mean stationary Gaussian random sequence, RS (the discrete equivalent of a random process), with autocorrelation
>
> $$\Psi(k_2 - k_1) = \sigma^2 e^{-\beta|k_2 - k_1|} \tag{1}$$
>
> where:
> Ψ is the autocorrelation value,
> β is the inverse of the correlation time (1/s), and
> σ is the standard deviation of the system noise.
>
> This type of RS can be modelled as the output of a linear system with input w_k begin a zero-mean white Gaussian noise with power spectral density, PSD, equal to unity.
>
> A difference equation model for this type of process can be defined as
>
> $$\begin{aligned} x_k &= \Phi x_{k-1} + G w_{k-1} \\ z_k &= x_k \end{aligned} \tag{2}$$
>
> where:
> Φ, is the transition matrix,
> G, is a Shaping matrix, and
> z_k, is the measurement.
>
> In order to use this model, we need to solve for the unknown parameters Φ and G as functions of the parameter β. To do so, we first multiply Equation 2 by x_{k-1} on both sides and take the expected values to obtain the equations
>
> $$\begin{aligned} E[x_k x_{k-1}] &= \Phi E[x_{k-1} x_{k-1}] + G E[w_{k-1} x_{k-1}] \\ \sigma^2 e^{-\beta} &= \Phi \sigma^2 \end{aligned} \tag{3}$$
>
> assuming the w_k are uncorrelated and $E[w_k] = 0$, so that $E[w_{k-1} x_{k-1}] = 0$. One obtains the solution,
>
> $$\Phi = e^{-\beta} \tag{4}$$
>
> Next, square the state variable defined by Equation 2 and take is expected value.

$$
\begin{aligned}
&E[x_k x_k] = \Phi^2 E[x_{k-1}x_{k-1}] + G^2 E[w_{k-1}w_{k-1}] \\
&\sigma^2 = \sigma^2\Phi^2 + G^2
\end{aligned} \tag{5}
$$

because the variance $E[w_{k-1}^2] = 1$ and the parameter $G = \sigma\sqrt{1-e^{-2\beta}}$. The complete model is then

$$
x_k = e^{-\beta}x_{k-1} + \sigma\sqrt{1-e^{-2\beta}}w_{k-1} \tag{6}
$$

with $E[w_k] = 0$ and $E[w_{k_1}w_{k_2}] = \Delta(k_2 - k_1)$.

1.2 The Model

The states are (in this order):
ϕ, latitude [rads]
λ, longitude [rads]
h, height [m]
v_n, velocity north [m/s]
v_e, velocity east [m/s]
v_{up}, velocity up [m/s]
dT, receiver clock offset [m]
$\dot{dT}$, receiver clock drift [m/s]

The state transition matrix, T or Φ, is:

$$
\Phi = \begin{bmatrix}
1 & 0 & 0 & (1-e^{-\beta_{v_n}\Delta t})/\beta_{v_n} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & (1-e^{-\beta_{v_e}\Delta t})/\beta_{v_e} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & (1-e^{-\beta_{v_{up}}\Delta t})/\beta_{v_{up}} & 0 & 0 \\
0 & 0 & 0 & e^{-\beta_{v_n}\Delta t} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-\beta_{v_e}\Delta t} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-\beta_{v_{up}}\Delta t} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & (1-e^{-\beta_{\dot{dT}}\Delta t})/\beta_{\dot{dT}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\beta_{\dot{dT}}\Delta t}
\end{bmatrix} \tag{7}
$$

The process noise matrix, Q, (refer Brown and Hwang (1997), p. 200-202), is described below.

For convenience:

$$
\begin{aligned}
e_{v_n} &= e^{(-\beta_{v_n}\Delta t)} \\
e_{v_e} &= e^{(-\beta_{v_e}\Delta t)} \\
e_{v_{up}} &= e^{(-\beta_{v_{up}}\Delta t)} \\
e_{\dot{dT}} &= e^{(-\beta_{\dot{dT}}\Delta t)} \\
q_{v_n} &= 2\sigma_{v_n}^2\beta_{v_n} \\
q_{v_e} &= 2\sigma_{v_e}^2\beta_{v_e} \\
q_{v_{up}} &= 2\sigma_{v_{up}}^2\beta_{v_{up}} \\
q_{\dot{dT}} &= 2\sigma_{\dot{dT}}^2\beta_{\dot{dT}}
\end{aligned} \tag{8}
$$

$$Q = \begin{bmatrix} q_{11} & 0 & 0 & q_{14} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & q_{25} & 0 & 0 & 0 \\ 0 & 0 & q_{33} & 0 & 0 & q_{36} & 0 & 0 \\ q_{41} & 0 & 0 & q_{44} & 0 & 0 & 0 & 0 \\ 0 & q_{52} & 0 & 0 & q_{55} & 0 & 0 & 0 \\ 0 & 0 & q_{63} & 0 & 0 & q_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{77} & q_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & q_{87} & q_{88} \end{bmatrix} \tag{9}$$

$$q_{11} = \frac{q_{v_n}}{\beta_{v_n}^2}[\Delta t - \frac{2}{\beta_{v_n}}(1 - e_{v_n}) + \frac{1}{2\beta_{v_n}}(1 - e_{v_n}^2)] \tag{10}$$

$$q_{22} = \frac{q_{v_e}}{\beta_{v_e}^2}[\Delta t - \frac{2}{\beta_{v_e}}(1 - e_{v_e}) + \frac{1}{2\beta_{v_e}}(1 - e_{v_e}^2)] \tag{11}$$

$$q_{33} = \frac{q_{v_{up}}}{\beta_{v_{up}}^2}[\Delta t - \frac{2}{\beta_{v_{up}}}(1 - e_{v_{up}}) + \frac{1}{2\beta_{v_{up}}}(1 - e_{v_{up}}^2)] \tag{12}$$

$$q_{14} = q_{41} = \frac{q_{v_n}}{\beta_{v_n}}[\frac{1}{\beta_{v_n}}(1 - e_{v_n}) - \frac{1}{2\beta_{v_n}}(1 - e_{v_n}^2)] \tag{13}$$

$$q_{25} = q_{52} = \frac{q_{v_e}}{\beta_{v_e}}[\frac{1}{\beta_{v_e}}(1 - e_{v_e}) - \frac{1}{2\beta_{v_e}}(1 - e_{v_e}^2)] \tag{14}$$

$$q_{25} = q_{52} = \frac{q_{v_{up}}}{\beta_{v_{up}}}[\frac{1}{\beta_{v_{up}}}(1 - e_{v_{up}}) - \frac{1}{2\beta_{v_{up}}}(1 - e_{v_{up}}^2)] \tag{15}$$

$$q_{44} = \frac{q_{v_n}}{2\beta_{v_n}}(1 - e_{v_n}^2) \tag{16}$$

$$q_{55} = \frac{q_{v_e}}{2\beta_{v_e}}(1 - e_{v_e}^2) \tag{17}$$

$$q_{66} = \frac{q_{v_{up}}}{2\beta_{v_{up}}}(1 - e_{v_{up}}^2) \tag{18}$$

$$q_{77} = \frac{q_{\dot{dT}}}{\beta_{\dot{dT}}^2}[\Delta t - \frac{2}{\beta_{\dot{dT}}}(1 - e_{\dot{dT}}) + \frac{1}{2\beta_{\dot{dT}}}(1 - e_{\dot{dT}}^2)] \tag{19}$$

$$q_{78} = q_{87} = \frac{q_{\dot{dT}}}{\beta_{\dot{dT}}}[\frac{1}{\beta_{\dot{dT}}}(1 - e_{\dot{dT}}) - \frac{1}{2\beta_{\dot{dT}}}(1 - e_{\dot{dT}}^2)] \tag{20}$$

$$q_{88} = \frac{q_{\dot{dT}}}{2\beta_{\dot{dT}}}(1 - e_{\dot{dT}}^2) \tag{21}$$

2 References

Brown, R.G. and P.Y.C. Hwang (1997). Introduction to Random Signals and Applied Kalman Filtering, Third Edition, John Wiley and Sons, Inc.

Grewal, M.S. and A.P. Andrews (2001). Kalman Filtering Theory and Practice ; using Matlab. 2nd Edition. John Wiley and Sons Inc.