A 3D and 2D with Height Constraint Closed-Form Solutions of Non-Linear GPS Pseudorange Equations

©2004 **O. Mezentsev, J. Collin and G. MacGougan**

1 Closed-Form 3D Solution with 4 Satellites

User-to-Satellite GPS pseudorange equation can be written as Equation 1.

$$\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}+ct=p_i \tag{1}$$

where (x, y, z) are user unknown coordinates, t is user clock bias, (x_i, y_i, z_i) are i^{th} satellite coordinates and c is the speed of light.

To solve for a 3D user position and a user clock bias, four observations are needed. Let us label 4 observable satellites as $\#1, \#2, \#3$ and $\#4$. Therefore, the system of four equations can be formed (Equation 2).

$$\begin{cases} \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}+ct=p_1 \\ \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2}+ct=p_2 \\ \sqrt{(x-x_3)^2+(y-y_3)^2+(z-z_3)^2}+ct=p_3 \\ \sqrt{(x-x_4)^2+(y-y_4)^2+(z-z_4)^2}+ct=p_4 \end{cases} \tag{2}$$

This system is non-linear with respect to user coordinates. To simplify derivations, new variables are defined (Equation 3).

$$\begin{array}{l} (x-x_1)=a \\ (y-y_1)=b \\ (z-z_1)=c \\ (p_1-ct)=d \end{array} \tag{3}$$

Substituting Equation 3 into Equation 2 and squarring both sides of all equations, one can get the following system of equations (Equation 4).

$$\begin{cases} a^2+b^2+c^2=d^2 \\ (a+(x_1-x_2))^2+(b+(y_1-y_2))^2+(c+(z_1-z_2))^2=(d+(p_2-p_1))^2 \\ (a+(x_1-x_3))^2+(b+(y_1-y_3))^2+(c+(z_1-z_3))^2=(d+(p_3-p_1))^2 \\ (a+(x_1-x_4))^2+(b+(y_1-y_4))^2+(c+(z_1-z_4))^2=(d+(p_4-p_1))^2 \end{cases} \tag{4}$$

To further simplify the equations, new variables are defines (Equation 5).

$$\begin{array}{llll} x_{12}=(x_1-x_2)\,; & y_{12}=(y_1-y_2)\,; & z_{12}=(z_1-z_2)\,; & p_{21}=(p_2-p_1)\,; \\ x_{13}=(x_1-x_3)\,; & y_{13}=(y_1-y_3)\,; & z_{13}=(z_1-z_3)\,; & p_{31}=(p_3-p_1)\,; \\ x_{14}=(x_1-x_4)\,; & y_{14}=(y_1-y_4)\,; & z_{14}=(z_1-z_4)\,; & p_{41}=(p_4-p_1)\,; \end{array} \tag{5}$$

Substituting Equations 5 into Equations 4, the following system of equations is obtained (Equation 6):

$$\begin{cases} a^2+b^2+c^2=d^2 \\ \left(a^2+2ax_{12}+x_{12}^2\right)+\left(b^2+2by_{12}+y_{12}^2\right)+\left(c^2+2cz_{12}+z_{12}^2\right) \\ \qquad =\left(d^2+2dp_{21}+p_{21}^2\right) \\ \left(a^2+2ax_{13}+x_{13}^2\right)+\left(b^2+2by_{13}+y_{13}^2\right)+\left(c^2+2cz_{13}+z_{13}^2\right) \\ \qquad =\left(d^2+2dp_{31}+p_{31}^2\right) \\ \left(a^2+2ax_{14}+x_{14}^2\right)+\left(b^2+2by_{14}+y_{14}^2\right)+\left(c^2+2cz_{14}+z_{14}^2\right) \\ \qquad =\left(d^2+2dp_{41}+p_{41}^2\right) \end{cases} \tag{6}$$

Subtracting the first equation in the system of Equations 6 from the rest, Equation 7 is obtained.

$$\begin{cases} (2ax_{12}+2by_{12}+2cz_{12})+\left(x_{12}^2+y_{12}^2+z_{12}^2-p_{21}^2\right)=2dp_{21} \\ (2ax_{13}+2by_{13}+2cz_{13})+\left(x_{13}^2+y_{13}^2+z_{13}^2-p_{31}^2\right)=2dp_{31} \\ (2ax_{14}+2by_{14}+2cz_{14})+\left(x_{14}^2+y_{14}^2+z_{14}^2-p_{41}^2\right)=2dp_{41} \end{cases} \tag{7}$$

The four unknowns now are (a, b, c, d). We will solve Equation 7 for (a, b, c) leaving d the fourth unknowns and then substitute (a, b, c) into the first equation of Equation 6. Then, it will be a quadratic equation for d. Then, solving for d will lead to two solutions; one of them can then easily be rejected. First, we will define:

$$\begin{aligned} k_1 &= \left(x_{12}^2+y_{12}^2+z_{12}^2-p_{21}^2\right) \\ k_2 &= \left(x_{13}^2+y_{13}^2+z_{13}^2-p_{31}^2\right) \\ k_3 &= \left(x_{14}^2+y_{14}^2+z_{14}^2-p_{41}^2\right) \end{aligned} \tag{8}$$

Substituting Equation 8 into Equation 7, one can get Equation 9.

$$\underbrace{\begin{pmatrix} 2x_{12} & 2y_{12} & 2z_{12} \\ 2x_{13} & 2y_{13} & 2z_{13} \\ 2x_{14} & 2y_{14} & 2z_{14} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 2p_{21} \\ 2p_{31} \\ 2p_{41} \end{pmatrix} d \tag{9}$$

Solving for (a, b, c) leads to Equation 10.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underbrace{A^{-1} \begin{pmatrix} 2p_{21} \\ 2p_{31} \\ 2p_{41} \end{pmatrix}}_{\mathbf{C}} d - \underbrace{A^{-1} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}}_{\mathbf{F}} \tag{10}$$

Matrices $\mathbf{C}$ and $\mathbf{F}$ are (3×1) column vectors. Therefore, (a, b, c) can be expressed as in Equation 11.

$$\begin{aligned} a &= c_1 d - f_1 \\ b &= c_2 d - f_2 \\ c &= c_3 d - f_3 \end{aligned} \tag{11}$$

After substituting Equation 11 into the first equation of Equation 6, Equation 30 results.

$$\underbrace{\left(c_1^2+c_2^2+c_3^2-1\right)}_{\mathbf{m_1}} d^2 \underbrace{-\left(2c_1f_1+2c_2f_2+2c_3f_3\right)}_{\mathbf{m_2}} d + \underbrace{\left(f_1^2+f_2^2+f_3^2\right)}_{\mathbf{m_3}} = 0 \tag{12}$$

Two solutions for the clock bias are obtained by solving Equation 30 (Equation 13). To return to the original clock bias variable, Equation 3 can be used.

$$d_{1,2} = \frac{-m_2 \pm \sqrt{m_2^2 - 4m_1m_3}}{2m_1} \tag{13}$$

Substituting found clock biases, d_1 and d_2, into Equation 10 and returning to original user coordinates (Equation 3), Equation 14 is obtained.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{1,2} = \mathbf{C}d_{1,2} - \mathbf{F} + \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{14}$$

To choose the right solution from the two, a simple check can be performed. The distance from the center of the Earth to the computed position should approximately be equal to the Earth radius ($\sim 6,371$ km).

$$\text{Solution} = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}_{i=1,2} : \text{ if } \quad \sqrt{x_i^2 + y_i^2 + z_i^2} \cong R_{Earth} \tag{15}$$

2 Closed-Form 2D Solution with 3 Satellites and a Given Height

Let us assume that only 3 satellites are in view and the height above the ellipsoid, h, is given ($H_{user} = R_E + h$), R_E is the distance from the user position to the center of the ellipsoid (6371 km for 1 iteration, then refined on 2nd iteration with an actual ellipsoidal value). Therefore, the pseudorange equations for 3 visible satellites and a height constraint can be written as in Equation 16.

$$\begin{cases} \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}+ct=p_1 \\ \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2}+ct=p_2 \\ \sqrt{(x-x_3)^2+(y-y_3)^2+(z-z_3)^2}+ct=p_3 \\ \sqrt{(x)^2+(y)^2+(z)^2}=H_{user} \end{cases} \tag{16}$$

This system is also non-linear with respect to user coordinates. To simplify derivations, new variables are defined (Equation 17).

$$\begin{aligned} (x-x_1)&=a \\ (y-y_1)&=b \\ (z-z_1)&=c \\ (p_1-ct)&=d \end{aligned} \tag{17}$$

Substituting Equation 17 into Equation 7 and squaring both sides of all equations, one can get the following system of equations (Equation 18).

$$\begin{cases} a^2+b^2+c^2=d^2 \\ (a+(x_1-x_2))^2+(b+(y_1-y_2))^2+(c+(z_1-z_2))^2=(d+(p_2-p_1))^2 \\ (a+(x_1-x_3))^2+(b+(y_1-y_3))^2+(c+(z_1-z_3))^2=(d+(p_3-p_1))^2 \\ (a+x_1)^2+(b+y_1)^2+(c+z_1)^2=H_{user}^2 \end{cases} \tag{18}$$

To further simplify the equations, new variables are defines (Equation 19).

$$\begin{aligned} &x_{12}=(x_1-x_2)\,; \quad y_{12}=(y_1-y_2)\,; \quad z_{12}=(z_1-z_2)\,; \quad p_{21}=(p_2-p_1)\,; \\ &x_{13}=(x_1-x_3)\,; \quad y_{13}=(y_1-y_3)\,; \quad z_{13}=(z_1-z_3)\,; \quad p_{31}=(p_3-p_1)\,; \end{aligned} \tag{19}$$

Substituting Equations 19 into Equations 18, the following system of equations is obtained (Equation 20):

$$\begin{cases} a^2+b^2+c^2=d^2 \\ \left(a^2+2ax_{12}+x_{12}^2\right)+\left(b^2+2by_{12}+y_{12}^2\right)+\left(c^2+2cz_{12}+z_{12}^2\right) \\ \qquad =\left(d^2+2dp_{21}+p_{21}^2\right) \\ \left(a^2+2ax_{13}+x_{13}^2\right)+\left(b^2+2by_{13}+y_{13}^2\right)+\left(c^2+2cz_{13}+z_{13}^2\right) \\ \qquad =\left(d^2+2dp_{31}+p_{31}^2\right) \\ \left(a^2+2ax_1+x_1^2\right)+\left(b^2+2by_1+y_1^2\right)+\left(c^2+2cz_1+z_1^2\right) \\ \qquad =H_{user}^2 \end{cases} \tag{20}$$

Subtracting the first equation in the system of Equations 20 from the rest, Equation 21 is obtained.

$$\begin{cases} (2ax_{12}+2by_{12}+2cz_{12})+\left(x_{12}^2+y_{12}^2+z_{12}^2-p_{21}^2\right)=2dp_{21} \\ (2ax_{13}+2by_{13}+2cz_{13})+\left(x_{13}^2+y_{13}^2+z_{13}^2-p_{31}^2\right)=2dp_{31} \\ (2ax_1+2by_1+2cz_1)+\left(x_1^2+y_1^2+z_1^2-H_{user}^2\right)=-d^2 \end{cases} \tag{21}$$

The four unknowns now are (a, b, c, d). We will solve Equation 21 for (a, b, c) leaving d the fourth unknowns and then substitute (a, b, c) into the first equation of Equation 20. Then, it will be a quartic equation for

d. Then, solving for d will lead to four solutions with the correct solution easily identified. First, we will define:

$$
\begin{aligned}
k_1 &= \left(x_{12}^2 + y_{12}^2 + z_{12}^2 - p_{21}^2\right)\\
k_2 &= \left(x_{13}^2 + y_{13}^2 + z_{13}^2 - p_{31}^2\right)\\
k_3 &= \left(x_1^2 + y_1^2 + z_1^2 - H_{user}^2\right)
\end{aligned}
\tag{22}
$$

Substituting Equation 22 into Equation 21, one can get Equation 23.

$$
\underbrace{\begin{pmatrix} 2x_{12} & 2y_{12} & 2z_{12} \\ 2x_{13} & 2y_{13} & 2z_{13} \\ 2x_1 & 2y_1 & 2z_1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 2p_{21}d \\ 2p_{31}d \\ -d^2 \end{pmatrix}
\tag{23}
$$

Let $C = A^{-1}$, then, solving for (a, b, c) leads to Equation 24.

$$
\begin{pmatrix} a \\ b \\ c \end{pmatrix} = C \begin{pmatrix} 2p_{31}d \\ 2p_{21}d \\ -d^2 \end{pmatrix} - \underbrace{C \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}}_{\mathbf{F}}
\tag{24}
$$

Matrix $\mathbf{F}$ is (3×1) column vector, (f_1, f_2, f_3). Now, we must solve (a, b, c) explicitly as in Equation 25.

$$
\begin{aligned}
a &= C_{11}(2p_{21}d) + C_{12}(2p_{31}d) - C_{13}(d^2) - f_1\\
b &= C_{21}(2p_{21}d) + C_{22}(2p_{31}d) - C_{23}(d^2) - f_2\\
c &= C_{31}(2p_{21}d) + C_{32}(2p_{31}d) - C_{33}(d^2) - f_3
\end{aligned}
\tag{25}
$$

Let us define new variables:

$$
\begin{aligned}
q &= C_{11}(2p_{21}) + C_{12}(2p_{31})\\
r &= C_{21}(2p_{21}) + C_{22}(2p_{31})\\
s &= C_{31}(2p_{21}) + C_{32}(2p_{31})
\end{aligned}
\tag{26}
$$

Substituting new variables defined in Equation 26, Equation 27 results:

$$
\begin{aligned}
a &= -C_{13}d^2 + qd - f_1\\
b &= -C_{23}d^2 + rd - f_2\\
c &= -C_{33}d^2 + sd - f_3
\end{aligned}
\tag{27}
$$

Squaring Equations 27 and rearranging the terms, Equation 28 is obtained.

$$
\begin{aligned}
a^2 &= (C_{13}^2)d^4 - (2C_{13}q)d^3 + (q^2 + 2C_{13}f_1)d^2 - (2qf_1)d + f_1^2\\
b^2 &= (C_{23}^2)d^4 - (2C_{23}r)d^3 + (r^2 + 2C_{23}f_2)d^2 - (2rf_2)d + f_2^2\\
c^2 &= (C_{33}^2)d^4 - (2C_{33}s)d^3 + (s^2 + 2C_{33}f_3)d^2 - (2sf_3)d + f_3^2
\end{aligned}
\tag{28}
$$

After substituting Equation 28 into the first equation of Equation 20, Equation 29 results.

$$
\begin{aligned}
&(C_{13}^2 + C_{23}^2 + C_{33}^2)d^4 - 2(C_{13}q + C_{23}r + C_{33}s)d^3 +\\
&(q^2 + r^2 + s^2 + 2(C_{13}f_1 + C_{23}f_2 + C_{33}f_3) - 1)d^2 -\\
&2(qf_1 + rf_2 + sf_3)d + f_1^2 + f_2^2 + f_3^2 = 0
\end{aligned}
\tag{29}
$$

Now forming the generalize quartic:

$$
d^4 + a_3d^3 + a_2d^2 + a_1d^1 + a_0 = 0;
\tag{30}
$$

where

$$\begin{aligned}
a_3 &= -2(C_{13}q + C_{23}r + C_{33}s)/(C_{13}^2 + C_{23}^2 + C_{33}^2) \\
a_2 &= (q^2 + r^2 + s^2 + 2(C_{13}f_1 + C_{23}f_2 + C_{33}f_3) - 1)/(C_{13}^2 + C_{23}^2 + C_{33}^2) \\
a_1 &= -2(qf_1 + rf_2 + sf_3)/(C_{13}^2 + C_{23}^2 + C_{33}^2) \\
a_0 &= (f_1^2 + f_2^2 + f_3^2)/(C_{13}^2 + C_{23}^2 + C_{33}^2)
\end{aligned} \tag{31}$$

The quartic can be solved to yield up to four real solutions by the method described at http://en.wikipedia.org/wiki/Quartic_equation. The correct solution is identified easily but requires the Doppler observations for the three satellites as well, which are always available if there is a pseudorange measurement. First, non-real roots are discarded. Next, a step is taken to adjust the pseudorange measurements input to the solution to limit the range of the receiver clock offset. It is known that for a user on the Earth, the range to a GPS satellite is within 21000-26000 km. Thus, the pseudorange measurements are adjusted but the difference in the mean pseudorange measurement values with a typical range, ($\sim 75ms * C$) to the satellites to limit the effective range of the receiver clock offset. This is accomplished by the following equation.

$$\begin{aligned}
k_{adj} &= 22500000 - \tfrac{p_1+p_2+p_3}{3} \\
p_{1adj} &= p1 + k_{adj} \\
p_{2adj} &= p2 + k_{adj} \\
p_{3adj} &= p3 + k_{adj}
\end{aligned} \tag{32}$$

This limits the effective range of the receiver clock offset in the solution to $\pm\ 15ms$. This check effectively removes solutions on the other side of the planet and limits the number of potential solutions to at most 2. The next step is to identify which of the two remaining solutions is correct. This is accomplished by comparing the predicted single differenced (between satellites) Doppler values with measured single differenced Doppler values. The predicted Doppler values and single differenced Doppler values are defined by the equation.

$$\begin{aligned}
i &= 1, 2, 3 \\
dx_i &= x_i - x \\
dy_i &= y_i - y \\
dz_i &= z_i - z \\
r_i &= \sqrt{dx_i^2 + dy_i^2 + dz_i^2} \\
Dp_i &= -1 * (x_i * dx + y_i * dy + z_i * dz)/r + clkdrift_i \\
SDp_{ij} &= Dp_i - Dp_j \\
SDm_{ij} &= Dm_i - Dm_j
\end{aligned} \tag{33}$$

where, x_i, y_i, z_i are satellite coordinates, x, y, z are the potential receiver coordinates, r_i is the geometric range between the receiver and the satellite, Dp_i is the predicted Doppler value, SDp_{ij} is the predicted single differenced Doppler value between satellites i and j ($i = 1, j = 2$, and $i = 1, j = 3$ are possible combinations), $clkdrift_i$ is the calculated satellite clock drift (based most likely on the broadcast clock correction calculations), Dm_i is a measured Doppler value, and SDm_i is the measured single differenced Doppler value. A simple cost function is then defined as follows.

$$cv = |SDp_{12} - SDm_{12}| + |SDp_{13} - SDm_{13}| \tag{34}$$

where, cv is the cost value. The correct solution has the smallest cost value.