Evaluating Graph Resilience: The Effects of Edge Perturbations on Connectivity and Centrality Metrics

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Abstract. This study investigates the impact of edge perturbations on graph structures and centrality metrics in undirected, weighted networks. Building on prior work focusing on node perturbations, we analyze edge failures using two probabilistic models: one based on edge betweenness and another on edge weights. A subset of edges is chosen randomly, and within this subset, each edge is removed with a 0-1 probability determined by either min-max or sum normalization. The study encompasses three graph types: a synthetic Erdős-Rényi graph, a kangaroo interaction network, and the karate club graph, each offering varying structural and contextual features. Key findings indicate that community-rich graphs, like the karate club graph, are particularly vulnerable to edge failures driven by edge betweenness, especially under min-max normalization. For edge weight-based failures, the degree of deformation is strongly influenced by the weight distribution, with min-max normalization causing more severe disruptions compared to sum normalization. These findings highlight the intricate relationship between network topology, failure models, and structural resilience, underscoring the importance of developing targeted strategies to enhance robustness in critical systems.

1 Introduction

In this work, we aim to explore how the adjacency matrix and centrality scores of graphs vary under perturbations. Specifically, our objective is to replicate the study conducted in Cavallaro et al. [1] but from a different perspective. While [1] investigates node perturbations in undirected and unweighted graphs, we focus instead on edge perturbations in undirected but weighted graphs.

In particular, we aim to replicate two perturbation settings similar to [2]: the first involves removing edges with probabilities proportional to their weights, and the second is based on their edge betweenness values. Both perturbation settings are analyzed under different values of τ, which represents the proportion of nodes that can fail. As in [1], we first sample a proportion τ of nodes and perform perturbations only on this subset, leaving the rest of the graph unchanged.

We simulate two different scenarios for edge removal. In the first scenario, similar to [1], each edge in the subset has a probability of being removed proportional to the ratio of its weight or betweenness to the sum of all weights or betweenness values (sum normalization). In the second scenario, not analyzed in [1], we apply a normalization of the weights or betweenness values to the range $[0, 1]$, where the edge with the highest weight or betweenness has a probability of 1 to be removed, and the edge with the lowest weight or betweenness has a probability of 0. This approach ensures that if the edge with the highest weight or betweenness is sampled, it is always removed, while the edge with the lowest value is never removed.

Once the perturbations are applied, we aim to analyze the extent of the perturbations introduced into the graph and determine whether a threshold value of τ exists at which these perturbations significantly alter the adjacency matrix, and therefore the overall connectivity structure. Finally, we examine how centrality metric vectors respond to these perturbations as a function of τ. To this end, we employ three different centrality metrics: weighted degree centrality, weighted eigenvector centrality, and weighted Katz centrality. Insights into the differences or similarities with [1] are also analyzed. To conduct our research, we use three different graphs:

- **A synthetic Erdős-Rényi (ER) graph**: This graph is generated with edge weights sampled from a uniform distribution, providing a controlled synthetic environment for analyzing the effects of perturbations.
- **The kangaroo interaction network**: This is a real-world weighted network representing interactions among kangaroos in their natural habitat. The weights represent the frequency or intensity of interactions.
- **The karate club graph**: This is a widely used social network graph where nodes represent members of a karate club and edges represent friendships or social ties. The weights are determined based on interaction strength or frequency.

The aim of this work is to uncover, in real-world scenarios, whether cutting a set of connections based on their importance significantly alters the importance of the nodes themselves. Such changes could lead to critical problems in networks, such as reduced robustness or loss of functionality in social, biological, or infrastructural systems. Understanding these dynamics allows us to propose strategies to mitigate the risks associated with targeted edge removal or failure.

2 Results

In this section, we present the outcomes of our simulations, with all results averaged across 100 runs.

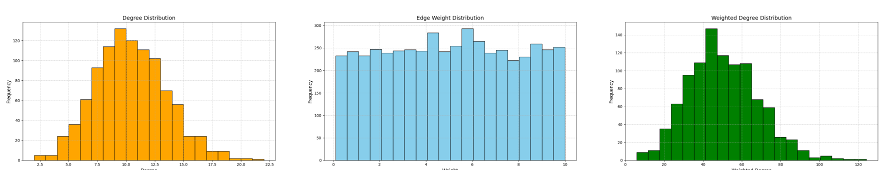
2.1 Insights and Properties of the Graphs

Synthetic graph The synthetic graph, with 1,000 nodes and 4,958 edges, is the largest and most uniform among the three networks, reflecting its origins in the Erdos-Rényi model. Its edge weights range narrowly between 0.103 and 9.99,

	Synthetic graph	Karate graph	Kangaroo graph
Number of nodes	1000	34	17
Number of edges	4958	78	91
Minimum edge weight	0.103	1.0	1.0
Maximum edge weight	9.99	7.0	54.0
Average edge weight	5.07	2.96	6.10
Graph density	0.01	0.14	0.67
Number of connected components	1	1	1
Sizes of connected components	1000	34	17
Global clustering coefficient	0.0046	0.2414	0.0705
Average path length	8.36	5.75	2.22
Diameter	24.59	13.0	4.0
Radius	13.34	7.0	3.0
Degree assortativity	-0.0239	-0.4756	-0.1934
Edge connectivity	2	1	1
Largest eigenvalue (λ_1)	57.64	21.68	124.79

Table 1. Network properties of the different graphs.

with an average weight of 5.07, suggesting a moderately uniform distribution. The graph is sparse, with a density of only 0.01, and forms a single connected component, emphasizing its cohesion. The global clustering coefficient is very low at 0.0046, consistent with the random nature of the graph, while the average path length is relatively long at 8.36, reflecting the sparse connections. The graph has a diameter of 24.59 and a radius of 13.34, indicating its extensive structure. Its degree assortativity is near zero (-0.0239), showing no preference for nodes to connect with others of similar degrees. The low edge connectivity of 2 means it is relatively fragile, as the removal of just two edges can disrupt connectivity.

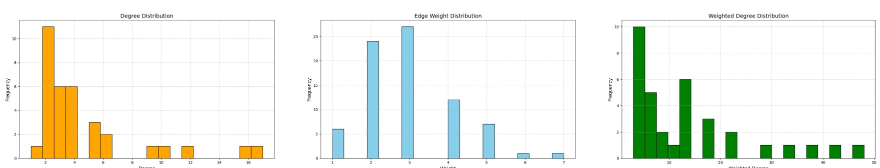


(a) (b) (c)

Fig. 1. Synthetic graph distributions of (a) degree, (b) edge weight and (c) weighted degree properties

In Figure 1(a), the distribution of the degrees of the nodes in the synthetic network is shown, with a distribution typical of a randomly generated graph, similar to Figure 1(c), which shows the weighted degree of the nodes in the graph. We can see that Figure 1(b) presents a distribution of the edge weights that is quite uniform.

Karate graph The Karate graph [3], a small social network with 34 nodes and 78 edges, displays properties typical of human interactions. Edge weights range from 1 to 7, with an average weight of 2.96, reflecting moderate variability in interaction strength. The graph is denser than the synthetic graph, with a density of 0.14, and forms a single connected component encompassing all nodes. Its global clustering coefficient is high at 0.2414, highlighting the presence of localized clusters, a hallmark of social networks. The average path length of 5.75 is shorter than that of the synthetic graph, demonstrating its small-world nature. The graph's diameter of 13 and radius of 7 underscore the proximity of nodes within the network. Negative degree assortativity (-0.4756) indicates a tendency for hubs to connect with low-degree nodes, further reinforcing its hierarchical structure. However, the graph's edge connectivity of 1 means it is vulnerable to disconnection from the removal of a single edge.

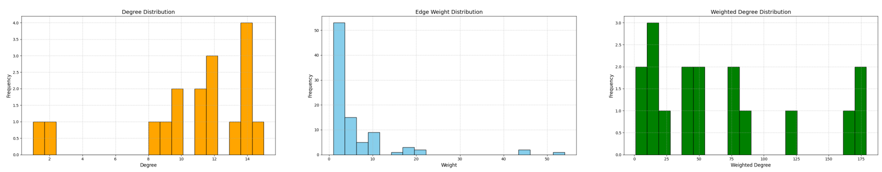


(a) (b) (c)

Fig. 2. Karate graph distributions of (a) degree, (b) edge weight and (c) weighted degree properties

Figure 2(a) shows that the majority of nodes have a low degree, so the network might have a few nodes with a high degree (hubs) that are highly connected, but most nodes are not connected to many others. In Figure 2(b), we can see that low-medium weights predominate. The network is mainly composed of weak relationships or links between the nodes. It is possible that the interactions between the nodes are not very significant or important, or that the nodes are connected to each other in a weak manner. In Figure 2(c), we can see that while some nodes may have a high weighted degree (i.e., being highly connected with strong relationships), the majority of nodes have few significant connections. This could suggest that the network has an imbalanced connection pattern, where some parts of the network are more important or influential than others

Kangaroo graph The Kangaroo graph [4], with 17 nodes and 91 edges, is the smallest and densest network, with a density of 0.67. Edge weights range widely from 1 to 54, with an average weight of 6.10, reflecting significant variability in interaction strengths, likely influenced by behavioral or environmental factors. Despite its small size, the graph forms a single connected component, ensuring that all nodes are part of a cohesive network. The global clustering coefficient is moderate at 0.0705, suggesting some localized clustering. The network is compact, with an average path length of 2.22, a diameter of 4, and a radius of 3, emphasizing its tightly connected nature. The degree assortativity is slightly negative (-0.1934), indicating a weak preference for nodes to connect with others of differing degrees. Like the Karate graph, the Kangaroo graph has an edge connectivity of 1, making it vulnerable to disconnection from the removal of a single edge.



(a) (b) (c)

Fig. 3. Kangaroo graph distributions of (a) degree, (b) edge weight and (c) weighted degree properties

In Figure 3(a), we can see that, in this case, most of the nodes have a relatively high degree, which suggests that the network is highly connected and dense, with many nodes connected to each other. This implies that the interactions are frequent and there is a high connectivity between the elements of the network. Figure 3(b) shows us that the majority of the connections between the nodes have a low weight, meaning they are connections with low intensity or strength. In Figure 3(c), we can see that the network has a fairly uniform distribution of weighted degree, meaning that the weights of the connections between the nodes are distributed similarly among most of the nodes. In other words, the nodes in the network have a similar level of connectivity in terms of the intensity or strength of their relationships.

2.2 Edge Betweenness-Based Failure Probability

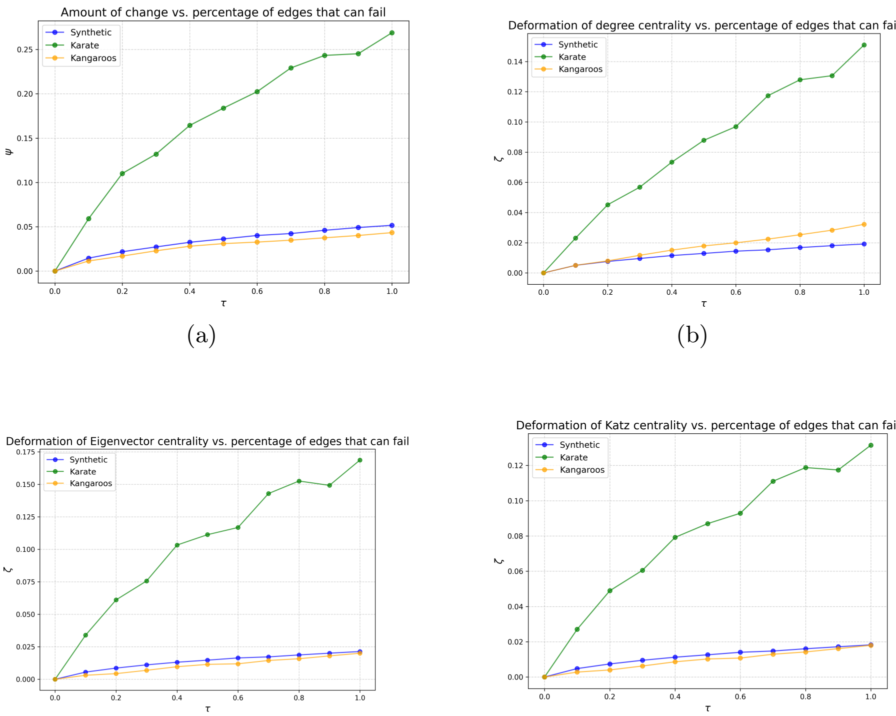
The results from Fig. 4 and Fig. 5 illustrate how the topology and centrality metrics of weighted networks are deformed as the percentage of edges that can fail (τ) increases under a failure probabilistic model based on the min-max and sum-normalization of the betweenness of the edges, respectively. Each panel provides insights into the sensitivity of different networks (Synthetic, Karate, and Kangaroo) under edge perturbations.

Min-max normalized In Fig. 4(a), the amount of deformation of the networks as τ increases is shown. It can be observed that the Karate network exhibits a greater increase in ψ compared to the other two networks as the percentage of affected edges increases. In other words, for the same percentage of affected edges, the Karate network is the most vulnerable to edge failures. On the other hand, the Kangaroos network shows the smallest increase in ψ across all values of τ, demonstrating robustness in its topology, followed by the synthetic network, which also exhibits similar robustness, likely due to its uniform structure and relatively high density.

In Figures 4(b)-4(d), a similar impact is observed in the deformation of the different centrality metrics analyzed, as seen for the amount of topological change. Fig.4(b) shows that the deformation of the weighted degree centrality with respect to the percentage of edges prone to failure τ follows a moderate growth for the synthetic networks and the Kangaroos network, with the deformation of the latter being slightly higher than that of the former. In contrast, for the Karate network, the deformation of degree centrality is much greater than for the other networks, likely due to its larger global clustering coefficient. In Figures 4(c)-4(d), the results are very similar to those obtained in Fig. 4(b), except that the deformation of the synthetic network is always slightly greater than that of the Kangaroos network.

Sum-normalized In Fig. 5(a), the amount of deformation of the networks as τ increases is shown. It can be observed that similar to Fig. 4(a) the Karate network exhibits a greater increase in ψ compared to the other two networks as the percentage of affected edges increases. On the other hand, the synthetic network shows the smallest increase in ψ across all values of τ, demonstrating robustness in its topology, followed by the Kangaroos network, which also exhibits similar robustness.

In Figures 5(b)-5(d), a similar impact is observed in the deformation of the different centrality metrics analyzed. All figures show that the deformation of the weighted centrality metrics with respect to the percentage of edges prone to failure τ follows a moderate growth for the synthetic networks and the Kangaroos network, with the deformation of the latter being always slightly higher than that of the former. We observe that both the amount of change in the topology and the deformation of centrality metrics show smaller values when using sum-normalization compared to min-max normalization in Fig. 4. This is likely due to the lower failure probability of each edge when using sum-normalization.



(a) (b) (c) (d)

Fig. 4. Amount of change (a) and deformation of centrality metrics (b)-(d) obtained by using a failure probabilistic model based on the min-max normalization of the betweenness of the edges.

2.3 Edge Weight-based Failure probability

The results from Fig. 6 and Fig. 7 illustrate how the topology and centrality metrics of weighted networks are deformed as the percentage of edges that can fail (τ) increases under a failure probabilistic model based on the min-max and sum-normalization of the weights of the edges, respectively.

Min-max normalized In Figure 6(a), the amount of change in the topology of the networks with respect to τ is shown, and we observe that the synthetic network exhibits the greatest change as the failure percentage increases, indicating that it is more sensitive to failures compared to the other networks under this type of model. This contrasts with the results obtained using a model based on edge betweenness in Figures 4(a) and 4(a), where the network that appears most vulnerable is the Karate network. This suggests that not only the topology of the network is important, as demonstrated in [1], but also that the model used determines how vulnerable a graph can be when perturbed. Here, our results are

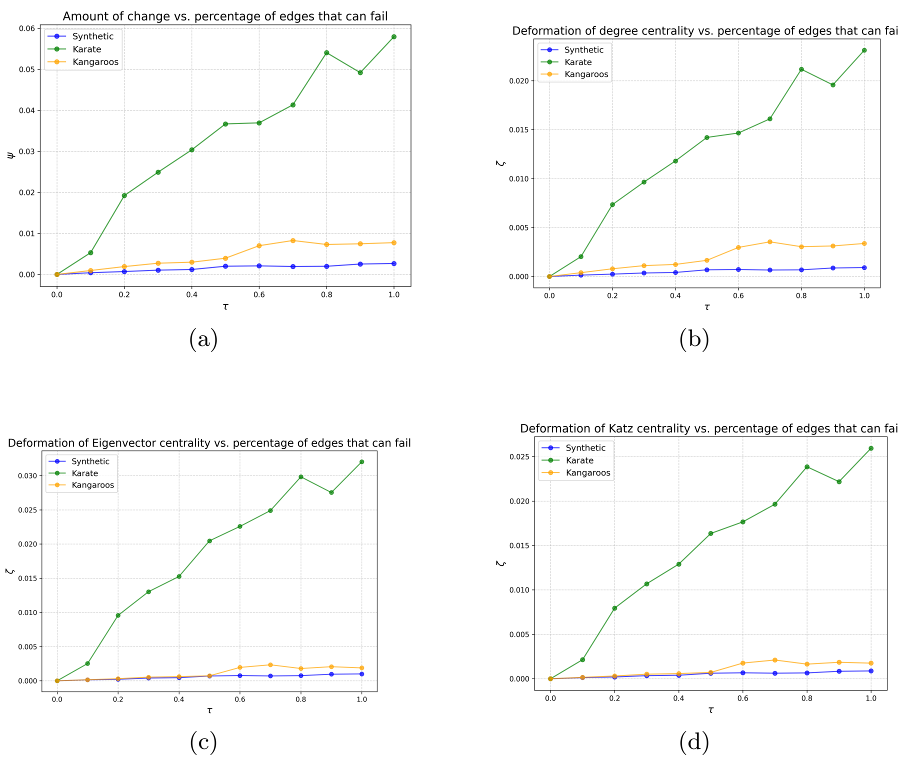


Fig. 5. Amount of change (a) and deformation of centrality metrics (b)-(d) obtained by using a failure probabilistic model based on the sum-normalization of the betweenness of the edges.

similar to those obtained in [1], where the ψ trend increases almost linearly as τ grows, but the rate slightly differs from one network to another.

Figures 6(b)-6(d) show a similar pattern, suggesting a relationship between the amount of deformation and the effect it has on the deformation of the studied centrality metrics. In Figures6(c) and 6(d), we observe that the centrality metrics of the synthetic network and the Karate network deform similarly when the percentage of edges that can fail is low. However, for high values of τ, the deformation of the metric in the synthetic network is more pronounced than in the Karate network. This phenomenon may be due to the fact that the weighted eigenvector centrality and Katz centrality metrics depend on the global structure, which initially remains stable, whereas the weighted degree centrality in Figure 6(b) is more sensitive to local changes. Initial failures thus impact homogeneous networks (synthetic) and heterogeneous networks (Karate) differently.

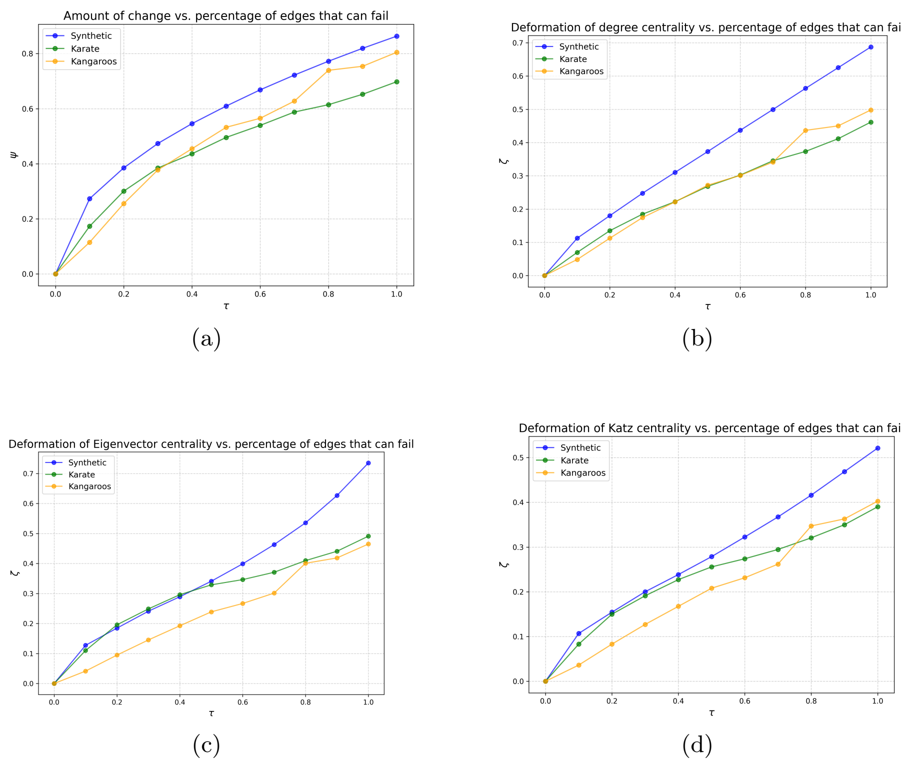


Fig. 6. Amount of change (a) and deformation of centrality metrics (b)-(d) obtained by using a failure probabilistic model based on the min-max normalization of the weights of the edges.

Sum-normalized Figure 7(a) shows that the topological deformation is lower in the synthetic network compared to the karate and kangaroo networks. This shows that network perturbation is influenced not only by its topology but also by the model used to determine the edge failure probability.

Meanwhile, Figures 7(b)- 7(d) reveal a consistent pattern: the deformation of the analyzed metrics increases as the percentage of candidate edges for failure grows. In Figures 7(c) and 7(d), the behavior of the karate and kangaroo networks is similar and differs significantly from the synthetic network. In contrast, these differences are less pronounced in Figure 7(b) where the differences between networks are more similar to the relationship they follow with the amount of change in the topology of Figure 7(a).

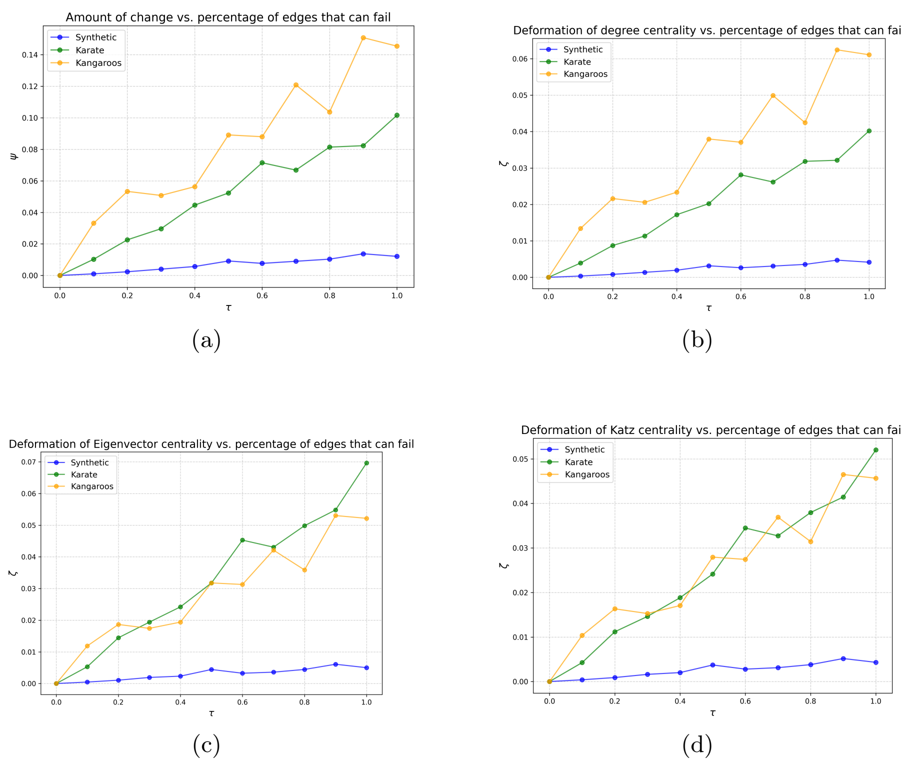


Fig. 7. Amount of change (a) and deformation of centrality metrics (b)-(d) obtained by using a failure probabilistic model based on the sum-normalization of the weights of the edges.

3 Discussion

3.1 Edge Betweenness Failure Probability

Normalization methods As shown in Figures 4 and 5, the normalization method applied to edge betweenness values significantly influences the results. While the trends in adjacency matrix perturbations and centrality metric deformations are consistent across normalization methods, the absolute values differ. In particular, min-max normalization produces more pronounced deformations, especially for the Karate graph, compared to sum normalization. Although the deformation for the Karate graph grows under sum normalization, its absolute value remains relatively negligible. For the Synthetic and Kangaroo graphs, the deformations are consistently much smaller regardless of the normalization method.

This difference arises because min-max normalization assigns significantly higher failure probabilities to edges with high betweenness, effectively neglecting edges with low betweenness by assigning them near-zero probabilities. In contrast, sum normalization distributes failure probabilities proportionally, reducing the emphasis on edges with high betweenness and resulting in milder deformations. Consequently, for edge betweenness-based failure, we focus on interpreting the results produced by min-max normalization, as it better highlights the structural impact of removing critical edges.

Adjacency Matrix Perturbation As illustrated in Figures 4 and 5, the adjacency matrix of the Karate graph is significantly affected by the removal of edges, unlike the Synthetic and Kangaroo graphs. This can be attributed to the Karate graph's well-defined community structure, reflected in its relatively high global clustering coefficient (Table 1).

Failure probabilities based on edge betweenness prioritize the removal of edges with high betweenness centrality, which often serve as bridges between communities. Removing these edges disrupts the connectivity between communities, leading to substantial perturbations in the adjacency matrix and a significant deformation of the largest eigenvalue (λ_1). Since the Frobenius norm of the adjacency matrix A is related to the square root of the sum of the squared eigenvalues, any significant change in the eigenvalues, particularly λ_1, will also affect the Frobenius norm.

Moreover, when communities are fragmented, intermediate eigenvalues $(\lambda_2, \lambda_3, \ldots)$ also undergo changes. Therefore, as a result, these changes contribute to amplifying the overall magnitude of deformation in the adjacency matrix. This highlights the vulnerability of graphs with strong community structures to targeted edge removals, where critical edges play a vital role in maintaining both local and global connectivity.

In contrast, the Synthetic and Kangaroo graphs exhibit higher global connectivity, as indicated by their larger λ_1 values in Table 1, combined with very low global clustering coefficients. These metrics suggest that these graphs lack strong community structures and are instead dominated by a few highly connected hubs. While these hubs maintain the overall connectivity of the graph, the nodes in the periphery remain sparsely connected.

As a result, the removal of edges with high betweenness has a limited impact on the global structure of the Synthetic and Kangaroo graphs. The absence of well-defined communities reduces the role of inter-community bridges, making these graphs less sensitive to targeted edge removals. Their robustness is maintained through centralized connectivity, with hubs playing a critical role in preserving global structure.

Given that it is not possible to determine a specific threshold value of τ for every graph, as it is closely tied to the connectivity and community structure of each individual graph, in our case, a potential threshold value could be 0.3.

Centrality Metrics Perturbation The perturbation of centrality metrics reflects similar trends to those observed in the adjacency matrix. In particular, the centrality of nodes in the Karate graph is the most affected, with eigenvector centrality showing the largest deformation. This supports the previous observation that the removal of critical edges with high betweenness significantly disrupts the global structure and connectivity in community-based graphs. Such disruptions lead to a more pronounced deformation of the adjacency matrix and its eigenvalues, which in turn greatly affects the relative importance of nodes, particularly in eigenvector centrality, as it is closely tied to the structure of the adjacency matrix. Notably, eigenvector centrality deformation demonstrates significant increases at $\tau = 0.4$ and $\tau = 0.7$, indicating structural changes in the graph's core connectivity.

On the other hand, the Synthetic and Kangaroo graphs exhibit much smaller deformations. The centrality metrics for these graphs remain stable, with ζ values consistently below 0.025 for all centrality measures and across all values of τ.

3.2 Edge Weight Failure Probability

Unlike the failure based on edge betweenness, Figures 6 and 7 reveal significant differences in both trends and the resulting values. Therefore, we will first analyze the results obtained using min-max normalization to assign the probability of edge removal, followed by those obtained using sum normalization.

Min-Max Edge Weight Normalization As shown in Figure 6(a), the adjacency matrix metric appears to be highly sensitive to this type of perturbation. Among the considered graphs, the synthetic ER graph exhibits the highest sensitivity to min-max edge weight normalization, followed by the kangaroos graph and, lastly, the karate club graph.

The kangaroos graph, while less sensitive at lower values of τ, shows a steeper increase in perturbation rates as τ rises, eventually approaching the sensitivity of the synthetic graph. This behavior can be explained by the characteristics of the weight distribution. Given that the weight distribution of the synthetic graph is uniform, applying min-max normalization results in relatively high probabilities of edge removal, as edges with weights above the mean acquire significantly higher normalized probabilities.

In the case of the kangaroos graph, the perturbation sensitivity is influenced by the presence of a few edges with very high weights relative to the rest. These high-weight edges, when removed at increasing values of τ (as their normalized probabilities rise), strongly affect the magnitude of the adjacency matrix, leading to substantial deformation. At smaller values of τ, the kangaroos graph initially appears less affected since the high-weight edges, which significantly deform the adjacency matrix, are less likely to be sampled. Instead, edges with lower weights dominate the perturbation process, causing minimal structural impact. For the karate club graph, the observed behavior aligns with its edge weight distribution.

Centrality metrics are also affected by this type of perturbation, as expected, with significant impacts observed on weighted degree centrality and eigenvector centrality. For degree centrality, the removal of high-weight edges strongly influences the degree of a node, especially if the removed edge significantly contributes to the node's total degree.

Eigenvector centrality is similarly impacted, as the perturbations deform the eigenvalues of the adjacency matrix. These changes lead to substantial alterations in the eigenvector centrality scores, suggesting that the graph's structure itself undergoes significant deformation due to the modifications of the adjacency matrix's spectral properties.

Notably, for values of τ greater than 0.5, an exponential increase in the rate of perturbation effects is observed. Among the graphs, the synthetic ER graph shows the highest sensitivity to centrality metric distortions, with the kangaroos and karate graphs showing less sensitivity at lower τ values but converging to similar deformation levels at higher τ values.

Edge Weight Sum-Normalization As shown in Figure 7, the results differ significantly from those obtained using min-max normalization. In this case, the probability of removing an edge with a high weight is considerably lower compared to the min-max normalization approach.

Under this normalization, the kangaroos graph exhibits the highest deformation of its adjacency matrix, followed by the karate graph, which experiences similar but less pronounced perturbations. In contrast, the synthetic ER graph appears to be robust to this type of perturbation, with minimal impact on its adjacency matrix.

A similar pattern is observed for centrality measures. Eigenvector centrality is again the most affected, showing a steep increase in deformation beyond a threshold of $\tau = 0.4$ for both the kangaroos and karate graphs. The synthetic graph remains largely unaffected, with negligible changes in eigenvector centrality.

Overall, the rate of deformation for centrality measures is relatively low and generally negligible, with the maximum observed perturbation being around 7%

for the karate graph at $\tau = 1$. Thus, all three graphs demonstrate robustness to edge weight sum-normalization perturbations in terms of centrality metrics.

3.3 Conclusions

In this work, we analyzed the deformation of the adjacency matrix and node centrality metrics—specifically degree centrality, eigenvector centrality, and Katz centrality—under edge perturbations in undirected, weighted graphs. Our focus was on understanding the effects of removing edges based on different criteria.

For edge removal based on edge betweenness, we observed that adjacency matrices tend to be robust to perturbations unless the graph has a strong community structure. In such cases, the removal of edges bridging communities, especially when selected with probabilities proportional to their normalized betweenness (via min-max normalization), results in notable structural deformation. This effect is particularly pronounced for eigenvector centrality, which shows significant sensitivity at higher values of the threshold parameter τ. However, the overall deformation rate becomes substantial only for large τ values.

When perturbations were based on edge weights, we found that using sum-normalized probabilities for edge removal had a limited impact on both the adjacency matrix and centrality metrics. This is because probabilities are proportional to the sum of edge weights, which may not produce sufficiently high removal probabilities. In contrast, min-max normalization provides probabilities that more accurately reflect the relative importance of edge weights. In this case, the extent to which the adjacency matrix and centrality metrics are affected depends heavily on the distribution of edge weights. The magnitude of deformation becomes more significant as τ increases.

Finally, our results do not fully align with those of [1], which suggests that node removal generally has a more pronounced effect on both the adjacency matrix and centrality metrics compared to edge removal under sum-normalized probabilities. However, when min-max normalization is applied, edge removal also produces significant effects on centrality measures.

It is important to note that it is not possible to establish a universal threshold value for τ as its impact depends heavily on the graph's structure.

4 Methods

In this section, we define the different models and implementations used in our study. The goal of this paper is to investigate how the norm of the centrality vectors of a graph varies as the proportion τ of candidate edges for failure changes, and to test the extent to which the results align with or differ from those of Cavallaro et al. [1]. We also aim to test whether threshold values of τ exist beyond which the graph structure is significantly affected by perturbation.

To achieve this, we ran our simulations using 100 iterations for each implementation. Our study focuses solely on weighted and undirected graphs: a synthetic graph generated using the Erdos-Renyi model, a graph based on interactions between free-ranging grey kangaroos, and a network containing social ties among the members of a university karate club collected by Wayne Zachary in 1977.

Similar to the process followed in [1] for the nodes of the graph, in our simulations, we consider only a subset of edges $E' \subseteq E$ as susceptible to failure. Thus, all our simulations are executed considering a fixed proportion $\tau \in [0, 1]$ of edges that can fail. That is, all edges that are not part of the subset E' are considered resistant to failure and those edges in E' have assigned a failure probability p based on different models (e.g., edge betweenness-based or weight-based models). The selection process for the edges to form part of this subset E' is entirely random.

Once this subset E' is selected, our simulations perturb the topology of the networks considering two distinct models: one that accounts for the edge betweenness of the edges, and another that considers the weight values of the edges to determine the probability of failure for each edge in E'.

Finally, we quantify the perturbations performed on the graph and evaluate how these perturbations affect the norm of the vectors corresponding to different centrality metrics.

4.1 Centrality metrics

4.2 Weighted Degree Centrality

The *Weighted Degree Centrality* extends the concept of traditional Degree Centrality to weighted graphs by accounting for the weights of the edges connected to each node. For a node i, its weighted degree d_i is the sum of the weights of the edges connecting i to its neighbors:

$$d_i = \sum_{j \in N(i)} w_{ij}, \tag{1}$$

where $N(i)$ is the set of neighbors of i, and w_{ij} is the weight of the edge between nodes i and j.

This metric privileges nodes that are not only well-connected but also connected through edges with high weights. It remains a *local measure* as it depends only on the direct connections of a node.

4.3 Weighted Eigenvector Centrality

The *Weighted Eigenvector Centrality* measures the importance of a node i not only based on the weights of its direct connections but also on the centrality of its neighbors. It is defined as the i-th component of the principal eigenvector of the adjacency matrix $\mathbf{W}$:

$$\mathbf{e} = \lambda_1^{-1} \mathbf{W} \mathbf{e}, \tag{2}$$

where $\mathbf{e}$ is the vector storing the Eigenvector Centrality scores, and λ_1 is the largest eigenvalue of $\mathbf{W}$.

As per the Perron-Frobenius theorem, for a connected and undirected graph, the principal eigenvector $\mathbf{e}$ has all positive components, and the i-th component e_i can be interpreted as the Eigenvector Centrality of node i. In weighted graphs, this centrality captures both the strength of connections (weights) and the influence of neighbors.

4.4 Weighted Katz Centrality

The *Weighted Katz Centrality* quantifies the centrality of a node i by considering all walks that start at i, with longer walks being attenuated by a factor β:

$$\mathbf{k} = (\mathbf{I} - \beta\mathbf{W})^{-1}\,\mathbf{1}, \tag{3}$$

where $\mathbf{k}$ is the Katz Centrality vector, $\mathbf{I}$ is the identity matrix, $\mathbf{W}$ is the weighted adjacency matrix, and β is the attenuation factor.

The parameter β must satisfy $\beta < 1/\lambda_1$, where λ_1 is the largest eigenvalue of $\mathbf{W}$. In our experiments, the attenuation factor was set to be 90% of $1/\lambda_1$. This ensures that the matrix $\mathbf{I} - \beta\mathbf{W}$ is positive definite and invertible.

In weighted graphs, Katz Centrality captures the influence of nodes by accounting for both the weights of connections and the global structure of the graph, where higher weights contribute more strongly to the centrality scores.

4.5 Probabilistic failure models

To alter the topology of our networks, we use two types of probabilistic failure models $\phi(e)$: (1) one based on the betweenness centrality of the edges in the network, and (2) another based on the weights of the edges. The former assumes that the probability of an edge being removed is proportional to the betweenness centrality of that edge, meaning that the higher the betweenness centrality of the edge, the lower its resistance to failure. The latter follows the same logic, considering only the original weight value of each edge.

Normalization Methods For each of these models, we use two distinct implementations to assess the susceptibility of an edge to be removed. In both, the entire network is considered to determine the probability of an edge's failure. The first implementation considers a normalized probability based on the minimum and maximum values of the model used (see Equation (4)), i.e., we min-max normalize the betweenness centrality and the weights of the edges in the network to determine the failure probability.

$$\phi_{min\_max\_norm}(e) = \frac{\phi(e) - \min_{e' \in E} \phi(e')}{\max_{e' \in E} \phi(e') - \min_{e' \in E} \phi(e')} \tag{4}$$

$$\phi_{sum\_norm}(e) = \frac{\phi(e)}{\sum_{e' \in E} \phi(e')} \tag{5}$$

The second implementation considers a sum-normalized failure probability (see Equation (5)), meaning it normalizes the probability of an edge failing relative to the sum of all edge weights in the network. In this case, when we apply min-max normalization, in large networks, edges with weights close to the minimum weight can be pushed very close to zero, resulting in a very low failure probability. On the other hand, in small networks, each edge represents a relatively large portion of the total, so each weight contributes significantly to the total. In contrast, in networks with many edges, the impact of a single edge's weight is much smaller, as the total is distributed across a large number of connections.

4.6 Evaluation metrics

The evaluation metrics for our experiments are the same as those defined in Cavallaro et al. [1] to quantify both the perturbation of a graph ψ and the deformation effect experienced by the centrality metrics analyzed ζ.

Amount of perturbation in a graph To achieve this, we define the perturbation of a graph as the ratio of the Frobenius norm of the perturbation matrix $\|\Delta A\|_F$ and the Frobenius norm of the original adjacency matrix $\|A\|_F$ (see Equation 6).

$$\psi = \frac{\|\Delta\mathbf{A}\|_F}{\|\mathbf{A}\|_F}, \quad \text{where} \quad \Delta\mathbf{A} = \tilde{\mathbf{A}} - \mathbf{A} \tag{6}$$

Given that $\tilde{\mathbf{A}}$ is the adjacency matrix of the perturbed graph.

Since the networks we use are undirected, as shown in [1], we can compute $\|\Delta\mathbf{A}\|_F$ as the square root of the trace of $\mathbf{A}^2$, i.e., $\|\mathbf{A}\|_F = \sqrt{Tr(\mathbf{A}^2)}$

Deformation of centrality metrics Following the same logic, to calculate the deformation of the different centrality metrics of our graph, we compute the ratio between the Euclidean norm of the deformation vector of our metric Δf_Θ and the Euclidean norm of the vector of the original graph's metric $f_\Theta(\mathbf{A})$ (see Equation 7). Here, $f_\Theta(\mathbf{A})$ represents the vector with the centrality values computed by a given centrality metric computed given a set of parameters Θ. The set of parameters used by each of the centrality metrics can be seen in Table 2.

$$\zeta = \frac{\|f_\Theta(\tilde{\mathbf{A}}) - f_\Theta(\mathbf{A})\|_F}{\|f_\Theta(\mathbf{A})\|_F} \tag{7}$$

Centrality metric	**Parameter**	**Value**	**Description**
Weighted eigenvector centrality	error tolerance	1.0e-6	Error tolerance (in Euclidean norm) used to check convergence in power iteration.
	max iterations	200	Maximum number of power iterations
Weighted Katz centrality	attenuation factor (β)	$\frac{0.9}{\lambda_1}$	Attenuation factor (β)
	neighborhood weight	1.0	Weight attributed to the immediate neighborhood.
	max iterations	1000	Maximum number of iterations in power method
	error tolerance	1.0e-6	Error tolerance used to check convergence in power method iteration

Table 2. Set of parameters Θ used for each of the centrality metrics.

References

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