Lecture 3 - The Finite Volume Method

Sections X.X (Not in Versteeg)

ME 2256/MEMS 1256 - Applications of Computational Heat and Mass Transfer

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Student Learning Objectives

At the end of the lecture, students should be able to:

- Understand how to formulate the finite volume diffusion equation for non-uniform grids;
- Construct grids using ANSYS ICEM.

Transport Equation

- Recall the transport equation in vector form:

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{Accumulation}} + \underbrace{\nabla \cdot (\rho\vec{V}\phi)}_{\text{Advection}} = \underbrace{\nabla \cdot (\Gamma\nabla\phi)}_{\text{Diffusion}} + \underbrace{S}_{\text{Generation}}$$

- $\phi \rightarrow$ conserved quantity;
- $\Gamma \rightarrow$ diffusion coefficient;
- $\rho \rightarrow$ density;
- We are interested in solving the diffusion term first.

1D Diffusion Equation - FVM

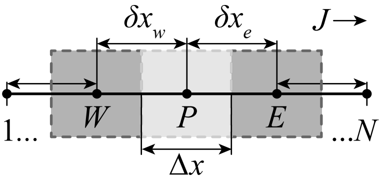
- Recall the 1D diffusion equation cast in the FVM:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E = \sum a_{nb}\phi_{nb}$$

$$\text{where } a_P = A_w\frac{\Gamma_w}{\delta x_w} + A_e\frac{\Gamma_e}{\delta x_e}; \quad a_W = A_w\frac{\Gamma_w}{\delta x_w};$$

$$\text{and } a_E = A_e\frac{\Gamma_e}{\delta x_e}$$

$$\text{with } a_P = a_W + a_E$$



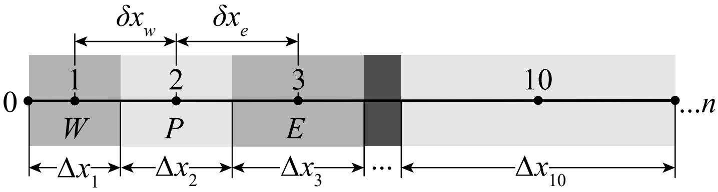
- This equation allows for the use of a non-uniform grid, i.e. where Δx, δx_w and δx_e are not a constant.

Example #1

- Reconsider Example #1 from Lecture 2. Keeping all values for L, Γ, $\phi(0)$, $\phi(L)$, A_e and A_w the same, now Δx is a linear function of length such that:

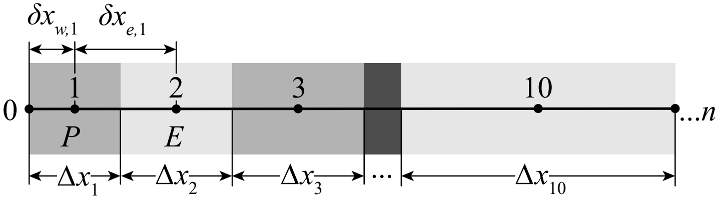
$$\Delta x = \frac{Lx}{N} + \frac{L}{2N}, \quad 0 \leq x \leq L$$

- Solve this for N=10 (i.e. there are 10 C.∀.) and compare the solution to that obtained using a uniform grid:



Example #1

- Starting with C.∀. 1:



- The diffusion equation is expressed as:

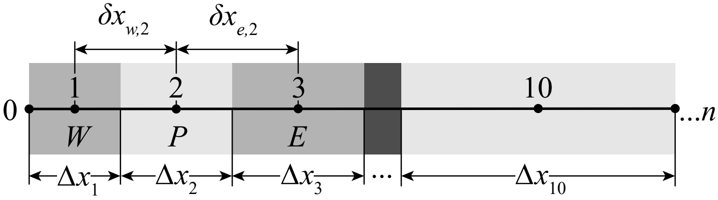
$$a_P(1)T(1) - a_E(1)T_2 - a_W(1)T_0 = 0$$

- T_0 is the temperature of the left boundary, and we have to modify δx_w to be one-half of Δx_1:

$$a_{W,1} = A_w \frac{\Gamma_w}{\left(\frac{\Delta x_1}{2}\right)} = A\frac{\lambda_w}{\delta x_{w,1}}; \quad a_{E,1} = A\frac{\lambda_e}{\delta x_{e,1}}$$

Example #1

- Moving to C.∀. 2:



- The diffusion equation is expressed as:

$$a_P(2)T(2) - a_E(2)T_3 - a_W(2)T_1 = 0$$

- with:

$$a_{W,2} = A\frac{\lambda_w}{\delta x_{w,2}}; \quad a_{E,2} = A\frac{\lambda_e}{\delta x_{e,2}}$$

Example #1

- We see the pattern for the interior C.∀.s and have the following:

$$\text{C.}\forall.\,3: \quad a_P(3)T(3) - a_E(3)T_4 - a_W(3)T_2 = 0$$

$$\text{with } a_{W,3} = A\frac{\lambda_w}{\delta x_{w,3}}; \quad a_{E,3} = A\frac{\lambda_e}{\delta x_{e,3}}$$

$$\text{C.}\forall.\,4: \quad a_P(4)T(4) - a_E(4)T_5 - a_W(4)T_3 = 0$$

$$\text{with } a_{W,4} = A\frac{\lambda_w}{\delta x_{w,4}}; \quad a_{E,4} = A\frac{\lambda_e}{\delta x_{e,4}}$$

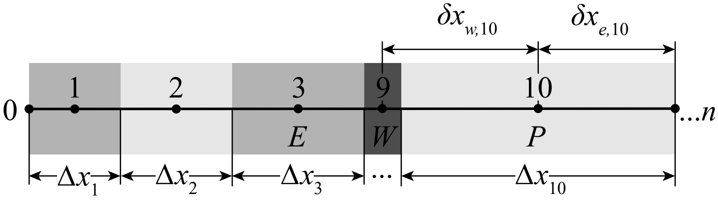
$$\vdots$$

$$\text{C.}\forall.\,9: \quad a_P(9)T(9) - a_E(9)T_{10} - a_W(9)T_8 = 0$$

$$\text{with } a_{W,9} = A\frac{\lambda_w}{\delta x_{w,9}}; \quad a_{E,9} = A\frac{\lambda_e}{\delta x_{e,9}}$$

Example #1

- Lastly, C.∀. 10:



- The diffusion equation is expressed as:

$$a_P(10)T(10) - a_E(10)T_{11} - a_W(10)T_9 = 0$$

- T_{11} is the temperature of the right boundary, and we have to modify δx_e to be one-half of Δx_{10}:

$$a_{W,10} = A\frac{\lambda_w}{\delta x_{w,10}}; \quad a_{E,10} = A_e\frac{\Gamma_e}{\left(\frac{\Delta x_{10}}{2}\right)} = A\frac{\lambda_e}{\delta x_{e,10}}$$

Example #1

- Putting this in matrix form:

$$\begin{bmatrix} a_P(1) & -a_E(1) & 0 & 0 & 0 \\ -a_W(2) & a_P(2) & -a_E(2) & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & & & 0 \\ 0 & 0 & -a_W(N-1) & a_P(N-1) & -a_E(N-1) \\ 0 & 0 & 0 & -a_W(N) & a_P(N) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N-1} \\ T_N \end{bmatrix}$$

$$= \begin{bmatrix} a_W(1)T_0 \\ 0 \\ \vdots \\ 0 \\ a_E(N)T_n \end{bmatrix}$$

- You can see the solution in the script title "L3Ex1.m".

Comparison to Lagrange Polynomials

- In the FDM, we can use Lagrange polynomials to construct the differencing equations for non-uniform grids. For instance, the two-point central difference equation for the first derivative is expressed as:

$$f'(x_{i+1}) = \frac{x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})} y_i + \ldots$$

$$\ldots + \frac{2x_{i+1} - x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} y_{i+1} + \ldots$$

$$\ldots + \frac{x_{i+1} - x_i}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} y_{i+2}$$

- The effort to employ said equation into a FD scheme is on the order of use defining δx_e and δx_w as a function of location.

Meshing

- Meshing is the process of generating the grid (domain) on which the numeric model is solved
- There are numerous meshing software available, ranging from commercial (ANSYS, Star-CMM, Comsol, etc.) to open-source (OpenFOAM, GMSH, etc.). We will focus on using ANSYS ICEM, for we have access to licensing.
- Instructional videos can be found on YouTube.