Lecture 4 - The Finite Volume Method in 1D

Sections 4.3, 8.2-8.4 (Versteeg)

ME 2256/MEMS 1256 - Applications of Computational Heat and Mass Transfer

Mechanical Engineering and Materials Science Department
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Student Learning Objectives

At the end of the lecture, students should be able to:

- Construct the 1D FVM for transient behavior of the diffusion equation.
- Construct the 1D diffusion equation with source terms.
- Formulate convective heat transfer boundary conditions for the 1D diffusion.

Generalized Transport Equation

- Recall the transport equation in vector form:

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{Accumulation}} + \underbrace{\nabla\cdot(\rho\vec{V}\phi)}_{\text{Advection}} = \underbrace{\nabla\cdot(\Gamma\nabla\phi)}_{\text{Diffusion}} + \underbrace{S}_{\text{Generation}}$$

- $\phi \rightarrow$ conserved quantity
- $\Gamma \rightarrow$ diffusion coefficient
- $\rho \rightarrow$ density
- Now, considering the accumulation and diffusion terms, we are left with:

$$\frac{\partial}{\partial t}(\rho\phi) = \nabla\cdot(\Gamma\nabla\phi)$$

The Four Rules of FVM

- For the finite volume method to be implemented properly, we must obey the four following rules:
  1. The flux across an interface between two adjacent control volumes must be represented by the same expression, and must be conserved, unless dictated by some other conservation equation.
  2. The coefficients for a_P and a_{nb} must be the same sign, such that an increase of the neighboring values leads to an increases in the value at P, or a decrease of the neighboring values leads to a decrease in the value at P.

The Four Rules of FVM

3. When linearizing a source term (to be discussed later), the slope must be negative or zero, such that the non-linear term has a zero or negative value, such that Rule 2 is obeyed.
4. We have already stated that $a_P = \sum a_{nb}$, and this must hold true if the dependent variable at P and nb are increased by some constant.

1D Transient Transport Equation

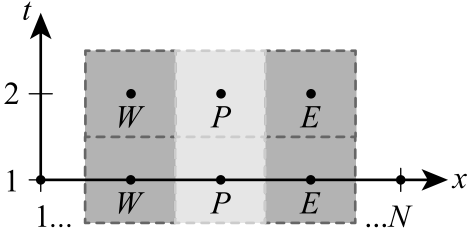
- Calling the diffusion coefficient κ and the conserved quantity T, and considering a system of constant density and specific heat, in one-dimension we can expression the transport equation as:

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right)$$

- Why is specific heat now added to the system of equations?

Temporal and Spatial Marching

- We have to march forward in time to resolve the change of temperature.
- We have to spatially resolve our temperature, based upon the east and west neighbors:



- Thus, we have to integrate over the time control volume and time interval.

Temporal Term

- Recalling Δx is the cell size and δx is the distance between cell centers

$$\rho C \int_w^e \int_t^{t+\Delta t} \frac{\partial T}{\partial t}\, dt\, dx = \int_t^{t+\Delta t} \int_w^e \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) dx\, dt$$

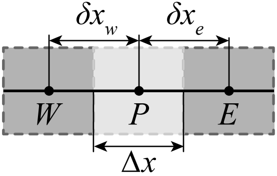
- The LHS can be solved as:

$$\rho C \int_w^e \int_t^{t+\Delta t} \frac{\partial T}{\partial t}\, dt\, dx$$

$$= \rho C\, x\Big|_w^e \Delta T\Big|_t^{t+\Delta t}$$

$$= \rho C(x_e - x_w)(T_P^{t+\Delta t} - T_P^t)$$

$$= \rho C \Delta x (T_P^2 - T_P^1)$$



Spatial Marching

- We have already constructed a method to solving the spatial integral of the diffusion term such that the RHS becomes:

$$\int_t^{t+\Delta t} \int_w^e \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) dx\, dt$$

$$= \int_t^{t+\Delta t} \left(\frac{\kappa_e (T_E - T_P)}{\delta x_e} - \frac{\kappa_w (T_P - T_W)}{\delta x_w} \right) dt$$

- We must assume how T_P, T_E and T_W vary between t and $t + \Delta t$:

$$\int_t^{t+\Delta t} T_P \, dt = \left(f T_P^2 + (1 - f) T_P^1 \right) \Delta t$$

- The weight factor is bounded such that $0 \leq f \leq 1$.

Temporal Advancement of Spatial Term

- Assuming a weighting factor, we can express the RHS as:

$$\int_t^{t+\Delta t}\left(\frac{\kappa_e(T_E - T_P)}{\delta x_e} - \frac{\kappa_w(T_P - T_W)}{\delta x_w}\right)dt$$

$$= f\left(\frac{\kappa_e(T_E^2 - T_P^2)}{\delta x_e} - \frac{\kappa_w(T_P^2 - T_W^2)}{\delta x_w}\right)\Delta t$$

$$+(1-f)\left(\frac{\kappa_e(T_E^1 - T_P^1)}{\delta x_e} - \frac{\kappa_w(T_P^1 - T_W^1)}{\delta x_w}\right)\Delta t$$

Temporal Advancement of Spatial Term

- Therefore, if we divide out Δt, our transient, 1D diffusion equation becomes:

$$\rho C \frac{\Delta x}{\Delta t}(T_P^2 - T_P^1) = f\left(\frac{\kappa_e(T_E^2 - T_P^2)}{\delta x_e} - \frac{\kappa_w(T_P^2 - T_W^2)}{\delta x_w}\right)$$

$$+(1-f)\left(\frac{\kappa_e(T_E^1 - T_P^1)}{\delta x_e} - \frac{\kappa_w(T_P^1 - T_W^1)}{\delta x_w}\right)$$

- Now we have to re-arrange for T_P^2.

Temporal Advancement of Spatial Term

- Call T_P^2 just T_P, and T_P^1 our initial time T_P^0, which is valid for both T_W and T_E, we have:

$$\rho C \frac{\Delta x}{\Delta t}(\cancelto{T_P}{T_P^2}) = f\left(\frac{\kappa_e(\cancelto{T_E}{T_E^2} - \cancelto{T_P}{T_P^2})}{\delta x_e} - \frac{\kappa_w(\cancelto{T_P}{T_P^2} - \cancelto{T_W}{T_W^2})}{\delta x_w}\right)$$

$$+(1-f)\left(\frac{\kappa_e(\cancelto{T_E^0}{T_E^1} - \cancelto{T_P^0}{T_P^1})}{\delta x_e} - \frac{\kappa_w(\cancelto{T_P^0}{T_P^1} - \cancelto{T_W^0}{T_W^1})}{\delta x_w}\right)$$

$$+\rho C \frac{\Delta x}{\Delta t}(\cancelto{T_P^0}{T_P^1})$$

Temporal Advancement of Spatial Term

- We will also define the following coefficients:

$$a_E = \frac{\kappa_e}{\delta x_e}; \qquad a_W = \frac{\kappa_w}{\delta x_w}$$

$$a_P^0 = \frac{\rho C\, \Delta x}{\Delta t}; \qquad a_P = f a_E + f a_W + a_P^0$$

$$\rho C \frac{\Delta x}{\Delta t}(T_P) = f\left(\cancelto{a_E}{\frac{\kappa_e}{\delta x_e}}(T_E - T_P) - \cancelto{a_W}{\frac{\kappa_w}{\delta x_w}}(T_P - T_W)\right)$$
$$+(1-f)\left(\cancelto{a_E}{\frac{\kappa_e}{\delta x_e}}(T_E^0 - T_P^0) - \cancelto{a_W}{\frac{\kappa_w}{\delta x_w}}(T_P^0 - T_W^0)\right)$$
$$+\cancelto{a_P^0}{\rho C\frac{\Delta x}{\Delta t}}(T_P^0)$$

FV Formulation

- Consolidating, we arrive at our final expression:

$$a_P T_P = a_E(fT_E + (1-f)T_E^0) + a_W(fT_W + (1-f)T_W^0)$$
$$+(a_P^0 - (1-f)a_E - (1-f)a_W)T_P^0$$

- The importance of f is seen in how we calculate the solution:
  1. If f=0, we are taking all T values from previous time-step (explicit method).
  2. If f=1, we are taking all T values at the current time-step (implicit method).
  3. If f=0.5, we have a blend of previous and current values (Crank-Nicolson).

Explicit Scheme

- If we set $f\!=\!0$:

$$a_P T_P = a_E(\cancelto{0}{fT_E}+(1-f)T_E^0)+a_W(\cancelto{0}{fT_W}+(1-f)T_W^0)$$
$$+(a_P^0-(1-f)a_E-(1-f)a_W)T_P^0$$

- We are left with:

$$a_P T_P = a_E T_E^0 + a_W T_W^0 + (a_P^0 - a_E - a_W)T_P^0$$

- Our temperature at the current time-step is completely, and only, dependent on the temperatures at the previous time-step.

Explicit Scheme - Limitations

- For this scheme to be of use, we have to introduce the rule of positive coefficients:
- *All coefficients (a_P and the neighbors a_{nb}) must always be positive.*
- That is, the increase in value of one grid point should lead to an *increase* in value of neighboring grid points.

$$a_P T_P = a_E T_E^0 + a_W T_W^0 + (a_P^0 - a_E - a_W) T_P^0$$

- What condition can result in T_P^0 becoming negative?

Explicit Stability Condition

- Recalling our coefficients:

$$a_P^0 = \frac{\rho C\,\Delta x}{\Delta t}; \quad a_E = \frac{\kappa_e}{\delta x_e}; \quad a_W = \frac{\kappa_w}{\delta x_w}$$

- Determining a condition where $a_P^0 > a_E + a_W$, assuming uniform conductivity and $\Delta x = \delta x_e = \delta x_w$

$$\frac{\rho C\,\Delta x}{\Delta t} > \frac{2\kappa}{\Delta x} \implies \Delta t < \frac{\rho C \Delta x^2}{2\kappa}$$

Crank-Nicolson

- When f=0.5, the scheme is referred to as unconditionally stable.
- That does not mean the result will be physically realistic, rather, a mathematical solution can be obtained.
- Looking at the coefficient of T_P^0 when f=0.5:

$$a_P^0 - \frac{a_E + a_W}{2}$$

- For a_P^0 to be less than the average of a_E and a_W:

$$\frac{\rho C \, \Delta x}{\Delta t} > \frac{\kappa}{\Delta x} \implies \Delta t < \frac{\rho C \Delta x^2}{\kappa}$$

Implicit Scheme

- If we set $f=1$:

$$a_P T_P = a_E(fT_E + \cancelto{0}{(1-f)T_E^0}) + a_W(fT_W + \cancelto{0}{(1-f)T_W^0})$$
$$+(a_P^0 - \cancelto{0}{(1-f)a_E} - \cancelto{0}{(1-f)a_W})T_P^0$$

- We are left with:

$$a_P T_P = a_E T_E + a_W T_W + a_P^0 T_P^0$$

- As it stands, a_P^0 is always positive.
- Now, this may seem like the best scheme, but for small time-steps, the Crank-Nicolson scheme is more accurate.

Full 1D Implicit Diffusion Equation

- We have arrived at the complete, 1D, implicit, transient diffusion equation with source terms:

$$a_P T_P = a_E T_E + a_W T_W + b$$

- With the following coefficients:

$$a_E = \frac{\kappa_e}{\delta x_e}; \quad a_W = \frac{\kappa_w}{\delta x_w}; \quad a_P^0 = \frac{\rho C \, \Delta x}{\Delta t}$$

$$b = S_C \Delta x + a_P^0 T_P^0$$

$$a_P = a_E + a_W + a_P^0 - S_P \Delta x$$

- We will introduce the components of our source terms, S_C and S_P

One-Dimensional Steady Diffusion

- Recall for a one-dimensional system:

$$\underbrace{a_P\phi_P = a_E\phi_E + a_W\phi_W}_{\text{“Homogeneous”}} + \underbrace{a_b\phi_b}_{\text{Dirichlet}} + \underbrace{q_{b,\text{given}}A_b}_{\text{Neumann}}$$

- where:

$$a_W = A_w\frac{\Gamma_w}{\delta x_w},\ a_E = A_e\frac{\Gamma_e}{\delta x_e},\ a_b = A_b\frac{2\Gamma_b}{\Delta x}$$

$$a_P = A_w\frac{\Gamma_w}{\delta x_w} + A_e\frac{\Gamma_e}{\delta x_e} + A_b\frac{2\Gamma_b}{\Delta x} = a_W + a_E + a_b$$

Volumetric Sources

- If we have a volumetric source, S:
  - $\phi \rightarrow$ energy, $S \rightarrow q'''$, volumetric heat generation rate
  - $\phi \rightarrow$ concentration, $S \rightarrow R$, volumetric reaction rate
  - $\phi \rightarrow$ turbulence energy, $S \rightarrow$ production/dissipation of turbulence
- In one-dimension:

$$\frac{\partial}{\partial x}\left(\Gamma A \frac{\partial \phi}{\partial x}\right) + S \cdot (\underbrace{A}_{\Delta y \, \Delta z}) = 0$$

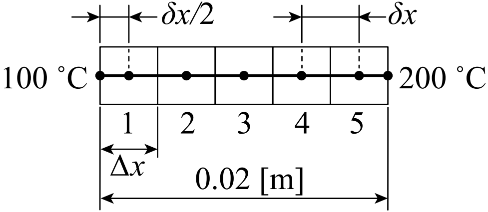
- Integrating in the x-direction:

$$\frac{\Gamma_e A_e}{\delta x}\phi_E - \frac{\Gamma_e A_e}{\delta x}\phi_P - \frac{\Gamma_w A_w}{\delta x}\phi_P + \frac{\Gamma_w A_w}{\delta x}\phi_W + \bar{S} \cdot (\underbrace{\forall}_{\Delta x \, \Delta \, \Delta z}) = 0$$

- $\bar{S}$ is the integrated (average) value of the volumetric source term in the cell.

Example #1

- Consider a slab with a thickness of 2 [cm] and a thermal conductivity λ=0.5 [W/m-K]. If there exists a uniform heat generation rate of 1,000 [kW/m^3], and with the left-hand side is held at 100 °C and the right hand side is held at 200 °C, determine the temperature profile within.
- Start by creating the numerical domain (5 C.∀.):



Example #1

- We will construct the one-dimensional diffusion equation, including the Dirichlet boundary conditions and volumetric source term:

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + a_b \phi_b + \bar{S} \forall$$

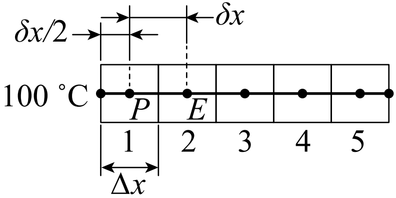
- The coefficients are expressed as, assuming $A_w = A_e = A_b = A = 1$, $\delta x_w = \delta x_e = \delta x$ and $\Gamma_e = \Gamma_w = \lambda$:

$$a_W = A \frac{\lambda}{\delta x}, \; a_E = A \frac{\lambda}{\delta x}, \; a_b = A \frac{2\lambda}{\Delta x}$$

$$a_P = a_W + a_E + a_b + \bar{S} A \Delta x$$

Example #1

- Starting with C.∀. 1:



$$a_P = \cancel{a_W}^{0} + a_E + a_b = A\frac{\lambda}{\delta x} + A\frac{2\lambda}{\Delta x}$$

$$\implies a_P = \frac{0.5}{0.004} + \frac{2(0.5)}{0.004} = 375$$

Example #1

- Populating the system of equations using the following:

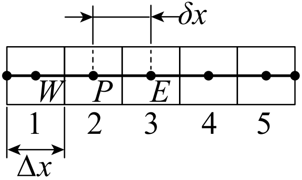
$$a_P\phi_P - a_W\phi_W - a_E\phi_E = a_b\phi_b + \bar{S}A\Delta x$$

$$\text{C.}\forall.\,1 : 375(T_1) - 0(T_0) - 125(T_1) = 250(100) + (1e6)(0.004)$$

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29,000 \\ \\ \\ \\ \\ \end{bmatrix}$$

Example #1

- Moving to C.∀. 2:



$$a_P = a_W + a_E + \cancelto{0}{a_P^0} = A\frac{\lambda}{\delta x} + A\frac{\lambda}{\delta x}$$

$$\implies a_P = \frac{0.5}{0.004} + \frac{0.5}{0.004} = 250$$

- This expression holds true for C.∀.s 2-4

Example #1

- Populating the system of equations using the following:

$$a_P\phi_P - a_W\phi_W - a_E\phi_E = a_b\phi_b + \bar{S}A\Delta x$$

$$\text{C.}\forall.\,2 : 250(T_2) - 125(T_1) - 125(T_3) = 0 + (1\text{e}6)(0.004)$$

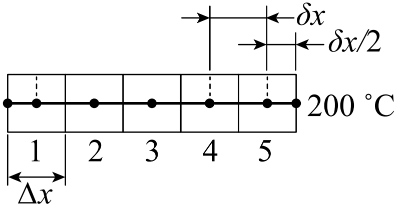
$$\text{C.}\forall.\,3 : 250(T_3) - 125(T_2) - 125(T_4) = 0 + (1\text{e}6)(0.004)$$

$$\text{C.}\forall.\,4 : 250(T_4) - 125(T_3) - 125(T_5) = 0 + (1\text{e}6)(0.004)$$

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ & & & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29,000 \\ 4,000 \\ 4,000 \\ 4,000 \\ \\ \end{bmatrix}$$

Example #1

- Moving to C.∀. 5:



$$a_P = a_W + \cancelto{0}{a_E} + a_b = A\frac{\lambda}{\delta x} + A\frac{2\lambda}{\Delta x}$$

$$\implies a_P = \frac{0.5}{0.004} + \frac{2(0.5)}{0.004} = 375$$

Example #1

- Populating the system of equations using the following:

$$a_P\phi_P - a_W\phi_W - a_E\phi_E = a_b\phi_b + \bar{S}A\Delta x$$

$$\text{C.}\forall.\,5: 375(T_5) - 125(T_4) - 0(T_6) = 250(200) + (1e6)(0.004)$$

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29,000 \\ 4,000 \\ 4,000 \\ 4,000 \\ 54,000 \end{bmatrix}$$

Example #1

- Solving this system of equations, the solution takes the form:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix}$$

Convective Boundary Conditions

- We have to reformulate our governing equation for convective boundary conditions.
- Recall Newton's law of cooling for a one-dimensional system:

$$\frac{d}{dx}\left(\lambda A\frac{dT}{dx}\right) = hA_s(T - T_\infty)$$

- The convective cooling term can be treated as a sink - the opposite of a source.
- We will define the variable n^2 as $(hA_s/\lambda A)$ such that, for constant A and λ:

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) - n^2(T - T_\infty) = 0$$

Convective Boundary Conditions

- We proceed with integration over the control volume, recognizing $T = \phi$ and $\Gamma = \lambda$:

$$\int_{\forall} \frac{d}{dx}\left(\frac{dT}{dx}\right) d\forall - \int_{\forall} n^2 (T - T_\infty)\, d\forall = 0$$

$$\Longrightarrow \left(A\frac{dT}{dx}\right)_e - \left(A\frac{dT}{dx}\right)_w - n^2 (T_P - T_\infty) A\delta x = 0$$

- In terms of cell-centered values using a backward difference:

$$\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_W}{\delta x}\right) - n^2 (T_P - T_\infty)\delta x = 0$$

- We must proceed with rearranging terms.

Convective Boundary Conditions

- Grouping like terms:

$$\left(\frac{1}{\delta x}+\frac{1}{\delta x}\right)T_P=\frac{1}{\delta x}T_W+\frac{1}{\delta x}T_E+n^2\delta xT_\infty-n^2\delta xT_P$$

- Thus, we can rewrite in the form:

$$a_PT_P=a_WT_W+a_ET_E+a_bT_b+\bar{S}$$

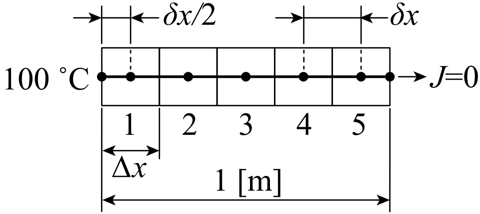
- with the coefficients:

$$a_P=a_W+a_E+a_b+n^2\delta x,\ a_W=\frac{1}{\delta x},\ a_E=\frac{1}{\delta x}$$

$$\bar{S}=n^2\delta xT_\infty$$

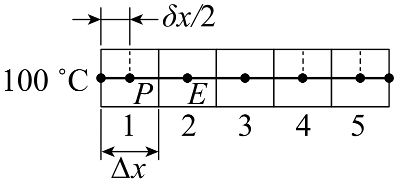
Example #2

- Consider a 1 [m] long circular fin with a constant cross-sectional area. If the base temperature is held at 100 °C, the end of the rod is insulated, and the exterior is exposed to a free-stream temperature of 20 °C, determine the temperature distribution given n^2=25.



Example #2

- Starting with C.∀. 1:



$$a_P = \cancelto{0}{a_W} + a_E + a_b + n^2\delta x$$

$$\implies a_P = \frac{1}{0.2} + \frac{2}{0.2} + 25(0.2) = 20$$

Example #2

- Populating the system of equations using the following:

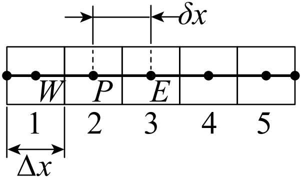
$$a_P T_P - a_W T_W - a_E T_E = a_b T_b + \bar{S}$$

$$\text{C.}\forall.\,1 : 20(T_1) - 0(T_0) - 5(T_2) = 10(100) + (25)(0.2)(20)$$

$$\begin{bmatrix} 20 & -5 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1,100 \\ \\ \\ \\ \\ \end{bmatrix}$$

Example #2

- Moving to C.∀. 2:



$$a_P = a_W + a_E + \cancelto{0}{a_b} + n^2 \delta x$$

$$\implies a_P = \frac{1}{0.2} + \frac{1}{0.2} + (25)(0.2) = 15$$

- This expression holds true for C.∀.s 2-4

Example #2

- Populating the system of equations using the following:

$$a_P T_P - a_W T_W - a_E T_E = a_b T_b + \bar{S}$$

$$\text{C.}\forall.\,2 : 15(T_2) - 5(T_1) - 5(T_3) = 0 + (25)(0.2)(20)$$

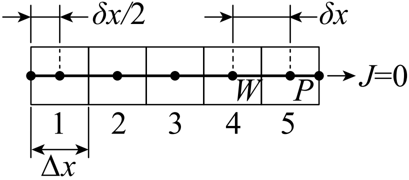
$$\text{C.}\forall.\,3 : 15(T_3) - 5(T_2) - 5(T_4) = 0 + (25)(0.2)(20)$$

$$\text{C.}\forall.\,4 : 15(T_4) - 5(T_3) - 5(T_5) = 0 + (25)(0.2)(20)$$

$$\begin{bmatrix} 20 & -5 & 0 & 0 & 0 \\ -5 & 15 & -5 & 0 & 0 \\ 0 & -5 & 15 & -5 & 0 \\ 0 & 0 & -5 & 15 & -5 \\ & & & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1,100 \\ 100 \\ 100 \\ 100 \\ \\ \end{bmatrix}$$

Example #2

- Moving to C.∀. 5:



$$a_P = a_W + \cancel{a_E}^{\,0} + \cancel{a_0}^{\,0} + n^2 \delta x$$

$$\implies a_P = \frac{1}{0.2} + 25(0.2) = 10$$

Example #2

- Populating the system of equations using the following:

$$a_P\phi_P - a_W\phi_W - a_E\phi_E = a_b\phi_b + \bar{S}$$

$$\text{C.}\forall.\,5 : 10(T_5) - 5(T_4) - 0(T_6) = 0 + (25)(0.2)(20)$$

$$\begin{bmatrix} 20 & -5 & 0 & 0 & 0 \\ -5 & 15 & -5 & 0 & 0 \\ 0 & -5 & 15 & -5 & 0 \\ 0 & 0 & -5 & 15 & -5 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1,100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

Example #2

- Solving this system of equations, the solution takes the form:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 64.2276 \\ 36.9106 \\ 26.5041 \\ 22.6016 \\ 21.3008 \end{bmatrix}$$

- How does this compare to the analytic solution?:

$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh(n(L - x))}{\cosh(nL)}$$