Lecture 6 - The Finite Volume Method in 2D

Sections 4.4, 7.1 (Versteeg)

ME 2256/MEMS 1256 - Applications of Computational Heat and Mass Transfer

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Student Learning Objectives

At the end of the lecture, students should be able to:

- Construct the FVM formulation for 2D diffusion problems.
- Learn the basics of ANSYS ICEM-CFD and ANSYS Fluent for 1D and 2D problems.

Generalized Transport Equation

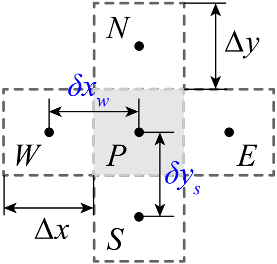
- Recall the transport equation in vector form:

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{Accumulation}} + \underbrace{\nabla \cdot (\rho \vec{V} \phi)}_{\text{Convection}} = \underbrace{\nabla \cdot (\Gamma \nabla \phi)}_{\text{Diffusion}} + \underbrace{S}_{\text{Generation}}$$

- $\phi \rightarrow$ conserved quantity
- $\Gamma \rightarrow$ diffusion coefficient
- $\rho \rightarrow$ density

2D FVM

- Let us extend our analysis to two dimensions, but including north and south neighbors to P.



- We note the unit depth is taken as unity.

2D FVM - Steady Diffusion

- Considering the steady diffusion equation, including source terms:

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}\right)+S=0$$

- Let us integrate over the C.∀. 1:

$$\int_{C.\forall.}\left\{\frac{\partial}{\partial x}\left(\Gamma\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial \phi}{\partial y}\right)+S=0\right\}$$

- Thus, integrating with respect to x, y and z:

$$\implies \left\{\left(\Gamma\frac{\partial \phi}{\partial x}\right)_e-\left(\Gamma\frac{\partial \phi}{\partial x}\right)_w\right\}\Delta y\Delta z+\left\{\left(\Gamma\frac{\partial \phi}{\partial y}\right)_n-\ldots\right.$$

$$\left.\ldots-\left(\Gamma\frac{\partial \phi}{\partial y}\right)_s\right\}\Delta x\Delta z+\bar{S}\Delta x\Delta y\Delta z=0$$

2D FVM - Steady Diffusion

- Assuming uniformly spaced cell-centers:

$$\frac{\Gamma_e \Delta y \Delta z}{\delta x_e}(\phi_E - \phi_P) - \frac{\Gamma_w \Delta y \Delta z}{\delta x_w}(\phi_P - \phi_W) + \ldots$$

$$\ldots + \frac{\Gamma_n \Delta x \Delta z}{\delta y_n}(\phi_N - \phi_P) - \frac{\Gamma_s \Delta x \Delta z}{\delta y_s}(\phi_P - \phi_S) + \bar{S}\Delta x \Delta y \Delta z = 0$$

- Expanding out the terms:

$$\frac{\Gamma_e \Delta y \Delta z}{\delta x_e}\phi_E - \frac{\Gamma_e \Delta y \Delta z}{\delta x_e}\phi_P - \frac{\Gamma_w \Delta y \Delta z}{\delta x_w}\phi_P + \frac{\Gamma_w \Delta y \Delta z}{\delta x_w}\phi_W + \ldots$$

$$\ldots + \frac{\Gamma_n \Delta x \Delta z}{\delta y_n}\phi_N - \frac{\Gamma_n \Delta x \Delta z}{\delta y_n}\phi_P - \frac{\Gamma_s \Delta x \Delta z}{\delta y_s}\phi_P + \frac{\Gamma_s \Delta x \Delta z}{\delta y_s}\phi_S + \ldots$$

$$\ldots + \bar{S}\Delta x \Delta y \Delta z = 0$$

2D FVM - Steady Diffusion

- Grouping like conserved quantities:

$$\phi_E\left(\frac{\Gamma_e\Delta y\Delta z}{\delta x_e}\right)+\phi_W\left(\frac{\Gamma_w\Delta y\Delta z}{\delta x_w}\right)+\phi_N\left(\frac{\Gamma_n\Delta x\Delta z}{\delta y_n}\right)+\ldots$$

$$\ldots+\phi_S\left(\frac{\Gamma_s\Delta x\Delta z}{\delta y_s}\right)+\phi_P\left(-\frac{\Gamma_e\Delta y\Delta z}{\delta x_e}-\frac{\Gamma_w\Delta y\Delta z}{\delta x_w}-\ldots\right.$$

$$\left.\ldots-\frac{\Gamma_n\Delta x\Delta z}{\delta y_n}-\frac{\Gamma_s\Delta x\Delta z}{\delta y_s}\right)+\bar{S}\Delta x\Delta y\Delta z=0$$

- We define the following coefficients:

$$a_e=\frac{\Gamma_e\Delta y\Delta z}{\delta x_e};\ a_w=\frac{\Gamma_w\Delta y\Delta z}{\delta x_w};\ a_n=\frac{\Gamma_n\Delta x\Delta z}{\delta y_n}$$

$$a_s=\frac{\Gamma_s\Delta x\Delta z}{\delta y_s};\ \bar{S}=S_c+S_P\phi_P;\ B=S_c\Delta x\Delta y\Delta z$$

2D FVM - Steady Diffusion

- Thus, the coefficient for C.∀. 1 is expressed as:

$$a_p = a_e + a_w + a_n + a_s - S_P \Delta x \Delta y \Delta z$$

- The transport equation is then expressed as:

$$a_p \phi_P = a_e \phi_E + a_w \phi_W + a_n \phi_N + a_s \phi_S + B$$

- Alternatively:

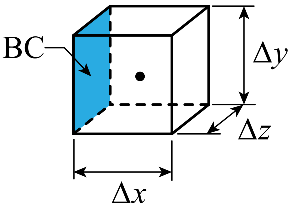
$$a_p \phi_P = \sum_{nb} a_{nb} \phi_{nb} + B$$

- B is the catch-all term (source, boundary, etc.)

Boundary Conditions in 2D

- Boundary conditions are handled the same in 2D as in 1D:
- If ϕ=constant

$$a_b = \frac{\Gamma_b \Delta y \Delta z}{\left(\frac{\delta x_b}{2}\right)} = \frac{2\Gamma_b \Delta y \Delta z}{\delta x_b}$$



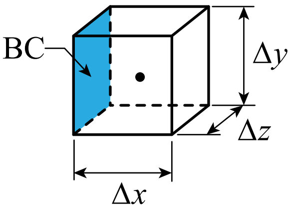
- The steps to include this into the solution are:
  1. $a_b \phi_b$ gets added to B
  2. a_b gets added to a_p
  3. Remove the coefficient in that direction (i.e. $a_w = 0$)

Boundary Conditions in 2D

- If J=constant

$$J = -\Gamma_b \frac{\partial \phi}{\partial x}\bigg|_b$$

$$= \frac{2\Gamma_b \Delta y \Delta z}{\delta x_b}$$



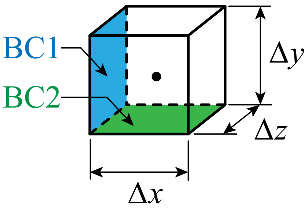
- The steps to include this into the solution are:
  1. $a_b = 0$
  2. $a_w = 0$
  3. $J\Delta y\Delta z$ get added to B

Boundary Conditions in 2D

- What if there are two boundary conditions on the C.∀. (i.e. corner)?
- We define two boundary coefficients:

$$a_{b_1} = \frac{2\Gamma_{B1}\Delta y \Delta z}{\delta x}$$

$$a_{b_2} = \frac{2\Gamma_{B2}\Delta x \Delta z}{\delta y}$$



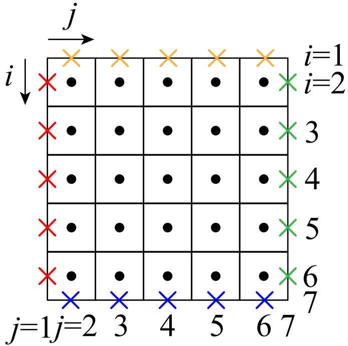
- Then we modify B such that:

$$B^* = B + a_{b_1}\phi_{B1} + a_{b_2}\phi_{B2}$$

- The same approach is taken for Neumann conditions.

2D System

- Say we have a 5x5 grid ($n = 5$), with cell-centers denoted by $\bullet$
- And there are 4 unique boundary conditions (constant ϕ), denoted by the colored $\times$, $\implies \phi(i,j)$=[7, 7]



Solutions to 2D Systems

- Employing Jacobi iteration:
- for j=2:n+1
    - for i=2:n+1

      $$\phi_P^{k+1} = \frac{1}{a_p}\left(a_w\phi_W^k + a_e\phi_E^k + a_s\phi_S^k + a_n\phi_N^k + B\right)$$

    - end
- end
- Employing Gauss-Seidel:
- for j=2:n+1
    - for i=2:n+1

      $$\phi_P^{k+1} = \frac{1}{a_p}\left(a_w\phi_W^{k+1} + a_e\phi_E^k + a_{s+1}\phi_S^k + a_n\phi_N^k + B\right)$$

    - end
- end

Solutions to 2D Systems

- Versteeg also mentions the TDMA can be applied in an iterative fashion to solve the equations representing a 2D system (as seen in 7.3).
- Doing such slows down convergence (because information is not being propagated through the domain quickly, i.e. you are only taking information from cells in one direction, then the other) and can lead to instability.
- The Penta-Diagonal Matrix Algorithm (PDMA) can be implemented, however it is scantly documented.

Example #1

- Consider a 2D, transient diffusion problem. A 40 [cm] by 40 [cm] plate, with a thermal diffusity of 0.97 [cm/s], is at an initial temperature of 100 °C. At time t=0, the sides at $x(0)$ and $x(L)$ are set to 0 °C, while the lateral sides are insulated. Plot $T(x, y, t)$ for t=[10, 15, 30, 50, 100] and determine the time it takes to reach a maximum temperature of 10 °C.

Example #1

- We will start with the transient diffusion equation:

$$\rho C \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T)$$

- In two dimensions:

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right)$$

- We must integrate over a time period $t : t + \Delta t$, and over the control volume ($d\forall = dx\, dy\, dz$), assumed $dz = 1$

Example #1

- Integrating of time and space:

$$\rho C \int_w^e \int_s^n \int_t^{t+\Delta t} \frac{\partial T}{\partial t} dt\, dy\, dx = \ldots$$

$$\ldots = \int_t^{t+\Delta t} \int_w^e \int_s^n \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) dy\, dx\, dt + \ldots$$

$$\ldots + \int_t^{t+\Delta t} \int_s^n \int_w^e \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) dx\, dy\, dt$$

- The temporal term becomes:

$$\rho C x\Big|_w^e y\Big|_s^n \Delta T\Big|_t^{t+\Delta t} = \rho C(x_e - x_w)(y_n - y_s)(T_P^{t+\Delta t} - T_P^t)$$

$$= \rho C \Delta x \Delta y (T_P^{t+\Delta t} - T_P^t)$$

Example #1

- The RHS of the diffusion equation (i.e. the spatial terms) becomes:

$$\int_t^{t+\Delta t}\left(\frac{\kappa_e(T_E-T_P)}{\delta x_e}-\frac{\kappa_w(T_P-T_W)}{\delta x_w}\right)\underbrace{(y_n-y_s)}_{\Delta y}\,dt+\ldots$$

$$\ldots+\int_t^{t+\Delta t}\left(\frac{\kappa_n(T_N-T_P)}{\delta x_n}-\frac{\kappa_s(T_P-T_S)}{\delta x_s}\right)\underbrace{(x_e-x_w)}_{\Delta x}\,dt$$

- We will re-introduce the weighting factor f, such that after temporal integration, we can choose a method (explicit, implicit, etc.).

Example #1

- In doing such, the temporal integration of the spatial terms yields:

$$f\left(\frac{\kappa_e(T_E^{t+\Delta t}-T_P^{t+\Delta t})}{\delta x_e}-\frac{\kappa_w(T_P^{t+\Delta t}-T_W^{t+\Delta t})}{\delta x_w}\right)\Delta y\,\Delta t+\ldots$$

$$\ldots+(1-f)\left(\frac{\kappa_e(T_E^{t}-T_P^{t})}{\delta x_e}-\frac{\kappa_w(T_P^{t}-T_W^{t})}{\delta x_w}\right)\Delta y\,\Delta t+\ldots$$

$$\ldots+f\left(\frac{\kappa_n(T_N^{t+\Delta t}-T_P^{t+\Delta t})}{\delta x_n}-\frac{\kappa_s(T_P^{t+\Delta t}-T_S^{t+\Delta t})}{\delta x_s}\right)\Delta x\,\Delta t+\ldots$$

$$\ldots+(1-f)\left(\frac{\kappa_n(T_N^{t}-T_P^{t})}{\delta x_n}-\frac{\kappa_s(T_P^{t}-T_S^{t})}{\delta x_s}\right)\Delta x\,\Delta t+\ldots$$

- We must set the LHS (temporal) equal to the RHS (spatial) and solve for $T_P^{t+\Delta t}$. We will divide both sides by Δt.

Example #1

- Grouping like terms:

$$T_P^{t+\Delta t}\left(\frac{\rho C\Delta x\Delta y}{\Delta t}+f\left(\frac{\kappa_e\Delta y}{\delta x_e}+\frac{\kappa_w\Delta y}{\delta x_w}+\frac{\kappa_n\Delta x}{\delta x_n}+\frac{\kappa_s\Delta x}{\delta x_s}\right)\right)=\ldots$$

$$\ldots=\frac{\kappa_e\Delta y}{\delta x_e}\left(fT_E^{t+\Delta t}+(1-f)T_E^t\right)+\frac{\kappa_w\Delta y}{\delta x_w}\left(fT_W^{t+\Delta t}+(1-f)T_W^t\right)+\ldots$$

$$\ldots+\frac{\kappa_n\Delta x}{\delta x_n}\left(fT_N^{t+\Delta t}+(1-f)T_N^t\right)+\frac{\kappa_s\Delta x}{\delta x_s}\left(fT_S^{t+\Delta t}+(1-f)T_S^t\right)+\ldots$$

$$\ldots+T_P^t\left(\frac{\rho C\Delta x\Delta y}{\Delta t}-(1-f)\left(\left(\frac{\kappa_e}{\delta x_e}+\frac{\kappa_w}{\delta x_w}\right)\Delta y+\left(\frac{\kappa_n}{\delta x_n}+\frac{\kappa_s}{\delta x_s}\right)\Delta x\right)\right)$$

- If explicit ($f=0$):

$$\frac{\rho C\Delta x\Delta y}{\Delta t}T_P^{t+\Delta t}=\frac{\kappa_e\Delta y}{\delta x_e}T_E^t+\frac{\kappa_w\Delta y}{\delta x_w}T_W^t+\frac{\kappa_n\Delta x}{\delta x_n}T_N^t+\frac{\kappa_s\Delta x}{\delta x_s}T_S^t+\ldots$$

$$\ldots+T_P^t\left(\frac{\rho C\Delta x\Delta y}{\Delta t}-\left(\left(\frac{\kappa_e}{\delta x_e}+\frac{\kappa_w}{\delta x_w}\right)\Delta y+\left(\frac{\kappa_n}{\delta x_n}+\frac{\kappa_s}{\delta x_s}\right)\Delta x\right)\right)$$

Example #1

- If we set $\kappa_e = \kappa_w = \kappa_n = \kappa_s = \kappa$, $\delta x_e = \delta x_w = \Delta x$, $\delta x_n = \delta x_s = \Delta y$:

$$\frac{\rho C \Delta x \Delta y}{\Delta t} T_P^{t+\Delta t} = \frac{\kappa \Delta y}{\Delta x}\left(T_E^t + T_W^t\right) + \frac{\kappa \Delta x}{\Delta y}\left(T_N^t + T_S^t\right) + \ldots$$
$$\ldots + T_P^t\left(\frac{\rho C \Delta x \Delta y}{\Delta t} - \frac{2\kappa \Delta y}{\Delta x} - \frac{2\kappa \Delta x}{\Delta y}\right)$$

- Defining $\alpha = \kappa \rho^{-1} C^{-1}$, and dividing the LHS by the leading coefficient:

$$T_P^{t+\Delta t} = \frac{\alpha \Delta t}{\Delta x^2}\left(T_E^t + T_W^t\right) + \frac{\alpha \Delta t}{\Delta y^2}\left(T_N^t + T_S^t\right) + T_P^t\left(1 - \frac{2\alpha \Delta t}{\Delta x^2} - \frac{2\alpha \Delta t}{\Delta y^2}\right)$$

- Run "L6_Ex1.m" for solution.