Chapter 8 - The Complete Response of RL and RC Circuits

Lecture 26

Sections 8.2-8.3

MEMS 0031 Electrical Circuits

Mechanical Engineering and Materials Science Department
University of Pittsburgh

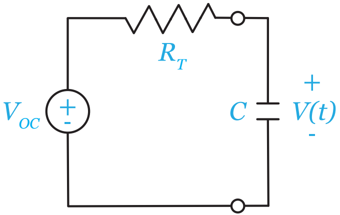
Student Learning Objectives

At the end of the lecture, students should be able to:

- Construct and apply the constitutive equations of RC and RL circuits to determine the natural and forced response of a circuit

RC Circuits

- Consider the following RC Circuit:



Recall the current through a capacitor:

$$i(t) = C\frac{dV(t)}{dt}$$

Applying KVL around the loop:

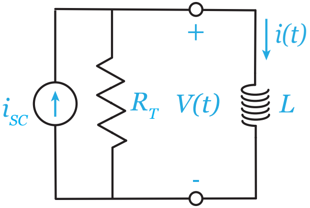
$$V_{oc} = i(t)R_T + V(t) = C\frac{dV(t)}{dt}R_T + V(t)$$

or

$$\frac{dV(t)}{dt} + \frac{V(t)}{R_T C} = \frac{V_{oc}}{R_T C}$$

RL Circuits

- Now consider the following RL Circuit:



Recall the voltage through an inductor:

$$V(t) = L\frac{d\,i(t)}{dt}$$

Applying KCL at the top node:

$$i_{SC} = \frac{V(t)}{R_T} + i(t) = \frac{L}{R_T}\frac{d\,i(t)}{dt} + i(t)$$

or

$$\frac{d\,i(t)}{dt} + \frac{R_T}{L}i(t) = \frac{R_T}{L}i_{SC}$$

Response of RC and RL Circuits

- The solution to these following equations is found as follows:
- Note the RC and RL circuits have a response in the same form, a 1st order differential equation:

$$\frac{d\,x(t)}{dt} + \frac{x(t)}{\tau} = k$$

- τ is a time constant, k is a constant
- We solve this by separation of variables

$$\frac{dx}{dt} = \frac{(k\tau - x)}{\tau} \implies \frac{dx}{x - k\tau} = -\frac{dt}{\tau}$$

- Integrate with respect to each derivative

$$\int \frac{1}{x - k\tau} dx = -\frac{1}{\tau} \int dt$$

Response of RC and RL Circuits Cont'd

- Evaluating the indefinite integrals

$$\ln(x - k\tau) = \frac{-t}{\tau} + c$$

- Exponentiating

$$x(t) = k\tau + Ae^{\frac{-t}{\tau}}$$

- $k\tau$ is the steady-state response (not dependent on time)
- $Ae^{\frac{-t}{\tau}}$ is the transient response (function of time)
- An initial condition must be supplied to the system:

$$x(t=0) = 0 \implies x(0) = k\tau + \cancelto{1}{Ae^{0}}$$
$$\implies A = x(0) - k\tau$$

Response of RC and RL Circuits Cont'd

- Therefore, the solution can be expressed as

$$x(t) = k\tau + [x(0) - k\tau]\mathrm{e}^{\frac{-t}{\tau}}$$

- As $t \to \infty$, we would reach the steady-state process

$$x(\infty) = \lim_{t\to\infty} = k\tau$$

- We still have an unknown variable, our time constant τ. What other information do we have about our system?

- The solution, as $t \to \infty$ is

$$x(t) = x(\infty) + [x(0) - x(\infty)]\mathrm{e}^{\frac{-t}{\tau}}$$

Response of RC and RL Circuits Cont'd

- If we differentiate $x(t)$ w.r.t. t, we can isolate τ

$$\frac{d\,x(t)}{dt} = \frac{-1}{\tau}[x(0) - x(\infty)]\mathrm{e}^{\frac{-t}{\tau}}$$

- At $t = 0$, we know our response should be that of the intial

$$\left.\frac{d\,x(t)}{dt}\right|_{t=0} = \frac{-1}{\tau}[x(0) - x(\infty)]$$

- Rearranging for τ

$$\tau = \frac{x(\infty) - x(0)}{\left.\frac{d\,x(t)}{dt}\right|_{t=0}}$$

Evaluation of τ for RC Circuit

- For a RC circuit, $x(t)$=$V(t)$, it follows (by comparison)

$$\frac{dV(t)}{dt} + \frac{V(t)}{R_T C} = \frac{V_{oc}}{R_T C}$$

- i.e. τ=$R_T C$ and k=$\frac{V_{oc}}{R_T C}$
- Our governing equation becomes

$$V(t) = V_{oc} + [V(0) - V_{oc}]\mathrm{e}^{\frac{-t}{R_T C}}$$

- The forced response is V_{oc} and the natural response is the remainder

Evaluation of τ for RL Circuit

- For a RL circuit, $x(t)=i(t)$, it follows (by comparison)

$$\frac{d\,i(t)}{dt} + \frac{R_T}{L} i(t) = \frac{R_T}{L} i_{SC}$$

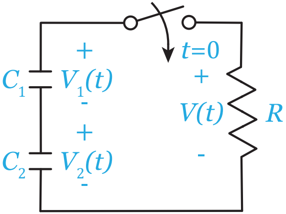
- i.e. $\tau = \dfrac{L}{R_T}$ and $k = \dfrac{L}{R_T} i_{SC}$
- Our governing equation becomes

$$i(t) = i_{SC} + [i(0) - i_{SC}]\mathrm{e}^{\frac{-t\,R_T}{L}}$$

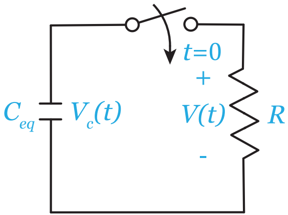
- The forced response is i_{SC} and the natural response is the remainder

Example #1

- Given R=250 [kΩ], C_1=5 [μF], C_2=20 [μF], $V_1(t<0)$=-4 [V] and $V_2(t<0)$=24 [V], determine $V_1(t)$, $V_2(t)$, $V(t)$ and $i(t)$ for $t \geq 0$ [s].



Example #1



Example #1

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Student Learning Objectives

At the end of the lecture, students should be able to:

- Construct and apply the constitutive equations of RC and RL circuits to determine the natural and forced response of a circuit
  - A series RC and parallel RL circuit behavior is described as

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = k$$

    where τ is the time constant, $R_T C$ for RC and L/R_T for RL circuits, and k is the forcing constant, $V_{oc}/R_T C$ for RC and $L i_{sc}/R_T$ for RL circuits.

Suggested Problems

- 8.3-1, 8.3-4, 8.3-5, 8.3-7, 8.3-10, 8.3-11, 8.3-17, 8.3-25