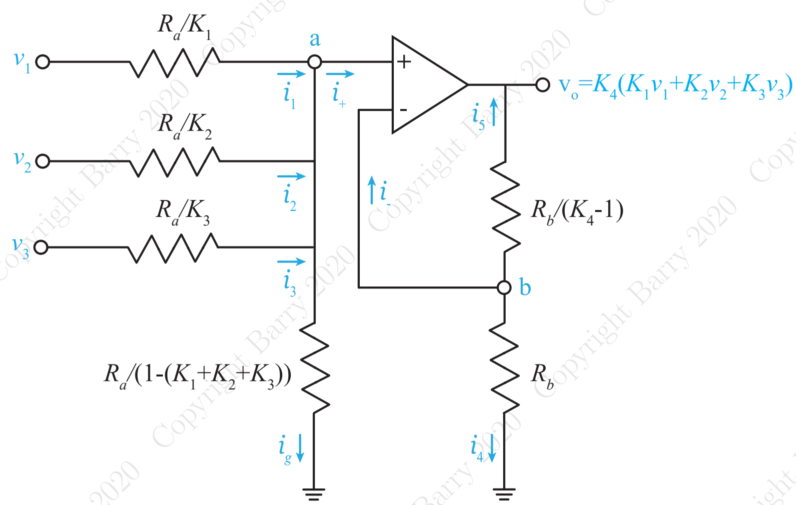
Homework #7

MEMS 0031 - Electrical Circuits

Assigned: June 29th, 2020
Due: July 1st, 2020 at 11:59 pm

Problem #1

Given the summing Op-amp as shown in Fig. 6.5-1 e) of the text, prove the equation as provided in Lecture 20 on Slide 8 is equal to that provided in the text.



We start by applying KCL at node a:

$$i_1 + i_2 + i_3 = \cancelto{0}{i_+} + i_g$$

Applying Ohm's law:

$$\frac{V_1 - V_a}{\left(\dfrac{R_a}{K_1}\right)} + \frac{V_2 - V_a}{\left(\dfrac{R_a}{K_2}\right)} + \frac{V_3 - V_a}{\left(\dfrac{R_a}{K_3}\right)} = \frac{V_a}{\left(\dfrac{R_a}{1-(K_1+K_2+K_3)}\right)}$$

Grouping like terms:

$$V_1\left(\frac{K_1}{R_a}\right) + V_2\left(\frac{K_2}{R_a}\right) + V_3\left(\frac{K_3}{R_a}\right) = V_a\left\{\left(\frac{1-(K_1+K_2+K_3)}{R_a}\right) + \frac{K_1}{R_a} + \frac{K_2}{R_a} + \frac{K_3}{R_a}\right\}$$

$$\implies V_1\left(\frac{K_1}{R_a}\right) + V_2\left(\frac{K_2}{R_a}\right) + V_3\left(\frac{K_3}{R_a}\right) = \frac{V_a}{R_a} \implies V_a = (V_1K_1 + V_2K_2 + V_3K_3)$$

Knowing $V_b = V_a$, we can then apply KCL at node b:

$$0 = \cancelto{0}{i_-} + i_4 + i_5$$

Applying Ohm's law:

$$\frac{V_b}{R_b} + \frac{V_b - V_o}{\left(\dfrac{R_b}{K_4 - 1}\right)} = 0$$

Solving for V_o

$$\frac{V_b}{R_b} + \frac{V_b(K_4-1)}{R_b} = \frac{V_o(K_4-1)}{R_b} \implies V_o = V_b\left(\frac{K_4}{K_4-1}\right)$$

Substituting in the expression for V_a for V_b

$$V_o = (V_1K_1 + V_2K_2 + V_3K_3)\left(\frac{K_4}{K_4-1}\right) = V_1\left(\frac{K_1K_4}{K_4-1}\right) + V_2\left(\frac{K_2K_4}{K_4-1}\right) + V_3\left(\frac{K_3K_4}{K_4-1}\right)$$

For there to be equal weights on the inputs, with a value of unity, we need

$$\frac{K_1K_4}{K_4-1} = \frac{K_2K_4}{K_4-1} = \frac{K_3K_4}{K_4-1} = 1$$

Therefore, we shall set $K_1 = K_2 = K_3 = K$. Additionally, we need the weight on R_b between node b and the output to be n the number of input

$$\frac{1}{K_4-1} = n \implies K_4 = 1 + \frac{1}{n}$$

Therefore,

$$\frac{KK_4}{K_4-1} = 1 \implies \frac{K\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)-1} = 1 \implies nK\left(1+\frac{1}{n}\right) = 1 \implies K(n+1) = 1 \implies K = \frac{1}{n+1}$$

The selection of K and K_4 then yields

$$\frac{KK_4}{K_4-1} = \frac{\left(\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)-1} = \frac{n}{n+1} + \frac{1}{n+1}$$

This expression is equal to 1 for all n. Thus, the output is

$$V_o = V_1 + V_2 + V_3$$

We can verify our solution using the Symbolics toolbox in MATLAB:

```
clear all
close all
clc

syms i1 i2 i3 i4 i5 ig V1 V2 V3 Va Vo Ra Rb K K1 K2 K3 K4 n

% Making assumptions to reduce the expression
K = 1/(n+1);
K1 = K;
K2 = K;
K3 = K;
K4 = (1 + (1/n));

% Branch currents for KCL at a
i1 = (V1 - Va)/(Ra/K1);
i2 = (V2 - Va)/(Ra/K2);
i3 = (V3 - Va)/(Ra/K3);
ig = Va/(Ra/(1 - (K1 + K2 + K3)));

KCL_a = i1 + i2 + i3 == ig;

Va_expr = expand(solve(KCL_a,Va));
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% We know Vb = Va. Branch currents for KCL at b
i4 = Va_expr/Rb;
i5 = (Va_expr - Vo)/(Rb/(K4-1));

KCL_b = i4 + i5 == 0;

Vo_expr = expand(solve(KCL_b,Vo));
disp(Vo_expr)
```
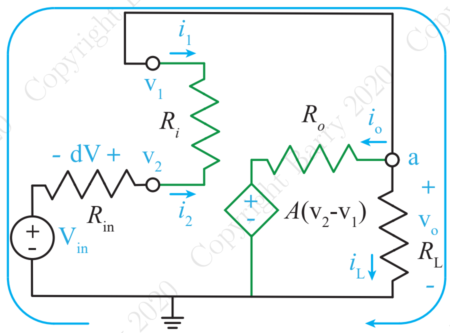

Problem #2

Reconsider Example #2 from Lecture 21. Repeat this problem to determine the ratio of the output voltage to the input voltage, V_o/V_{in}, using the given resistances and gain, however, you are to *not* ignore i_1.



Start by applying KCL at node as:

$$i_1 + i_o + i_L = 0$$

Apply Ohm's law, assuming the internal resistance R_i of the device behaves linearly:

$$\frac{V_o - V_{\text{in}}}{R_{\text{in}} + R_i} + \frac{V_o - A(V_2 - V_1)}{R_o} + \frac{V_o}{R_{\text{L}}} = 0$$

Solving in terms of $\Delta V = V_2 - V_1$

$$\Delta V = \frac{V_o}{A}\left(1 + \frac{R_o}{R_L}\right) + \frac{R_o(V_o - V_{\text{in}})}{R_{\text{in}} + R_i}$$

Applying KVL around the loop:

$$-V_{\text{in}} - dV + \Delta V + V_o = 0 \implies -V_{in} - \left(\frac{V_o - V_{\text{in}}}{R_{\text{in}} + R_i}\right)R_{\text{in}} + \frac{V_o}{A}\left(1 + \frac{R_o}{R_L}\right) + \frac{R_o(V_o - V_{\text{in}})}{R_{\text{in}} + R_i} + V_o = 0$$

Solving for the ratio V_o/V_{in}:

$$\frac{V_o}{V_{\text{in}}} = \frac{\left(1 - \frac{R_{\text{in}}}{R_{\text{in}} + R_i} + \frac{R_o}{R_{\text{in}} + R_i}\right)}{\left(1 - \frac{R_{\text{in}}}{R_{\text{in}} + R_i} + \frac{1}{A}\left(1 + \frac{R_o}{R_L}\right) + \frac{R_o}{R_{\text{in}} + R_i}\right)}$$

Substituting in R_{in}=1 [kΩ], R_{L}=10 [kΩ], R_i=100 [kΩ], R_o=100 [Ω] and A=10^5

$$\frac{V_o}{V_{\text{in}}} = 0.999\,989$$

In comparison to our previous solution, this reflects a 9.08e-6 percent difference.