Chapter 4 - Integral Form for a Control Volume

Lecture 14
Section 4.3

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

- Use the Conservation of Mass and Linear Momentum to formulate the Bernoulli Equation
- Understand the assumptions made to formulate the Bernoulli Equation

Reynolds Transport Theorem

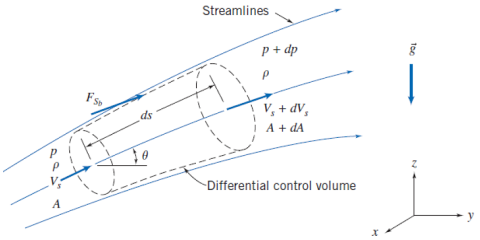
- Recall RTT

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{\delta}{\delta t}\int_{C.\forall.} b\rho d\forall + \int_{\text{C.S.}} b\rho\vec{V}\cdot d\vec{A}$$

- The LHS is the rate of change of the extensive property of system
- The first term on the RHS is the rate of change of the extensive property of the system within the C.∀.
- The second term on the RHS is the rate at which the extensive property of the system is exiting the C.∀. through the C.S.

Formulation

- We can combine the COM and COLM equations to relative velocity and pressure in one equation, the Bernoulli equation
- Imagine we have a C.∀., SS, ρ=c, no flow across streamlines



Formulation

- Looking at the COM

$$0 = \int_{C.S.} \rho \vec{V} \cdot d\vec{A} = \sum_{out} \rho \vec{V} \cdot \vec{A} - \sum_{in} \rho \vec{V} \cdot \vec{A}$$

- Expressing this in terms of density, velocity across the surface V_s and area

$$[\rho(V_s + dV_s)(A + dA)] - \rho V_s A = 0$$

- Multiply out terms

$$\rho\{\cancel{V_s A} + V_s dA + dV_s A + dV_s dA\} - \cancel{\rho V_s A} = 0$$

- Comparing orders of magnitude

$$\boxed{V_s dA + A dV_s = 0}$$

Formulation

- Looking at the COLM

$$\vec{F} = \vec{F}_b + \vec{F}_s = \frac{\delta}{\delta t}\int_{C.\forall.} \rho \vec{V} d\forall + \int_{C.S.} \rho \vec{V}(\vec{V}\cdot\vec{n})dA$$

- SS, break down each term

$$F_s = PA - (P + dP)(A + dA) + \left(P + \frac{dP}{2}\right)dA$$

- First term on RHS is the force on the left face of the C.∀
- Second term on RHS is the force on the right face of the C.∀
- Third term on RHS is the average pressure acting on the C.S.

Formulation

- Multiply out terms

$$F_s = \cancel{PA} - \cancel{PA} - \cancel{PdA} - AdP - dPdA + \cancel{PdA} + \frac{dPdA}{2}$$

- Therefore

$$\boxed{F_s = -AdP - \frac{dPdA}{2}}$$

- The body force terms

$$\boxed{F_b = \rho g d\forall = \rho(\text{-}g\sin\theta)\left(A + \frac{dA}{2}\right)ds = \text{-}\rho g\left(A + \frac{dA}{2}\right)dz}$$

Formulation

- Lastly, the momentum across the control surfaces
- Applying COLM to LHS

$$\int_{C.S.} \rho\vec{V}(\vec{V}\cdot\vec{n})dA = V_s(\rho V_s A)$$

- RHS

$$\int_{C.S.} \rho\vec{V}(\vec{V}\cdot\vec{n})dA=(V_s+dV_s)\boxed{(\rho(V_s+dV_s)(A+dA))}$$

- The term in blue is simply equal to $\rho V_s A$ from our COM formulation

$$\int_{C.S.} \rho\vec{V}(\vec{V}\cdot\vec{n})dA = (V_s+dV_s)(\rho V_s A)$$

Formulation

- RHS-LHS

$$\boxed{\int_{C.S.} \rho \vec{V}(\vec{V}\cdot\vec{n})dA = \rho V_s A dV_s}$$

- Substituting all boxed (red) equations back into COLM

$$\boxed{-AdP - \frac{dPdA}{2}} - \boxed{\rho g\left(A + \frac{dA}{2}\right)dz} = \boxed{\rho V_s A dV_s}$$

- The terms in red are F_s, those in green F_b and those in blue $\int_{C.S.}(\;)$

Formulation

- Multiplying out F_b terms

$$-AdP - \frac{dPdA}{2} - \rho g A dz - \frac{\rho g dA dz}{2} = \rho V_s A dV_s$$

- Divide by ρA and compare orders of magnitude

$$-\frac{dP}{\rho} - \cancel{\frac{dPdA}{2\rho A}} - gdz - \cancel{\frac{gdAdz}{2A}} = V_s dV_s = d\left(\frac{V_s^2}{2}\right)$$

- This leaves us with

$$\frac{dP}{\rho} + \frac{d(V_s^2)}{2} + gdz = 0$$

Formulation

- Indefinite integral

$$\int \frac{dP}{\rho} + \int \frac{d(V_s^2)}{2} + g \int dz = 0$$

- Leaves us with the Bernoulli equation

$$\boxed{\boxed{\frac{P}{\rho} + \frac{V_s^2}{2} + gz = c}}$$

- The pressure energy plus kinetic energy plus potential energy must be constant along a streamline

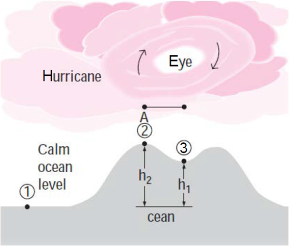
The Bernoulli equation

- The main assumptions used in the derivation (they must be satisfied for this equation to be used)

1. The fluid must be incompressible
2. There can be no mechanical work introduced into the system between two points of interest within the fluid region
3. No heat is transferred into our out of the fluid
4. There is no energy loss due to friction (i.e. very low viscosity)
5. Thee flow must be along a streamline

Example #1

- A hurricane forms over the ocean due to low atmospheric pressure. Hurricanes create storm surges as they approach land. A class 5 hurricane can have winds exceeding 156 mph (70 [m/s]), with the eye of the hurricane having very low speeds.



Example #1

- Take P_{atm} to be 322 [km] away from the eye 1, with P_{atm}=101.325 [kPa]
- The pressure at the eye is 74.5 [kPa]
- Take the density of the sea to be 1,029 [kg/m^3], that of air to be 1.225 [kg/m^3]
- Find the swell at points 2 and 3 with the wind velocity at 2 being 70 [m/s]

Example #1

- Solution:

Example #1

- Solution:

Example #1

- Solution:

Example #1

- Solution: