Homework #8

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: November 3rd, 2019
Due: November 8th, 2019

Problem #1

If the x- and y-components of a steady velocity field of a incompressible fluid are given as:

$$u = 2y; \qquad v = 4x$$

Determine the stream function describing this velocity field.
Recall from the definition of a stream function:

$$u = \frac{\partial \psi}{\partial y} = 2y$$

Separating and integrating:

$$\partial \psi = 2y\, \partial y \implies \psi = y^2 + g(x)$$

Recall from the definition of a stream function:

$$v = -\frac{\partial \psi}{\partial x}$$

Inserting the value for v and differentiating ψ, as found from u, with respect to x:

$$4x = -g'(x)$$

Integrating:

$$g(x) = -2x^2$$

Thus, the stream function is expressed as:

$$\psi(x, y) = y^2 - 2x^2$$

Verifying this satisfies continuity:

$$\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x\, \partial y} = \frac{\partial^2}{\partial x\, \partial y}(y^2 - 2x^2) - \frac{\partial^2}{\partial x\, \partial y}(y^2 - 2x^2) = 0$$

Problem #2

Consider the following stream function:

$$\psi(x, y) = y^2 - x$$

Determine the velocity of the flow for the position y=1 [m] and along the streamline $\psi(x, y)$=2 [m^2/s]. Furthermore, show the stream function satisfies continuity.

The x-component of velocity is:

$$u = \frac{\partial \psi}{\partial y} = 2y$$

The y-component of velocity is:

$$v = -\frac{\partial \psi}{\partial x} = 1$$

The velocity of the flow is simply:

$$\vec{V} = \langle 2,\, 1 \rangle \; [\text{m/s}]$$

This would be the velocity at point (-1, 1), given:

$$\psi(x, y = 1) = (1)^2 - x = 2 \implies x = -1$$

Verifying this satisfies continuity:

$$\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x\, \partial y} = \frac{\partial^2}{\partial x\, \partial y}(y^2 - x) - \frac{\partial^2}{\partial x\, \partial y}(y^2 - x) = 0$$

Problem #3

If a uniform flow field is described by the following velocity field

$$u = U\sin(\theta); \qquad v = -U\cos(\theta)$$

where U is a constant, determine the stream function for this flow, and verify that it satisfies continuity.
Recall from the definition of a stream function:

$$u = U\sin(\theta) = \frac{\partial \psi}{\partial y} \implies \partial \psi = U\sin(\theta)\, \partial y$$

Thus, solving for ψ:

$$\partial \psi = U\sin(\theta)\, \partial y \implies \psi = (U\sin(\theta))y + g(x)$$

Recall from the definition of a stream function:

$$v = -U\cos(\theta) = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\Big\{(U\sin(\theta))y + g(x)\Big\} \implies U\cos(\theta) = g'(x)$$

Integrating to solve for $g(x)$:

$$g(x) = U\cos(\theta)x + C$$

Setting the constant of integration to zero, the stream function is:

$$\psi(x, y) = U(\sin(\theta)y + \cos(\theta)x)$$

Verifying this satisfies continuity:

$$\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x\, \partial y} = \frac{\partial^2 \psi}{\partial x\, \partial y}(U(\sin(\theta)y + \cos(\theta)x)) - \frac{\partial^2 \psi}{\partial x\, \partial y}(U(\sin(\theta)y + \cos(\theta)x)) = 0$$

Problem #4

If the streamline for a fluid flowing around a corner is given as

$$\psi(x, y) = 5xy \, [\mathrm{m}^2/\mathrm{s}]$$

Determine the x- and y-components of velocity at the point (2, 3).

The x-component of velocity is:

$$u = \frac{\partial \psi}{\partial y} = 5x$$

The y-component of velocity is:

$$v = -\frac{\partial \psi}{\partial x} = -5y$$

The velocity at (2, 3) is:

$$\vec{V} = \langle u,\, v \rangle = \langle 10,\, -15 \rangle \, [\mathrm{m/s}]$$

Verifying this satisfies continuity:

$$\frac{\partial^2 \psi}{\partial x \, \partial y} - \frac{\partial^2 \psi}{\partial x \, \partial y} = \frac{\partial^2}{\partial x \, \partial y}(5xy) - \frac{\partial^2}{\partial x \, \partial y}(5xy) = 0$$

Problem #5

If the velocity field for a flow is given as

$$\vec{V} = 6y\hat{\imath} + 3x\hat{\jmath} \, [\mathrm{m/s}]$$

Determine the equation of the streamline, and evaluate said streamline for the point (1, 2).

From the definition of a streamline:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{3x}{6y} = \frac{x}{2y}$$

Separating like variables and integrating:

$$\int 2y \, dy = \int x \, dx \implies y^2 = \frac{x^2}{2} + c$$

Using the given point (1, 2), we can solve for the constant of integration:

$$(2)^2 - \frac{(1)^2}{2} = c \implies c = 3.5$$

Therefore

$$y = \sqrt{\frac{x^2}{2} + 3.5}$$

Problem #6

If an inviscid fluid is steadily flowing around a sphere with radius r, where the velocity profile is given as

$$\vec{V} = V_o\left(1 + \frac{r^3}{x^3}\right)$$

where V_o is the free-stream velocity. Considering the flow in the x-direction only, where the flow can be broken down into three regions:

$$\vec{V} = \begin{cases} V_o\hat{\imath} & x < -10 \\ V\hat{\imath} & -10 \leq x < r \\ 0 & x = -r \end{cases}$$

Determine the pressure variation along the streamline in the x-direction from $-\infty$ to $-r$. Hint: energy must be conserved along a streamline.

Since the flow is along a streamline, it is invscid and steady, we can employ the Bernoulli equation to the non-trivial region of interest:

$$\frac{P_o}{\rho g} + \frac{V_o^2}{2g} + z_o = \frac{P}{\rho g} + \frac{V^2}{2g} + z \implies \frac{P - P_o}{\rho g} = -\frac{V^2 - V_o^2}{2g} - \cancelto{0}{(z - z_o)}$$

Substituting in the value for the velocity:

$$\Delta P(x) = P - P_o = -\frac{\rho}{2}\left\{\left(V_o\left(1 + \frac{r^3}{x^3}\right)\right)^2 - V_o^2\right\} = -\frac{\rho}{2}\left\{V_o^2\left(1 + \frac{2r^3}{x^3} + \frac{r^6}{x^6}\right) - V_o^2\right\}$$

Therefore, the pressure variation in the x-direction is:

$$\Delta P(x) = -\frac{\rho V_o^2}{2}\left(\frac{2r^3}{x^3} + \frac{r^6}{x^6}\right)$$

Problem #7

Given the following velocity field

$$u = a(x^2 - y^2); \qquad v = -2axy$$

Determine the stream function describing it.

From the x-component of velocity:

$$u = \frac{\partial \psi}{\partial y} \implies \psi = \int u\,\partial y = \int a(x^2 - y^2)\,\partial y = ax^2y - \frac{ay^3}{3} + g(x)$$

From the y-component of velocity (noting the minus sign on both sides of the equation cancels out):

$$v = -\frac{\partial \psi}{\partial x} = 2axy + g'(x) = 2axy \implies g'(x) = 0$$

Thus, $g(x)$ is a constant, which we can set equal to zero. Therefore:

$$\psi(x, y) = ax^2y - \frac{ay^3}{3}$$