Homework #10

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: November 17$^{\text{th}}$, 2019
Due: November 22$^{\text{nd}}$, 2019

Problem #1

The Navier-Stokes equation in cylindrical coordinates is expressed as the following:
a) Rigorously show the viscous dissipation terms in the r-direction are equivalent.
<u>r-direction</u>:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

Applying the product rule to the second term on the right hand side:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)=\frac{\partial}{\partial r}\left(\frac{1}{r}\left(r\frac{\partial v_r}{\partial r}+v_r\cancelto{1}{\frac{\partial r}{\partial r}}\right)\right)=\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right)$$

$$=\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\left(r\frac{\partial v_r}{\partial r}-v_r\cancelto{1}{\frac{\partial r}{\partial r}}\right)=\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}$$

Substituting this in to the viscous dissipation term:

$$\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)$$

Alternatively, it can be expressed as:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

Applying the product rule to the second term on the right hand side:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)=\frac{1}{r}\left(r\frac{\partial^2 v_r}{\partial r^2}+\frac{\partial v_r}{\partial r}\cancelto{1}{\frac{\partial r}{\partial r}}\right)=\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}$$

Substituting this in to the viscous dissipation term:

$$\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)$$

<u>θ-direction</u>:

b) Rigorously show the viscous dissipation terms in the θ-direction are equivalent.

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

Applying the product, and subsequently quotient, rule to the second term on the right hand side:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\left(r\frac{\partial v_\theta}{\partial r} + v_\theta \cancelto{1}{\frac{\partial r}{\partial r}}\right)\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\left(r\frac{\partial v_\theta}{\partial r} + v_\theta\right)\right) = \frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right)$$

$$= \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\left(r\frac{\partial v_\theta}{\partial r} - v_\theta \cancelto{1}{\frac{\partial r}{\partial r}}\right) = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}$$

Substituting this back in to the viscous dissipation term:

$$\mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right)$$

Alternatively, it can be expressed as:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

Applying the produce rule to the second term on the right hand side:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) = \frac{1}{r}\left(r\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial v_\theta}{\partial r}\cancelto{1}{\frac{\partial r}{\partial r}}\right) = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r}$$

Substituting this back in to the viscous dissipation term:

$$\mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right)$$

Problem #2

Consider two concentric, infinitely long cylinders. The cylinders are oriented such that the center-line is along the z-axis, and the radii exist in the r-direction. The inner cylinder has a radius of r_a and the outer cylinder has a radius r_b. The inner cylinder rotates with an angular velocity of ω whereas the outer cylinder is stationary. There is no pressure gradient applied nor gravity. The fluid contained between the cylinders is assumed to be Netwonian, incompressible, isotropic and isothermal. The flow of the fluid is assumed steady and laminar. Construct an expression for the θ-component of the velocity, assuming no-slip boundary conditions on the cylinders. Also determine the force required per unit area to cause rotation.

Starting with the continuity equation for a steady-state, incompressible fluid:

$$\cancelto{0,\ 1}{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancelto{0,\ 2}{\frac{\partial u_z}{\partial z}} = 0$$

With the following assumptions:

1 – no velocity in the r-direction

2 – no velocity in the z-direction

Thus, the continuity equation indicates the flow is fully developed:

$$\frac{\partial u_\theta}{\partial \theta} = 0$$

Writing the θ-component of the momentum equation, with the following assumptions:

$$\rho\left(\cancelto{0,\,3}{\frac{\partial u_\theta}{\partial t}}+\cancelto{0,\,1}{u_r\frac{\partial u_\theta}{\partial r}}+\cancelto{0,\,4}{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}+\cancelto{0,\,1}{\frac{u_r u_\theta}{r}}+\cancelto{0,\,2}{u_z\frac{\partial u_\theta}{\partial z}}\right)=-\cancelto{0,\,5}{\frac{1}{r}\frac{\partial P}{\partial \theta}}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\cancelto{0,\,4}{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}}+\cancelto{0,\,1}{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}}+\cancelto{0,\,6}{\frac{\partial^2 u_\theta}{\partial z^2}}\right)+\cancelto{0,\,7}{\rho g_\theta}$$

3 – steady state

4 – consequence of continuity

5 – no pressure gradient

6 – u_θ not a function of the z-direction

7 – no gravity in the z-direction

Thus, the momentum equation reduces to:

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r}=0$$

This can be re-expressed as:

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}=0$$

Assuming the solution takes the form:

$$u_\theta = r^n$$

Substituting this back into the second-order ODE:

$$\frac{\partial^2 r^n}{\partial r^2}+\frac{1}{r}\frac{\partial r^n}{\partial r}-\frac{r^n}{r^2}=0$$

$$\implies n(n-1)r^{n-2}+\frac{nr^{n-1}}{r}-\frac{r^n}{r^2}=0$$

Factoring:

$$r^{n-2}(n(n-1)+n-1)=0 \implies n^2-n+n-1=0 \implies n^2=1$$

$$\implies n_{1,2}=\pm 1$$

Therefore, we assume the solution takes the form:

$$u_\theta = c_1 r^{n_1}+c_2 r^{n_2}=c_1 r+\frac{c_2}{r}$$

To determine the constants of integration, we apply the following boundary conditions:

$$u_\theta(r=r_a)=\omega r_a \implies c_1 r_a+\frac{c_2}{r_a}=\omega r_a$$

$$u_\theta(r=r_b)=0 \implies c_1 r_b+\frac{c_2}{r_b}=0$$

Constructing a system of equations:

$$\begin{bmatrix} r_a & \frac{1}{r_a} \\ r_b & \frac{1}{r_b}\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}=\begin{bmatrix}\omega r_a \\ 0\end{bmatrix}$$

Solving for the constants:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} r_a & \dfrac{1}{r_a} \\ r_b & \dfrac{1}{r_b} \end{bmatrix}^{-1} \begin{bmatrix} \omega r_a \\ 0 \end{bmatrix} = \frac{1}{\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} \begin{bmatrix} \dfrac{1}{r_b} & -\dfrac{1}{r_a} \\ -r_b & r_a \end{bmatrix} \begin{bmatrix} \omega r_a \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{r_b\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} & \dfrac{-1}{r_a\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} \\ \dfrac{-r_b}{\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} & \dfrac{r_a}{\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} \end{bmatrix} \begin{bmatrix} \omega r_a \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\omega r_a}{r_b\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} \\ \dfrac{-\omega r_a r_b}{\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} \end{bmatrix}$$

Therefore, we can express the velocity profile as:

$$u_\theta = \frac{\omega r_a r}{r_b\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)} - \frac{\omega r_a r_b}{r\left(\dfrac{r_a}{r_b} - \dfrac{r_b}{r_a}\right)}$$

Say $r_a = 1$, $r_b = 3$ and $\omega = 5$, we can plot the solution in Matlab:

```
clear all
close all
clc

%% Problem #2

r = linspace(1,3,100);

ra = 1;
rb = 3;
omega = 5;

c1 = (omega*ra)/(rb*((ra/rb) - (rb/ra)));
c2 = (-omega*ra*rb)/((ra/rb) - (rb/ra));

u = @(r) c1.*r + c2./r;

plot(r,u(r))
xlabel('r')
ylabel('u_{\theta}(r)')
```

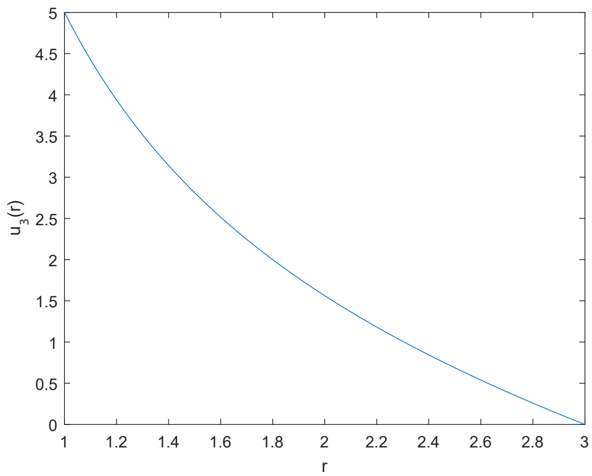

Figure 1: Solution to Problem #1 assuming $r_a = 1$, $r_b = 3$ and $\omega = 5$

As per the force per unit area, we must evaluate the shear stress tensor:

$$\tau = \begin{bmatrix} 2\mu \cancelto{0,1}{\frac{\partial u_r}{\partial r}} & \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\cancelto{0,1}{\frac{\partial u_r}{\partial \theta}}\right) & \mu\left(\cancelto{0,1}{\frac{\partial u_r}{\partial z}} + \cancelto{0,2}{\frac{\partial u_z}{\partial r}}\right) \\ \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\cancelto{0,1}{\frac{\partial u_r}{\partial \theta}}\right) & 2\mu\left(\frac{1}{r}\cancelto{0,4}{\frac{\partial u_\theta}{\partial \theta}} + \cancelto{0,1}{\frac{u_r}{r}}\right) & \mu\left(\cancelto{0,6}{\frac{\partial u_\theta}{\partial z}} + \frac{1}{r}\cancelto{0,2}{\frac{\partial u_z}{\partial \theta}}\right) \\ \mu\left(\cancelto{0,1}{\frac{\partial u_r}{\partial z}} + \cancelto{0,2}{\frac{\partial u_z}{\partial r}}\right) & \mu\left(\cancelto{0,6}{\frac{\partial u_\theta}{\partial z}} + \frac{1}{r}\cancelto{0,2}{\frac{\partial u_z}{\partial \theta}}\right) & 2\mu\cancelto{0,2}{\frac{\partial u_z}{\partial z}} \end{bmatrix}$$

Thus, the only remaining term is for τ results in the following:

$$\tau = \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right) = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

Substituting in the expression for u_θ:

$$\tau = \mu\left(\left(\frac{\omega r_a}{r_b\left(\frac{r_a}{r_b} - \frac{r_b}{r_a}\right)} + \frac{\omega r_a r_b}{r^2\left(\frac{r_a}{r_b} - \frac{r_b}{r_a}\right)}\right) - \left(\frac{\omega r_a}{r_b\left(\frac{r_a}{r_b} - \frac{r_b}{r_a}\right)} - \frac{\omega r_a r_b}{r^2\left(\frac{r_a}{r_b} - \frac{r_b}{r_a}\right)}\right)\right)$$

Therefore:

$$\tau = \mu\left(\frac{2\omega r_a r_b}{r^2\left(\frac{r_a}{r_b} - \frac{r_b}{r_a}\right)}\right)$$

Problem #3

Consider two concentric, infinitely long cylinders. The cylinders are oriented such that the center-line is along the z-axis, and the radii exist in the r-direction. The inner cylinder has a radius of r_a and the outer cylinder has a radius r_b. The inner cylinder moves in the positive z-direction with a velocity W while the outer cylinder is held stationary. The fluid contained between the cylinders is assumed to be Netwonian, incompressible, isotropic and isothermal. The flow of the fluid is assumed steady and laminar. Construct an expression for the z-component of the velocity, assuming no-slip boundary conditions on the cylinders, and determine the force required per unit area to cause translation.

Starting with the continuity equation for a steady-state, incompressible fluid:

$$\cancelto{0,\,1}{\frac{1}{r}\frac{\partial(r u_r)}{\partial r}} + \cancelto{0,\,2}{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} = 0$$

With the following assumptions:

1 – no velocity in the r-direction

2 – no velocity in the θ-direction

Thus, the continuity equation indicates the flow is fully developed:

$$\frac{\partial u_z}{\partial z} = 0$$

Writing the z-component of the momentum equation, with the following assumptions:

$$\rho\left(\cancelto{0,\,3}{\frac{\partial u_z}{\partial t}} + \cancelto{0,\,1}{u_r\frac{\partial u_z}{\partial r}} + \cancelto{0,\,2}{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}} + \cancelto{0,\,4}{u_z\frac{\partial u_z}{\partial z}}\right) = -\cancelto{0,\,5}{\frac{\partial P}{\partial z}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \cancelto{0,\,6}{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}} + \cancelto{0,\,4}{\frac{\partial^2 u_z}{\partial z^2}}\right) + \cancelto{0,\,7}{\rho g_z}$$

3 – steady state

4 – consequence of continuity

5 – no pressure gradient

6 – u_z not a function of the θ-direction

7 – no gravity in the z-direction

Thus, the momentum equation reduces to:

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = 0$$

This can be re-expressed as:

$$r\frac{\partial^2 u_z}{\partial r^2} + \frac{\partial u_z}{\partial r} = 0$$

Assuming the solution to the Euler-Cauchy equation takes the form:

$$u_z = r^n$$

Substituting this back into the second-order ODE:

$$r\frac{\partial^2 r^n}{\partial r^2} + \frac{\partial r^n}{\partial r} = 0$$

$$\implies rn(n-1)r^{n-2} + nr^{n-1} = 0$$

Factoring:

$$r^{n-1}(n(n-1)+n) = 0 \implies n^2 - n + n = 0 \implies n^2 = 0$$

$$\implies n_{1,2} = 0$$

With one, repeated, real root, we assume the solution takes the form:

$$u_z = c_1 r^{n_1} \ln(r) + c_2 r^{n_2}$$

To determine the constants of integration, we apply the following boundary conditions:

$$u_z(r = r_a) = W \implies c_1 \ln(r_a) + c_2 = W$$

$$u_\theta(r = r_b) = 0 \implies c_1 \ln(r_b) + c_2 = 0$$

Constructing a system of equations:

$$\begin{bmatrix} \ln(r_a) & 1 \\ \ln(r_b) & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

Solving for the constants:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \ln(r_a) & 1 \\ \ln(r_b) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \omega r_a \\ 0 \end{bmatrix} = \frac{1}{\ln(r_a) - \ln(r_b)} \begin{bmatrix} 1 & -1 \\ -\ln(r_b) & \ln(r_a) \end{bmatrix} \begin{bmatrix} W \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\ln(r_a) - \ln(r_b)} & \dfrac{-1}{\ln(r_a) - \ln(r_b)} \\ \dfrac{-\ln(r_b)}{\ln(r_a) - \ln(r_b)} & \dfrac{\ln(r_a)}{\ln(r_a) - \ln(r_b)} \end{bmatrix} \begin{bmatrix} W \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{W}{\ln(r_a) - \ln(r_b)} \\ \\ \dfrac{-W\ln(r_b)}{\ln(r_a) - \ln(r_b)} \end{bmatrix} = \begin{bmatrix} \dfrac{W}{\ln\left(\dfrac{r_a}{r_b}\right)} \\ \dfrac{-W\ln(r_b)}{\ln\left(\dfrac{r_a}{r_b}\right)} \end{bmatrix}$$

Therefore, we can express the velocity profile as:

$$u_z = \frac{W\ln(r)}{\ln\left(\dfrac{r_a}{r_b}\right)} - \frac{W\ln(r_b)}{\ln\left(\dfrac{r_a}{r_b}\right)}$$

Say $r_a = 1$, $r_b = 3$ and $W = 5$, we can plot the solution in Matlab:

```
clear all
close all
clc

%% Problem #3

r = linspace(1,3,100);

ra = 1;
rb = 3;
W = 5;

c1 = W/log(ra/rb);
c2 = (W*log(rb))/log(ra/rb);

u = @(r) c1.*log(r) - c2;

```

```
plot(r,u(r))
xlabel('r')
ylabel('u_{z}(r)')
ylim([0 5])
```

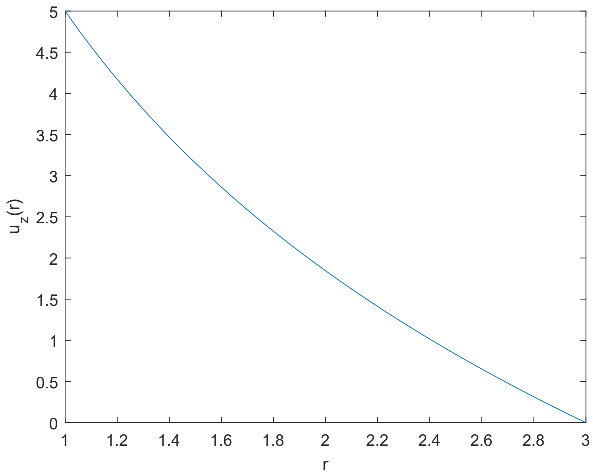

Figure 2: Solution to Problem #2 assuming $r_a = 1$, $r_b = 3$ and $W = 5$

As per the force per unit area, we must evaluate the shear stress tensor:

$$\tau = \begin{bmatrix} 2\mu \cancelto{0,1}{\frac{\partial u_r}{\partial r}} & \mu\left(r\frac{\partial}{\partial r}\left(\cancelto{0,2}{\frac{u_\theta}{r}}\right) + \frac{1}{r}\cancelto{0,1}{\frac{\partial u_r}{\partial \theta}}\right) & \mu\left(\cancelto{0,1}{\frac{\partial u_r}{\partial z}} + \frac{\partial u_z}{\partial r}\right) \\ \mu\left(r\frac{\partial}{\partial r}\left(\cancelto{0,2}{\frac{u_\theta}{r}}\right) + \frac{1}{r}\cancelto{0,1}{\frac{\partial u_r}{\partial \theta}}\right) & 2\mu\left(\frac{1}{r}\cancelto{0,2}{\frac{\partial u_\theta}{\partial \theta}} + \cancelto{0,1}{\frac{u_r}{r}}\right) & \mu\left(\cancelto{0,2}{\frac{\partial u_\theta}{\partial z}} + \frac{1}{r}\cancelto{0,6}{\frac{\partial u_z}{\partial \theta}}\right) \\ \mu\left(\cancelto{0,1}{\frac{\partial u_r}{\partial z}} + \frac{\partial u_z}{\partial r}\right) & \mu\left(\cancelto{0,2}{\frac{\partial u_\theta}{\partial z}} + \frac{1}{r}\cancelto{0,6}{\frac{\partial u_z}{\partial \theta}}\right) & 2\mu\cancelto{0,4}{\frac{\partial u_z}{\partial z}} \end{bmatrix}$$

Thus, the only remaining term is for τ results in the following:

$$\tau = \mu\left(\frac{\partial u_z}{\partial r}\right)$$

Substituting in the expression for u_z:

$$\tau = \mu\left(\frac{W}{r\ln\left(\frac{r_a}{r_b}\right)}\right)$$

Problem #4

Consider two concentric, infinitely long cylinders. The cylinders are oriented such that the center-line is along the z-axis, and the radii exist in the r-direction. The inner cylinder has a radius of r_a and the outer cylinder has a radius r_b. A pressure gradient exists in the positive z-direction such that $\partial P/\partial z = -c$. The fluid contained between the cylinders is assumed to be Netwonian, incompressible, isotropic and isothermal. The flow of the fluid is assumed steady and laminar. Construct an expression for the z-component of the velocity, assuming no-slip boundary conditions on the stationary cylinders. Also determine the force required per unit area to cause the aforementioned translation.

Starting with the continuity equation for a steady-state, incompressible fluid:

$$\cancelto{0,\,1}{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}} + \cancelto{0,\,2}{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} = 0$$

With the following assumptions:

1 – no velocity in the r-direction

2 – no velocity in the θ-direction

Thus, the continuity equation indicates the flow is fully developed:

$$\frac{\partial u_z}{\partial z} = 0$$

Writing the z-component of the momentum equation, with the following assumptions:

$$\rho\left(\cancelto{0,\,3}{\frac{\partial u_z}{\partial t}} + \cancelto{0,\,1}{u_r\frac{\partial u_z}{\partial r}} + \cancelto{0,\,2}{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}} + \cancelto{0,\,4}{u_z\frac{\partial u_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \cancelto{0,\,5}{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}} + \cancelto{0,\,4}{\frac{\partial^2 u_z}{\partial z^2}}\right) + \cancelto{0,\,6}{\rho g_z}$$

3 – steady state

4 – consequence of continuity

5 – u_z not a function of the θ-direction

6 – no gravity in the z-direction

Thus, the momentum equation reduces to:

$$\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right) = \frac{\partial P}{\partial z}$$

This can be re-expressed as:

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{r}{\mu}\frac{dP}{dz}$$

Integrating:

$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\mu}\frac{dP}{dz} + c_1$$

Re-expressing this as:

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu}\frac{dP}{dz} + \frac{c_1}{r}$$

Integrating:

$$u_z = \frac{r^2}{4\mu}\frac{dP}{dz} + c_1\ln(r) + c_2$$

To determine the constants of integration, we apply the following boundary conditions:

$$u_z(r = r_a) = 0 \implies c_1\ln(r_a) + c_2 = -\frac{r_a^2}{4\mu}\frac{dP}{dz}$$

$$u_z(r=r_b)=0 \implies c_1\ln(r_b)+c_2=-\frac{r_b^2}{4\mu}\frac{dP}{dz}$$

Constructing a system of equations:

$$\begin{bmatrix}\ln(r_a) & 1\\ \ln(r_b) & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}-\frac{r_a^2}{4\mu}\frac{dP}{dz}\\ -\frac{r_b^2}{4\mu}\frac{dP}{dz}\end{bmatrix}$$

Solving for the constants:

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\ln(r_a) & 1\\ \ln(r_b) & 1\end{bmatrix}^{-1}\begin{bmatrix}\omega r_a\\ 0\end{bmatrix}=\frac{1}{\ln(r_a)-\ln(r_b)}\begin{bmatrix}1 & -1\\ -\ln(r_b) & \ln(r_a)\end{bmatrix}\begin{bmatrix}-\frac{r_a^2}{4\mu}\frac{dP}{dz}\\ -\frac{r_b^2}{4\mu}\frac{dP}{dz}\end{bmatrix}$$

$$\implies \begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\frac{1}{\ln\left(\frac{r_a}{r_b}\right)} & \frac{-1}{\ln\left(\frac{r_a}{r_b}\right)}\\ \frac{-\ln(r_b)}{\ln\left(\frac{r_a}{r_b}\right)} & \frac{\ln(r_a)}{\ln\left(\frac{r_a}{r_b}\right)}\end{bmatrix}\begin{bmatrix}-\frac{r_a^2}{4\mu}\frac{dP}{dz}\\ -\frac{r_b^2}{4\mu}\frac{dP}{dz}\end{bmatrix}$$

$$\therefore \begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\frac{r_b^2-r_a^2}{4\mu\ln\left(\frac{r_a}{r_b}\right)}\left(\frac{dP}{dz}\right)\\ \frac{\ln(r_b)r_a^2-\ln(r_a)r_b^2}{4\mu\ln\left(\frac{r_a}{r_b}\right)}\left(\frac{dP}{dz}\right)\end{bmatrix}$$

Therefore, we can express the velocity profile as:

$$u_z=\frac{1}{4\mu}\left(\frac{dP}{dx}\right)\left(r^2+\frac{(r_b^2-r_a^2)\ln(r)+\ln(r_b)r_a^2-\ln(r_a)r_b^2}{\ln\left(\frac{r_a}{r_b}\right)}\right)$$

Say $r_a=1$, $r_b=3$, $dP/dz=-5$ and $\mu=1$, we can plot the solution in Matlab:

```
clear all
close all
clc

%% Problem #4

r = linspace(1,3,100);

ra = 1;
rb = 3;
dP = -5;
mu = 1;

c1 = ((rb^2 - ra^2)/(4*mu*log(ra/rb)))*dP;
c2 = ((log(rb)*ra^2 - log(ra)*rb^2)/(4*mu*log(ra/rb)))*dP;

```

```
u = @(r) (1/(4*mu))*dP*(r.^2 + ((rb^2 - ra^2)*log(r) + log(rb)*ra^2 - log(ra)*
    rb^2)/log(ra/rb));

plot(r,u(r))
xlabel('r')
ylabel('u_{z}(r)')
```

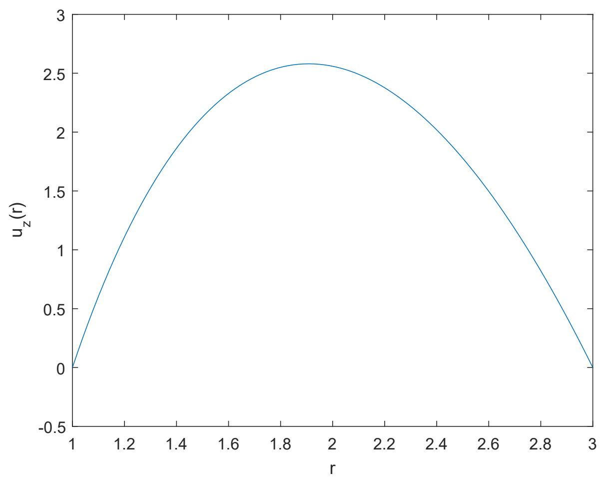

Figure 3: Solution to Problem #3 assuming $r_a = 1$, $r_b = 3$, $dP/dz = -5$ and $\mu = 1$.

As per the force per unit area, we must evaluate the shear stress tensor:

$$\tau = \begin{bmatrix} 2\mu\cancelto{0,1}{\frac{\partial u_r}{\partial r}} & \mu\left(r\frac{\partial}{\partial r}\cancelto{0,2}{\left(\frac{u_\theta}{r}\right)} + \frac{1}{r}\cancelto{0,1}{\frac{\partial u_r}{\partial \theta}}\right) & \mu\left(\cancelto{0,1}{\frac{\partial u_r}{\partial z}} + \frac{\partial u_z}{\partial r}\right) \\ \mu\left(r\frac{\partial}{\partial r}\cancelto{0,2}{\left(\frac{u_\theta}{r}\right)} + \frac{1}{r}\cancelto{0,1}{\frac{\partial u_r}{\partial \theta}}\right) & 2\mu\left(\frac{1}{r}\cancelto{0,2}{\frac{\partial u_\theta}{\partial \theta}} + \cancelto{0,1}{\frac{u_r}{r}}\right) & \mu\left(\cancelto{0,2}{\frac{\partial u_\theta}{\partial z}} + \frac{1}{r}\cancelto{0,5}{\frac{\partial u_z}{\partial \theta}}\right) \\ \mu\left(\cancelto{0,1}{\frac{\partial u_r}{\partial z}} + \frac{\partial u_z}{\partial r}\right) & \mu\left(\cancelto{0,2}{\frac{\partial u_\theta}{\partial z}} + \frac{1}{r}\cancelto{0,5}{\frac{\partial u_z}{\partial \theta}}\right) & 2\mu\cancelto{0,4}{\frac{\partial u_z}{\partial z}} \end{bmatrix}$$

Thus, the only remaining term is for τ results in the following:

$$\tau = \mu\left(\frac{\partial u_z}{\partial r}\right)$$

Substituting in the expression for u_z:

$$\tau = \mu\left(\frac{r}{8\mu}\left(\frac{dP}{dz}\right) + \frac{1}{4\mu r}\left(\frac{dP}{dz}\right)\frac{r_b^2 - r_a^2}{\ln\left(\frac{r_a}{r_b}\right)}\right)$$