Chapter 4 - Integral Form for a Control Volume

Lecture 16
Section 4.3

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

- Understand the formulation of the Conservation of Energy equation in an RTT framework

Reynolds Transport Theorem

- Recall RTT

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{\delta}{\delta t}\int_{C.\forall.} b\rho d\forall + \int_{\text{C.S.}} b\rho \vec{V} \cdot d\vec{A}$$

- The LHS is the rate of change of the extensive property of system
- The first term on the RHS is the rate of change of the extensive property of the system within the C.∀.
- The second term on the RHS is the rate at which the extensive property of the system is exiting the C.∀. through the C.S.

Conservation of Energy

- Recall our definition of energy

$$E = \int_m e dm$$

where

$$e = u + \frac{V^2}{2} + gz$$

- The First Law of Thermodynamics states that the rate of heat supplied to the system less the rate of work done by the system is equal to the rate of change of energy of the system

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

Conservation of Energy

- There can be multiple terms for the rate of work done by the C.∀.

$$\dot{W} = \dot{W}_{shaft} + \dot{W}_{normal} + \dot{W}_{shear} + \dot{W}_{other}$$

- Shaft work is the rate of work transferred out of the C.∀. and is expressed as

$$\dot{W}_{shaft} = \vec{F} \cdot \vec{V} = 2\pi T\dot{n}$$

Conservation of Energy

- There is also the rate of work done by the normal stress acting on the fluid (normal stress times the differential area of the fluid element, recalling $\nu=1/\rho$)

$$\dot{W}_{normal} = -\int_{C.S.} \sigma_n(\vec{V}\cdot\vec{n})dA \approx \int_{C.S.} (P\nu)(\vec{V}\cdot\vec{n})dA$$

Conservation of Energy

- The rate of work done the shear stress acting on the fluid (shear stress times the differential area of the fluid element)

$$\dot{W}_{shear} = -\int_{C.S.} \vec{\tau} \cdot \vec{V} dA$$

- If we choose our C.S. to be perpendicular to inlet/outlet flows, we know the dot product of two perpendicular vectors is zero, thus

$$\dot{W}_{shear} = 0$$

Conservation of Energy

- We can substitute our system variable E for B and e for b

$$\dot{Q} - \dot{W}_{shaft} + \dot{W}_{normal} + \dot{W}_{other} = \frac{\delta}{\delta t}\int_{C.\forall.} e\rho d\forall + \int_{C.S.} (e + P\nu)\rho(\vec{V}\cdot\vec{n})dA$$

- This is expressed as

$$\dot{Q}_{net,in} - \dot{W}_{net,out} = \frac{\delta}{\delta t}\int_{C.\forall.}\left(u + \frac{V^2}{2} + gz\right)\rho d\forall + \int_{C.S.}\left(u + P\nu + \frac{V^2}{2} + gz\right)\rho(\vec{V}\cdot\vec{n})dA$$

Conservation of Energy

- Noting

$$\dot{m} = \int_A \rho(\vec{V} \cdot \vec{n}) dA$$

- This is expressed as

$$\dot{Q}_{net,in} - \dot{W}_{net,out} = \frac{\delta}{\delta t} \int_{C.\forall.} \left(u + \frac{V^2}{2} + gz \right) \rho d\forall + \sum_{out} \dot{m} \left(u + \frac{P}{\rho} + \frac{V^2}{2} + gz \right) - \sum_{in} \dot{m} \left(u + \frac{P}{\rho} + \frac{V^2}{2} + gz \right)$$

Conservation of Energy

- Noting the definition of enthalpy

$$h = u + P\nu = u + \frac{P}{\rho}$$

- This is expressed as

$$\dot{Q}_{net,in} - \dot{W}_{net,out} = \frac{\delta}{\delta t}\int_{C.\forall.}\left(u + \frac{V^2}{2} + gz\right)\rho d\forall + \sum_{out}\dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in}\dot{m}\left(h + \frac{V^2}{2} + gz\right)$$

Conservation of Energy

- If steady-state, this is expressed as

$$\dot{Q}_{net,in} - \dot{W}_{net,out} = \sum_{out} \dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(h + \frac{V^2}{2} + gz\right)$$

- If single-stream

$$\dot{Q}_{net,in}\text{-}\dot{W}_{net,out} = \dot{m}\left(h_2\text{-}h_1 + \frac{V_2^2\text{-}V_1^2}{2} + g(z_2\text{-}z_1)\right)$$

- Look familiar?

Conservation of Energy

- If we substitute u and $P\nu$ in for enthalpy

$$\frac{\dot{W}_{net,in}}{\dot{m}} + \frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 + \frac{\dot{Q}_{net,in}}{\dot{m}}\right)$$

Conservation of Energy

- If we have an ideal flow with no irreversibilities, mechanical energy must be conserved such that

$$u_2 - u_1 = \frac{\dot{Q}_{net,in}}{\dot{m}}$$

- If u_2-u_1>$q_{net,in}$, there is irreversible conversion of mechanical to thermal energy

$$e_{mech,loss} = u_2 - u_1 - q_{net,in}$$

Conservation of Energy

- Therefore

$$\dot{w}_{net,in} + \frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 =$$

$$\frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 + e_{mech,loss}$$

Conservation of Energy

- If our flow is incompressible and we specify the next specific work in is the difference between pump (in) less turbine (out), and our mechanical losses are those of the pump, turbine and piping

$$\dot{m}\left(\frac{P_1}{\rho}+\frac{V_1^2}{2}+gz_1\right)+\dot{W}_{pump}=\dot{m}\left(\frac{P_2}{\rho}+\frac{V_2^2}{2}+gz_2\right) +\dot{W}_{turbine}+\dot{E}_{mech,loss}$$

Conservation of Energy

- Separating mechanical losses from irreversibilities, our general energy equation, based upon prior assumptions, can be expressed as

$$\boxed{\frac{P_1}{\rho g}+\frac{V_1^2}{2g}+z_1+h_{pump}=\frac{P_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_{turbine}+h_{loss}}$$

  h stands for head [m], and we have pump head, turbine head and head losses

- The COE is a modification to the Bernoulli equation - mass and momentum (energy) must be conserved

Conservation of Energy

- Defining η as our mechanical efficiency, $0 \leq \eta \leq 1$
- Pump head is expressed as

$$h_{pump} = \frac{W_{pump}}{g} = \frac{\dot{W}_{pump}}{\dot{m}g} = \frac{\eta \dot{W}_{pump}}{\dot{m}g}$$

- Note that if η is less than 1, the pump produces less head, i.e. it can not pump a fluid as high as a more efficient pump
- Turbine head is expressed as

$$h_{turbine} = \frac{W_{turbine}}{g} = \frac{\dot{W}_{turbine}}{\dot{m}g} = \frac{\dot{W}_{turbine}}{\eta \dot{m}g}$$

- Since $h_{turbine}$ is the extracted head from the fluid, η less than 1 indicates a greater head needs to be provided to produce the same work as a more efficient turbine

Conservation of Energy

- Lastly, head losses are defined as such

$$h_{loss} = \frac{\dot{E}_{mech,loss}}{\dot{m}g}$$

- This notation is not convenient for evaluation. In MEMS 1071, you will evaluated k-factors and frictional losses in piping systems to populate this term.