Chapter 9 - Differential Analysis of Fluid Flow

Lecture 22
Sections 9.4 and 9.5

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

- Construct the Conservation of Momentum, i.e. the Navier-Stokes equations, in a differential framework;
- Understand the implications and use of the Navier-Stokes equations within a two-dimensional framework.

Momentum

- Applying Newton's Second Law to a particle, the force being applied to a particle is the time rate of change of momentum, which is simply the mass time acceleration:

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\int_m \vec{V}\,dm$$

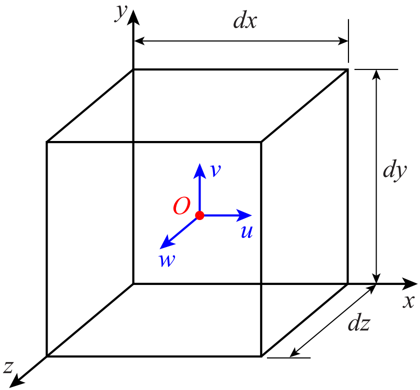
- Evaluating the material derivative on a differential mass:

$$d\vec{F} = dm\frac{D\vec{V}}{Dt} = dm\left(\frac{\partial \vec{V}}{\partial t} + \nabla \cdot \vec{V}\right)$$

- We recall the force acting on a particle is the sum of the surface and body forces.

Forces Acting on a Fluid Particle

- Imagine we have a differential volume, $d\forall$, with side of length dx, dy and dz.
- There can be forces (as a result of stresses) acting on each face, $d\vec{F}_s$ and $d\vec{F}_b$.

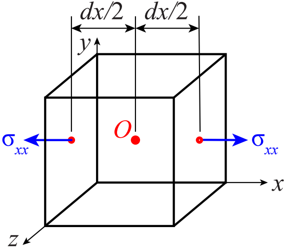


Forces Acting on a Fluid Particle

- Considering the stresses acting on the faces in the x-direction, there exists normal stresses σ_{xx} on the left and right faces. These are expressed on the faces by doing a Taylor series expansion about point O:

$$\sigma_{xx}|_{x+dx/2} = \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2} + \text{...H.O.T.}$$

$$\sigma_{xx}|_{x-dx/2} = \sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2} + \text{...H.O.T.}$$



Forces Acting on a Fluid Particle

- There are shear stresses on the top and bottom faces that exist in the x-direction, τ_{yx}, and those that exist on the front and back faces that exist in the x-direction, τ_{zx}. These are expressed on the faces by doing a Taylor series expansion about point O:

$$\tau_{yx}|_{y+dy/2} = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2} + \text{...H.O.T.}$$

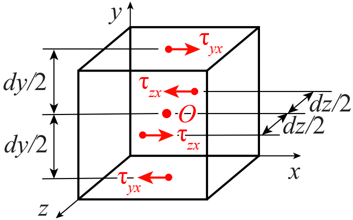
$$\tau_{yx}|_{y-dy/2} = \tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2} + \text{...H.O.T.}$$

$$\tau_{zx}|_{z+dz/2} = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2} + \text{...}$$

...H.O.T.

$$\tau_{zx}|_{z-dz/2} = \tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2} + \text{...}$$

...H.O.T.



Forces Acting on a Fluid Particle

- If we sum the surface forces ($dF_{s,x}$) acting in the x-direction:

$$\begin{aligned}
&\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy\,dz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy\,dz \\
&+\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right)dx\,dz \\
&+\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy - \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy \\
\hline
&\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dx\,dy\,dz
\end{aligned}$$

- The body force ($dF_{b,x}$) in the x-direction are given as:

$$\rho g_x\,dx\,dy\,dz$$

Differential Momentum Equations

- Therefore, the net forces in the x-direction (surface plus body) can be equated to the time rate of change of momentum of the fluid particle (recalling $dm = \rho\, dx\, dy\, dz$):

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x\right)$$

- The same equations can be constructed for the y- and z-directions —this exercise is left to the student.
- Now in terms of measurable quantities, this expression does not do us much good - how are we supposed to measure the shear and normal stresses on an imaginary differential volume within the flow field? We can at least express our shear stresses in terms of a velocity gradient. For a Newtonian fluid, the viscous stress is proportional to the angular deformation rate.

Newtonian Fluid - N.S. Equations

- The shear stresses are defined as:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

- The normal stresses are defined as:

$$\sigma_{xx} = -P - \frac{2\mu\nabla\cdot\vec{V}}{3} + 2\mu\left(\frac{\partial u}{\partial x}\right)$$

$$\sigma_{yy} = -P - \frac{2\mu\nabla\cdot\vec{V}}{3} + 2\mu\left(\frac{\partial v}{\partial y}\right)$$

$$\sigma_{xx} = -P - \frac{2\mu\nabla\cdot\vec{V}}{3} + 2\mu\left(\frac{\partial w}{\partial x}\right)$$

- Thus, substituting in our expressions for shear and normal stress into the momentum equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}\left(-P - \frac{2\mu\nabla\cdot\vec{V}}{3} + 2\mu\frac{\partial u}{\partial x}\right)\dots$$

$$\dots + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + \rho g_x$$

Newtonian Fluid - N.S. Equations

- Evaluating the partial derivatives:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\left(\frac{\partial u}{\partial x}-\frac{\nabla\cdot\vec{V}}{3}\right)\right)\ldots$$

$$\ldots+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right)$$

- Once we evaluate the y- and z-directions, we can consolidate this equation into the following form using tensor notation:

$$\rho\left(\frac{\partial\vec{V}}{\partial t}+\vec{V}\cdot\nabla\vec{V}\right)=-\nabla P+\nabla\cdot\left(\mu\left(\nabla\vec{V}+(\nabla\vec{V})^{\mathrm{T}}\right)\ldots\right.$$

$$\left.\ldots-\frac{2\mu}{3}\left(\nabla\cdot\vec{V}\right)\vec{I}\right)+\rho\vec{g}$$

- This is the <u>complete</u> Navier-Stoke equation.

Newtonian Fluid - N.S. Equations

- If our fluid is incompressible, $\nabla \cdot \vec{V} = 0$:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}\right) = -\nabla P + \nabla\cdot\left(\mu\left(\nabla\vec{V} + (\nabla\vec{V})^{\mathrm{T}}\right)\right) + \rho\vec{g}$$

- Furthermore, if we assume the fluid is *isotropic* - the value of the property of interest does not vary spatially - the strain rate tensor $\boldsymbol{\epsilon}$ is one half $\nabla\vec{V} + (\nabla\vec{V})^{\mathrm{T}}$, and the shear stress tensor $\boldsymbol{\tau} = \mu\boldsymbol{\epsilon}$, as via Stokes's stress constitutive equation:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}\right) = -\nabla P + \nabla\cdot(\boldsymbol{\tau}) + \rho\vec{g}$$

- A result of isotropy is $\nabla\cdot\boldsymbol{\tau} = 2\mu\nabla\cdot\boldsymbol{\epsilon} = \mu\nabla^2\vec{V}$.

Newtonian Fluid - N.S. Equations

- We are finally left with, assuming our fluid is incompressible, Newtonian and isotropic, the Navier-Stokes equations:

$$\boxed{\underbrace{\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}\right)}_{\text{Total acceleration}} = \underbrace{-\nabla P}_{\substack{\text{Pressure}\\\text{gradient}}} + \underbrace{\mu\nabla^2\vec{V}}_{\substack{\text{Viscous}\\\text{force}}} - \underbrace{\rho\vec{g}}_{\substack{\text{Body}\\\text{force}}}}$$

- This system of equations (three second-order non-linear partial differential equations) must be coupled with the continuity equation to solve for u, v, w and P.

Newtonian Fluid - N.S. Equations

- We can decompose this equation into x-, y-, and z-directions by evaluating the operators.
- The gradient operator is :

$$\nabla\vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{bmatrix}$$

- The Laplacian operator $(\nabla \cdot \nabla)$ is expressed as:

$$\nabla^2\vec{V} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix}$$

- The divergence operator is expressed as

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}$$

Newtonian Fluid - N.S. Equations

- Thus, the Navier-Stokes and continuity equations in x-, y- and z-directions are as follows:

x-direction:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

y-direction:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

z-direction:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

Continuity equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$