Chapter 4 - Integral Form for a Control Volume

Lecture 10

Section 4.3

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

- Understand the formulation of the Conservation of Mass equation
- Analyze steady-state and transient systems using the COM

Conservation of Mass

- The conservation of mass is an intuitive law: mass cannot be created or destroyed (except for relativistic systems)
- The rate at which mass is entering or exiting a C.$\forall$. must be the rate at which the mass inside the C.$\forall$. is increasing or decreasing, respectively
- Our extensive system property is M, therefore $B{=}M$
- Our intensive system property is b, which is B/M, or unity

RTT COM

- Substituting in our known values into RTT

$$\left.\frac{dM}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{C.\forall.} \rho d\forall + \int_{\text{C.S.}} \rho \vec{V} \cdot d\vec{A}$$

- If the mass of the system remains constant

$$\frac{\partial}{\partial t}\int_{C.\forall.} \rho d\forall + \int_{\text{C.S.}} \rho \vec{V} \cdot d\vec{A} = 0$$

- The first term is the rate of change of mass within the C.∀. and the second is the net mass flux out of the C.∀. through the C.S.

Mass Flux

- If we use the identities $V_n = V\cos\alpha = \vec{V}\cdot\vec{n}$, $d\vec{A} = \vec{n}dA$, assuming we have uniform flow through the C.S., and for steady, incompressible flows

$$\int_{\text{C.S.}} \rho V_n dA = 0$$

- If we break the net mass outflux into the mass outflux minus the mass influx

$$\int_{\text{C.S., out}} \rho V_n dA - \int_{\text{C.S., in}} \rho V_n dA = 0$$

- If we break the net mass outflux into the mass outflux minus the mass influx and sum all potential streams

$$\sum_{out} \int_{\text{C.S.}} \rho V_n dA - \sum_{in} \int_{\text{C.S.}} \rho V_n dA = 0$$

Mass Flux Cont'd

- Evaluating the term within the integrand, we see the the time rate of change of the amount of mass flowing through a differential area, also known as the mass flow rate, $\dot{m}$ is

$$\partial\dot{m} = \rho V_n dA$$

- The total mass flow rate flow through a surface is

$$\dot{m} = \int_A \partial\dot{m} = \int_A \rho V_n dA = \rho V_n A \left[\frac{\text{kg}}{\text{s}}\right]$$

- Substituting this back into our expression for net mass outflux

$$\sum_{out}\dot{m} - \sum_{in}\dot{m} = 0$$

Steady-State Mass Flow Rate

- That is, under **steady-state** conditions for an incompressible substance, the mass into the system must equal the mass out of the system

$$\boxed{\sum_{in} \dot{m} = \sum_{out} \dot{m}}$$

Transient Mass Flow Rate

- Under transient conditions, and using the identity time rate of change of the mass within the C.∀. remains

$$\frac{\partial}{\partial t}\int_{C.\forall.} \rho d\forall + \sum_{out}\dot{m} - \sum_{in}\dot{m} = 0$$

- The mass within the C.∀. is simply the density times the volume

$$m_{C.\forall.} = \rho\forall = \int_{C.\forall.} \rho d\forall$$

- Thus, the time rate of change of the mass of the C.∀. can be expressed using the following

$$\boxed{\frac{dm_{C.\forall.}}{dt} + \sum_{out}\dot{m} - \sum_{in}\dot{m} = 0}$$

Volumetric Flow Rate

- A useful identity is the volumetric flow rate, $\dot{\forall}$

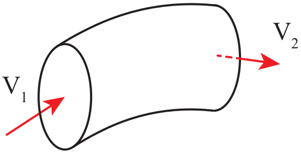
$$\dot{\forall} = \int_A \vec{V} \cdot d\vec{A} = V_n A$$

- Another useful identity is the average velocity

$$V_{avg} = \frac{1}{A} \int_A V_n dA$$

Example #1

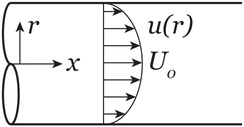
- Consider the stream tube (3D representation of streamlines) depicted below.



- Write the conservation of mass equation for the steady flow of an incompressible substance through the tube and an expression volumetric flow rate, $\dot{\forall}$:

Example #2

- For a steady, viscous flow through a circular pipe, the velocity profile can be expressed as



$$u = U_o\left(1 - \frac{r}{R}\right)^m$$

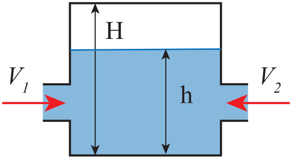
- Assuming no slip boundary conditions, determine the magnitude of the average velocity:

Example #2

- Solution:

Example #3

- A tank as depicted below is being filled by two separate inlets. The inlet diameter and velocity of the first inlet is 1 [in] and 3 [ft/s], respectively. The inlet diameter and velocity of the second inlet is 3 [in] and 2 [ft/s], respectively. The tank has an area of 2 [ft^2]. The air within the top of the tank cannot escape. Find and compute the change in water height as a function of time, dh/dt.



Example #3

- Solution:

Example #3

- Solution: