Chapter 2 - Fundamental Concepts

Lecture 19
Sections 4.2, 4.4-4.5

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Learning Objectives

4.2 Flow Patterns and Flow Visualization

4.4 Other Kinematic Descriptions

4.5 Vorticity and Rotationality

Student Learning Objectives

Students should be able to:

- Identify streamlines, streaklines and pathlines;
- Identify the four fundamental kinematic properties;
- Determine the relationship between strain rate and shear stress in terms of viscosity;
- Identify rotational and irrotational flows.

Streamlines

- Flow visualization is a representation of the numerical data describing a flow field.
- A **streamline** is a curve that is everywhere tangent to the instantaneous local velocity vector:

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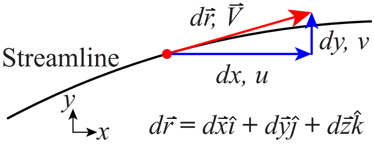
Streamlines

- Considering an infinitesimal arc length $d\vec{r}$ that defines our streamline:



- Since the streamline must be parallel to the velocity vector, it can be shown using similar triangles and comparing magnitudees that the streamline:

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \implies \boxed{\left.\frac{dy}{dx}\right|_{\text{streamline}} = \frac{v(x,y)}{u(x,y)}}$$

- It is noted fluid is not able to cross the streamline, i.e. mass must be conserved between two streamlines.

Example #1

- Given our velocity vector below, determine and plot the streamlines:

$$\vec{V}(u,v) = (0.5 + 0.8x)\hat{\imath} + (1.5 - 0.8y)\hat{\jmath}$$

- The equation for a streamline is found as:

$$\left.\frac{dy}{dx}\right|_{\text{streamline}} = \frac{v}{u} = \frac{1.5 - 0.8y}{0.5 + 0.8x}$$

- Getting like variables on each side of the equation:

$$\frac{dy}{1.5 - 0.8y} = \frac{dx}{0.5 + 0.8x} \implies \int \frac{dy}{1.5 - 0.8y} dy = \int \frac{dx}{0.5 + 0.8x}$$

- Evaluating the indefinite integrals:

$$-1.25\log(1.5 - 0.8y) + c = 1.25\log(0.8x + 0.5) + c$$

Example #1

- Rearranging, noting a constant divided by a constant is still a constant:

$$\log(1.5 - 0.8y) = -\log(0.8x + 0.5) + c$$

- Exponentiating, noting that exponentiating a constant yields a constant:

$$1.5 - 0.8y = \frac{c}{0.8x + 0.5}$$

- Solving for y, noting that the negative of a constant is a constant:

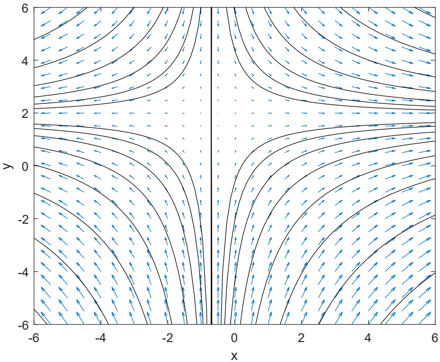
$$y = \frac{c}{0.8(0.8x + 0.5)} + \frac{1.5}{0.8}$$

- By varying the value of c, we can populate different streamlines.

Example #1

- Overlaying the streamlines with the velocity field:

```
x_streamline = linspace(-6,6,101);
c = logspace(0,2,11);
for i = 1:length(c)
    y_streamline_pos = c(i)./(0.8*(0.8.*x_streamline + 0.5)) + 1.875;
    y_streamline_neg = -c(i)./(0.8*(0.8.*x_streamline + 0.5)) + 1.875;
    plot(x_streamline,y_streamline_pos,'k',x_streamline,y_streamline_neg,'k')
end
```



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Pathlines

- A **pathline** is the actual path traveled by an individual fluid particle over some time period.
- A pathline provides a Lagrangian description of the flow field.

- Pathlines are commonly used in PIV, and are great for unsteady flow and experimental set-ups, but have many practical limitations (tremendous amounts of data are required to represent a complex flow field).
- For steady flows, steamlines and pathlines are identical.

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Streaklines

- A **streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow field.
- A **timeline** is a set of adjacent fluid particles that were marked at the same instant in time.
- If the flow is steady, streamlines, pathlines and streaklines are identical; if the flow is unsteady, pathlines and streaklines (which have a time-history) will not reflect the instantaneous flow pattern of a streamline.

A contour plot of Kármán vortex shedding

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Types of Motion or Deformation

- A fluid is able to undergo four fundamental types of motion or deformation:
  1. Translation
  2. Rotation
  3. Linear strain
  4. Shear strain
- These deformations are expressed in terms of corresponding rates:
  1. Velocity
  2. Angular velocity
  3. Linear strain rate
  4. Shear strain rate

1. The **rate of translation vector** is simply the velocity vector:

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} = \langle u,\, v,\, w \rangle$$

Rate of Rotation

2. The **rate of rotation**, or **angular velocity**, is the average rotation rate of two initially perpendicular lines that intersect at a point of interest expressed as:

$$\vec{\omega} = \frac{\nabla \times \vec{V}}{2}$$

- This introduces the curl operator (which is the same as the cross-product):

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

- Therefore, the angular velocity vector is:

$$\boxed{\vec{\omega} = \frac{1}{2}\left\langle \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \right\rangle}$$

Linear and Volumetric Strain Rate

3. The **linear strain rate** is the rate of increases in length per unit length and can expressed in terms of the three principle directions:

$$\boxed{\epsilon_{xx} = \frac{\partial u}{\partial x}; \quad \epsilon_{yy} = \frac{\partial v}{\partial y}; \quad \epsilon_{zz} = \frac{\partial w}{\partial z}}$$

- The **volumetric strain rate** is the rate of increase of a fluid element per unit volume using the divergence operator:

$$\frac{1}{\forall}\frac{D\forall}{Dt} = \frac{1}{\forall}\frac{d\forall}{dt} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \equiv \nabla \cdot \vec{V}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle u, v, w \rangle = \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

- If the volumetric strain rate is zero, the fluid is incompressible.

Shear Strain Rate

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4. The **shear strain rate** is half of the rate of the decreases of the angle between two initially perpendicular lines that intersect at a point of interest:

$$\boxed{\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)}$$

$$\boxed{\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)}$$

Strain Rate Tensor

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- Combining the linear and shear strain rates, we have the **strain rate tensor**:

$$\underline{\underline{\epsilon}}_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

- This array is known as a second order tensor.

Vorticity and Rotationality

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- The **vorticity vector**, which represents the vorticity of a fluid, or the measure of the rotation of a fluid particle, is expressed as:

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = 2\vec{\omega}$$

$$\boxed{\vec{\zeta} = \left\langle \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), -\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right), \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \right\rangle}$$