MEMS 0071
Fall 2019
Final
12/12/19

Name (Print): ______________________________

This exam contains 5 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes. A calculator is permitted on this exam.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If there is not enough information to answer the question, state what is needed and how you would approach solving the problem.
- **BONUS:**
- 6 pts: OMET response rate was 96.36%.
- 5 pts: December 12^{th}, 1901 marks the first transatlantic transmission via what device?

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	25	
6	25	
7	25	
8	25	
Total:	140	

1. (10 points) Given the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0,$$

if the flow is incompressible and the velocity field is given as

$$\vec{V} = (a_1 x + b_1 y + c_1 z)\hat{\imath} + (a_2 x + b_2 y + c_2 z)\hat{\jmath} + (a_3 x + b_3 y + c_3 z)\hat{k},$$

where a_1, b_1, etc. are constant, what conditions are required for the velocity field to satisfy continuity?

2. (10 points) Given a vector field $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$, show the gradient of the divergence, expressed as

$$\nabla(\nabla \cdot \vec{V}) - \nabla \times (\nabla \times \vec{V})$$

is equal to the Laplacian of $\vec{V}$, which is expressed as $\nabla^2 \vec{V}$

3. (10 points) Determine the result of taking the divergence of the angular velocity vector:

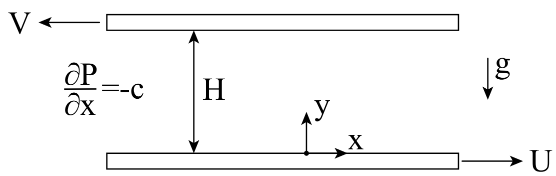
$$\nabla \cdot (\nabla \times \vec{\omega})$$

4. (10 points) Given the x-component of a velocity field as

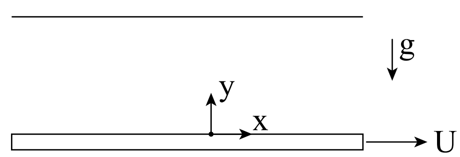
$$u = ax^2 - bxy$$

where a and b are constant. Let $v = 0$ for all x when $y = 0$, that is, $v = 0$ along the x-axis. Generate an expression for the stream function.

5. (25 points) Consider a situation where a fluid exists between two infinite, parallel plates are moving in opposite directions. The bottom plate is moving in the positive x-direction with a velocity of U, and the top plate is moving in the negative x-direction with a velocity V. The two plates are separated by a distance H. A pressure gradient is applied in the positive x-direction. Gravity is acting in the negative y-direction. Construct an expression for the velocity profile in the x-direction. Assume the flow is compressible, turbulent and non-Newtonian.

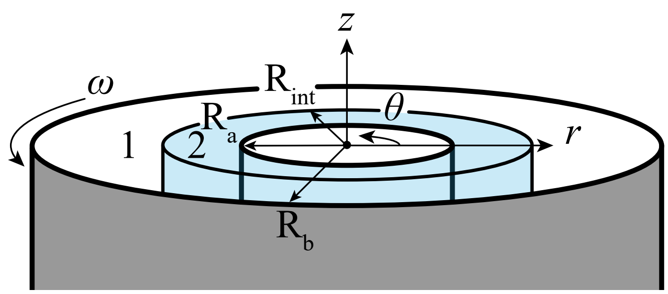


6. (25 points) Consider a situation where a fluid exists above an infinitely long and wide plate. The fluid is initially at rest. At time t=0, the bottom plate moves with a velocity U in the positive x-direction. Gravity is acting in the negative y-direction. Construct an expression for the velocity profile of the fluid, based upon the notes below. Assume the flow is incompressible, steady-state, laminar and Newtonian. Notes:



- Once the diffusion equation is obtained, assume the solution is separable, i.e. u(y,t) = Y(y)T(t). Substitute the expression for u(y,t) into your PDE and differentiate accordingly.
- Once differentiated, group like terms (i.e. Y's and T's), and recognize that since one side of the equation is purely dependent on Y, while the other is purely dependent on T, that the equation must be constant, i.e. both sides are equal to -λ^2
- Construct two ODEs, one as a function of Y, and the other as a function of T, in terms of the constant -λ^2
- It is seen the solution for Y is that of simple harmonic motion. Assume the solution for this ODE has the form Y(y)=cos(λy) + sin(λy).

7. (25 points) Consider a situation where two immiscible fluids exist between two infinitely long, concentric cylinders, where the cylinders have radii $\mathrm{R_a}$ and $\mathrm{R_b}$. The fluid interface is denoted as $\mathrm{R_{int}}$. The fluids are denoted by numbers 1 and 2, each with a unique dynamic viscosity μ and density ρ. The outer cylinder rotates with an angular velocity ω whereas the inner cylinder is stationary.



Given the continuity equation as:

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

and the momentum equations as:
r-direction:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r}$$
$$+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho g_r$$

θ-direction:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta}$$
$$+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

z-direction:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}$$
$$+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

<u>shear-stress tensor</u>:

$$\tau = \begin{bmatrix} 2\mu\dfrac{\partial u_r}{\partial r} & \mu\left(r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)+\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) & \mu\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) \\ \mu\left(r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)+\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) & 2\mu\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & 2\mu\dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

construct an expression for the velocity profile of each fluid. Assume the flow is incompressible, steady-state, laminar and Newtonian.

8. (25 points) Steady, incompressible, laminar flow for a Newtonian fluid is occurring in an infinitely long pipe with diameter D. The pipe is inclined some angle γ. Gravity acts downward and there is no applied pressure gradient. Determine the expression for the velocity profile in the z-direction. Use the continuity and momentum equations from the previous problem.