# Chapter 3 - Fluid Statics

## Lecture 2
## Sections 2.1, 3.1-3.3

MEMS 0071 - Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

# Student Learning Objectives



Students should be able to:

- Distinguish between intensive and extensive properties
- Know the relationship of specific gravity and density
- Distinguish between gage, absolute and atmospheric pressure
- Determine the pressure variation in fluid as a function of height

# Density





- Density ($\rho$) is defined as the mass per unit volume [kg/m$^3$]:

$$\rho = \frac{m}{\forall}$$

- For water at STP, $\rho$=998 [kg/m$^3$]
- Water is incompressible, that is, if the pressure is increased from 1 to 100 atm, the density increases to 1,003 [kg/m$^3$] - insignificant
- For air at STP, $\rho$=1.225 [kg/m$^3$]

# Specific Gravity





- Another common property is specific gravity (SG) [-]:

$$\text{SG} = \frac{\rho_{fluid}}{\rho_{\mathrm{H_2O}}(4°C)} = \frac{\rho_{fluid}}{1,000[\mathrm{kg/m^3}]}$$

- This is quite convenient to use water as a reference, and many fluids have their density given as SG

# Specific Weight





- Another common property is specific weight ($\gamma$) [N/m$^3$]:

$$\gamma = \frac{mg}{\forall} = \rho g$$

- This is quite convenient to use water as a reference, and many fluids have their density given as specific weight

# Intensive vs. Extensive

- Density, SG and $\gamma$ are intensive properties - they are independent of the mass or size of the system
- Can you think of any other intensive properties besides those listed?

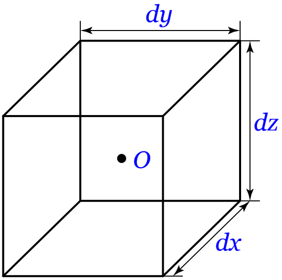
- Extensive properties are the opposite - they depend on the mass or the size of the system
- Can you think of any other extensive properties besides those listed?

# Body Force - Gravity





- Consider a small differential volume of fluid with dimensions $dx$, $dy$ and $dz$
- The center is $O$, and each of the six faces have centers also




- The differential body force (gravity) acting on $O$:

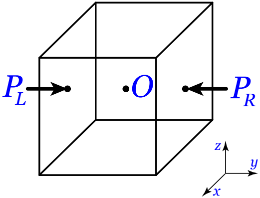
$$d\vec{F}_b = \vec{g}\,dm = \vec{g}\,\rho\,d\forall = \rho\,\vec{g}\,dx\,dy\,dz$$

# Scalar Forces - Pressure





- Considering the y-direction, there exists two unique scalar forces, pressure, acting on the left and right hand side of $d\forall$, namely $P_L$ and $P_R$:




- The pressures act uniformly on these surfaces, but since $dA \to 0$, we assume they are acting on a point

# Pressure about $O$





- To determine the pressure acting on $O$, $P$ we employ a Taylor series expansion about $O$
- This allows for the determination of the pressure at locations $\pm dy/2$, i.e. $P_R$ and $P_L$.
- The general form of the Taylor series expansion:

$$P_x = P_a + \frac{\partial P}{\partial x}(x-a) + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}(x-a)^2 + ...$$

- Neglecting higher order terms and setting $x{=}y \pm dy/2$ and $a{=}y$, the magnitudes are:

$$P_L = P + \frac{\partial P}{\partial y}\left(\left(y - \frac{dy}{2}\right) - y\right) = P - \frac{\partial P}{\partial y}\frac{dy}{2}$$

$$P_R = P + \frac{\partial P}{\partial y}\left(\left(y + \frac{dy}{2}\right) - y\right) = P + \frac{\partial P}{\partial y}\frac{dy}{2}$$

# Pressure Forces





- Summing all scalar surface forces in the y-direction:

$$\sum P_y = P_L + P_R = \left(P - \frac{\partial P}{\partial y}\frac{dy}{2}\right) + \left(P + \frac{\partial P}{\partial y}\frac{dy}{2}\right)$$

- Recalling force is pressure times area, and is a vector, i.e. there is directionality associated with it's magnitude:

$$\sum d\vec{F}_{y,s} = \left(P-\frac{\partial P}{\partial y}\frac{dy}{2}\right)(dx\,dz)\hat{\jmath}+\left(P+\frac{\partial P}{\partial y}\frac{dy}{2}\right)(dx\,dz)(\text{-}\hat{\jmath})$$

- We introduce the unit vector, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ represent the positive $x$, $y$ and $z$ directions, respectively

# Entirety of Pressure

- This process can be repeated for the front and back, and top and bottom faces (left to the reader)
- Summing all the surface pressure forces acting on the $d\forall$

$$\sum d\vec{F}_s =$$
$$\left(P-\frac{\partial P}{\partial x}\frac{dx}{2}\right)(dy\,dz)\hat{\imath}+\left(P+\frac{\partial P}{\partial x}\frac{dx}{2}\right)(dy\,dz)(\text{-}\hat{\imath})+$$
$$\left(P-\frac{\partial P}{\partial y}\frac{dy}{2}\right)(dx\,dz)\hat{\jmath}+\left(P+\frac{\partial P}{\partial y}\frac{dy}{2}\right)(dx\,dz)(\text{-}\hat{\jmath})+$$
$$\left(P-\frac{\partial P}{\partial z}\frac{dz}{2}\right)(dx\,dy)\hat{k}+\left(P+\frac{\partial P}{\partial z}\frac{dz}{2}\right)(dx\,dy)(\text{-}\hat{k})$$

# Formulation



- Executing the math in the $x$ direction:

$$\begin{aligned} d\vec{F}_{x,s} &= \left(P - \frac{\partial P}{\partial x}\frac{dx}{2}\right)(dy\,dz)\hat{\imath} + \left(P + \frac{\partial P}{\partial x}\frac{dx}{2}\right)(dy\,dz)(\text{-}\hat{\imath}) \\ &= \left(P - \frac{\partial P}{\partial x}\frac{dx}{2}\right)(dy\,dz)\hat{\imath} - \left(P + \frac{\partial P}{\partial x}\frac{dx}{2}\right)(dy\,dz)\hat{\imath} \\ &= -\frac{\partial P}{\partial x}(dx\,dy\,dz)\hat{\imath} \end{aligned}$$

- Thus, summing in all directions:

$$\begin{aligned} \sum d\vec{F}_s &= -\left(\frac{\partial P}{\partial x}\hat{\imath} + \frac{\partial P}{\partial y}\hat{\jmath} + \frac{\partial P}{\partial z}\hat{k}\right)dx\,dy\,dz \\ &= -\nabla P\,(dx\,dy\,dz) = -\nabla P\,d\forall \end{aligned}$$

- We have introduced the gradient operator

$$\boxed{\nabla = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)}$$

# Surface and Body Forces





- Adding both the surface and body forces in a force balance

$$\begin{aligned} d\vec{F} &= d\vec{F}_s + d\vec{F}_b = (-\nabla P + \rho\vec{g})(dx\,dy\,dz) \\ &= (-\nabla P + \rho\vec{g})d\forall \end{aligned}$$

- Applying this on a per-volume basis and recalling a static system has zero acceleration:

$$\frac{d\vec{F}}{d\forall} = -\nabla P + \rho\vec{g} = 0 \implies -\nabla P + \rho\vec{g} = 0$$

- Note this in in tensor notation (contains all three dimensions)

# Pressure Variation in 3D



- Breaking this down into three dimensions:

$$\text{x-dir.:} \; -\frac{\partial P}{\partial x} + \rho g_x = 0 \implies \frac{\partial P}{\partial x} = 0$$

$$\text{y-dir.:} \; -\frac{\partial P}{\partial y} + \rho g_y = 0 \implies \frac{\partial P}{\partial y} = 0$$

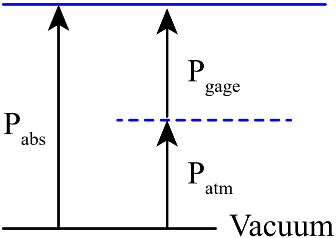
$$\text{z-dir.:} \; -\frac{\partial P}{\partial z} + \rho g_z = 0 \implies \frac{\partial P}{\partial z} = -\rho g = -\gamma$$

- <u>Assumptions</u>: 1) rigid body motion, 2) gravity is the only body force, 3) $z$-axis is opposite direction of gravity
- <u>Result</u>: change of pressure with respect to $z$ has a constant, negative slope - linearly increases the "deeper" you go in a fluid

# Pressure

- Pressure, force per unit area [N/m$^2$], is expressed as Pascals [Pa]
- If our reference pressure is a vacuum, the measured pressure is **absolute pressure**, $P_{abs}$
- If our reference pressure is atmosphere, the measured pressured is **gage pressure**, $P_{gage}$

$$P_{gage} = P_{abs} - P_{atm}$$



# Standard Atmospheric Pressure



- The standard atmospheric pressure is our reference point taken at sea level and has the following properties:
  1. $T$=15°C
  2. $P$=101.3 [kPa]
  3. $\rho$=1.225 [kg/m$^3$]
  4. $\mu$=1.789·10$^{-5}$ [Pa-s]
- The most important variables are pressure and density; often temperature can be taken at 20 or 25°C
- We will use these values unless otherwise stated

# Pressure Variation in Static Fluid





- If analyzing an incompressible liquid ($\rho$=c), and gravity is constant, our pressure versus depth is:

$$\frac{\partial P}{\partial z} = -\rho g = c$$

- Since $P$ is only a function of $z$, we can convert from a partial to full derivative and separate like terms:

$$dP = -\rho g dz$$

- Integrating between bounds of $P_o$ and $P$ for pressure, and $z_o$ and z for depth, where $P_o$ is pressure at $z_o$, and $P$ is pressure at $z$:

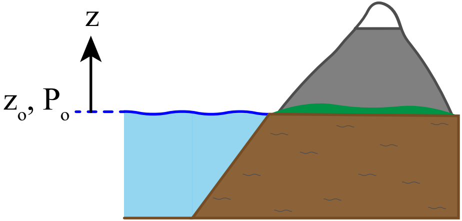
$$\int_{P_o}^{P} dP = -\rho g \int_{z_o}^{z} dz \implies (P - P_o) = \rho g (z_o - z)$$

# Pressure Differential



- If we define $h$ (height) as the difference between our reference location and our point of interest, $h$=$z_o$-$z$, then:

$$\boxed{\Delta P = P - P_o = \rho g h}$$



## Pascal's Law

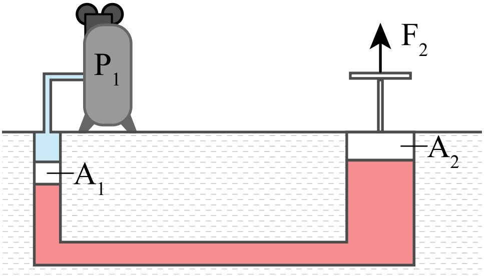




- An interesting consequence of pressure only varying with depth is Pascal's Law - the pressure applied to a confined fluid increases the pressure throughout by the same amount
- The fact that a fluid exerts a pressure normal to the surface walls, and does not vary laterally, allows us to construct amazing devices


# Pascal's Law Example

- Derive a relationship between force and area of the following hydraulic lift:



# Student Learning Objectives - Completed





Students should be able to:

- Distinguish between intensive and extensive properties
  - Independent and dependent of mass, respectively
- Know the relationship of specific gravity and density
  - SG is the density of fluid of interest per that of water at it's densest state
- Distinguish between gage, absolute and atmospheric pressure
  - Gage pressure is measured relative to atmospheric - absolute is measured relative to a perfect vacuum

# Student Learning Objectives - Completed





- Determine the pressure variation in fluid as a function of height
  - Pressure linearly increases with increasing depth and linearly decreases with increasing height