Chapter 4 - Integral Form for a Control Volume

Lecture 9

Section 4.2

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

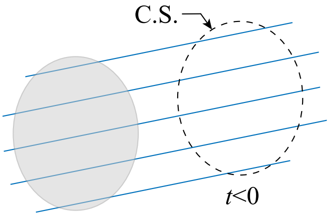
- Understand the Material Derivative
- Use the conservation equations and understand their formulation through the Reynolds Transport Theorem using a Control Volume formulation

What is RTT?

- The relationship between the time rate of change of an extensive property for a system, and for a control volume, which contains our system of interest, is known as the **Reynolds Transport Theorem** (**RTT**)
- Osborne Reynolds (1842-1912) - by the end of this course, his name will be ingrained in your memory

Definition of C.S.

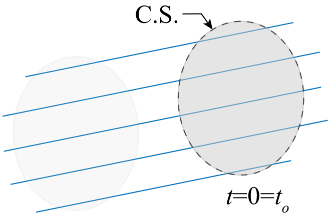
- Consider a Control Surface, C.S., that defines our Control Volume, C.∀.



- Before a time t_o, a quantity of our system is approaching the C.S., following a streamline, a concept we will later introduce

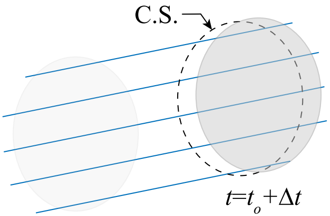
System of Interest

- At time t_o, the system is completely within the C.S., establishing a C.∀.



Change of System of Interest

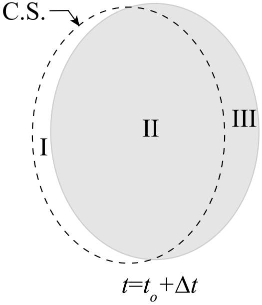
- After some small time interval, Δt, the system moves in the direction of flow and moves out of the C.$\forall$.



- We have three quantities: that of which the C.$\forall$. is not occupied by the system, that which the system is still in the C.$\forall$., and that which the system left the C.$\forall$.

Definition of Regions

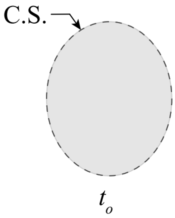
- Regions I and II are within the C.∀. for some t, however, at time $t_o + \Delta t$, the part of the system within the C.∀ is only within region II
- Region III is the part of the system that is outside the C.∀. at time $t_o + \Delta t$



System Property B

- If we represent our extensive property as B (say, banana), we can express B of the system at times t_o and $t_o + \Delta t$
- At time t_o, the extensive property of the system is the same as that of the control volume

$$\boxed{B_{sys}|_{t_o} = B_{C.\forall.}|_{t_o}}$$



Change of B with δt

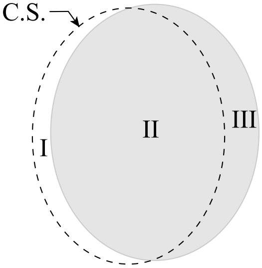
- At time $t_o + \Delta t$, the extensive property of the system is that of regions II and III

$$B_{sys}|_{t_o+\Delta t} = B_{\mathrm{II}}|_{t_o+\Delta t} + B_{\mathrm{III}}|_{t_o+\Delta t}$$

- Region II is the C.∀. less region I

$$B_{II}|_{t_o+\Delta t} = B_{C.\forall.}|_{t_o+\Delta t} - B_{I}|_{t_o+\Delta t}$$

$$\Longrightarrow \boxed{B_{sys}|_{t_o+\Delta t} = (B_{C.\forall.} - B_{\mathrm{I}} + B_{\mathrm{III}})|_{t_o+\Delta t}}$$



$t = t_o + \Delta t$

Time Rate of Change of B

- Recalling the definition of a derivative

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{B_{sys}|_{t_o+\Delta t} - B_{sys}|_{t_o}}{\Delta t}$$

- Substituting in our expressions for $B_{sys})_{t_o}$ and $B_{sys}|_{t_o+\Delta t}$ (red boxes)

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{(B_{C.\forall.} - B_{\text{I}} + B_{\text{III}})|_{t_o+\Delta t} - B_{C.\forall.}|_{t_o}}{\Delta t}$$

RTT in Rate Basis

- The limit of a sum is equal to the sum of the limit such that

$$\left.\frac{dB}{dt}\right|_{sys} = \overbrace{\lim_{\Delta t \to 0} \frac{B_{C.\forall.}|_{t_o+\Delta t} - B_{C.\forall.}|_{t_o}}{\Delta t}}^{\text{term 1}} + \underbrace{\lim_{\Delta t \to 0} \frac{B_{\text{III}}|_{t_o+\Delta t}}{\Delta t}}_{\text{term 2}} - \underbrace{\lim_{\Delta t \to 0} \frac{B_{\text{I}}|_{t_o+\Delta t}}{\Delta t}}_{\text{term 3}}$$

- Let us look at each individual term

ROC of B to Integral Formulation

- The first term
$$\lim_{\Delta t \to 0} \frac{B_{C.\forall.}|_{t_o+\Delta t} - B_{C.\forall.}|_{t_o}}{\Delta t} = \frac{\delta B_{C.\forall.}}{\delta t}$$
is merely the time rate of change of the property, by the definition of a derivative
- It is our goal to take system rate equation and convert it into a control volume equation
- The intensive property of our system, on a per mass basis, $b{=}B/M$, can be expressed as
$$B_{sys} = \int_{M_{sys}} b\,dM = \int_{\forall_{sys}} b\,\rho\,d\forall$$

ROC of Intensive System Property b

- Thus, the first term can be expressed as

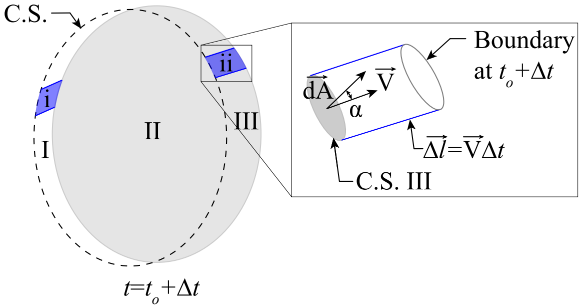
$$\boxed{\frac{\delta B_{C.\forall.}}{\delta t} = \frac{\delta}{\delta t}\int_{C.\forall.} b\,\rho\,d\forall}$$

- The time rate of change of our system property B is equal to the time rate of change of the total amount of the property within the control volume.

- If the term is positive, our system property B is increasing, and vice versa

Outflux Through C.S.

- Term II is solved for by analyzing subregion ii:

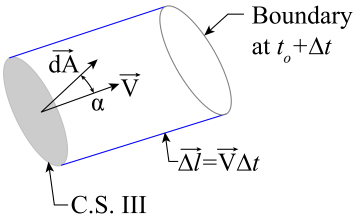


- This is the net rate of outflow from the C.∀. through the control surface, C.S. III
- The flow with velocity $\vec{V}$ is crossing the differential area dA, which has a unit normal $\vec{n}$, which can be represented as the area vector $d\vec{A}$, at some angle α

Volume Outflux $d\forall$ Through C.S.

- The amount of fluid (magnitude) crossing C.S. III, which can be represented as a volume $d\forall$, is the differential length of the projection of the C.S. at time t_o to $t_o+\Delta t$, $\Delta\vec{l}$, times (dot product) the differential area vector $d\vec{A}$

$$d\forall = \Delta\vec{l} \cdot d\vec{A}$$



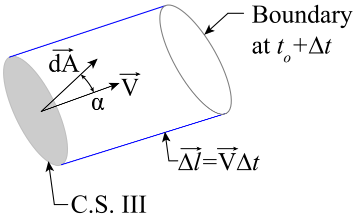
Magnitude of $d\forall$

- The differential length is simply the velocity of the fluid $\vec{V}$ times the time interval Δt

$$\Delta \vec{l} = \vec{V} \Delta t$$

- Thus, the amount of fluid crossing C.S. III is

$$d\forall = \vec{V} \cdot d\vec{A} \Delta t$$



Integral Formulation of $d\forall$

- Thus, reconstructing the second term

$$B_{\text{III}}|_{t_o+\Delta t} = \int_{C.\forall.} b\rho\forall|_{t_o+\Delta t}$$
$$\implies dB_{\text{III}}|_{t_o+\Delta t} = b\rho d\forall|_{t_o+\Delta t}$$
$$\implies dB_{\text{III}}|_{t_o+\Delta t} = b\rho\vec{V}\cdot d\vec{A}\Delta t$$

- We need to integrate over C.S. III to determine the total amount of the property leaving the C.∀.

$$\lim_{\Delta t\to 0}\frac{B_{\text{III}}|_{t_o+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{\text{C.S. III}} B_{\text{III}}|_{t_o+\Delta t}}{\Delta t} =$$
$$\lim_{\Delta t\to 0}\frac{\int_{\text{C.S. III}} b\rho\vec{V}\cdot d\vec{A}\Delta t}{\Delta t} = \int_{\text{C.S. III}} b\rho\vec{V}\cdot d\vec{A}$$

Net $d\forall$ Outflux/Influx

- Therefore

$$\boxed{\lim_{\Delta t \to 0} \frac{B_{\mathrm{III}}|_{t_o+\Delta t}}{\Delta t} = \int_{\text{C.S. III}} b\rho\vec{V} \cdot d\vec{A}}$$

- The same approach can be applied to a subregion of region I.
- It is noted the velocity vector is into the C.$\forall$, opposing the unit normal of dA, thus

$$\boxed{\lim_{\Delta t \to 0} \frac{B_{\mathrm{I}}|_{t_o+\Delta t}}{\Delta t} = -\int_{\text{C.S. I}} b\rho\vec{V} \cdot d\vec{A}}$$

Reynolds Transport Theorem

- Combining our expressions for terms 1 through 3, we have the following

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\delta}{\delta t}\int_{C.\forall.} b\rho d\forall + \int_{\text{C.S. III}} b\rho\vec{V}\cdot d\vec{A} - \int_{\text{C.S. I}} b\rho\vec{V}\cdot d\vec{A}$$

- Noting C.S. I and C.S. III constitute the entire control surface, we are left with RTT:

$$\boxed{\left.\frac{dB}{dt}\right|_{sys} = \frac{\delta}{\delta t}\int_{C.\forall.} b\rho d\forall + \int_{\text{C.S.}} b\rho\vec{V}\cdot d\vec{A}}$$

RTT Term by Term

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\delta}{\delta t}\int_{C.\forall.} b\rho d\forall + \int_{\text{C.S.}} b\rho\vec{V}\cdot d\vec{A}$$

- The LHS is the rate of change of the extensive property of system
- The first term on the RHS is the rate of change of the extensive property of the system within the C.∀.
- The second term on the RHS is the rate at which the extensive property of the system is exiting the C.∀. through the C.S.

Application to Conservation Laws

- RTT is used to derive our conservation equations
- $B{=}M$, $b{=}1 \implies$ Conservation of Mass (COM)
- $B{=}\vec{P}$, $b{=}\vec{V} \implies$ Conservation of Linear Momentum (COLM)
- $B{=}\vec{H}$, $b{=}\vec{r}\times\vec{V} \implies$ Conservation of Angular Momentum (COAM)
- $B{=}E$, $b{=}e \implies$ Conservation of Energy (COE), 1^{st} Law of Thermodynamics
- $B{=}S$, $b{=}s \implies$ 2^{nd} Law of Thermodynamics