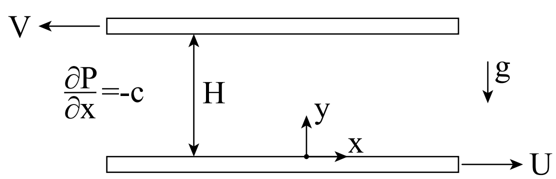
Homework #9

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: November 10^{th}, 2019
Due: November 15^{th}, 2019

Problem #1

Consider a situation where a fluid exists between two infinite, parallel plates that are moving in opposite directions. The bottom plate is moving in the positive x-direction with a velocity of U, and the top plate is moving in the negative x-direction with a velocity V. The two plates are separated by a distance H. A pressure gradient is applied in the positive x-direction. Gravity is acting in the negative y-direction. Construct an expression for the velocity profile in the x-direction. Assume the flow is incompressible, steady-state, laminar, isotropic and Newtonian.



Starting with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{V} = 0$$

Assuming steady-state and an incompressible fluid:

$$\frac{\partial u}{\partial x} + \cancelto{0,\ 1}{\frac{\partial v}{\partial y}} + \cancelto{0,\ 2}{\frac{\partial w}{\partial z}} = 0$$

Where we have the following assumptions:

1 – no y-component of velocity

2 – no z-component of velocity

Thus, we can say the flow is fully developed:

$$\frac{\partial u}{\partial x} = 0$$

Writing the momentum equation in the x-direction:

$$\rho\left(\cancelto{0,\ 3}{\frac{\partial u}{\partial t}} + u\cancelto{0,\ 4}{\frac{\partial u}{\partial x}} + \cancelto{0,\ 1}{v\frac{\partial u}{\partial y}} + \cancelto{0,\ 2}{w\frac{\partial u}{\partial w}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancelto{0,\ 4}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{0,\ 5}{\frac{\partial^2 u}{\partial z^2}}\right) + \cancelto{0,\ 6}{\rho g_x}$$

Where we have the following assumptions:

3 – steady state

4 – consequence of continuity

5 – u is not a function of z

6 – no gravity in the x-direction

Thus, we are left with the following:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial P}{\partial x}$$

Integrating twice, recalling the pressure gradient is a constant:

$$u(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x}y^2 + c_1 y + c_2$$

Applying the following boundary conditions, we can solve for the constants of integration:

$$u(y=0) = U = \frac{1}{2\mu}\frac{\partial P}{\partial x}(0)^2 + c_1(0) + c_2 \implies c_2 = U$$

$$u(y=H) = -V = \frac{1}{2\mu}\frac{\partial P}{\partial x}(H)^2 + c_1(H) + U \implies c_1 = -\frac{V}{H} - \frac{1}{2\mu}\frac{\partial P}{\partial x}H - \frac{U}{H}$$

Therefore, the velocity profile in the x-direction is found as:

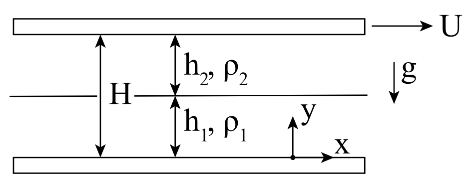
$$u(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x}y^2 - \left(\frac{V}{H} + \frac{1}{2\mu}\frac{\partial P}{\partial x}H + \frac{U}{H}\right)y + U$$

Written concisely:

$$\boxed{u(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x}y(y-H) - \frac{y}{H}(V+U) + U}$$

Problem #2

Consider a Couette flow were there are two immiscible fluids existing between two infinite, parallel plates. The top plate moves with a velocity magnitude U in the positive x-direction while the bottom plate is stationary. The plates are separated by a distance H. Fluid 1 has a density ρ_1, viscosity μ_1 and has a height h_1. Fluid 2, on top of fluid 1, has a density ρ_2, viscosity μ_2 and a height of h_2. It is noted $h_1+h_2=H$. The interface between the two liquids is assumed parallel to the top and bottom plates. Gravity acts in the negative y-direction. Assume the flow is steady-state and laminar, and the fluids are incompressible, isotropic and Newtonian. Construct an expression for the velocity profile in the x-direction. Hint: think about the two boundary conditions existing at the fluid interface.



Starting with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{V} = 0$$

Assuming steady-state and an incompressible fluid:

$$\frac{\partial u}{\partial x} + \cancelto{0,\,(1)}{\frac{\partial v}{\partial y}} + \cancelto{0,\,(2)}{\frac{\partial w}{\partial z}} = 0$$

Where we have the following assumptions:

(1) – no y-component of velocity

(2) – no z-component of velocity

Thus, we can say the flow is fully developed:

$$\frac{\partial u}{\partial x} = 0$$

Writing the momentum equation in the x-direction for fluid 1:

$$\rho_1\left(\cancelto{0,\,(3)}{\frac{\partial u_1}{\partial t}} + u_1\cancelto{0,\,(4)}{\frac{\partial u_1}{\partial x}} + \cancelto{0,\,(1)}{v_1\frac{\partial u_1}{\partial y}} + \cancelto{0,\,(2)}{w_1\frac{\partial u_1}{\partial w}}\right) = -\cancelto{0,\,(5)}{\frac{\partial P}{\partial x}} + \mu_1\left(\cancelto{0,\,(4)}{\frac{\partial^2 u_1}{\partial x^2}} + \frac{\partial^2 u_1}{\partial y^2} + \cancelto{0,\,(6)}{\frac{\partial^2 u_1}{\partial z^2}}\right) + \cancelto{0,\,(7)}{\rho_1 g_x}$$

Where we have the following assumptions:

(3) – steady state

(4) – consequence of continuity

(5) – no pressure gradient in the x-direction

(6) – u is not a function of z

(7) – no gravity in the x-direction

Thus, for fluid 1, we have the following:

$$\frac{\partial^2 u_1}{\partial y^2} = 0$$

Integrating twice:

$$u_1 = c_1 y + c_2$$

The same momentum expression can be written for fluid 2:
Writing the momentum equation in the x-direction for fluid 1:

$$\rho_1\left(\cancelto{0,\,(3)}{\frac{\partial u_2}{\partial t}} + u_2\cancelto{0,\,(4)}{\frac{\partial u_2}{\partial x}} + \cancelto{0,\,(1)}{v_1\frac{\partial u_2}{\partial y}} + \cancelto{0,\,(2)}{w_1\frac{\partial u_2}{\partial w}}\right) = -\cancelto{0,\,(5)}{\frac{\partial P}{\partial x}} + \mu_1\left(\cancelto{0,\,(4)}{\frac{\partial^2 u_2}{\partial x^2}} + \frac{\partial^2 u_2}{\partial y^2} + \cancelto{0,\,(6)}{\frac{\partial^2 u_2}{\partial z^2}}\right) + \cancelto{0,\,(7)}{\rho_2 g_x}$$

Thus, for fluid 1, we have the following:

$$\frac{\partial^2 u_2}{\partial y^2} = 0$$

Integrating twice:

$$u_2 = c_3 y + c_4$$

We have a system of two ordinary differential equations, for which we need to solve for four constants of integration. Applying the following boundary conditions for the bottom and top plates:

$$u_1(y=0) = 0 = c_1(0) + c_2 \implies c_2 = 0$$

$$u_2(y=H) = U = c_3(H) + c_4 \implies c_3(h_1 + h_2) + c_4 = U$$

At the interface of the two fluids, the velocities of both fluids have to be equal, as well as the shear stress. Thus, we have the following boundary conditions:

$$u_1(y=h_1) = u_2(y=h_1) \implies c_1 h_1 = c_3 h_1 + c_4$$

$$\mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=h_1} = \mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=h_1} \implies \mu_1 c_1 = \mu_2 c_3$$

We have three equations and three unknowns (c_1, c_2 and c_3). Solving for the constants:

$$c_1 = \frac{\mu_2 U}{\mu_1 h_2 + \mu_2 h_1}$$

$$c_3 = \frac{\mu_1 U}{\mu_1 h_2 + \mu_2 h_1}$$

$$c_4 = U\left(\frac{\mu_2 h_1 - \mu_1 h_1}{\mu_1 h_2 + \mu_2 h_1}\right)$$

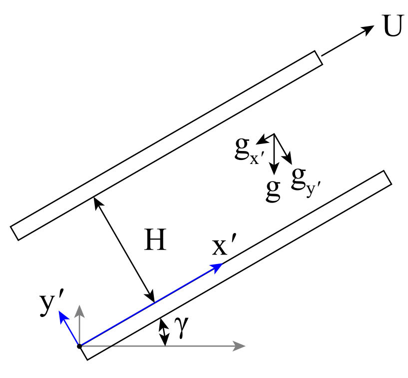
Thus, the velocities for fluids 1 and 2 can be expressed as:

$$\boxed{u_1(y) = \left(\frac{\mu_2 U}{\mu_1 h_2 + \mu_2 h_1}\right) y}$$

$$\boxed{u_2(y) = \left(\frac{\mu_1 U}{\mu_1 h_2 + \mu_2 h_1}\right) y + U\left(\frac{\mu_2 h_1 - \mu_1 h_1}{\mu_1 h_2 + \mu_2 h_1}\right)}$$

Problem #3

Consider a situation where a fluid exists between two infinite, parallel plates that are inclined above the x-axis by some angle γ. The bottom plate is stationary, whereas the top plate is moving with some velocity U. The plates are separated by a distance H. Gravity is acting in the negative y-direction. Construct an expression for the velocity profile of the fluid. Assume the flow is incompressible, steady-state, laminar and Newtonian.



Rotate the coordinate system as shown. Starting with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{V} = 0$$

Assuming steady-state and an incompressible fluid:

$$\frac{\partial u}{\partial x} + \cancelto{0,\ (1)}{\frac{\partial v}{\partial y}} + \cancelto{0,\ (2)}{\frac{\partial w}{\partial z}} = 0$$

Where we have the following assumptions:

1 – no y-component of velocity

2 – no z-component of velocity

Thus, we can say the flow is fully developed:

$$\frac{\partial u}{\partial x} = 0$$

Writing the momentum equation in the x-direction:

$$\rho\left(\overset{0,\,3}{\cancel{\frac{\partial u}{\partial t}}} + u\overset{0,\,4}{\cancel{\frac{\partial u}{\partial x}}} + \overset{0,\,1}{\cancel{v\frac{\partial u}{\partial y}}} + \overset{0,\,2}{\cancel{w\frac{\partial u}{\partial w}}}\right) = -\overset{0,\,5}{\cancel{\frac{\partial P}{\partial x}}} + \mu\left(\overset{0,\,4}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \overset{0,\,6}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right) + \rho g_x$$

Where we have the following assumptions:

3 – steady state

4 – consequence of continuity

5 – no pressure gradient in the x-direction

6 – u is not a function of z

Gravity is acting in the negative x-direction, i.e. g_x has a negative value. Thus, we are left with the following:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\rho g_x$$

Integrating twice, recalling the gravity in the negative x-direction is a constant:

$$u(y) = \frac{1}{2\mu}\rho g_x y^2 + c_1 y + c_2$$

Applying the following boundary conditions, we can solve for the constants of integration:

$$u(y=0) = 0 = \frac{1}{2\mu}\rho g_x (0)^2 + c_1(0) + c_2 \implies c_2 = 0$$

$$u(y=H) = U = \frac{1}{2\mu}\rho g_x (H)^2 + c_1(H) \implies c_1 = \frac{U}{H} - \frac{H}{2\mu}\rho g_x$$

Therefore, the velocity profile in the x-direction is found as:

$$u(y) = \frac{1}{2\mu}\rho g_x y^2 + \left(\frac{U}{H} - \frac{H}{2\mu}\rho g_x\right) y$$

Written concisely:

$$\boxed{u(y) = \frac{\rho g_x y}{2\mu}\left(y - H\right) + \frac{U}{H} y}$$