Homework #7

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: October 27th, 2019
Due: November 1st, 2019

Problem #1

Given an incompressible fluid that flows past a sphere with radius r. If the fluid velocity is

$$\vec{V} = V_o\left(1 + \frac{r^3}{x^3}\right)\hat{\imath}$$

where V_o is the free-stream velocity, determine the acceleration vector of the fluid.
Calculating the acceleration, recognizing the flow is steady, and there are no y- or z-components of flow:

$$\vec{a} = \cancelto{0}{\frac{\partial \vec{V}}{\partial t}} + u\frac{\partial \vec{V}}{\partial x} + \cancelto{0}{v\frac{\partial \vec{V}}{\partial y}} + \cancelto{0}{w\frac{\partial \vec{V}}{\partial z}} = u\frac{\partial \vec{V}}{\partial x} = \left(V_o\left(1 + \frac{r^3}{x^3}\right)\right)\frac{\partial (V_o r^3 x^{-3})}{\partial x} = \left(V_o\left(1 + \frac{r^3}{x^3}\right)\right)(-3V_o r^3 x^{-4})\hat{\imath}$$

Problem #2

For a two-dimensional steady flow give as

$$\vec{V} = \left(\frac{V_o}{L}\right)(-x\hat{\imath} + y\hat{\jmath})$$

where V_o and L are constants, determine the acceleration vector of this flow.
Calculating the acceleration, recognizing the flow is steady, and there is no z-components of flow:

$$\vec{a} = \cancelto{0}{\frac{\partial \vec{V}}{\partial t}} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \cancelto{0}{w\frac{\partial \vec{V}}{\partial z}} = \left(\frac{V_o}{L}\right)\left\{-x\left(-\frac{V_o}{L}\right)\hat{\imath} + y\left(\frac{V_o}{L}\right)\hat{\jmath}\right\} = \left(\frac{V_o^2}{L^2}\right)(x\hat{\imath} + y\hat{\jmath})$$

Problem #3

Fluid is steadily flowing through a nozzle, which has a shape described as

$$\frac{y}{L} = \pm\frac{0.5}{\left(1 + \frac{x}{L}\right)}$$

which is valid for the ranges -0.5<y/L<0.5 and 0<x/L<1. If the pressure field of the fluid is described as

$$P - P_o = -\left(\frac{\rho V_o^2}{2}\right)\left(\frac{x^2 + y^2}{L^2} + \frac{2x}{L}\right)$$

and V_o and P_o are the velocity and pressure at the origin, determine the time rate of change of the pressure field through the nozzle.
To determine the time rate of change of pressure, we will take the material derivative of pressure. The flow and pressure fields are steady and two-dimensional, i.e. no z-component.

$$\frac{DP}{dt} = \cancelto{0}{\frac{\partial P}{\partial t}} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + \cancelto{0}{w\frac{\partial P}{\partial w}} = u\left((-\rho V_o^2)\left(\frac{x}{L^2} + \frac{1}{L}\right)\right) + v\left(\frac{-\rho V_o^2 y}{L^2}\right)$$

The x- and y-components of velocity are ascertained from the shape of the nozzle:

$$u \approx V_o\left(1 + \frac{x}{L}\right); \qquad v \approx -V_o\left(\frac{y}{L}\right)$$

Therefore:

$$\frac{DP}{dt} = V_o\left(1 + \frac{x}{L}\right)\left((-\rho V_o^2)\left(\frac{x}{L^2} + \frac{1}{L}\right)\right) - V_o\left(\frac{y}{L}\right)\left(\frac{-\rho V_o^2 y}{L^2}\right)$$

Problem #4

For a two-dimensional steady flow, if the velocity field is given as

$$\vec{V} = (x^2 + y^2)\hat{\imath} - 2xy\hat{\jmath}$$

determine if the flow is rotational or irrotational.
The angular velocity vector provides the x-, y- and z-components of rotation. The x-component of velocity has dependence on x and y; the y-component of velocity has dependence on x and y; there is no z-component of velocity.

$$\vec{\omega} = \frac{1}{2}\left\{\left(\cancelto{0}{\frac{\partial w}{\partial y}} - \cancelto{0}{\frac{\partial v}{\partial z}}\right)\hat{\imath} + \left(\cancelto{0}{\frac{\partial u}{\partial z}} - \cancelto{0}{\frac{\partial w}{\partial x}}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right\}$$

Thus:

$$\vec{\omega} = \frac{1}{2}\left(-2y - 2y\right) = -2y$$

The flow is rotational.

Problem #5

Given the following velocity field for an incompressible fluid

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine the z-component of velocity that satisfies continuity.
The continuity equation, for an incompressible flow, is given as:

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

We have the following:

$$\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial v}{\partial y} = x + z$$

Thus, for the continuity equation to be equal to zero, we must have the following:

$$\frac{\partial w}{\partial z} = -(3x+z) \implies w = -3xz - \frac{z^2}{2} + f(x,y)$$

The function of x and y is a product of integrating with respect to a partial derivative; w can be a function of x and y.

Problem #6

If the velocity field of planar flow between two plates is described as

$$u = 0.002(1 - 10(10^3)y)\,[\text{m/s}] = 0.002 - 20y\,[\text{m/s}])$$

which is valid in the range of -10<y [mm]<10, determine the vorticity and shear strain rate when y=5 [mm]. The vorticity vector is twice the angular velocity vector, which is simply the curl of the velocity field. There exists only an x-component of velocity, which is a function of y.

$$\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{V} = \left\{\left(\cancelto{0}{\frac{\partial w}{\partial y}} - \cancelto{0}{\frac{\partial v}{\partial z}}\right)\hat{\imath} + \left(\cancelto{0}{\frac{\partial u}{\partial z}} - \cancelto{0}{\frac{\partial w}{\partial x}}\right)\hat{\jmath} + \left(\cancelto{0}{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y}\right)\hat{k}\right\} = 20\hat{k}\,[\text{s}^{-1}]$$

Note: the shear is independent of y-location. The shear strain rate, in two dimensions, is:

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -20\,[\text{s}^{-1}]$$

Problem #7

Consider a situation of planar Couette flow, where the top plate is moving in the positive x-direction with a velocity in the positive x-direction of 0.32 [m/s] and the bottom plate is stationary. If the velocity profile between the two plates is described as

$$u = (40y - 800y^2)\,[\text{m/s}]$$

where y is the height of the channel, taken as 10 [mm], determine the shear stress acting on the bottom. Take the dynamic viscosity of the fluid to be 897 μPa-s.
Shear stress is defined as:

$$\tau = \mu \frac{du}{dy}\bigg|_{y=0} = (897 \cdot 10^{-6}\,[\text{Pa-s}])(40 - 1,600(0))\,[\text{s}^{-1}] = 35.88\,[\text{mPa}]$$