Chapter 2 - Fundamental Concepts

Lecture 21
Sections 6.3 and 2.6

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

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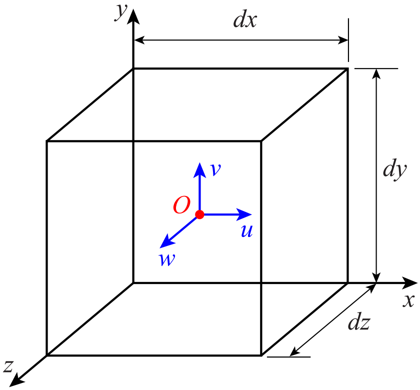
Student Learning Objectives

Students should be able to:

- Construct the Conservation of Mass in Cartesian coordinates;
- Evaluate the Conservation of Mass for steady and unsteady problems;

Formulation of Density

- Imagine we have a differential volume, $d\forall$, with sides of length dx, dy and dz, where the center of the $d\forall$ is taken as O. The density at O is taken as ρ. The velocity field at O is taken as $\vec{V} = \langle u, v, w \rangle$.

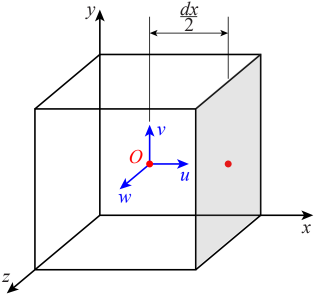


Formulation of Density

- We are interested in evaluating how ρ and $\vec{V}$ change when we move a differential distance $dx/2$, $dy/2$ or $dz/2$, from O. Employing a Taylor series expansion about point O:

$$\rho|_{x+dx/2} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} + \left(\frac{\partial^2 \rho}{\partial x^2}\right)\frac{1}{2!}\left(\frac{dx}{2}\right)^2 + \text{...H.O.T.}$$

$$u|_{x+dx/2} = u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2} + \left(\frac{\partial^2 u}{\partial x^2}\right)\frac{1}{2!}\left(\frac{dx}{2}\right)^2 + \text{...H.O.T.}$$

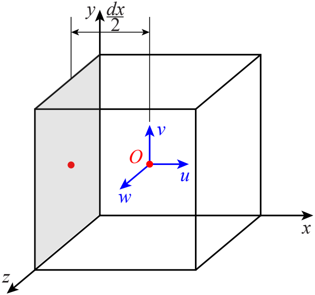


Formulation of Density

- Employing a Taylor series expansion about point O on the negative x-face:

$$\rho|_{x-dx/2} = \rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} + \left(\frac{\partial^2 \rho}{\partial x^2}\right)\frac{1}{2!}\left(\frac{dx}{2}\right)^2 + \text{...H.O.T.}$$

$$u|_{x-dx/2} = u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2} + \left(\frac{\partial^2 u}{\partial x^2}\right)\frac{1}{2!}\left(\frac{dx}{2}\right)^2 + \text{...H.O.T.}$$



- We will ignore $\partial^2/\partial x^2$ and H.O.T. - why?

Reynolds Transport Theorem

- Recall from the Reynolds Transport Theorem that the time rate of change of mass is equal to the time rate of change of mass within the control volume plus the net mass out outflux of the control volume through the control surface:

$$\frac{dm_{sys}}{dt} = \frac{\partial}{\partial t}\int_{C.\forall.} \rho d\forall + \int_{C.S.} \rho(\vec{V}\cdot\vec{n})dA$$

- Under steady-state, the net outflux must equal the net influx. Evaluating the control surface integral on the right face (positive x-direction), we have density times velocities times a differential area:

$$\left(\rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}\right)\left(u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right) dy\, dz$$

$$= \rho u\, dy\, dz + \frac{1}{2}\left(u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right) dx\, dy\, dz$$

Net Mass Influx

- Evaluating the control surface integral on the left face (negative x-direction):

$$-\left(\rho-\left(\frac{\partial\rho}{\partial x}\right)\frac{dx}{2}\right)\left(u-\left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right)dy\,dz$$

$$=-\rho u\,dy\,dz+\frac{1}{2}\left(u\left(\frac{\partial\rho}{\partial x}\right)+\rho\left(\frac{\partial u}{\partial x}\right)\right)dx\,dy\,dz$$

- Summing the two control surface integrals:

$$\begin{array}{r} \rho u\,dy\,dz+\frac{1}{2}\left(u\left(\frac{\partial\rho}{\partial x}\right)+\rho\left(\frac{\partial u}{\partial x}\right)\right)dx\,dy\,dz \\ +(-\rho u\,dy\,dz)+\frac{1}{2}\left(u\left(\frac{\partial\rho}{\partial x}\right)+\rho\left(\frac{\partial u}{\partial x}\right)\right)dx\,dy\,dz \\ \hline \left(u\left(\frac{\partial\rho}{\partial x}\right)+\rho\left(\frac{\partial u}{\partial x}\right)\right)dx\,dy\,dz \end{array}$$

Net Mass Influx

- We can repeat the same procedure for the y- and z-directions. Evaluating the net mass outflux taking into consideration all six faces:

$$\int_{C.S.} \rho(\vec{V} \cdot \vec{n}) dA = \left(\left\{ u \left(\frac{\partial \rho}{\partial x} \right) + \rho \left(\frac{\partial u}{\partial x} \right) \right\} + ... \right.$$

$$\left. ... \left\{ v \left(\frac{\partial \rho}{\partial y} \right) + \rho \left(\frac{\partial v}{\partial y} \right) \right\} + \left\{ w \left(\frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial w}{\partial z} \right) \right\} \right) dx\, dy\, dz$$

- We notice this is the product rule of ρ and $\vec{V}$:

$$\int_{C.S.} \rho(\vec{V} \cdot \vec{n}) dA = \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) dx\, dy\, dz$$

Change of Mass in C.∀.

- Next, we must evaluate the time rate of change of mass within the C.∀.:

$$\frac{\partial}{\partial t}\int_{C.\forall.}\rho d\forall = \frac{\partial \rho}{\partial t}\,dx\,dy\,dz$$

- Combining the expression for change of mass in the C.∀. and net mass outflux:

$$\frac{\partial \rho}{\partial t}\,dx\,dy\,dz + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right)dx\,dy\,dz = 0$$

- Recalling the divergence operator:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$

- For steady flow:

$$\boxed{\nabla \cdot (\rho \vec{V}) = 0}$$

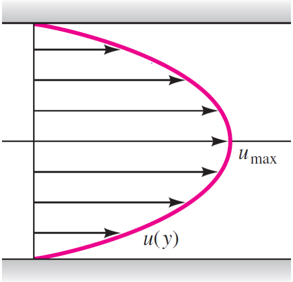
- If incompressible:

$$\boxed{\nabla \cdot \vec{V} = 0}$$

Example #1 - Plane Poiseuille Flow

- For plane Poiseuille flow, the velocity field is given as:

$$\vec{V} = \frac{C}{2r} y(y-d) = V(y)\,\hat{\imath}$$



- Does this obey continuity?

Example #1 - Plane Poiseuille Flow

- Solution:

Example #1 - Plane Poiseuille Flow

- Solution:

Exampe #2 - Unsteady Continuity

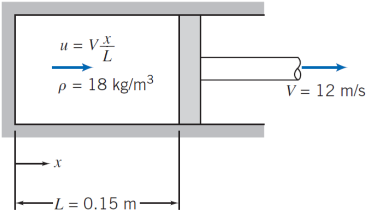
- Image a gas-filled piston-cylinder.
- When the piston is 0.15 [m] from the bottom of the cylinder, ρ=18 [kg/m^3].
- The piston then moves away from the bottom of the cylinder at a velocity of 12 [m/s], where the velocity is linear and expressed as

$$u = V\frac{x}{L}$$

where it is 0 [m/s] at the end and u=V as the piston.
- Find the time rate of change of the density at this instant and express the average density as a function of time.

Unsteady Continuity

This is illustrated as follows:



Solution:

The continuity equation is expressed as:

$$\frac{\delta\rho}{\delta t} + \nabla \cdot (\rho\vec{V}) = 0$$

Unsteady Continuity

Expanding the divergence of $\rho\vec{V}$:

$$\frac{\delta\rho}{\delta t}+\frac{\delta\rho u}{\delta x}+\frac{\delta\rho v}{\delta y}+\frac{\rho w}{\delta z}=0$$

The only component of velocity is in the x-direction:

$$\frac{\delta\rho}{\delta t}+\frac{\delta\rho u}{\delta x}+\cancelto{0}{\frac{\delta\rho v}{\delta y}}+\cancelto{0}{\frac{\rho w}{\delta z}}=0$$

Applying the product rule, ρ=constant:

$$\frac{\delta\rho}{\delta t}+\rho\frac{\delta u}{\delta x}+u\cancelto{0}{\frac{\delta\rho}{\delta x}}=0$$

Unsteady Continuity

Recalling the expression for u and evaluating the derivative of u with respect to x:

$$u = V\frac{x}{L} \implies \frac{\delta u}{\delta x} = \frac{V}{L}$$

The distance between the bottom of the cylinder to the piston is linear and expressed as L=L_o+Vt, i.e. at t=0, L=L_o=0.15 [m]

Substituting in the expression for $\delta u/\delta x$:

$$\frac{\delta \rho}{\delta t} + \rho\frac{V}{L} = 0 \implies \frac{\delta \rho}{\delta t} + \rho\frac{V}{L_o + Vt} = 0$$

Unsteady Continuity

Grouping like variables

$$\frac{\delta\rho}{\rho} + \frac{V\delta t}{L_o + Vt} = 0 \implies \frac{\delta\rho}{\rho} = -\frac{V\delta t}{L_o + Vt}$$

Integrate time between 0 and t, where density at t=0 is ρ_o and density at t is ρ

$$\int_{\rho_o}^{\rho} \frac{1}{\rho} d\rho = -\int_0^t \frac{V}{L_o + Vt} dt$$

Thus

$$\ln\left(\frac{\rho}{\rho_o}\right) = \ln\left(\frac{L_o}{L_o + Vt}\right)$$

Unsteady Continuity

Exponentiating and solving for ρ

$$\rho(t) = \rho_o \left(\frac{1}{1 + \frac{Vt}{L_o}} \right)$$

At the instant t=0, the rate of change of density is

$$\frac{\delta \rho}{\delta t} = -\rho_o \frac{V}{L}$$

Substitute values ρ_o=18 [kg/m^3], V=12 [m/s] and L=0.15 [m]

$$\frac{\delta \rho}{\delta t} = -1,440 \left[\frac{\mathrm{kg}}{\mathrm{m}^3\,\mathrm{s}} \right]$$