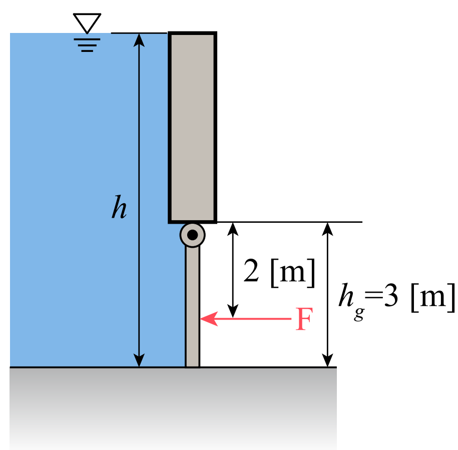
# Homework #3

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: September 14$^{\text{th}}$, 2019
Due: September 20$^{\text{th}}$, 2019

## Problem #1

The tank shown below has a width, $b$, of 1 [m]. The height of the rectangular gate, $h_g$, is 3 [m]. The fluid is water at 20°C. A force $F$ of maximum value 1 [MN] is applied at 2 [m] below the top of the gate. Determine the maximum depth of the fluid $h$ that the gate can hold.




We know the expression for $h_c$ with h defined as the total depth of the fluid is:

$$h_c = (h - h_g) + y_c,$$

where $y_c = 1.5$ [m], but because $h$ is unknown, we keep $h_c$ expressed in terms of $h$ such that $h_c$=$h$-1.5 [m]. Since the gate is at $\theta = 90°$, and that atmospheric pressure is acting on both side, we can simply use the equation for the force:

$$F_R = P_c A = (\rho g h_c)(b h_g).$$

Determining the location for which $F_R$ exists:

$$y' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_c A}.$$

For a rectangular plate, this becomes:

$$y' = h_c + \frac{b h_g^3}{12 h_c (b h_g)} = h_c + \frac{h_g^2}{12 h_c}$$

We must apply a moment balance about the hinge, setting it equal to zero as to create an expression for the height of water that counteracts the applied 1 [MN] force. Summing the moment at the hinge:

$$\sum M_{\text{hinge}} = 0 \implies F(y_F) - F_R\left(y' - (h - h_g)\right)$$

where $F$ is the 1 [MN] force and $y_F$ is the location of the application of $F$, i.e. 2 [m] below the hinge. Recall $y'$ is measured from the fluid level, and the hinge is located $h - h_g$ from the fluid level, with the difference of there terms yielding the moment arm for which $F_R$ acts on. Substituting in expressions for $F_R$ and $y'$m and recalling $h - h_g$=$h_c$-$y_c$:

$$F(y_F) = (\rho g h_c)(b h_g)\left(\frac{h_g^2}{12 h_c} + y_c\right)$$

Solving for $hc$:

$$h_c = \frac{F(y_F)}{\rho g h_g b y_c} - \frac{h_g^2}{12 y_c}$$

Plugging in the numeric values:

$$h_c = \frac{(1 \cdot 10^6\ [\text{N}])(2\ [\text{m}])}{\left(998\ \left[\frac{\text{kg}}{\text{m}^2}\right]\right)\left(9.81\ \left[\frac{\text{m}}{\text{s}^2}\right]\right)(3\ [\text{m}])(1\ [\text{m}])(1.5\ [\text{m}])} - \frac{(3\ [\text{m}])^2}{(12)\,(1.5\ [\text{m}])} = 44.89\ [\text{m}]$$

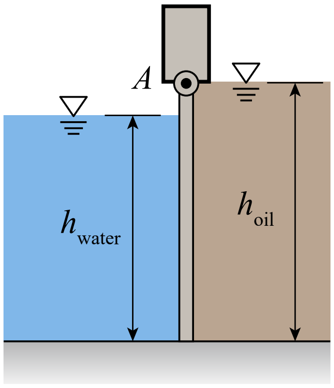
And so:

$$h = h_c + h_g - y_c = 44.89\ [\text{m}] + 3\ [\text{m}] - 1.5\ [\text{m}]$$

$$\boxed{h = 46.39\ [\text{m}]}$$

## Problem #2

Consider an open tank with a partition separating oil and water. A gate is located at the bottom of the partition, with a height and width of 2 [m], hinged as $A$. For a height of 2 [m] of oil, determine the height of water needed to keep the gate closed. Take the density of water to be 998 [kg/m$^3$] and that of oil to be 900 [kg/m$^3$].




Starting with the oil, the distance to the centroid of the gate with respect to the fluid level is:

$$h_{c,\,\text{oil}} = h + y_c = 1\ [\text{m}]$$

The resultant force is calculated as:

$$F_{R,\,\text{oil}} = P_c A = \rho_{\text{oil}} g h_{c,\,\text{oil}} A = (900\ [\text{kg/m}^3])(9.81\ [\text{m/s}^2])(1\ [\text{m}])(4\ [\text{m}^2]) = 35.32\ [\text{kN}]$$

The vertical distance $y'_{\text{oil}}$ for where $F_{R,\,\text{oil}}$ acts is found as:

$$y'_{\text{oil}} = h_{c,\,\text{oil}} + \frac{I_{\hat{x}\hat{x}}}{h_{c,\,\text{oil}}A} = 1\,[\text{m}] + \frac{(2\,[\text{m}])(2\,[\text{m}])^3}{12(1\,[\text{m}])(4\,[\text{m}^2])} = 1.\bar{3}\,[\text{m}]$$

We repeat the same process, however recognize that the water level is some distance below the hinge. Or in other words, the water level is (2 [m] - $h_{\text{water}}$) below hinge $A$. This will be important when we calculate the moment. Starting with the distance to the centroid of the gate with respect to the fluid:

$$h_{c,\,\text{water}} = \frac{h_{\text{water}}}{2}$$

The resultant force is calculated as:

$$F_{R,\,\text{water}} = P_cA = \rho_{\text{water}}gh_{c,\,\text{water}}A = (998\,[\text{kg/m}^3])(9.81\,[\text{m/s}^2])\left(\frac{h_{\text{water}}}{2}\right)(h_{\text{water}})(2\,[\text{m}]) = 9.79h^2_{\text{water}}\,[\text{kN}]$$

That is, $y'$ is at the centroid of the triangle representing the pressure/force distribution on the gate. The vertical distance $y'_{\text{water}}$ for where $F_{R,\,\text{water}}$ acts is found as:

$$y'_{\text{water}} = h_{c,\,\text{water}} + \frac{I_{\hat{x}\hat{x}}}{h_{c,\,\text{water}}A} = \frac{h_{\text{water}}}{2} + \frac{(2\,[\text{m}])(h_{\text{water}})^3}{12\left(\frac{h_{\text{water}}}{2}\right)(h_{\text{water}})(2\,[\text{m}])} = \frac{2h_{\text{water}}}{3}\,[\text{m}]$$

Or in terms of vertical distance from the hinge, denoted as $r$, taking into account the gate height:

$$r = \left(2 - \frac{h_{\text{water}}}{3}\right)[\text{m}]$$

Taking the moment about the hinge at point $A$, recalling the offset of the water level:

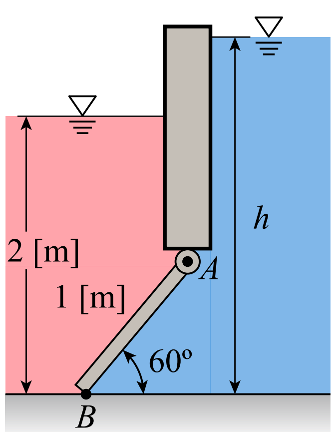
$$\sum M_A = 0 \implies F_{R,\,\text{water}}\left(2\,[\text{m}] - \frac{h_{\text{water}}}{3}\right) = F_{R,\,\text{oil}}y'_{\text{oil}}$$

$$(9.79h^2_{\text{water}}\,[\text{kN}])\left(2\,[\text{m}] - \frac{h_{\text{water}}}{3}\right) = (35.32\,[\text{kN}])(1.\bar{3}\,[\text{m}])$$

$$\implies h_{\text{water}} = 1.87\,[\text{m}]$$

## Problem #3

Consider a gate separating mercury and water. If the gate is 1 [m] wide, a length of 1 [m], the depth of mercury is 2 [m], and the SG of mercury is 13.6, determine the height of water that would generate a zero moment about point $A$



First, determine $y_c$ of the gate then $h_c$:

$$y_c = \frac{\sin(60^\circ)}{2}\,[\text{m}]$$

Then, from the fluid level of the mercury, the vertical distance to the centroid is found as:

$$h_c = (2 - y_c)\,[\text{m}] = 1.567\,[\text{m}]$$

Then, determine $F_R$:

$$F_R = P_c A = \rho g h_c A = (13,600\,[\text{kg/m}^3])(9.81\,[\text{m/s}^2]))(1.567\,[\text{m}])(1\,[\text{m}^2]) = 209.1\,[\text{kN}]$$

Determining the line of action $y'$ (note $x'$=W/2):

$$y' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_c A} = 1.567\,[\text{m}] + \frac{(1\,[\text{m}])(1\,[\text{m}])^3(\sin(60^\circ))}{12(1.567\,[\text{m}])(1\,[\text{m}^2])} = 1.613\,[\text{m}]$$

Next, to determine the force of the water, we leave it in terms of $h$

$$h_{c,\,\text{water}} = h - \frac{\sin(60^\circ)\,[\text{m}]}{2} = (h - 0.433)\,[\text{m}]$$

Determining $F_R$ of the water

$$F_{R,\,\text{water}} = \rho_{\text{water}} g h_{c,\,\text{water}} A = 9.79 h_{c,\,\text{water}}\,[\text{kN}]$$

Determining the line of action $y'$ (note $x'$=W/2) for the water:

$$y'_{\text{water}} = h_{c,\,\text{water}} + \frac{I_{\hat{x}\hat{x}}}{h_{c,\,\text{water}} A} = h_{c,\,\text{water}} + \frac{(1\,[\text{m}])(1\,[\text{m}])^3(\sin(60^\circ))}{12(h_{c,\,\text{water}})(1\,[\text{m}])^2} = h_{c,\,\text{water}} + \frac{0.072}{h_{c,\,\text{water}}}$$

Taking the moments about point $A$, recalling the forces act normal to the surface (thus the moment arm is along the length of the gate), we can solve for the height. $F_R$ acts 1.613 [m] below the surface of the mercury, or 0.387 [m] from the bottom. The plate has a length of sin(60°) [m]. Thus, $F_R$ acts (sin(60°)-0.387) [m] vertically below $A$. The moment arm (i.e. length along the gate) is (sin(60°)-0.387)/sin(60°), or 0.553 [m]. The same reasoning is applied to $F_{R,\,\text{water}}$ acting at $y'_{\text{water}}$:

$$\sum M_A = 0 \implies (209.1\,[\text{kN}])(0.553\,[\text{m}]) = (9.79 h_{c,\,\text{water}}\,[\text{kN}])\left(\frac{\sin(60^\circ) - (h - y'_{\text{water}})}{\sin(60^\circ)}\right)[\text{m}]$$

Not expressing units:

$$115.63 = 9.79 h_{c,\,\text{water}} - \frac{9.79 h_{c,\,\text{water}} h}{\sin(60^\circ)} + \frac{9.79 h_{c,\,\text{water}} y'_{\text{water}}}{\sin(60^\circ)}$$

Recalling $h_{c,\,\text{water}} = h - 0.433$ [m]:

$$115.63 = 9.79 h_{c,\,\text{water}} - \left(\frac{9.79}{\sin(60^\circ)}\right)(h_{c,\,\text{water}})(h_{c,\,\text{water}} + 0.433) + \ldots$$

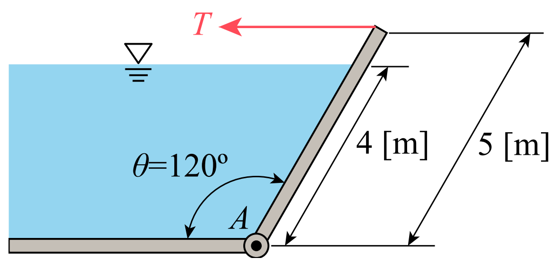
$$\ldots + \left(\frac{9.79}{\sin(60^\circ)}\right)(h_{c,\,\text{water}})\left(h_{c,\,\text{water}} + \frac{0.072}{h_{c,\,\text{water}}}\right)$$

$$\implies h_{c,\,\text{water}} = 23.46\,[\text{m}]$$

$$\therefore h = 23.89\,[\text{m}]$$

## Problem #4

A 5 [m] by 5 [m] rectangular gate is hinged at point $A$ and is supported by a cable, as shown in the figure below. The gate is holding back water, with a density of 998 [kg/m$^3$]. Determine the tension $T$ in the cable, neglecting the weight of the gate.




Determine $h_c$:

$$h_c = h + y_c = \frac{4\sin(60°)}{2}\,[\text{m}] = \sqrt{3}\,[\text{m}]$$

Calculate the resultant force:

$$F_R = P_c A = \rho g h_c A = (998\,[\text{kg/m}^3])(9.81\,[\text{m/s}^2])(\sqrt{3}\,[\text{m}])(20\,[\text{m}^2]) = 339.15\,[\text{kN}]$$

Calculate $y'$:

$$y' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_c A} = \sqrt{3}\,[\text{m}] + \frac{(5\,[\text{m}])(4\,[\text{m}])^3\sin(60°)}{12(\sqrt{3}\,[\text{m}])(20\,[\text{m}^2])} = 2.40\,[\text{m}]$$

Summing the moments about point $A$:

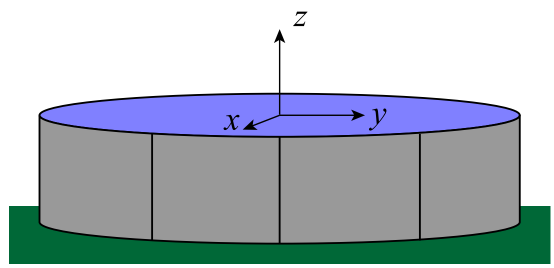
$$\sum M_A = 0 \implies (339.15\,[\text{kN}])(4 - 2.77\,[\text{m}]) = T\cos(30°)(5\,[\text{m}])$$

$$\implies T = 96.37\,[\text{kN}]$$

## Problem #5

Consider an above-ground pool with a diameter of 10 [m] and a depth of 1.5 [m]. The x- and y- directions can be taken as the lateral directions, whereas the positive y-direction is upward and perpendicular to the surface of the water. Using the formulation of forces acting on curved surfaces, determine:

a) The net horizontal force acting on the pool structure in the x-direction;
b) The net horizontal force acting on the pool structure in the y-direction;
c) The net vertical force acting on the pool structure in the z-direction.




a) It is simply seen via symmetry that the net forces acting in the x-direction are zero.

b) It is simply seen via symmetry that the net forces acting in the y-direction are zero.

c) It is simply seen that the next forces acting in the y-direction are that due to the weight of the water above the base of the pool:

$$F_z = mg = \rho \forall g = (998\,[\mathrm{kg/m^3}])(\pi(5\,[\mathrm{m}])^2))(1.5\,[\mathrm{m}])(9.81\,[\mathrm{m/s^2}]) = 1.15\,[\mathrm{MN}]$$

## Problem #6

Consider a sphere with a radius of 0.5 [m]. The sphere is submerged in water to a depth of 3 [m] (from the surface of the water to the center of the sphere). The x- and y- directions can be taken as the lateral directions, whereas the positive y-direction is upward and perpendicular to the surface of the water. Using the formulation of forces acting on curved surfaces, determine:

a) The net horizontal force acting on the sphere in the x-direction;
b) The net horizontal force acting on the sphere in the y-direction;
c) The net vertical force acting on the sphere in the z-direction.

a) It is simply seen via symmetry that the net forces acting in the x-direction are zero.

b) It is simply seen via symmetry that the net forces acting in the y-direction are zero.

c) For the vertical force, we project the sphere to a horizontal plane beneath. In doing such, we calculate the force acting upward due to the pressure generated on this plane, $F_{y+}$, and subtract the weight of the water above the plane, $F_{y-}$, (i.e. cylinder less sphere):

$$F_{y+} = P_c A = \rho g h_c A = (998\,[\mathrm{kg/m^3}])(9.81\,[\mathrm{m/s^2}])(3.5\,[\mathrm{m}])\pi(0.5\,[\mathrm{m}])^2 = 26.91\,[\mathrm{kN}]$$

The weight of fluid about the bottom horizontal plane, $F_{y-}$ is:

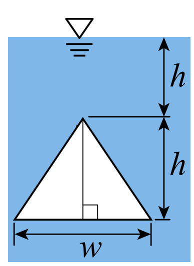
$$F_{y-} = (998\,[\mathrm{kg/m^3}])(9.81\,[\mathrm{m/s^2}])\left(\pi(0.5\,[\mathrm{m}])^2(3.5\,[\mathrm{m}]) - \frac{4\pi(0.5\,[\mathrm{m}])^3}{3}\right) = 21.79\,[\mathrm{kN}]$$

Thus, the net vertical force is the summation of forces $(F_{y+} + F_{y-})$, which yields 5.12 [kN]

## Problem #7

Consider a triangular body submerged in water with a base of $w$, a height of $h$, and a length into or out of the page of unity, where the top the triangle is $h$ below the surface of the water. Determine:

a) The x-component of the net resultant force acting on the inclined surfaces of the body;
b) The y-component of the net resultant force acting on the inclined surfaces of the body;
c) The y-component of the net resultant force acting on the bottom of the body;
d) The net force acting on the body in the y-direction via the summation of b) and c);
e) The buoyant force acting on the body;
f) The weight of the body;
g) The net force acting on the body in the y-direction via the summation of e) and f).




To be completed 9/23