Chapter 4 - Integral Form for a Control Volume

Lecture 12
Section 4.3

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

- Understand the formulation of the Conservation of Linear Momentum equation in an RTT framework
- Understand the momentum flux correction factor and how it applies to our uniform velocity assumption

Reynolds Transport Theorem

- Recall RTT

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{\delta}{\delta t}\int_{C.\forall.} b\rho d\forall + \int_{\text{C.S.}} b\rho \overrightarrow{V} \cdot d\overrightarrow{A}$$

- The LHS is the rate of change of the extensive property of system
- The first term on the RHS is the rate of change of the extensive property of the system within the C.∀.
- The second term on the RHS is the rate at which the extensive property of the system is exiting the C.∀. through the C.S.

Forces on C.∀.

- A fluid within a C.∀. experiences body and surface forces
- Surface forces create reactionary forces at points of contact between the fluid and a surface (i.e. pressure)
- Body forces are things like gravity, electromagnetic fields, etc.
- The total force acting on a C.∀. is expressed as

$$\sum \vec{F} = \sum \vec{F}_b + \sum \vec{F}_s$$

Forces on C.∀.

- The body force of interest will be the weight of the fluid within the C.∀.

$$\vec{F}_b = \int_{C.\forall.} \rho \vec{g} d\forall = \vec{W}_{C.\forall.} = m\vec{g}$$

- The totality of surfaces forces is expressed as

$$\vec{F}_s = \int_A \boldsymbol{\sigma}_{ij} \cdot \vec{n} dA$$

Forces on C.∀.

- The surface forces of interest are almost always due to the formation of hydrostatic pressure, and we can neglect any viscous forces, reducing our equation to

$$\vec{F}_s = \int_A -P d\vec{A}$$

- The minus sign indicates the pressure is acting on (toward) the surface

Momentum

- Recall from physics the definition of force in relation to momentum for a system

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\vec{P}}{dt}\Bigg)_s$$

- The momentum can alternatively be expressed as

$$\vec{P} = \int_m \vec{V}dm = \int_{C.\forall.} \rho\vec{V}d\forall$$

- Therefore, the summation of forces acting on the C.∀. can be expressed as

$$\sum \vec{F} = \frac{d}{dt}\int_{C.\forall.} \rho\vec{V}d\forall$$

Conservation of Linear Momentum

- Substituting momentum $\vec{P}$ in for our system variable B, and $\vec{V}$ for b,

$$\sum \vec{F} = \frac{d\vec{P}}{dt}\bigg)_s = \frac{\delta}{\delta t}\int_{C.\forall.} \rho \vec{V} d\forall + \int_{C.S.} \rho \vec{V}(\vec{V} \cdot \vec{n}) dA$$

- Recalling our force is the summation of surface and body forces

$$\Sigma \vec{F}_b + \Sigma \vec{F}_s = \frac{\delta}{\delta t}\int_{C.\forall.} \rho \vec{V} d\forall + \int_{C.S.} \rho \vec{V}(\vec{V} \cdot \vec{n}) dA$$

Conservation of Linear Momentum

- The sum of all external forces acting on the C.∀. is equal to the time rate of change of the linear momentum of the fluid within the C.∀. plus the net flow rate of the linear momentum out of the C.∀. through the C.S.

$$\boxed{\Sigma\vec{F}_b + \Sigma\vec{F}_s = \frac{\delta}{\delta t}\int_{C.\forall.} \rho\vec{V}d\forall + \int_{C.S.} \rho\vec{V}(\vec{V}\cdot\vec{n})dA}$$

Conservation of Linear Momentum

- For steady-state flow

$$\boxed{\Sigma\vec{F}_b + \Sigma\vec{F}_s = \int_{C.S.} \rho\vec{V}(\vec{V}\cdot\vec{n})dA}$$

- Recall $\vec{V}\cdot\vec{n}$ gives us the velocity component normal to the surface, V_n

$$\int_{C.S.} \rho\vec{V}(\vec{V}\cdot\vec{n})dA \equiv \int_{C.S.} \rho\vec{V}V_n dA$$

- Recall that the area integral of ρV_n gives us mass flow rate

$$\dot{m} = \int_A \rho V_n dA$$

Conservation of Linear Momentum

- Substituting in our expression for $\dot{m}$ and assuming our velocity across the C.S. is uniform ($\vec{V} = \vec{V}_{avg}$), i.e. a constant we can pull out of the integral)

$$\int_{C.S.} \rho \vec{V} V_n dA \equiv \dot{m}\vec{V}_{avg}$$

- Therefore, the COLM for steady-state flow can be expressed as

$$\boxed{\Sigma \vec{F} = \dot{m}\vec{V}_{avg}}$$

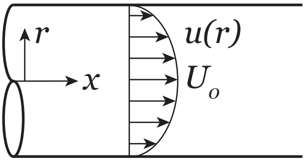
- However, this expression relies on the assumption that the flow across the C.S. is uniform

Example

https://www.youtube.com/watch?v=VX9gv3kS9CM

Momentum Flux Correction Factor

- Consider the velocity profile from our previous example of calculating average velocity



- We saw the average velocity was expressed as

$$V_{avg} = \frac{2U_o}{(1+m)(2+m)}$$

Momentum Flux Correction Factor

- We must introduce the dimensionless correction factor β to transform a non-uniform velocity profile to a compatible form that can be used in our algebraic formulation of our momentum flux across the C.S.
- β accounts for the variation of $\vec{V}$ across the C.S. by computing the exact flux across the C.S. and setting it equal to a flux based on the average velocity across the C.S.

Momentum Flux Correction Factor

- For constant density and $\vec{V}$ acting in the same direction as $\vec{V}_{avg}$

$$\beta = \frac{\int_A \rho V (\vec{V} \cdot \vec{n}) dA}{\dot{m} V_{avg}}$$

- We are tasked with creating a general expression for β

Momentum Flux Correction Factor

- Expressing our denominator in terms of density, velocity and area

$$\beta = \frac{\int_A \rho V(\vec{V}\cdot\vec{n})dA}{(\rho V_{avg}A)V_{avg}}$$

- Our normal velocity is just our velocity

$$\beta = \frac{\int_A \rho V V dA}{(\rho V_{avg}A)V_{avg}}$$

- Thus

$$\boxed{\beta = \frac{1}{A}\int_A \left(\frac{V}{V_{avg}}\right)^2 dA}$$

Example #1

- Consider a flow where the velocity profile is given as

$$u = U_o\left(1 - \frac{r}{R}\right)$$

where it will later be shown that it is equal to

$$u = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$$

for turbulent flow conditions

- Determine the value for β

Example #1

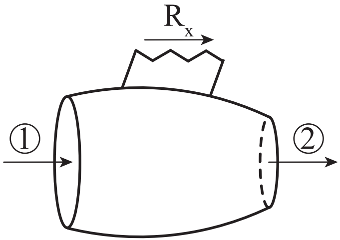
- Solution:

Example #1

- Solution:

Example #2

- The figure below shows a jet test stand. Air enters the engine at 20°C and 101 [kPa], with a cross-sectional flow area of 0.5 [m^2] and a velocity of 250 [m/s]. The fuel-to-air ratio is 1:30. The exhaust then exits the engine with a velocity of 900 [m/s] through a reduced cross-sectional area of 0.4 [m^2]. What is the horizontal reactionary for R_x required to hold the engine in place?



Example #2

- Solution:

Example #2

- Solution: