Chapter 3 - Fluid Statics

Lecture 4
Section 3.4

Introduction to Fluid Mechanics

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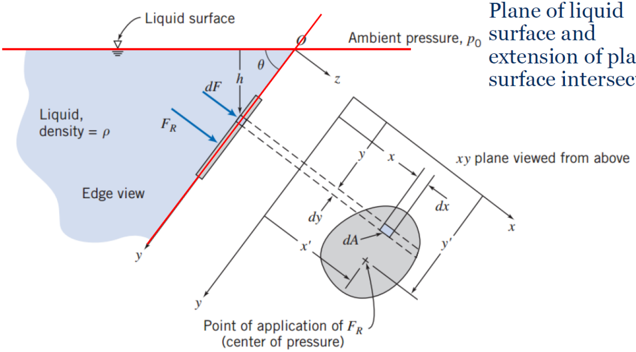
Student Learning Objectives

Students should be able to determine the:

- Magnitude of a force acting on a submerged planar surface;
- Direction of a force acting on a submerged planar surface;
- The line of action of a force acting on a submerged planar surface.

Submerged Planar Surfaces

- Imagine there is a submerged disc. We want to determine the magnitude of force acting on the surface, the direction in which the force is acting and the "line of action"



Plane of liquid surface and extension of plane surface intersect

Resultant Force Formulation

- The magnitude of the force is F_R with coordinates x' and y' - this is where on the x-y plane it is interacting with the submerged surface, i.e. center of dA
- The only force acting on our $d\forall$ is the *normal force* - same holds true for a surface $dA{=}dxdy$
- Recall $P{=}F/A$ - the force acting on our differential areas is $dF{=}PdA$
- We must add up all the forces acting on all the infinitesimally small areas:

$$F_R = \int_A PdA$$

Pressure Formulation

- P is expressed in terms of atmospheric pressure and depth:

$$P = P_o + \rho g h$$

- h is the vertical distance from the surface of the liquid to the center of dA
- Therefore:

$$F_R = \int_A (P_o + \rho g h) \, dA$$

- The value of h changes since we are on an inclined surface, and depends on the location of our y-axis

$$h = y \sin(\theta)$$

Resultant Force in terms of Pressure

- Therefore:

$$F_R = \int_A (P_o + \rho g y \sin(\theta))\, dA$$

- We know the integral of two added terms is the respective integral of each term. Bringing out the constants:

$$F_R = P_o \int_A dA + \rho g \sin(\theta) \int_A y\, dA$$

- The first term on the RHS is $P_o A$ and the second integral term is the centroid of area:

$$\int_A y\, dA = y_c A$$

Resultant Force Magnitude

- Therefore:

$$\boxed{F_R = (P_o + \rho g \sin(\theta) y_c) A}$$

- Notice the terms within () is the expression for absolute pressure at the centroid of A:

$$F_R = P_c A$$

- The **magnitude** of F_R acting on the surface of a completely submerged plate in a constant density fluid is equal to the product of the pressure at the centroid of the surface, P_c, and the area of the surface
- If P_o is acting on the other side of the plate, it can be ignored in the formulation

Line of Action

- We still have to determine the direction in which F_R is acting - the "**line of action**"
- Recall F_R is acting on (x', y') - determine y'
- The vertical location of the line of action is determined by equating the moment of the resultant force to the moment of the distributed pressure force about the x-axis

Line of Action - Sum of Moments

- Taking the sum of moments about the moments about the x-axis:

$$y' F_R = \int_A y P \, dA$$

- Expressing P=$P(y)$

$$P = P_o + \rho g h = P_o + \rho g y \sin(\theta)$$

Second Moment of Inertia

- Then

$$y' F_R = \int_A y(P_o + \rho g y \sin(\theta))\, dA$$
$$= P_o \int_A y dA + \rho g \sin(\theta) \int_A y^2\, dA$$

- The first term on the RHS is simply $y_c A$
- The second term on the RHS is the second moment of area about the x-axis, I_{xx}, which can be found in many engineering texts. We need it expressed in terms of our centroidal axis, $\hat{x}$.
- Using the parallel axis theorem

$$I_{xx} = I_{\hat{x}\hat{x}} + y_c^2 A$$

y-Location of F_R

- Then

$$y' F_R = P_o y_c A + \rho g \sin(\theta)(I_{\hat{x}\hat{x}} + y_c^2 A)$$

- Solving for y', the location F_R is acting

$$\boxed{y' = y_c + \frac{\rho g \sin(\theta) I_{\hat{x}\hat{x}}}{F_R}}$$

- If the ambient pressure is acting on the opposite side, we neglect P_o and are left with

$$y' = y_c + \frac{I_{\hat{x}\hat{x}}}{A y_c}$$

- Does $y' > y_c$ make physical sense?

x-Location of F_R Formulation

- Similarly, we have to find x'

$$x' F_R = \int_A x P \, dA$$

- The same procedure is followed, except that $h{=}xysin(\theta)$

$$\begin{aligned} x' F_R &= \int_A x(P_o + \rho g h) \, dA \\ &= \int_A (P_o x + \rho g x y \sin(\theta)) \, dA \\ &= P_o \int_A x \, dA + \rho g \sin(\theta) \int_A xy \, dA \end{aligned}$$

y-Location of F_R

- The first integral looks like what we saw before, $x_c A$, whereas the second is

$$\int_A xy\,dA = I_{xy} = I_{\hat{x}\hat{y}} + x_c y_c A$$

- Substituting in our known expressions and solving for x'

$$\boxed{x' = x_c + \frac{\rho g \sin(\theta) I_{\hat{x}\hat{y}}}{F_R}}$$

- Once again, if atmospheric is acting on the other side

$$x' = x_c + \frac{I_{\hat{x}\hat{y}}}{A y_c}$$

Solution Algorithm

- To solve for the forces on submerged planar surfaces, we will follow the algorithm below:

1. Determine the centroid y_c of the planar surface (often represented as h_c if the edge of surface is below the surface of the fluid surface)

$$h_c = y + y_c$$

where y is the distance the object is below the surface of the water. Note if y=0, h_c=y_c

Solution Algorithm Cont'd

2. Determine F_R

$$F_R = P_cA = (P_o + \rho g \sin(\theta)\cancelto{h_c}{y})A$$

If P_o is acting on the opposite side of the surface (i.e. a real sluice gate), P_o=0. If θ=90°, $\rho g\sin(\theta)h_cA$=$\rho g h_c A$=P_cA

3. Determine the line of action y'

$$y' = h_c + \frac{\rho g \sin(\theta) I_{\hat{x}\hat{x}}}{F_R}$$

If P_o is acting on the opposite side of the surface (i.e. a real sluice gate)

$$y' = h_c + \frac{I_{\hat{x}\hat{x}}}{h_cA}$$