Chapter 9 - Differential Analysis of Fluid Flow

Lecture 23
Sections 9.5 and 9.6

Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
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Student Learning Objectives

Students should be able to:

- Solve two-dimensional steady-state planar Couette flow using the Navier-Stokes equations.

Newtonian Fluid - N.S. Equations

- When assuming our fluid is incompressible, Newtonian and isotropic, the Navier-Stokes equations is reduced to:

$$\boxed{\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{V}\cdot\nabla\vec{V}\right)=-\nabla P+\mu\nabla^2\vec{V}-\rho\vec{g}}$$

- This system of equations (three non-linear partial differential equations) must be coupled with the continuity equation to solve for u, v, w and P. Note since the equation is second order, we need to provide two spatial boundary conditions. Also, since the equation is time-dependent, we need to specify an initial condition.

Newtonian Fluid - N.S. Equations

- Recall the following mathematical operators:
- The gradient operator is :

$$\nabla\vec{V} = \begin{bmatrix} \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z} \end{bmatrix}$$

- The Laplacian operator $(\nabla \cdot \nabla)$ is expressed as:

$$\nabla^2\vec{V} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix}$$

- The divergence operator is expressed as

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}$$

Newtonian Fluid - N.S. Equations

- The Navier-Stokes and continuity equations in x-, y- and z-directions are as follows:

x-direction:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

y-direction:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

z-direction:

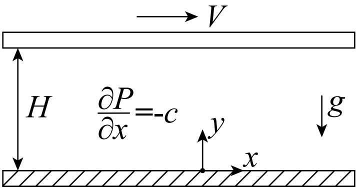
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

Continuity equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0$$

Example #1

- Consider a simple case of planar, shear-driven flow. Assume the flow is steady-state, incompressible, and laminar. Assume the fluid is Newtonian. Calculate the velocity and pressure fields and estimate the shear force acting on the bottom plate.



Plane Couette Flow

Schematic:

Plane Couette Flow

<u>Solution:</u>

Plane Couette Flow

Solution:

Plane Couette Flow

Solution:

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Plane Couette Flow

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