Introductory Material

Section 4.1 from Çengel and Cimbala, 3rd Edition

MEMS 0071 - Introduction to Fluid Mechanics

Mechanical Engineering and Materials Science
University of Pittsburgh

Student Learning Objectives

Students should be able to:

- Understand the Material Derivative;
- Distinguish between a Lagrangian and Eulerian description of fluid flow.

Fluid Kinematics

- The Lagrangian (up to <6:00) description is when we follow a fluid particle's position (x, y, z) and velocity $(\vec{V})$ as a function of time in reference to an initial, fixed position.
- This method is quite cumbersome - think of the number of position and velocity vectors needed to characterize a gas (1 mol = 6.022E23 atoms).
- The Eulerian (>6:00) description is where we define a C.$\forall$ and look at how **field variable** changed within and across the C.S.:
  1. Pressure field: $P = P(x, y, z, t)$
  2. Velocity field: $\vec{V} = \vec{V}(x, y, z, t)$
  3. Acceleration field: $\vec{a} = \vec{a}(x, y, z, t)$
- Note a velocity vector is defined as $\vec{V} = \langle u, v, w \rangle$.

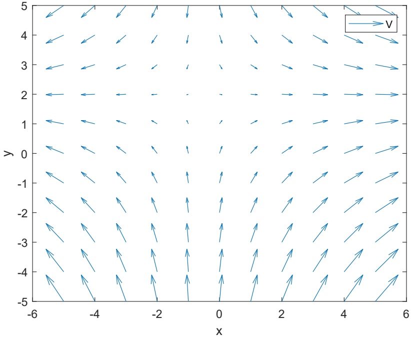
Example #1

- Consider a velocity field given as:

$$\vec{V}(u,v) = (0.5 + 0.8x)\hat{\imath} + (1.5 - 0.8y)\hat{\jmath}$$

- Using MATLAB, a) plot the velocity vectors to visualize the flow field, b) and determine if there is a point within the flow field where the velocity is equal to zero (i.e. the **stagnation point**).

```
% a)
clear all; close all; clc
[x,y]=meshgrid(-5:1:5,-5:1:5);
u = 0.5 + 0.8*x;
v = 1.5 - 0.8*y;
figure
quiver(x,y,u,v)
legend('V')
xlabel('x')
ylabel('y')
```



Example #1

- b) The stagnation point appears to be around $y = 2$ and $x = -1$ graphically, but we can solve for this analytically:

$$\vec{V}(u,v) = (0.5 + 0.8x)\hat{\imath} + (1.5 - 0.8y)\hat{\jmath} = 0$$

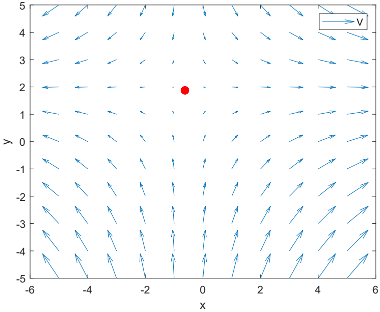
- This implies that both the x- and y-components of velocity are simultaneously zero:

$$u = 0.5 + 0.8x = 0$$

$$\implies x = \frac{-0.5}{0.8} = -0.625$$

$$v = 1.5 - 0.8y = 0$$

$$\implies y = \frac{1.5}{0.8} = 1.875$$



Example #2

- Calculate the velocity field given $\vec{V} = (0.5 + 0.8)\hat{\imath} + (1.5 + 2.5\sin(\omega t) - 0.8y)\hat{\jmath}$ and $\omega = 2\pi$ and plot in MATLAB for $0 < t < 2$.

Acceleration Fields

- The acceleration field is the time rate of change of the velocity field, $\vec{V} = \vec{V}(x, y, z, t)$:

$$\vec{a} = \frac{D\vec{V}}{Dt}$$

- Looking at a fluid particle moving within a flow field, we can say the velocity of the particle is the same as the local velocity of the flow field.

- Applying the chain rule to the velocity field derivative:

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz}{dt}$$

Material Derivative

- Noting the change of displacement with respect to time is velocity:

$$\frac{dx}{dt} = u; \quad \frac{dy}{dt} = v; \quad \frac{dz}{dt} = w$$

- Therefore, the acceleration field is expressed as:

$$\boxed{\vec{a} = \frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}$$

- D/Dt is the **material derivative** - it represents the sum of the time derivative and the control volume (convective) derivative.

Gradient Operator

- To simplify the expression of the material derivative, we introduce the gradient operator, which is defined as:

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

- Therefore:

$$\vec{a} = \frac{D\vec{V}}{Dt} \implies \boxed{\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}}$$

- The first term on the RHS of the second equals sign is the **local acceleration** and is equal to zero if the flow is steady.

- The second term on the RHS of the second equals sign is the **convective acceleration**, which is the acceleration due to convection or movement of the fluid particle to a different part of the flow field.

Convective Acceleration

- The term $(\vec{V} \cdot \vec{\nabla})\vec{V}$ can be thought of as:

$$(\vec{V} \cdot \vec{\nabla})\vec{V} = \vec{V}_j \frac{\partial \vec{V}_i}{\partial x_j}$$

- This is the Einstein summation convention which is short-hand for vector/tensor arithmetic.
- The subscripts i and j are indices in Matlab - they must be looped over from $i = 1:3$ and $j = 1:3$:

$$\vec{a} = \begin{cases} a_x = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} \\[2ex] a_y = \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} \\[2ex] a_z = \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} \end{cases}$$

Example #3

- Consider a velocity field given as:

$$\vec{V}(u,v) = (0.5+0.8x)\hat{\imath} + (1.5-0.8y)\hat{\jmath}$$

- Determine, at $x=2$ and $y=3$, and plot over a range of x and y, the acceleration field for the given $\vec{V}$.
- Since our velocity field is two-dimensional, we will consider the x- and y-components. Starting with x:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ &= \frac{\partial}{\partial t}(0.5+0.8x) + (0.5+0.8x)\frac{\partial}{\partial x}(0.5+0.8x) + \ldots \\ &\quad (1.5-0.8y)\frac{\partial}{\partial y}(0.5+0.8x) + (0)\frac{\partial}{\partial z}(0.5+0.8x) \\ &= 0 + (0.5+0.8x)(0.8) + (1.5-0.8y)(0) + 0 \\ &= 0.4 + 0.64x \end{aligned}$$

Example #3

- Proceeding with the y-component:

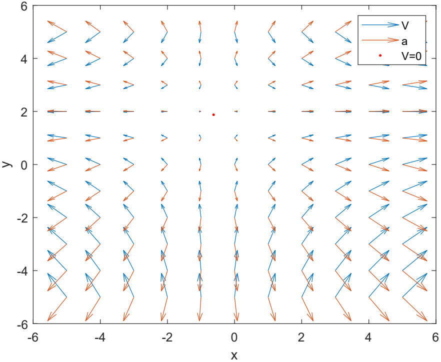
$$
\begin{aligned}
a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\
&= \frac{\partial}{\partial t}(1.5 - 0.8y) + (0.5 + 0.8x)\frac{\partial}{\partial x}(1.5 - 0.8y) + \ldots \\
&\quad (1.5 - 0.8y)\frac{\partial}{\partial y}(1.5 - 0.8y) + (0)\frac{\partial}{\partial z}(1.5 - 0.8y) \\
&= 0 + (0.5 + 0.8x)(0) + (1.5 - 0.8y)(-0.8) + 0 \\
&= -1.2 + 0.64y
\end{aligned}
$$

- a_z is obviously zero-valued.

Example #3

- We can plot this in MATLAB using the gradient command:

```
V = u + v;
[ax,ay] = gradient(V);
quiver(x,y,u,v)
hold on
quiver(x,y,u.*ax,v.*ay)
plot(-0.625,1.875,'.r')
xlim([-6 6])
ylim([-6 6])
legend('V','a','V=0')
xlabel('x')
ylabel('y')
```



Example #4

- Calculate the acceleration field given $\vec{V} = (0.5 + 0.8)\hat{\imath} + (1.5 + 2.5\sin(\omega t) - 0.8y)\hat{\jmath}$ and $\omega = 2\pi$ and plot in MATLAB for $0 < t < 2$.