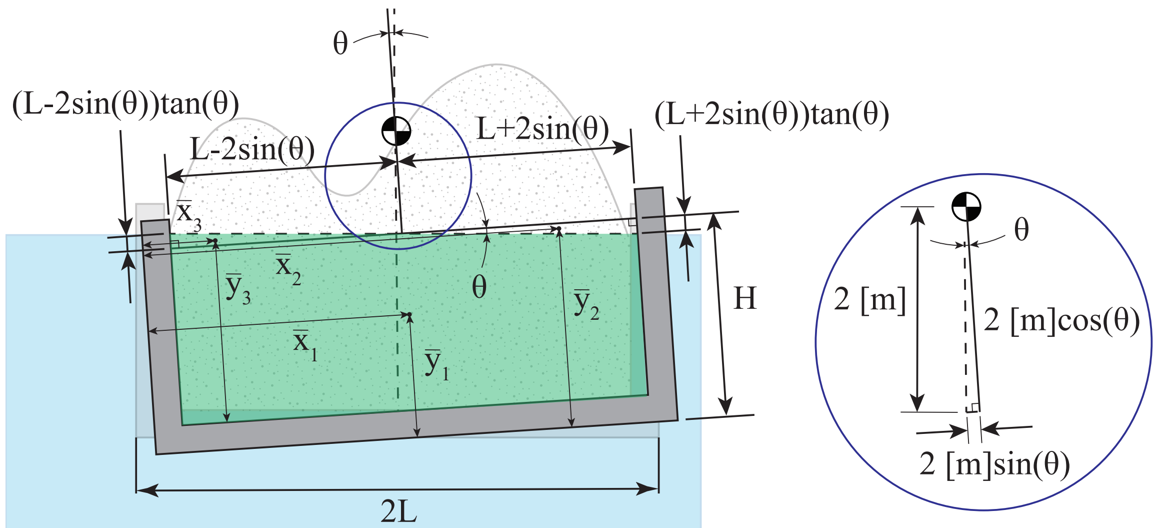
Homework #4

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: September 28$^{\text{th}}$, 2019
Due: October 4$^{\text{th}}$, 2019

Problem #1

A coal barge that is 15 [m] wide (2L), that has a draft of 4 [m] (H), is loaded such that the center of gravity is 2 [m] above the waterline and 7.5 [m] (L) from either side. Assuming a unit length of the barge, determine if this configuration stable. Additionally, determine at what angle of θ that the barge becomes unstable.



In it's original configuration, the center of displacement, found with respect to the bottom left corner of the barge, is **B**=(7.5 [m], 2 [m]). Since the center of gravity **G**=(7.5 [m], 6 [m]) is directly above the center of displacement, we cannot determine the metacentric height. Let us perturb the barge by rotating it some angle θ about **G**. The volume (assuming depth of unity) of the fluid displaced (light green) is calculated as:

$$\begin{aligned}\forall_{\text{disp}} &= \forall_{\text{originally submerged}} - \forall_{\text{above}} + \forall_{\text{below}} \\ &= 2HL - \frac{1}{2}(L+2\sin(\theta))((L+2\sin(\theta))\tan(\theta)) + \frac{1}{2}(L-2\sin(\theta))((L-2\sin(\theta))\tan(\theta)) \\ &= 2L(H - 2\sin(\theta)\tan(\theta))\end{aligned}$$

Next we must determine the centroid, i.e. B′, of the displaced volume. We will do such by taking the moment of the original volume ($\bar{x}_1 A_1$, $\bar{y}_1 A_1$) less that of the volume above the surface ($\bar{x}_2 A_2$, $\bar{y}_2 A_2$), plus that what is now submerged ($\bar{x}_3 A_3$, $\bar{y}_3 A_3$), per the differences of volume:

$$\bar{x} = \frac{\sum_{i=1}^{2} \bar{x}_i A_i}{\sum_{i=1}^{2} A_i} = \left\{ L(2HL) - \left(2L - \frac{(L+2\sin(\theta))}{3}\right)\left(\frac{(L+2\sin(\theta))(L+2\sin(\theta))\tan(\theta))}{2}\right) + ... \right.$$

$$\left. ... + \left(L - \frac{2\sin(\theta)}{3}\right)\left(\frac{(L-2\sin(\theta))((L-2\sin(\theta))\tan(\theta))}{2}\right)\right\} \Big/ \left\{2L(H-2\sin(\theta)\tan(\theta))\right\}$$

$$\bar{y} = \frac{\sum_{i=1}^{2} \bar{y}_i A_i}{\sum_{i=1}^{2} A_i} = \left\{ \left(\frac{H}{2}\right)(2HL) - \left(H - \frac{(L+2\sin(\theta))\tan(\theta)}{3}\right)\left(\frac{(L+2\sin(\theta))(L+2\sin(\theta))\tan(\theta))}{2}\right) + \ldots\right.$$

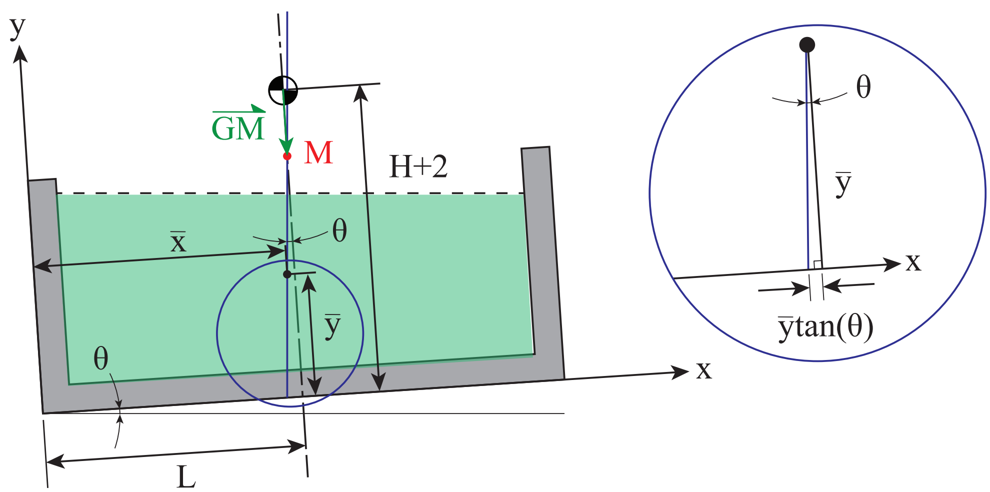
$$\left.\ldots + \left(H + \frac{(L-2\sin(\theta))\tan(\theta)}{3}\right)\left(\frac{(L-2\sin(\theta))((L-2\sin(\theta))\tan(\theta))}{2}\right)\right\} \Big/ \left\{2L(H - 2\sin(\theta)\tan(\theta))\right\}$$

Applying the small angle approximation ($\sin(\theta)\approx \theta$ up to 14° and $\tan(\theta)\approx \theta$ up to 10°):

$$\bar{x} = \frac{L(2HL) - \left(2L - \frac{(L+2\theta)}{3}\right)\left(\frac{\theta(L+2\theta)^2}{2}\right) + \left(L - \frac{2\theta}{3}\right)\left(\frac{\theta(L-2\theta)^2}{2}\right)}{2L(H-2\theta^2)}$$

$$\bar{y} = \frac{\left(\frac{H}{2}\right)(2HL) - \left(H - \frac{\theta(L+2\theta)}{3}\right)\left(\frac{\theta(L+2\theta)^2}{2}\right) + \left(H + \frac{L-2\theta^2}{3}\right)\left(\frac{\theta(L-2\theta)^2}{2}\right)}{2L(H-2\theta^2)}$$

To determine M, the point of intersection of the vertical line through the center of gravity and the line through B′, we need to construct an expression for said lines.



Working in a rotated Cartesian coordinate system, that is rotated counter-clockwise by some angle θ, the line representing the line of action of the center of gravity **G** is simply x=L. The line representing the line of action for the center of buoyancy is expressed as y=mx+b. The slope m is expressed as:

$$m = \frac{\bar{y}}{\bar{y}\tan(\theta)} = \frac{1}{\tan(\theta)} = \frac{1}{\theta}$$

To solve for the x-intercept, we do the following

$$\bar{y}(x) = \mathrm{m}\bar{x} + \mathrm{b} \implies b = \left(\bar{y} - \frac{\bar{x}}{m}\right) = \left(\bar{y} - \frac{\bar{x}}{\theta}\right)$$

Thus, the equation for the line of action of the buoyant force is:

$$y = \frac{x}{\theta} + \left(\bar{y} - \frac{\bar{x}}{\theta}\right)$$

Now that we have to equations for the two lines, to find their point of intersection, we set them equal. That is we substitute x=L into the expression for the line of action of the buoyant force:

$$y = \frac{L}{\theta} + \left(\bar{y} - \frac{\bar{x}}{\theta}\right) = \bar{y} + \frac{L - \bar{x}}{\theta}$$

Thus, the point M can be described as:

$$M = \left(L,\, \bar{y} + \frac{L - \bar{x}}{\theta}\right)$$

The metacentric height represented by vector **GM** is **M-G** such that:

$$\mathbf{GM} = \left(L,\, \bar{y} + \frac{L - \bar{x}}{\theta}\right) - (L,\, H + 2) = \left\langle 0,\, \left(\bar{y} + \frac{L - \bar{x}}{\theta}\right) - (H + 2)\right\rangle$$

We note that having θ in the denominator yields an undefined value when $\theta = 0$. For small angles, $0^\circ \leq \theta < 1^\circ$, the configuration is unstable. Thus, without any angle of tilt, it is unstable, and will be unstable for any angle of tilt.

Problem #2

Fluid is flowing through a control surface with a velocity $\vec{V}=\langle 8,\, 1,\, -2\rangle$ [m/s]. The control surface is described via the cross product of the following two vectors: $\vec{a}=\langle 0.5,\, 10,\, -1\rangle$ [m] and $\vec{b}=\langle 0.75,\, 2,\, 6\rangle$ [m]. Determine the normal component of velocity with respect to the control surface in terms of its Cartesian components and magnitude.

We must first determine the unit normal of the control surface by taking the cross-product of $\vec{a}$ and $\vec{b}$ and then determining the unit vector thereof:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0.5 & 10 & -1 \\ 0.75 & 2 & 6 \end{vmatrix} = 62\hat{\imath} - 3.75\hat{\jmath} - 6.5\hat{k}$$

$$\vec{n} = \frac{\langle 62,\, -3.75,\, -6.5\rangle}{\sqrt{62^2 + 3.75^2 + 6.5^2}} = \langle 0.9928,\, -0.06,\, -0.1041\rangle$$

The magnitude of the normal velocity is the velocity vector dotted with the unit normal vector, while the Cartesian components of the normal velocity are the magnitude of the normal velocity times the unit normal vector:

$$V_n = \vec{V} \cdot \vec{n} = (8\hat{\imath} + 1\hat{\jmath} - 2\hat{k}) \cdot (0.9928\hat{\imath} - 0.06\hat{\jmath} - 0.1041\hat{k}) = 8.0906\,[\text{m/s}]$$

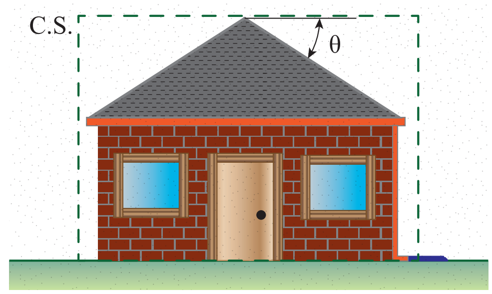
$$\vec{V_n} = (8.0906\,[\text{m/s}])\langle 0.9928,\, -0.06,\, -0.1041\rangle = \langle 8.0323,\, -0.4854,\, -0.8422\rangle\,[\text{m/s}]$$

As a check V_n is indeed less than $|\vec{V}|$ which is 8.3066 [m/s].

Problem #3

A home, with a footprint of 10 [m] by 10 [m], and roof with an angle θ=35°, is experiencing heavy rainfall. If the rain is falling at a rate of 50 [mm/hr], and the single downspout has a cross-sectional flow area of 3,750 [mm^2], determine:

a) The velocity of the water exiting the downspout;
b) The volumetric flow rate through the downspout.



The basal area of the home is 100 [m^2]. The surface area of the roof does not matter, for if we define a control surface that is based upon the projection of the roof onto a horizontal plane above, the area for which the rain enters if 100 [m^2]. Thus, the quantity of rain entering the control volume through the top control surface is:

$$\dot{\forall} = AV = (100\,[\mathrm{m}^2])(0.05\,[\mathrm{m/hr}]) = 5\,[\mathrm{m}^3/\mathrm{hr}]$$

Since water is incompressible, the volumetric flow rate entering the C.S. must be the same as what is exiting the C.S., that is, we already solved for part b). To determine the velocity, we recognize equivalency of volumetric flow rates:

$$\dot{\forall}_{\text{in}} = \dot{\forall}_{\text{out}} \implies 5\,[\mathrm{m}^3/\mathrm{hr}] = (0.003\,75\,[\mathrm{m}^2])V_{\text{out}}$$

$$V_{\text{out}} = \frac{5\,[\mathrm{m}^3/\mathrm{hr}]}{0.003\,75\,[\mathrm{m}^2]} = 1,333.\bar{3}\,[\mathrm{m/hr}] = 0.37\,[\mathrm{m/s}]$$

Problem #4

A circular balloon is being filled through the inlet, which has a cross-sectional area denoted as A_1. The air enters the balloon with a velocity V_1 and a density ρ_1. If the balloon radius is denoted as R, and the density within is taken as an average value ρ_{avg}, construct an expression for the time rate of change of mass within the balloon.

Defining our control surface around the balloon, with the inlet velocity normal to our surface, we can express the Conservation of Mass as follows:

$$\left.\frac{dm}{dt}\right|_{sys} = \frac{\partial}{\partial t}\int_{C.\forall.} \rho\, d\forall + \int_{C.S.} \rho\vec{V}\cdot\vec{n}\, dA$$

There is no mass outflux, only influx, and assuming the air enters inline with the unit normal of the C.S., the control surface integral reduces to an algebraic equation. Taking the density within the control volume as a constant (ρ_{avg}), and recalling volume is $(4/3)\pi R^3$, the control volume integral reduces to:

$$\left.\frac{dm}{dt}\right|_{sys} = \frac{\partial}{\partial t}\left(\frac{4\rho_{\text{avg}}\pi R^3}{3}\right) - \rho V_1 A_1$$

Since the mass of our system must be conserved, we can express the time rate of change of the mass in the control volume to that entering the volume such that:

$$\frac{\partial}{\partial t}\left(\rho_{\text{avg}} R^3\right) = \frac{3\rho V_1 A_1}{4\pi}$$

It is noted that the radius changes as a function of time (the balloon expands), and the average density may change, thus we keep said terms within the derivative.

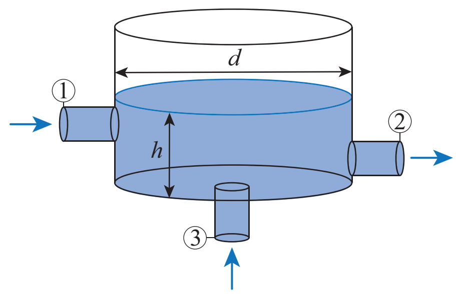
Problem #5

Water enters the tank with diameter d from (1) and (3) and exits at (2). The cross-sectional area of pipe (1) is 50 [mm^2], that of pipe (2) is 70 [mm^2]. The volumetric flow rate into the system through pipe (3) is 0.01 [m^3/s]. Determine the following:

a) An anlytical expression for the change of water height, dh/dt in terms of the volumetric flow rates $\dot{\forall}_1$, $\dot{\forall}_2$ and $\dot{\forall}_3$;

b) Once the system has reached steady-state, determine the exit velocity of the fluid at (2) given V_1=3 [m/s] and $\dot{\forall}_3$=0.01 [m^3/s];

c) The time it takes to reach steady-state.



The conservation of mass, in non-steady form is:

$$\frac{dm}{dt} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$$

The mass of the system is the density times the cross-sectional area times height:

$$\rho A \frac{dh}{dt} = \dot{m}_1 + \dot{m}_3 - \dot{m}_2$$

In terms of volumetric flow rates:

$$\frac{dh}{dt} = \frac{\rho(\dot{\forall}_1 + \dot{\forall}_3 - \dot{\forall}_2)}{\rho A}$$

$$\implies \frac{dh}{dt} = \frac{\dot{\forall}_1 + \dot{\forall}_3 - \dot{\forall}_2}{\left(\frac{\pi d^2}{4}\right)}$$

Once the system has reached steady-state, we know the time-rate-of-change of mass is zero

$$\frac{dm}{dt} = 0 = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$$

$$\implies \dot{m}_2 = \rho A_2 V_2 = \rho(A_1 V_1 + A_3 V_3)$$

$$\implies V_2 = \frac{A_1 V_1 + A_3 V_3}{A_2} = \frac{(5 \cdot 10^{-5}\,[\text{m}^2])(3\,[\text{m/s}]) + (0.01\,[\text{m}^3/\text{s}])}{(7 \cdot 10^{-5}\,[\text{m}^2])}$$

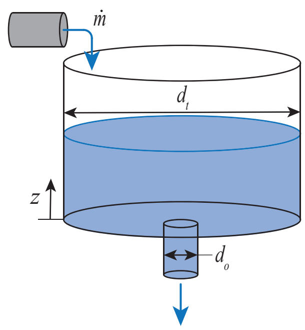
$$\therefore V_2 = 145\,[\text{m/s}]$$

For the time to reach steady-state, see Problem #6. The same methodology is employed.

Problem #6

Water is entering a tank with diameter d_t with a constant mass flow rate of $\dot{m}$. The tank has an opening at the bottom with diameter d_o. Assume the exit velocity is $\sqrt{2gh}$. If the tank is empty, determine:

a) The maximum height the water can reach in the tank;
b) A relation for the water height, z, as a function of time;
c) The time it takes for the tank to reach steady-state.



To determine the maximum height the water can reach, we will look at the steady-state formulation of the continuity equation (i.e. when the maximum height is reached):

$$\frac{dm}{dt} = 0 = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}$$

Expressing the mass flows in terms of velocities, with the outlet having that expressed by the Torricelli equation:

$$\rho A_{\text{in}} V_{\text{in}} = \rho A_{\text{out}} V_{\text{out}} \implies \left(\frac{\pi d_t^2}{4}\right) V_{\text{in}} = \left(\frac{\pi d_o^2}{4}\right)\sqrt{2gh}$$

Solving for h:

$$V_{\text{in}}\left(\frac{d_t}{d_o}\right)^2 = \sqrt{2gh} \implies h = \left(\frac{d_t}{d_o}\right)^4\left(\frac{V_{\text{in}}^2}{2g}\right)$$

To determine the height of the water, z, as a function of time, we have to evaluate the non-steady continuity equation:

$$\frac{dm}{dt} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}$$

Grouping like terms:

$$dm = (\dot{m}_{\text{in}} - \dot{m}_{\text{out}})dt$$

Recalling a differential mass is equal to density times constant cross-sectional area time differential height, $dm = \rho A dz$, and that the outlet mass flow is the velocity, as given by Torricelli's equation, times the density and cross-sectional flow area, we have:

$$\rho\left(\frac{\pi d_t^2}{4}\right)dz = \left(\dot{m}_{\text{in}} - \frac{\rho \pi d_o^2}{4}\sqrt{2gz}\right)dt$$

Grouping all the constants on the LHS as a variable called a, and all constants on the RHS, excluding the z term, as a variable called b, we have:

$$a dz = (\dot{m}_{\text{in}} - b\sqrt{z})dt$$

Getting like terms on each side:

$$dt = \frac{a}{\dot{m}_{\text{in}} - b\sqrt{z}}dz$$

Integrating between bounds of 0 and t, for which the water is going from a height of 0 to z:

$$\implies \int_0^t dt = \int_0^z \frac{a}{\dot{m}_{\text{in}} - b\sqrt{z}} dz$$

Using an integration utility, we could arrive at;

$$t = \frac{-2a(\dot{m}_{\text{in}}\log(\dot{m}_{\text{in}} - b\sqrt{z}) + b\sqrt{z})}{b^2} + c$$

where c is a constant. To determine the height as a function of time, we would have to vary values of z and plot the t response.

Problem #7

Calculate the Reynolds number for the following scenarios and determine if the flow is laminar, transitional or turbulent:

a) Air (ρ=1.225 [kg/m^3], μ=16.82 [μPa-s]) flowing through a rectangular duct of 25 [cm] by 40 [cm] at a velocity of 1.5 [m/s]

$$Re = \frac{\rho V D}{\mu} = \frac{(1.225\,[\text{kg/m}^3])(1.5\,[\text{m/s}])(0.308\,[\text{m}])}{16.82 \cdot 10^{-6}\,[\text{Pa-s}]} = 33,650, \text{ turbulent}$$

b) Water (ρ=998 [kg/m^3], μ= 8.90$\cdot$10^{-4} [Pa-s]) flowing through a cylindrical spillway tunnel with a diameter of 15.24 [m] at 53.5 [m/s]

$$Re = \frac{\rho V D}{\mu} = \frac{(998\,[\text{kg/m}^3])(53.5\,[\text{m/s}])(15.24\,[\text{m}])}{8.9 \cdot 10^{-4}\,[\text{Pa-s}]} = 9.14 \cdot 10^8, \text{ turbulent}$$

c) A Boeing 757 flying at 10,000 [m] through air (ρ=0.4135 [kg/m^3], μ=1.458$\cdot$10^{-5} [Pa-s]) at a speed of 858 [km/hr] where the chord length of the wing is taken as 8.20 [m] at the fuselage and 1.73 [m] at the wingtip

$$Re = \frac{\rho U_\infty x}{\mu} = \frac{(0.4135\,[\text{kg/m}^3])(238.3\,[\text{m/s}])(8.2\,[\text{m}])}{1.458 \cdot 10^{-5}\,[\text{Pa-s}]} = 5.54 \cdot 10^7, \text{ turbulent}$$

$$Re = \frac{\rho U_\infty x}{\mu} = \frac{(0.4135\,[\text{kg/m}^3])(238.3\,[\text{m/s}])(1.73\,[\text{m}])}{1.458 \cdot 10^{-5}\,[\text{Pa-s}]} = 1.17 \cdot 10^8, \text{ turbulent}$$