Homework #10

MEMS 0071 - Introduction to Fluid Mechanics

Assigned: November 17$^{\text{th}}$, 2019
Due: November 22$^{\text{nd}}$, 2019

Problem #1

The Navier-Stokes equation in cylindrical coordinates is expressed as the following:
<u>r-direction</u>:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

Alternatively, it can be expressed as:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

a) Rigorously show the viscous dissipation terms in the r-direction are equivalent.
<u>θ-direction</u>:

$$\rho\left(\frac{\delta v_\theta}{\delta t}+v_r\frac{\delta v_\theta}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_\theta}{\delta \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\delta v_\theta}{\delta z}\right)=-\frac{1}{r}\frac{\delta P}{\delta \theta}+\mu\left(\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta(rv_\theta)}{\delta r}\right)+\frac{1}{r^2}\frac{\delta^2 v_\theta}{\delta \theta^2}+\frac{2}{r^2}\frac{\delta v_r}{\delta \theta}+\frac{\delta^2 v_\theta}{\delta z^2}\right)+\rho g_\theta$$

Alternatively, it can be expressed as:

$$\rho\left(\frac{\delta v_\theta}{\delta t}+v_r\frac{\delta v_\theta}{\delta r}+\frac{v_\theta}{r}\frac{\delta v_\theta}{\delta \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\delta v_\theta}{\delta z}\right)=-\frac{1}{r}\frac{\delta P}{\delta \theta}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

b) Rigorously show the viscous dissipation terms in the θ-direction are equivalent.

Problem #2

Consider two concentric, infinitely long cylinders. The cylinders are oriented such that the center-line is along the z-axis, and the radii exist in the r-direction. The inner cylinder has a radius of r_a and the outer cylinder has a radius r_b. The inner cylinder rotates with an angular velocity of ω whereas the outer cylinder is stationary. There is no pressure gradient applied nor gravity. The fluid contained between the cylinders is assumed to be Netwonian, incompressible, isotropic and isothermal. The flow of the fluid is assumed steady and laminar. Construct an expression for the θ-component of the velocity, assuming no-slip boundary conditions on the cylinders. Also determine the force required per unit area to cause rotation.

Problem #3

Consider two concentric, infinitely long cylinders. The cylinders are oriented such that the center-line is along the z-axis, and the radii exist in the r-direction. The inner cylinder has a radius of r_a and the outer cylinder has a radius r_b. The inner cylinder moves in the positive z-direction with a velocity W while the outer cylinder is held stationary. The fluid contained between the cylinders is assumed to be Netwonian, incompressible, isotropic and isothermal. The flow of the fluid is assumed steady and laminar. Construct an expression for the z-component of the velocity, assuming no-slip boundary conditions on the cylinders, and determine the force required per unit area to cause translation. The shear stress tensor is given as:

$$\tau = \begin{bmatrix} 2\mu\dfrac{\partial u_r}{\partial r} & \mu\left(r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)+\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) & \mu\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) \\ \mu\left(r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)+\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) & 2\mu\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & 2\mu\dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

Problem #4

Consider two concentric, infinitely long cylinders. The cylinders are oriented such that the center-line is along the z-axis, and the radii exist in the r-direction. The inner cylinder has a radius of r_a and the outer cylinder has a radius r_b. A pressure gradient exists in the positive z-direction such that $\partial P/\partial z = -c$. The fluid contained between the cylinders is assumed to be Netwonian, incompressible, isotropic and isothermal. The flow of the fluid is assumed steady and laminar. Construct an expression for the z-component of the velocity, assuming no-slip boundary conditions on the stationary cylinders. Also determine the force required per unit area to cause the aforementioned translation.