MEMS 0071
Fall 2019
Final
12/12/19

Name (Print): ______________________________

This exam contains 11 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes. A calculator is permitted on this exam.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If there is not enough information to answer the question, state what is needed and how you would approach solving the problem.
- **BONUS:**
- 6 pts: OMET response rate was 96.36%.
- 5 pts: December 12$^{\text{th}}$, 1901 marks the first transatlantic transmission via what device? Radio

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	25	
6	25	
7	25	
8	25	
Total:	140	

1. (10 points) Given the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0,$$

if the flow is incompressible and the velocity field is given as

$$\vec{V} = (a_1 x + b_1 y + c_1 z)\hat{\imath} + (a_2 x + b_2 y + c_2 z)\hat{\jmath} + (a_3 x + b_3 y + c_3 z)\hat{k},$$

where a_1, b_1, etc. are constant, what conditions are required for the velocity field to satisfy continuity?

No velocity component is a function of time, and the flow is incompressible. Thus, the divergence of the velocity field becomes:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = a_1 + b_2 + c_3$$

Thus, for the flow to satisfy continuity:

$$a_1 + b_2 + c_3 = 0$$

2. (10 points) Given a vector field $\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$, show the gradient of the divergence, expressed as

$$\nabla(\nabla \cdot \vec{V}) - \nabla \times (\nabla \times \vec{V})$$

is equal to the Laplacian of $\vec{V}$, which is expressed as $\nabla^2 \vec{V}$

The divergence of $\vec{V}$ is given as:

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The gradient of the divergence is then:

$$\nabla(\nabla \cdot \vec{V}) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\hat{\imath} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\hat{\jmath} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\hat{k}$$

Expanding out the derivatives:

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right)\hat{\imath} + \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z}\right)\hat{\jmath} + \left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial z^2}\right)\hat{k}$$

The curl of $\vec{V}$ is:

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

Taking the curl once more:

$$\nabla \times (\nabla \times \vec{V}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) & -\left(\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\right) & \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(-\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\right)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\right)\hat{\jmath}$$
$$+\left(\frac{\partial}{\partial x}\left(-\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\right)\hat{k}$$

Expanding out the derivatives:

$$= \left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 u}{\partial z^2}\right)\hat{\imath} - \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 w}{\partial z \partial y} + \frac{\partial^2 v}{\partial z^2}\right)\hat{\jmath} + \left(-\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z}\right)\hat{k}$$

Subtracting the curl of the curl of $\vec{V}$ from the gradient of the divergence of $\vec{V}$:

$$\begin{array}{ccc} \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial x \partial y} + \dfrac{\partial^2 w}{\partial x \partial z}\right)\hat{\imath} & \left(\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 w}{\partial y \partial z}\right)\hat{\jmath} & \left(\dfrac{\partial^2 u}{\partial z \partial x} + \dfrac{\partial^2 v}{\partial z \partial y} + \dfrac{\partial^2 w}{\partial z^2}\right)\hat{k} \\ -\left(\dfrac{\partial^2 v}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 w}{\partial x \partial z} - \dfrac{\partial^2 u}{\partial z^2}\right)\hat{\imath} & +\left(\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial^2 w}{\partial z \partial y} + \dfrac{\partial^2 v}{\partial z^2}\right)\hat{\jmath} & -\left(-\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial z} - \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 v}{\partial y \partial z}\right) \\ \hline \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)\hat{\imath} & \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)\hat{\jmath} & \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)\hat{k} \end{array}$$

3. (10 points) Determine the result of taking the divergence of the curl of angular velocity vector:

$$\nabla \cdot (\nabla \times \vec{\omega})$$

The divergence of the curl is:

$$\frac{\partial}{\partial x}\left(\frac{\partial \vec{\omega}}{\partial y} - \frac{\partial \vec{\omega}}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \vec{\omega}}{\partial x} - \frac{\partial \vec{\omega}}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \vec{\omega}}{\partial x} - \frac{\partial \vec{\omega}}{\partial y}\right) = 0$$

4. (10 points) Given the x-component of a velocity field as

$$u = ax^2 - bxy$$

where a and b are constant. Let $v = 0$ for all x when $y = 0$, that is, $v = 0$ along the x-axis. Generate an expression for the stream function.

Staring with the x-component of velocity:

$$u = \frac{\partial \psi}{\partial y} \implies \partial \psi = u\,\partial y \implies \psi = ayx^2 - \frac{bxy^2}{2} + g(x)$$

The y-component of velocity:

$$v = -\frac{\partial \psi}{\partial x} = -2axy + \frac{by^2}{2} - g'(x) = 0 \implies g'(x) = -2axy + \frac{by^2}{2}$$

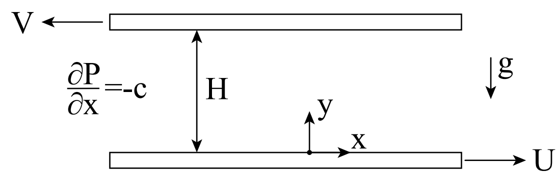
Integrate $g'(x)$ with respect to x:

$$g(x) = -ayx^2 + \frac{bxy^2}{2}$$

Substituting $g(x)$ back into the original expression:

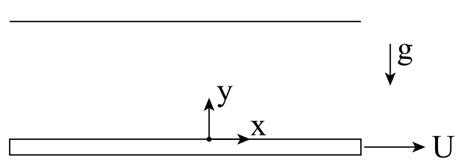
$$\psi = 0$$

5. (25 points) Consider a situation where a fluid exists between two infinite, parallel plates are moving in opposite directions. The bottom plate is moving in the positive x-direction with a velocity of U, and the top plate is moving in the negative x-direction with a velocity V. The two plates are separated by a distance H. A pressure gradient is applied in the positive x-direction. Gravity is acting in the negative y-direction. Construct an expression for the velocity profile in the x-direction. Assume the flow is compressible, turbulent and non-Newtonian.



Since the fluid is compressible, turbulent and non-Newtonian, we cannot use the formulation of Navier-Stokes as derived in class. Thus, this problem is not solvable. Stating such yields full credit.

6. (25 points) Consider a situation where a fluid exists above an infinitely long and wide plate. The fluid is initially at rest. At time t=0, the bottom plate moves with a velocity U in the positive x-direction. Gravity is acting in the negative y-direction. Construct an expression for the velocity profile of the fluid, based upon the notes below. Assume the flow is incompressible, steady-state, laminar and Newtonian. Notes:



- Once the diffusion equation is obtained, assume the solution is separable, i.e. u(y,t) = Y(y)T(t). Substitute the expression for u(y,t) into your PDE and differentiate accordingly.

- Once differentiated, group like terms (i.e. Y's and T's), and recognize that since one side of the equation is purely dependent on Y, while the other is purely dependent on T, that the equation must be constant, i.e. both sides are equal to -λ^2
- Construct two ODEs, one as a function of Y, and the other as a function of T, in terms of the constant -λ^2
- It is seen the solution for Y is that of simple harmonic motion. Assume the solution for this ODE has the form Y(y)=cos(λy) + sin(λy).

Starting with the continuity equation for an incompressible fluid:

$$\frac{\partial u}{\partial x} + \overbrace{\cancel{\frac{\partial v}{\partial y}}}^{0,\,1} + \overbrace{\cancel{\frac{\partial w}{\partial z}}}^{0,\,2} = 0$$

1 - no y-component of flow
2 - no z-component of flow

$$\implies \frac{\partial u}{\partial x} = 0$$

Thus, the flow is fully developed. Evaluating the momentum equation in the x-direction:

$$\rho\left(\frac{\partial u}{\partial t} + \overbrace{\cancel{u\frac{\partial u}{\partial x}}}^{0,\,3} + \overbrace{\cancel{v\frac{\partial u}{\partial y}}}^{0,\,1} + \overbrace{\cancel{w\frac{\partial u}{\partial z}}}^{0,\,4}\right) = -\overbrace{\cancel{\frac{\partial P}{\partial x}}}^{0,\,5} + \mu\left(\overbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}^{0,\,3} + \frac{\partial^2 u}{\partial y^2} + \overbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}^{0,\,6}\right) + \overbrace{\cancel{\rho g_x}}^{0,\,7}$$

3 - consequence of continuity
4 - no z-component of velocity
5 - no pressure gradient in the x-direction
6 - x-component of velocity not a function of z
7 - no gravity in the x-direction

Recalling the definition of kinematic viscosity, the momentum equation become:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

Assuming the solution has the form u(y,t) = Y(y)T(t):

$$\frac{\partial}{\partial t}Y(y)T(t) = \nu\frac{\partial^2}{\partial y^2}Y(y)T(t) \implies T'(t)Y(y) = \nu Y''(y)T(t)$$

Grouping like variables and equating them by the constant $-\lambda^2$

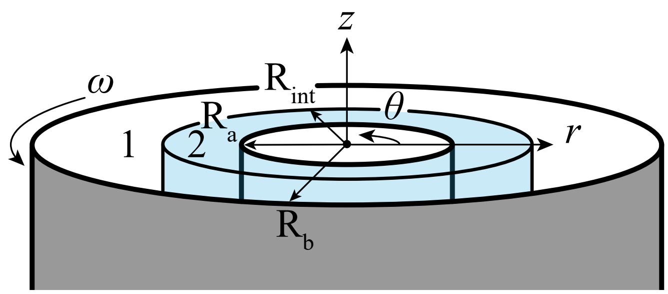
$$\frac{T'(t)}{T(t)} = \nu\frac{Y''(y)}{Y(y)} = -\lambda^2$$

Constructing two ODEs in terms of $-\lambda^2$

$$T'(t) + \lambda^2 T(t) = 0$$

$$Y''(y) + \frac{\lambda^2}{\nu} Y(y) = 0$$

7. (25 points) Consider a situation where two immiscible fluids exist between two infinitely long, concentric cylinders, where the cylinders have radii $\mathrm{R_a}$ and $\mathrm{R_b}$. The fluid interface is denoted as $\mathrm{R_{int}}$. The fluids are denoted by numbers 1 and 2, each with a unique dynamic viscosity μ and density ρ. The outer cylinder rotates with an angular velocity ω whereas the inner cylinder is stationary.



Given the continuity equation as:

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

and the momentum equations as:
<u>r-direction</u>:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}$$
$$+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

<u>θ-direction</u>:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}$$
$$+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

<u>z-direction</u>:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}$$
$$+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

shear-stress tensor:

$$\tau = \begin{bmatrix} 2\mu\dfrac{\partial u_r}{\partial r} & \mu\left(r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)+\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) & \mu\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) \\ \mu\left(r\dfrac{\partial}{\partial r}\left(\dfrac{u_\theta}{r}\right)+\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) & 2\mu\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \mu\left(\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right) & \mu\left(\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & 2\mu\dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

construct an expression for the velocity profile of each fluid. Assume the flow is incompressible, steady-state, laminar and Newtonian.

Starting with the continuity equation:

$$\cancelto{0,\,1}{\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r}} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \cancelto{0,\,2}{\frac{\partial(\rho v_z)}{\partial z}} = 0$$

Since the flow is incompressible, the flow in the radial direction does not vary, thus it is fully developed:

$$\implies \frac{\partial(v_\theta)}{\partial \theta} = 0$$

1 - no radial component of velocity

2 - no axial component of velocity.

Evaluating the momentum equation in the theta-direction:

$$\rho\left(\cancelto{0,\,3}{\frac{\partial v_\theta}{\partial t}} + \cancelto{0,\,1}{v_r\frac{\partial v_\theta}{\partial r}} + \cancelto{0,\,4}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0,\,1}{\frac{v_r v_\theta}{r}} + \cancelto{0,\,2}{v_z\frac{\partial v_\theta}{\partial z}}\right) = \cancelto{0,\,5}{-\frac{1}{r}\frac{\partial P}{\partial \theta}}$$

$$+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \cancelto{0,\,4}{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancelto{0,\,1}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancelto{0,\,6}{\frac{\partial^2 v_\theta}{\partial z^2}}\right) + \cancelto{0,\,7}{\rho g_\theta}$$

3 - steady-state

4 - consequence of continuity

5 - no pressure gradient in the theta-direction

6 - theta-component of velocity not a function of z

7 - no gravity in the theta-direction

Thus

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) = 0$$

Recall, we can expand this out through the use of the product rule and subsequently the quotient rule:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}$$

Assuming the solution takes the form:

$$u_\theta = r^n$$

Substituting this back into the second-order ODE:

$$\frac{\partial^2 r^n}{\partial r^2} + \frac{1}{r}\frac{\partial r^n}{\partial r} - \frac{r^n}{r^2} = 0$$

$$\implies n(n-1)r^{n-2} + \frac{nr^{n-1}}{r} - \frac{r^n}{r^2} = 0$$

Factoring:

$$r^{n-2}(n(n-1) + n - 1) = 0 \implies n^2 - n + n - 1 = 0 \implies n^2 = 1$$

$$\implies n_{1,2} = \pm 1$$

Therefore, we assume the solution takes the form:

$$v_\theta = c_1 r^{n_1} + c_2 r^{n_2} = c_1 r + \frac{c_2}{r}$$

Thus, for both fluids:

$$v_{\theta,2} = c_1 r^{n_1} + c_2 r^{n_2} = c_1 r + \frac{c_2}{r}$$

$$v_{\theta,1} = c_3 r^{n_1} + c_4 r^{n_2} = c_3 r + \frac{c_4}{r}$$

To determine the constants of integration, we apply the following boundary conditions. The velocity of the fluid 2 on the inner cylinder has zero velocity:

$$\text{BC1: } v_{\theta,2}(r = R_a) = 0$$

The velocities of fluids 1 and 2 are the same at R_{int}:

$$\text{BC2: } v_{\theta,2}(r = R_{int}) = v_{\theta,1}(r = R_{int})$$

The shear-stress of fluids 1 and 2 are the same at R_{int}:

$$\text{BC3: } \tau_2|_{r=R_{int}} = \tau_1|_{r=R_{int}}$$

The velocity of fluid 1 on the outter cylinder is equal to ωR_b:

$$\text{BC4: } v_{\theta,1}(r = R_b) = \omega R_b$$

Before substituting in these values, we will reduce the shear-stress tensor:

$$\tau = \begin{bmatrix} 2\mu\cancelto{0,1}{\frac{\partial v_r}{\partial r}} & \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cancelto{0,1}{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}\right) & \mu\left(\cancelto{0,1}{\frac{\partial v_r}{\partial z}} + \cancelto{0,2}{\frac{\partial v_z}{\partial r}}\right) \\ \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cancelto{0,1}{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}\right) & 2\mu\left(\cancelto{0,4}{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0,1}{\frac{v_r}{r}}\right) & \mu\left(\cancelto{0,6}{\frac{\partial v_\theta}{\partial z}} + \cancelto{0,2}{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}\right) \\ \mu\left(\cancelto{0,1}{\frac{\partial v_r}{\partial z}} + \cancelto{0,2}{\frac{\partial v_z}{\partial r}}\right) & \mu\left(\cancelto{0,6}{\frac{\partial v_\theta}{\partial z}} + \cancelto{0,2}{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}\right) & \cancelto{0,2}{2\mu\frac{\partial v_z}{\partial z}} \end{bmatrix}$$

Applying BC1:

$$v_{\theta,1} = c_3 R_A + \frac{c_4}{R_A} = 0$$

Applying BC2:

$$c_1 R_{int} + \frac{c_2}{R_{int}} = c_3 R_{int} + \frac{c_4}{R_{int}}$$

Applying BC3:

$$\mu_2 \left(r \frac{\partial}{\partial r} \left(\frac{v_{\theta,2}}{r} \right) \right) \bigg|_{r=R_{int}} = \mu_1 \left(r \frac{\partial}{\partial r} \left(\frac{v_{\theta,1}}{r} \right) \right) \bigg|_{r=R_{int}} \implies \mu_2 v_{\theta,2} = \mu_1 v_{\theta,1}$$

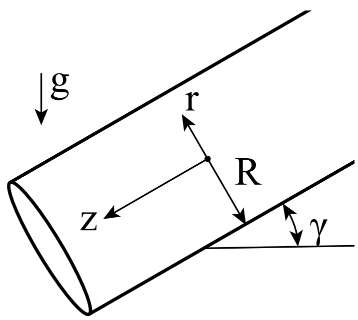
$$\implies \mu_2 \left(c_1 R_{int} + \frac{c_2}{R_{int}} \right) = \mu_1 \left(c_3 R_{int} + \frac{c_4}{R_{int}} \right)$$

Applying BC4:

$$v_{\theta,1} = c_3 R_b + \frac{c_4}{R_B} = \omega R_B$$

Solving for the constants is an exercise left up the reader.

8. (25 points) Steady, incompressible, laminar flow for a Newtonian fluid is occurring in an infinitely long pipe with diameter D. The pipe is included some angle γ. Gravity acts downward and there is no applied pressure gradient. Determine the expression for the velocity profile in the z-direction. Use the continuity and momentum equations from the previous problem.



Starting with the continuity equation:

$$\cancel{\frac{1}{r}} \xcancel{\frac{\partial(r\rho v_r)}{\partial r}}^{0,\,1} + \cancel{\frac{1}{r}} \xcancel{\frac{\partial(\rho v_\theta)}{\partial \theta}}^{0,\,2} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Since the flow is incompressible, the flow in the radial direction does not vary, thus it is fully developed:

$$\implies \frac{\partial v_z}{\partial z} = 0$$

1 - no radial component of velocity

2 - no theta-component of velocity.

Evaluating the momentum equation in the z-direction: z-direction:

$$\rho\left(\overset{0,\,3}{\cancel{\frac{\partial v_z}{\partial t}}} + \overset{0,\,1}{\cancel{v_r\frac{\partial v_z}{\partial r}}} + \overset{0,\,2}{\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}} + \overset{0,\,4}{\cancel{v_z\frac{\partial v_z}{\partial z}}}\right) = -\overset{0,\,5}{\cancel{\frac{\partial P}{\partial z}}}$$

$$+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \overset{0,\,6}{\cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}} + \overset{0,\,4}{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}\right) + \rho g_z$$

3 - steady-state
4 - consequence of continuity
5 - no pressure in the z-direction
6 - z-component of velocity not a function of theta
Thus, taking into acount the angle γ

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\rho g}{\mu \sin(\gamma)}$$

Multiplying by r:

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{\rho g r \sin(\gamma)}{\mu}$$

Integrating

$$r\frac{\partial v_z}{\partial r} = \frac{\rho g r^2 \sin(\gamma)}{2\mu} + c_1$$

Dividing by r:

$$\frac{\partial v_z}{\partial r} = \frac{\rho g r}{2\mu \sin(\gamma)} + \frac{c_1}{r}$$

Integrating:

$$v_z = \frac{\rho g r^2 \sin(\gamma)}{4\mu} + c_1 \ln(r) + c_2$$

We have the following boundary conditions. Applying a no-slip boundary condition on the pipe wall:

$$v_z(r=R) = 0 \implies \frac{\rho g R^2 \sin(\gamma)}{4\mu} + c_1 \ln(R) + c_2 = 0$$

Assuming the maximum velocity occurs at the centerline:

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=0} + \frac{c_1}{r} = 0 \implies c_1 = 0$$

Thus

$$c_2 = -\frac{\rho g R^2 \sin(\gamma)}{4\mu}$$

Therefore, the velocity in the z-direction as a function of r is:

$$v_z = \frac{\rho g (r^2 - R^2)\sin(\gamma)}{4\mu}$$