Attribute and Variable Sampling Plan

Design and Operation

Mathews Malnar and Bailey, Inc.
466 West Jackson Street, Painesville, OH 44077
Phone: 440-350-0911
Website: www.mmbstatistical.com
E-mail: paul@mmbstatistical.com
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These notes are posted at *www.mmbstatistical.com/Notes/ AttributeAndVariableSamplingPlansForDefectives_20190508.pdf*.

Agenda

1. Review acceptance sampling fundamentals
2. Attributes sampling plan design and operation
3. Variables sampling plan design and operation
4. Comparison of attributes and variables sample sizes
5. Comments on sampling standards

Sampling Plan Goal

- The goal of any sampling plan is to distinguish good lots from bad lots.
- The observations may be attribute or variable.
- The formal hypotheses being tested are:

$$H_0 : p = p_0 \text{ (the lot is good)}$$

$$H_A : p > p_0 \text{ (the lot is bad)}$$

  where p is the lot's true fraction defective.
- When the goal of an experiment is expressed in terms of reliability instead of fraction defective, just replace p with $p = 1 - R$ or R with $R = 1 - p$.

Acceptance Sampling Plan Risks

- Like any other statistical method, acceptance sampling is based on sampling from a large population which is subject to decision risks:
  - Type 1 error, manufacturer's risk, false alarm: Rejecting a good lot
  - Type 2 error, consumer's risk, missed alarm: Accepting a bad lot
- At the time that a sampling error occurs we never know if we've committed an error; however, we can control the rates that such errors occur.

Acceptance Sampling Use

Acceptance sampling is used in:

- Incoming inspection
- In-process inspection
- Final inspection
- The risk requirements that determine the sample size and acceptance criterion in the different applications may differ

Attribute Sampling Plan

- In attribute sampling each unit inspected is judged to be good or bad.
- Attribute sampling plans are characterized by their sample size n and an acceptance number c, i.e. the largest number of defectives allowed in the sample to accept the lot.
- Attribute sampling plan operation:
  1. Define lots:
     a. The boundaries between lots are special cause events
     b. Product quality within each lot is homogeneous
  2. Draw a random sample of size n from the lot.
  3. Inspect and count the number of defective units D in the sample.
  4. If $\dot{D} > c$ then reject the lot. If $D \leq c$ then accept the lot.

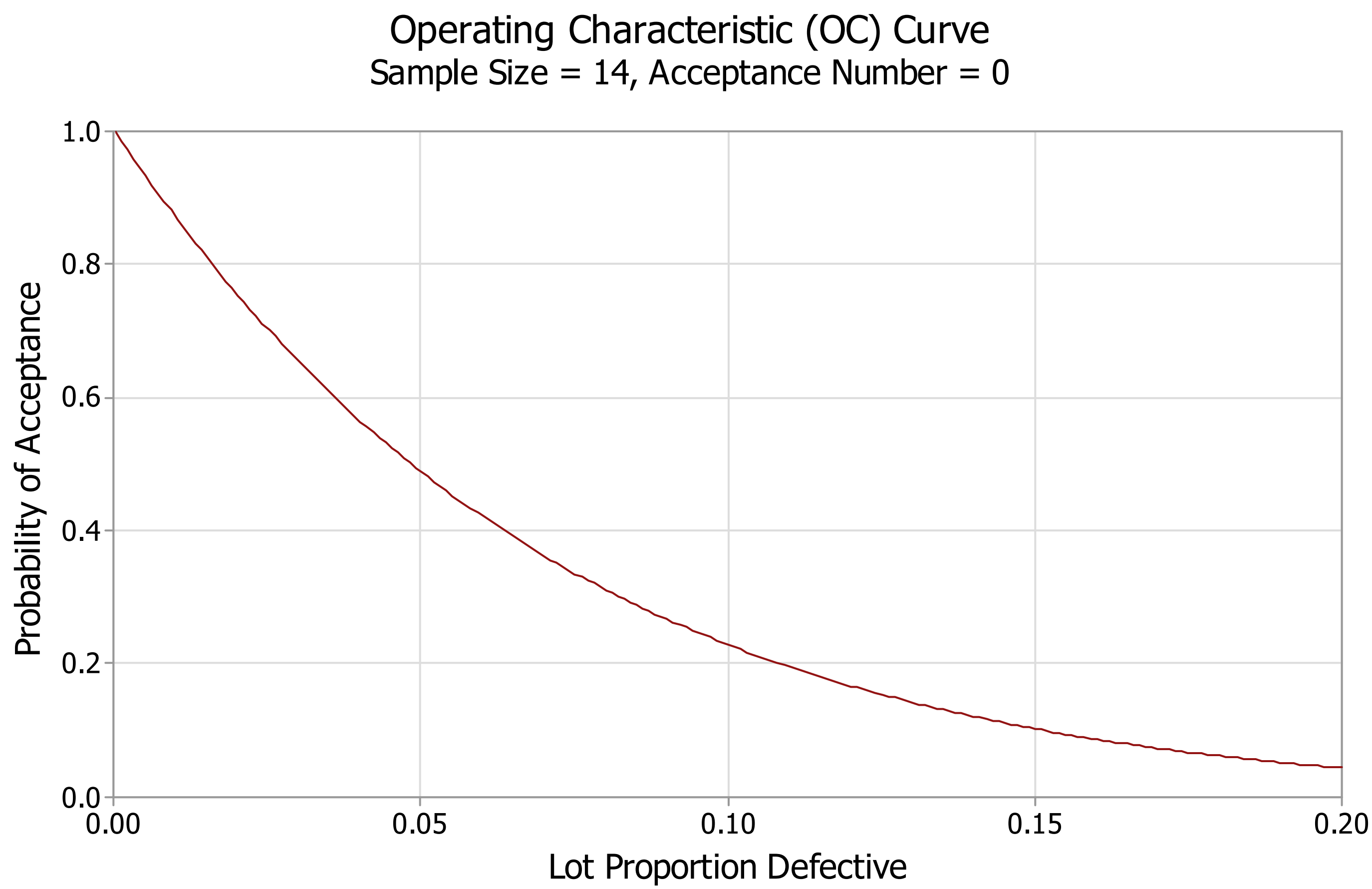
Attribute Sampling Plan: Example

Example: What decision should be made if an attribute sampling plan with $n = 198$ and $c = 4$ finds the following number of defectives in random samples?

1. $D = 0$
2. $D = 1$
3. $D = 4$
4. $D = 5$
5. $D = 8$

Attribute Sampling Plan: OC Curve

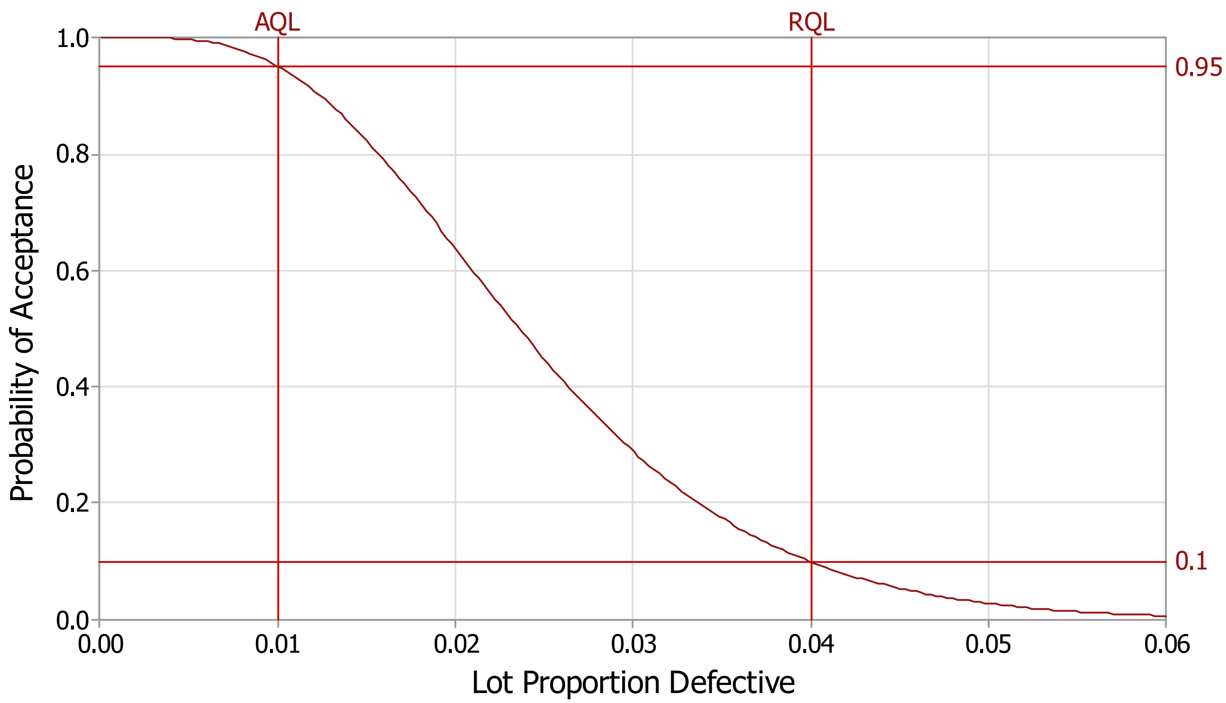
Sampling plans are characterized by their Operating Characteristic (OC) curves - a plot of the probability of accepting lots as a function of the fraction defective.



Attribute Sampling Plan Design

We can design an attribute sampling plan by choosing two points on its OC curve:

- Acceptable Quality Level (AQL) condition: We want the plan to have a high probability of accepting lots with $p = AQL$.
- Rejectable Quality Level (RQL) condition: We want the plan to have a low probability of accepting lots with $p = RQL$.



Attribute Sampling Plan Design

- The *AQL* and *RQL* conditions provide two equations with two unknowns (n and c):

$$b(c; n, p = AQL) = 1 - \alpha$$

$$b(c; n, p = RQL) = \beta$$

  - $b(c; n, p)$ is the cumulative binomial distribution
  - α is the type 1 error rate (the probability of rejecting good lots)
  - β is the type 2 error rate (the probability of accepting bad lots).
- We want both error rates to be low but the cost of type 1 errors (internal failures) is usually different from type 2 errors (external failures) so their values should be chosen independently based on the cost consequences of each failure type.
- The simultaneous solution to the two equations gives unique values for n and c.

Attribute Sampling Plan Design: Example

Problem: What sample size and acceptance number are required to accept 95% of lots with 1% defective and 10% of lots with 4% defective?

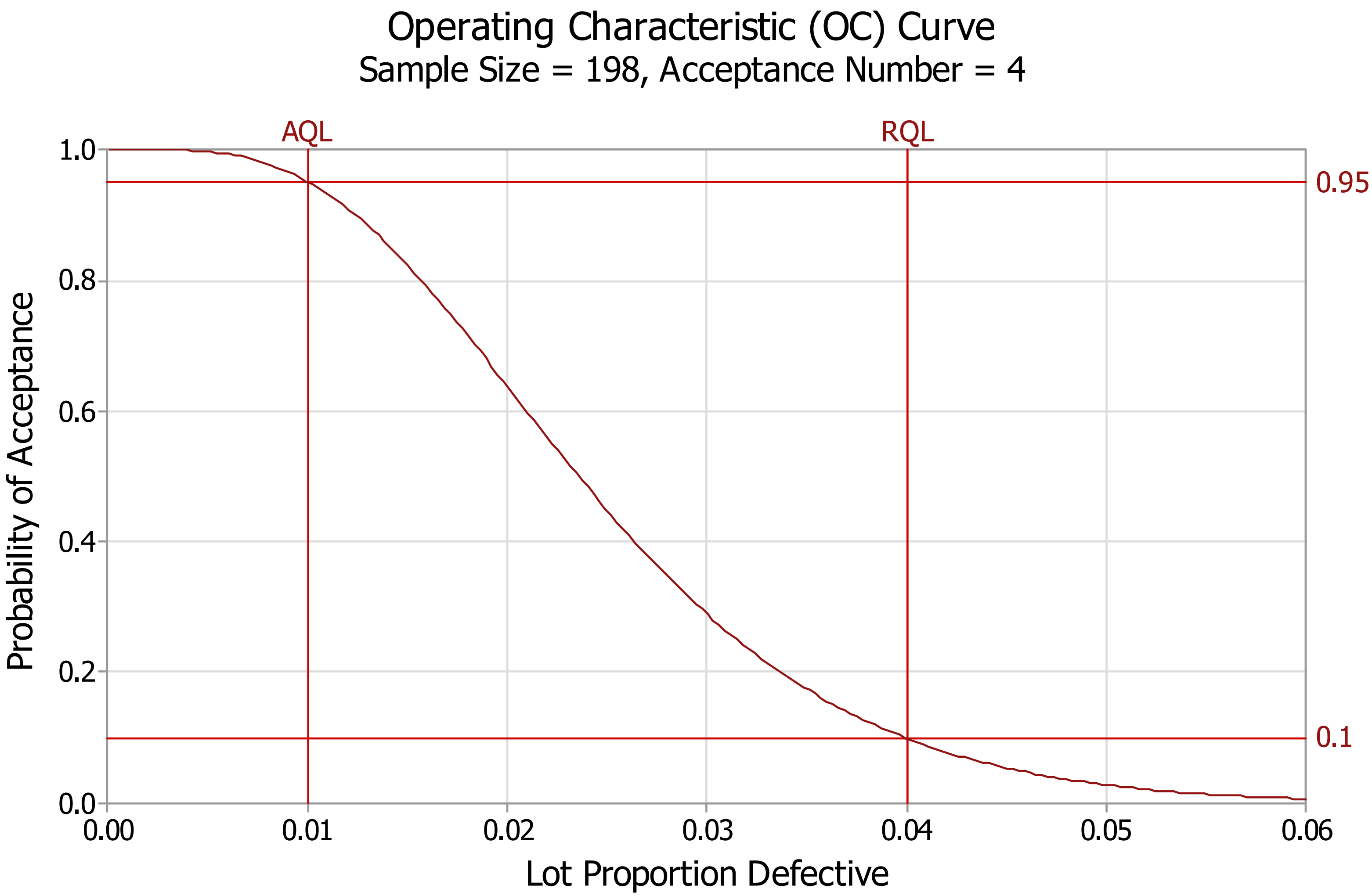
Solution: The simultaneous solution to:

$$b(c; n, p = AQL = 0.01) = 0.95$$

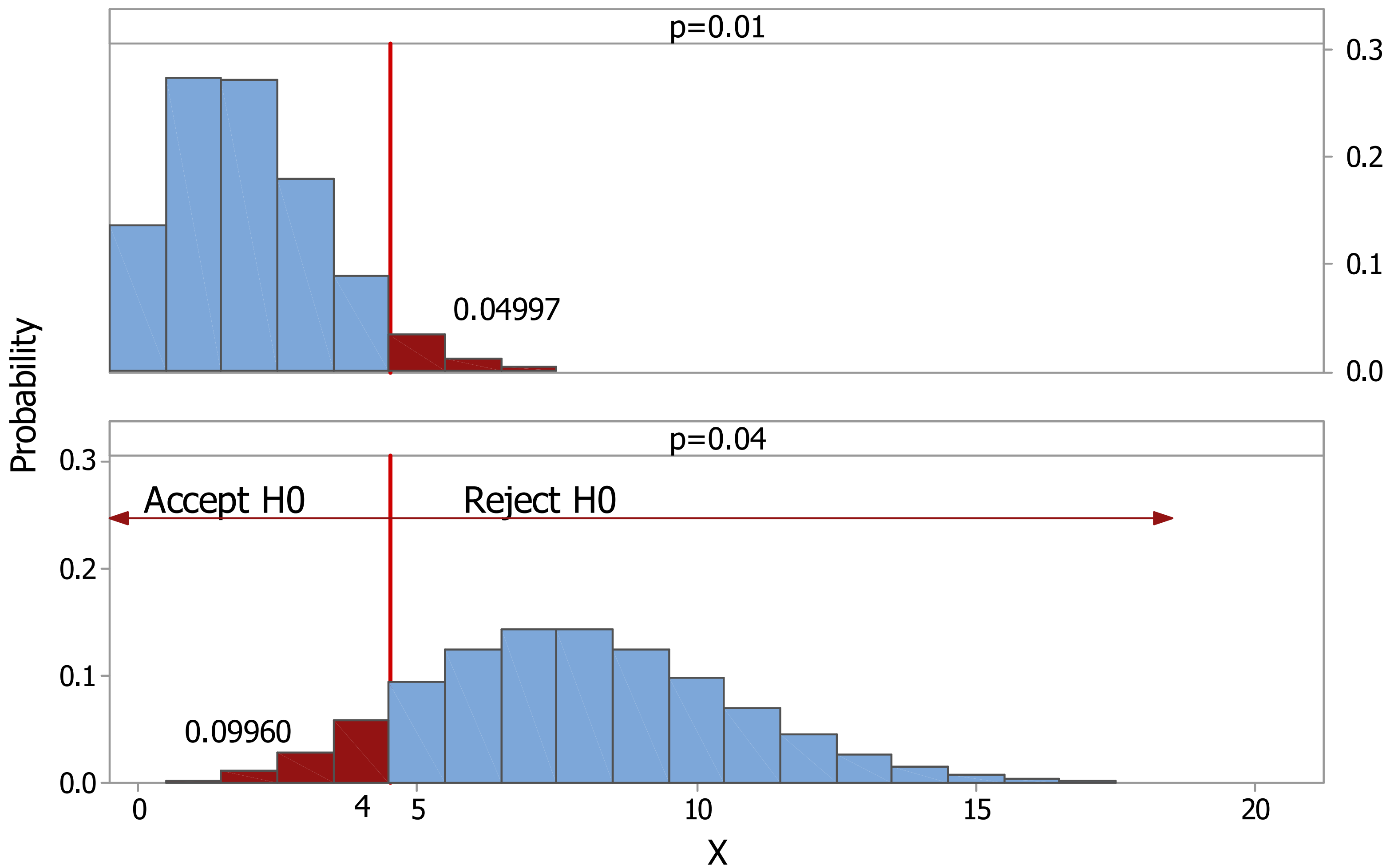
$$b(c; n, p = RQL = 0.04) = 0.10$$

can be determined by manual calculation (very painful), Larson's nomogram, or appropriate software.

Attribute Sampling Plan Design: Example

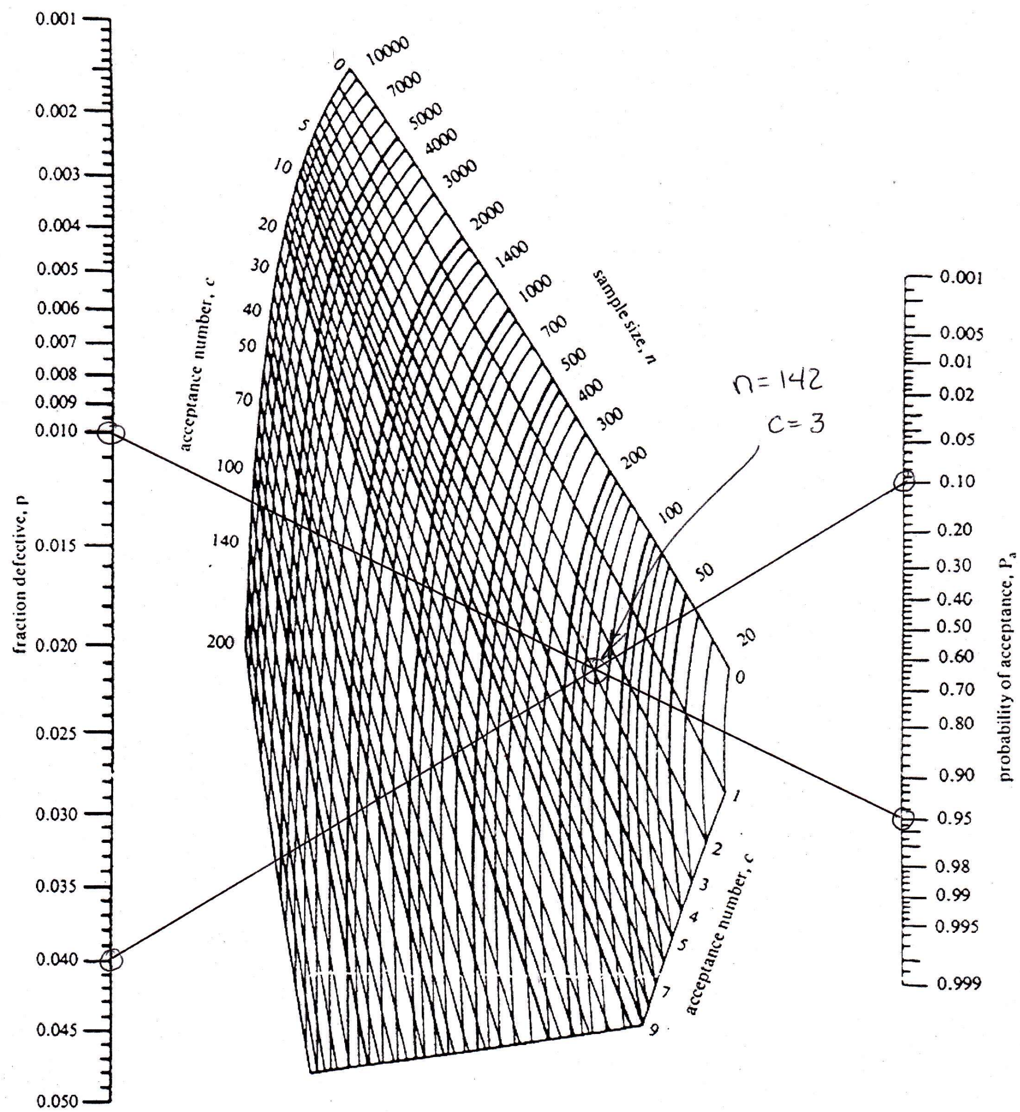


Attribute Sampling Plan Design: Example

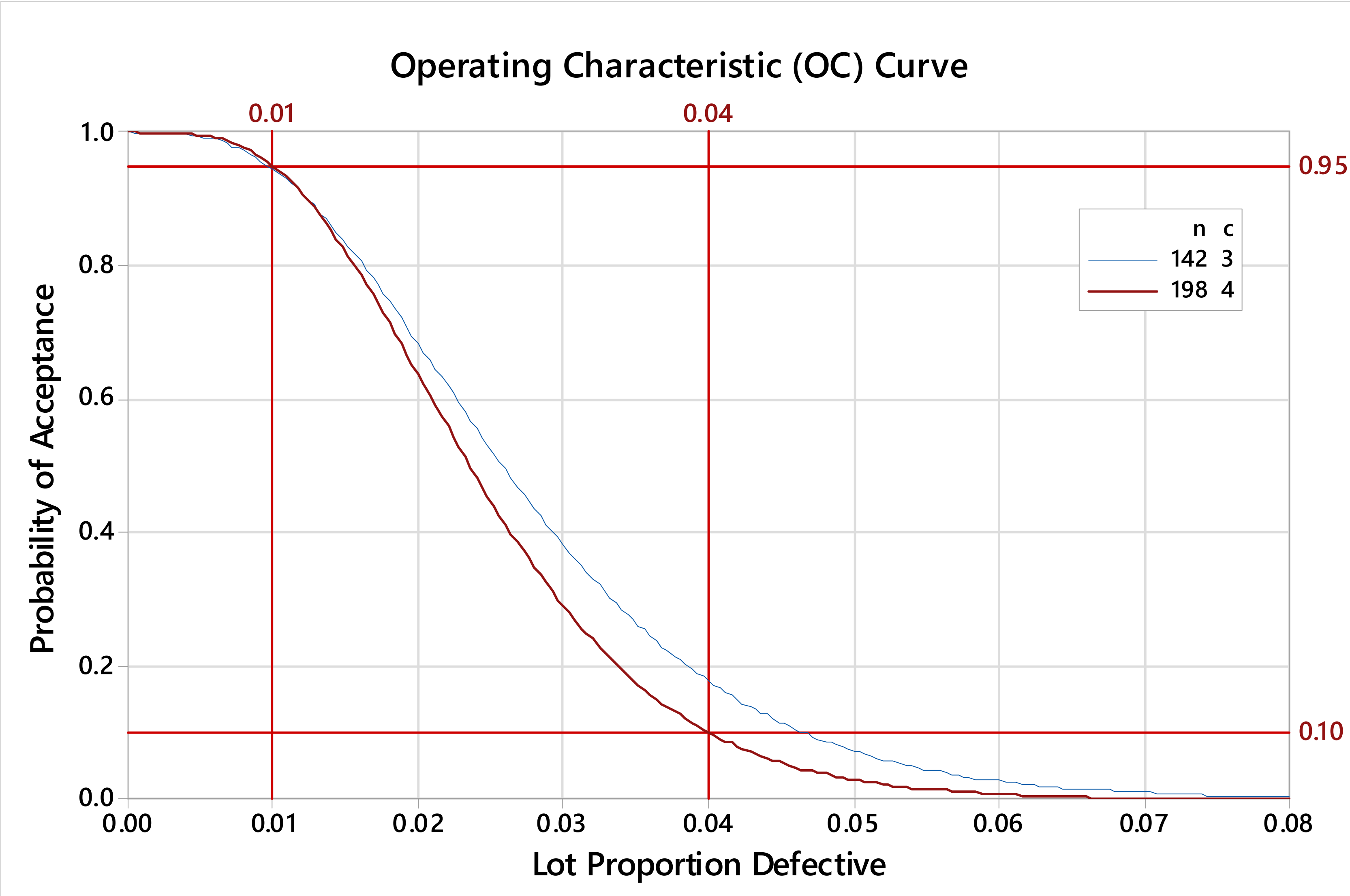


Attribute Sampling Plan Design: Example

Solution using Larson's nomogram:



Attribute Sampling Plan Design: Example



Attribute Sampling Plan Design: $c = 0$

- The sampling plan design strategy previously presented used specification of two points on the OC curve, $(p = AQL, \alpha)$ and $(p = RQL, \beta)$, to determine the sampling plans unique sample size n and acceptance number c.
- There is another family of attribute sampling plans, the *zero acceptance number sampling plans*, which uses one of either the AQL or RQL point plus the $c = 0$ condition:
  - $c = 0$ plus $(p = AQL, \alpha)$
  - $c = 0$ plus $(p = RQL, \beta)$
- The $c = 0$ sampling plans can be determined using software but there are some very simple formulas to calculate the sample size.

Attribute Sampling Plan Design: $c = 0$

- The **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes**> **Create a Sampling Plan** menu's inputs are the *AQL* and *RQL* points.
- The MINITAB menu does not have a provision for choosing $c = 0$ AND the *AQL* or the *RQL* point; however, you can trick it into doing those calculations.

Attribute Sampling Plan Design: RQL **and** $c = 0$

To obtain the $c = 0$ sampling plan for the $(p = RQL, \beta)$ point on the OC curve:

1. Set AQL to a very small value, e.g. 0.000001
2. Set RQL to its desired value
3. Set $\alpha = 0.05$
4. Set β to its desired value

Attribute Sampling Plan Design: AQL **and** $c = 0$

Use the following method to obtain the $c = 0$ sampling plan for the $(p = AQL, 1 - \alpha)$ point on the OC curve. This trick neuters the menu's AQL input and uses the RQL inputs to specify the target AQL point.

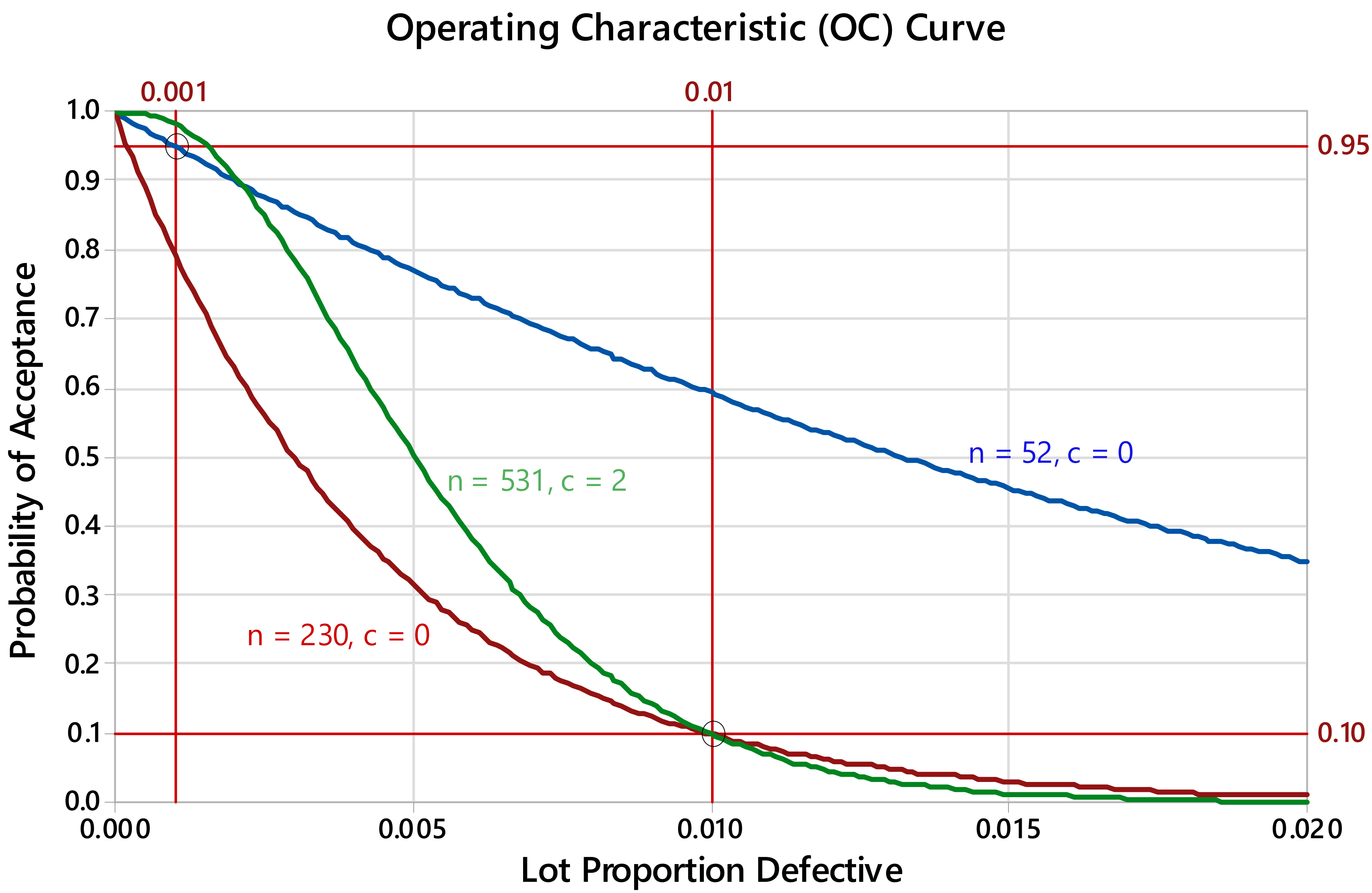
1. Set AQL to a very small value, e.g. 0.000001
2. Set RQL to the target AQL value
3. Set α to a small value, e.g. 0.001
4. Set $\beta = 1 - \alpha$

Attribute Sampling Plan Design: $c = 0$

You can also use the following special equations to calculate approximate sample sizes for $c = 0$ plans:

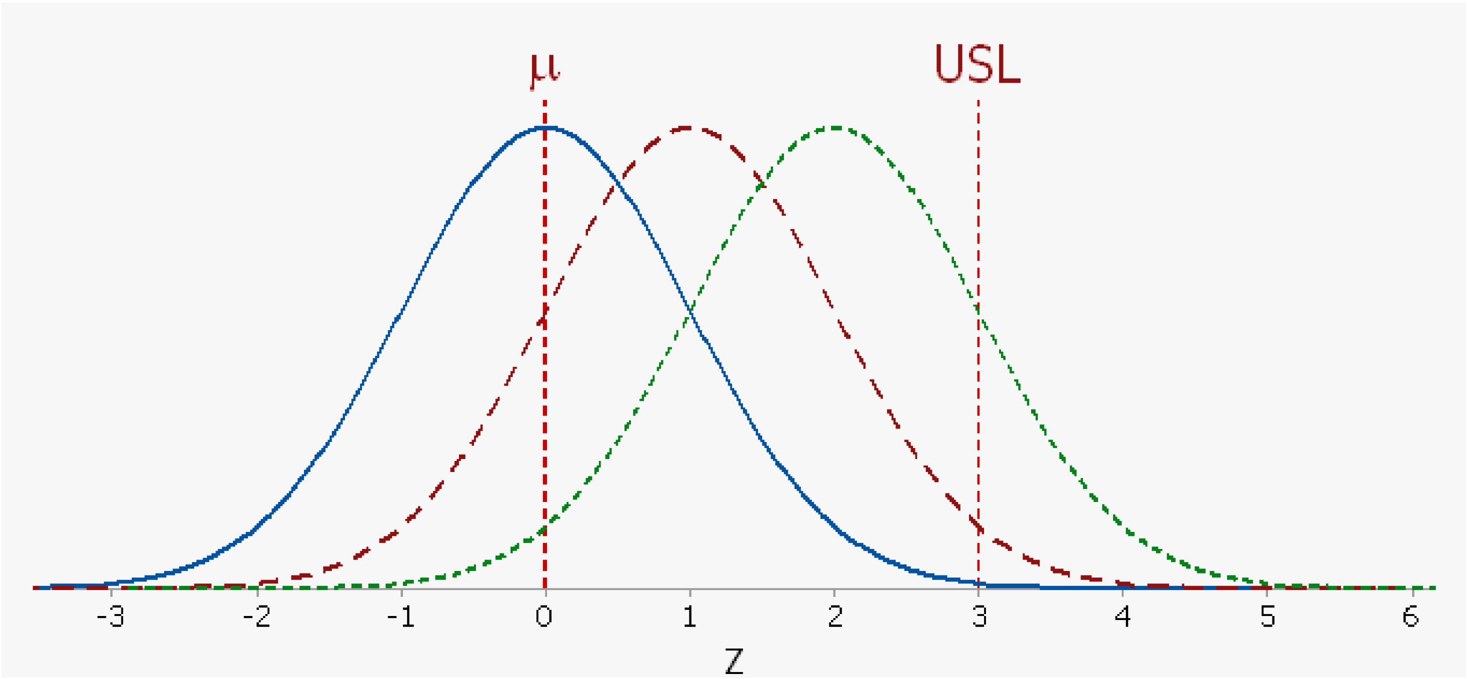
- To obtain the $c = 0$ and $(p = RQL, \beta)$ sampling plan:
  - $n = 2.3/RQL$ for $\beta = 0.10$
  - $n = 3/RQL$ for $\beta = 0.05$
  - $n = 4.6/RQL$ for $\beta = 0.01$.
- To obtain the $c = 0$ and $(p = AQL, 1 - \alpha)$ sampling plan use $n = \alpha/AQL$

Attribute Sampling Plan Design: Example



Variables Sampling Plans

- Variables sampling plans (VSP) have the same goal as atttribute plans:
  - Accept lots with low fraction defective.
  - Reject lots with high fraction defective.
- Variables sampling plans use variables or measurement data instead of attribute data.
- The defective rate varies with the mean, standard deviation, and distribution shape.

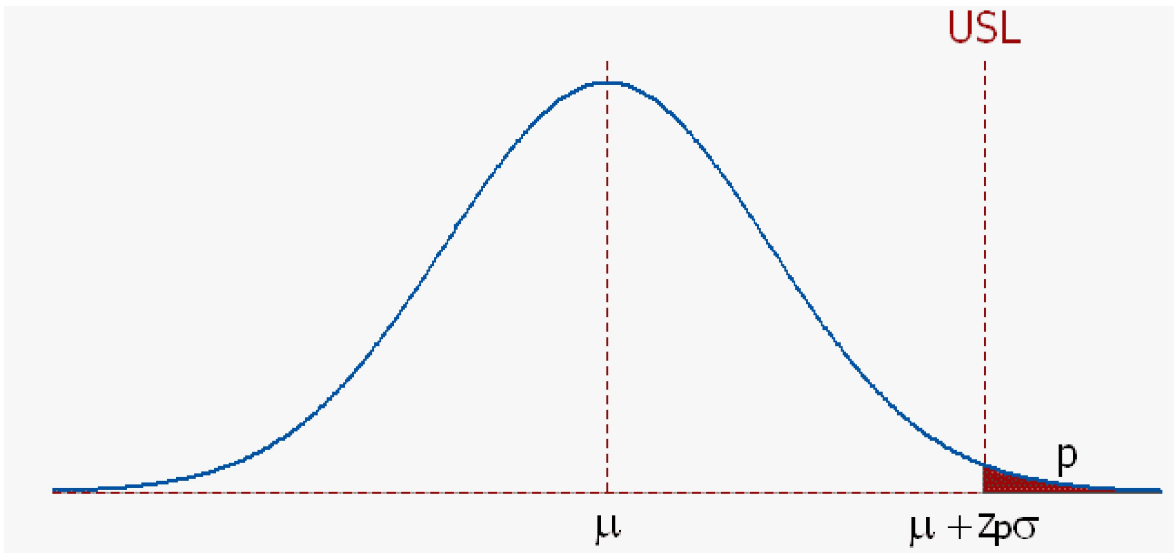


Variables Sampling Plans

When μ and σ are known and the distribution is normal the fraction defective p relative to the one-sided upper specification limit USL is

$$z_p = \frac{USL - \mu}{\sigma}$$

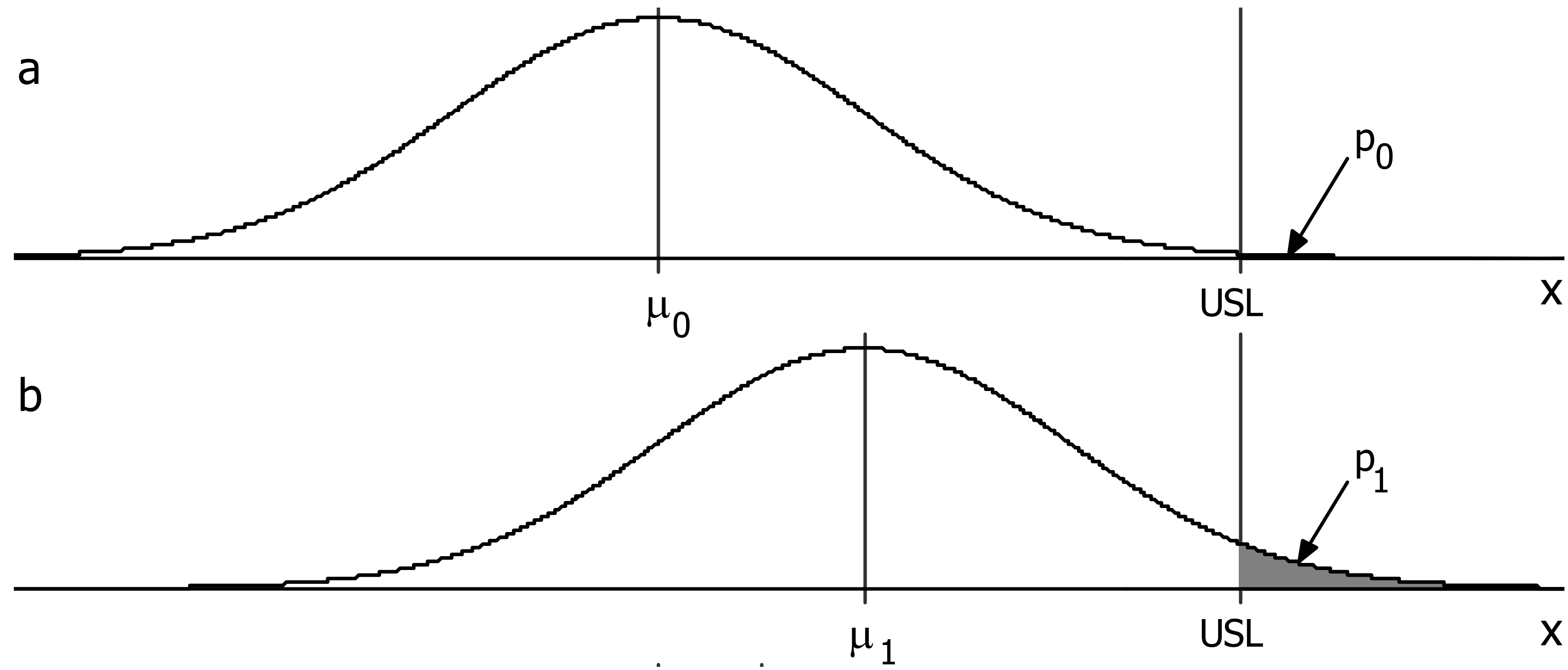
where p is the tail area under the normal curve.



The random sample in a VSP is used to estimate the population mean ($\bar{x}$ estimates μ) and maybe the standard deviation (s approximates σ).

Variables Sampling Plan: Design

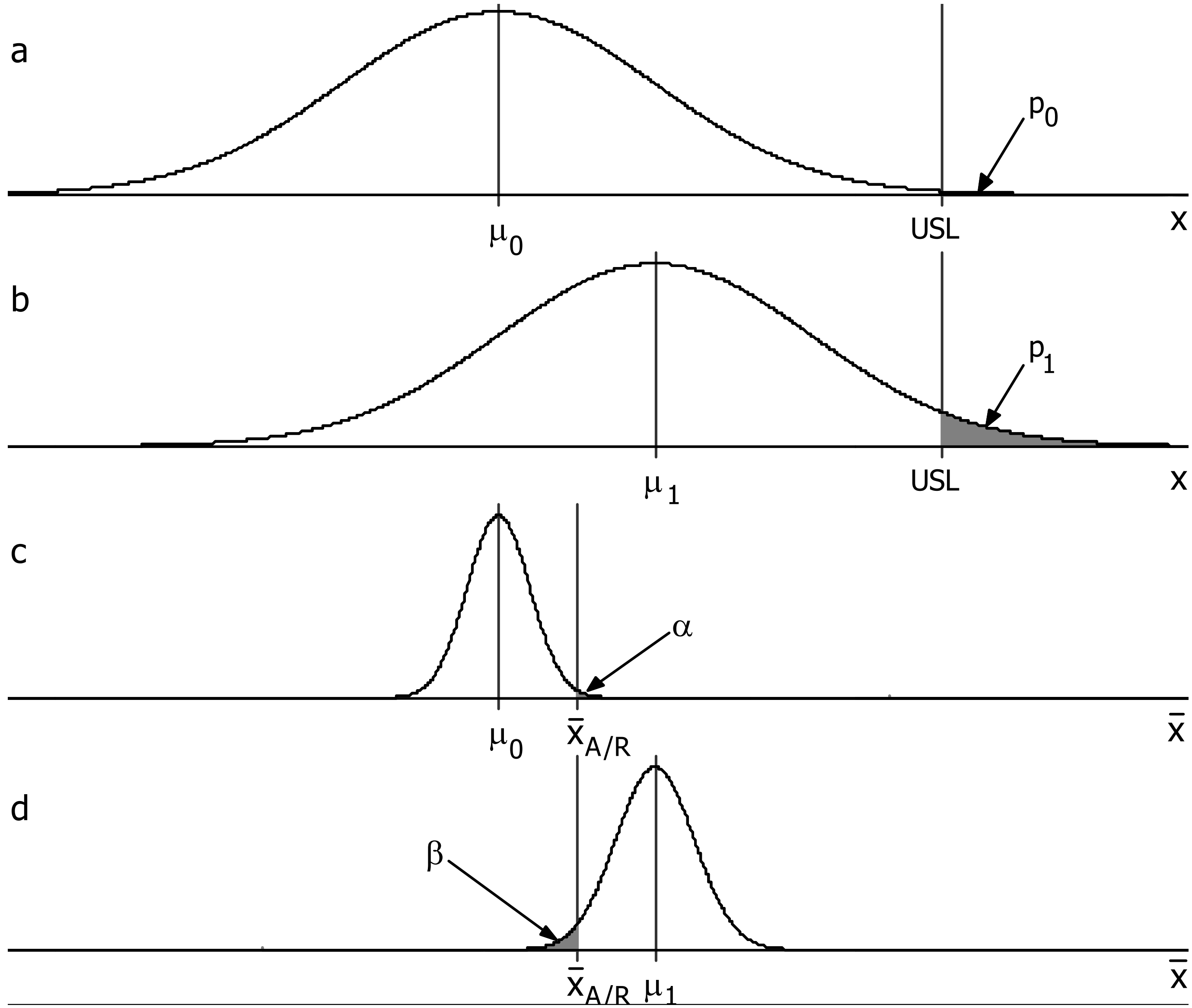
Suppose that we define $AQL\ (p_0)$ and $RQL\ (p_1)$ conditions:



If we know σ_x then at USL we can write

$$USL = \mu_0 + z_{p_0}\sigma_x = \mu_1 + z_{p_1}\sigma_x$$

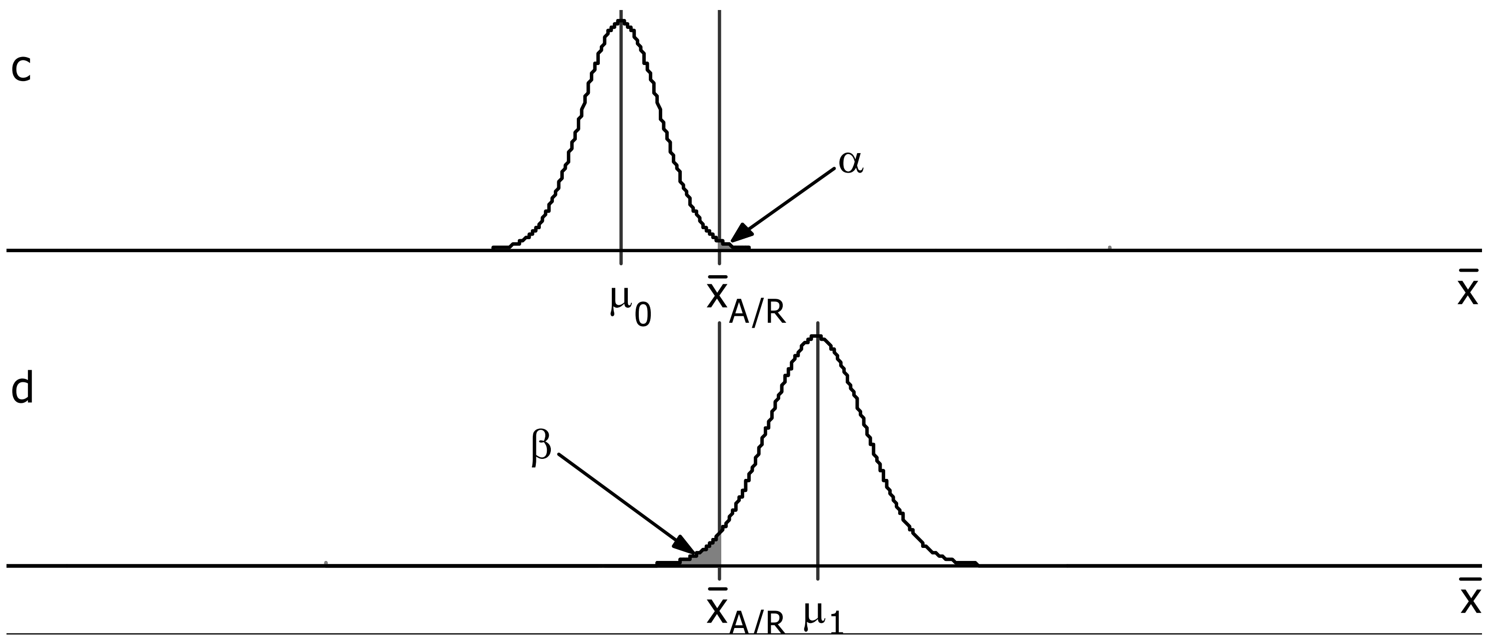
Variables Sampling Plan: Design



Variables Sampling Plan Design

From the distributions of sample means at $\bar{x}_{A/R}$ we can write:

$$\bar{x}_{A/R} = \mu_0 + z_\alpha \sigma_{\bar{x}} = \mu_1 - z_\beta \sigma_{\bar{x}}$$



Variables Sampling Plan: Design

If we solve the two equations:

$$USL = \mu_0 + z_{p_0}\sigma_x = \mu_1 + z_{p_1}\sigma_x$$

$$\bar{x}_{A/R} = \mu_0 + z_\alpha \sigma_{\bar{x}} = \mu_1 - z_\beta \sigma_{\bar{x}}$$

for the sample size n where $\sigma_{\bar{x}} = \sigma_x/\sqrt{n}$ we obtain:

$$n = \left(\frac{z_\alpha + z_\beta}{z_{p_0} - z_{p_1}} \right)^2$$

Variables Sampling Plan: Design

The decision to accept or reject lots is made by comparing the distance between USL and the sample mean $\bar{x}$ expressed in z units

$$z = \frac{USL - \bar{x}}{\sigma}$$

to the critical distance between USL and $\bar{x}_{A/R}$ in z units given by

$$\begin{aligned} k &= \frac{USL - \bar{x}_{A/R}}{\sigma_x} \\ &= z_{p_0} - z_\alpha/\sqrt{n} \end{aligned}$$

(Note that MINITAB uses a slightly different formula for k so its values might differ slightly from manual calculations.)

Accept lots for which $\bar{x}$ is far away from the specification limit, i.e. lots that have

$$z > k$$

and reject lots for which $\bar{x}$ is too close to the specification limit, i.e. lots that have

$$z < k$$

Variables Sampling Plan: Example

Problem: Find the variables sampling plan that will accept 95% of the lots with 1% defectives and reject 90% of the lots with 4% defectives when $\sigma = 30$ and the specification is one-sided with $USL = 700$.

Variables Sampling Plan: Example

Solution: The two specified points on the OC curve are

$$(p_0, 1-\alpha) = (0.01, 0.95)$$

$$(p_1, \beta) = (0.04, 0.10)$$

The required sample size is

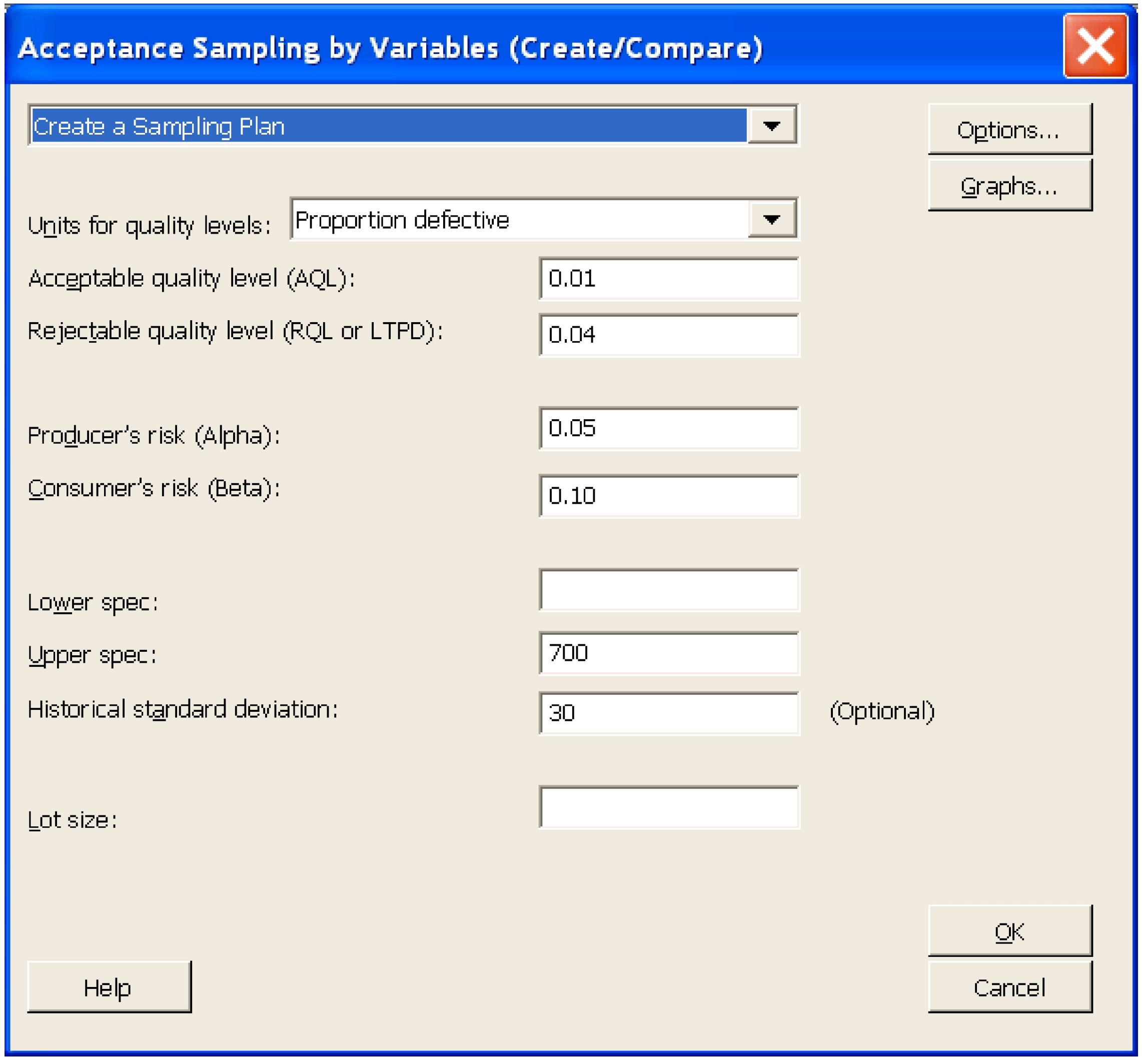
$$\begin{aligned} n &= \left(\frac{z_\alpha + z_\beta}{z_{p_0} - z_{p_1}}\right)^2 \\ &= \left(\frac{1.645 + 1.282}{2.33 - 1.75}\right)^2 \\ &= 26 \end{aligned}$$

and the critical k value is

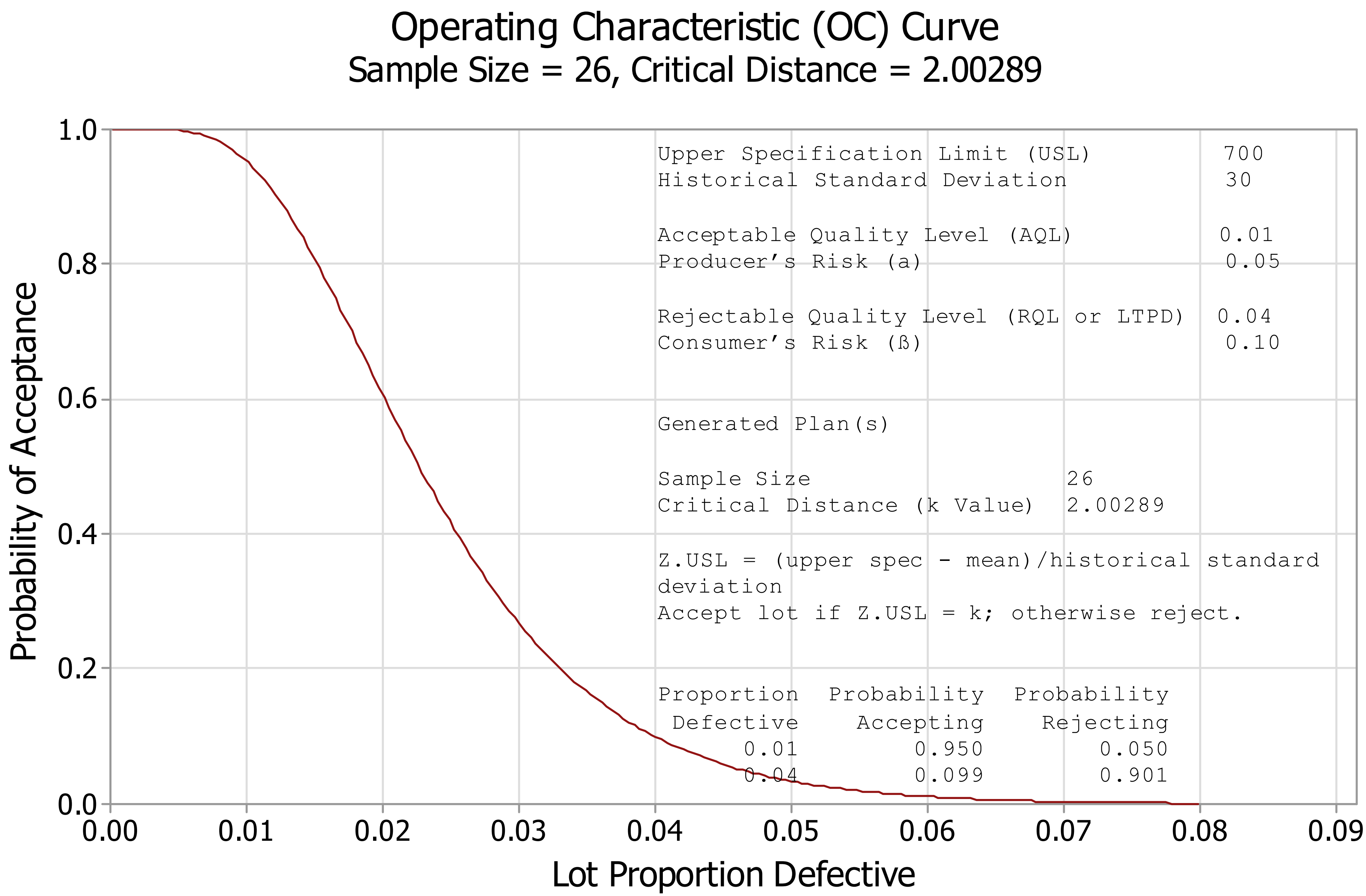
$$\begin{aligned} k &= z_{p_0} - z_\alpha / \sqrt{n} \\ &= 2.326 - 1.645/\sqrt{26} \\ &= 2.0034 \end{aligned}$$

Variables Sampling Plan: Example

Solution: The analytical solution can be confirmed in MINITAB:



Variables Sampling Plan: Example



Variables Sampling Plan - σ Unknown

- When the population standard deviation σ_x is not known it must be estimated from the sample
- Then the sample size must be increased by a factor of $1 + \frac{1}{2}k^2$, i.e.

$$n_{\sigma\ unknown} = n_{\sigma\ known}\left(1 + \frac{k^2}{2}\right)$$

  where k is the same critical value as in the σ known case.
  - The 1 term in the parentheses accounts for the precision of the estimate for the population mean
  - The $\frac{k^2}{2}$ term, which is much larger, accounts for the precision of the estimate for the population standard deviation.
- The consequence of not knowing the population standard deviation is that you must use a much larger sample size.

Variables Sampling Plan: Example

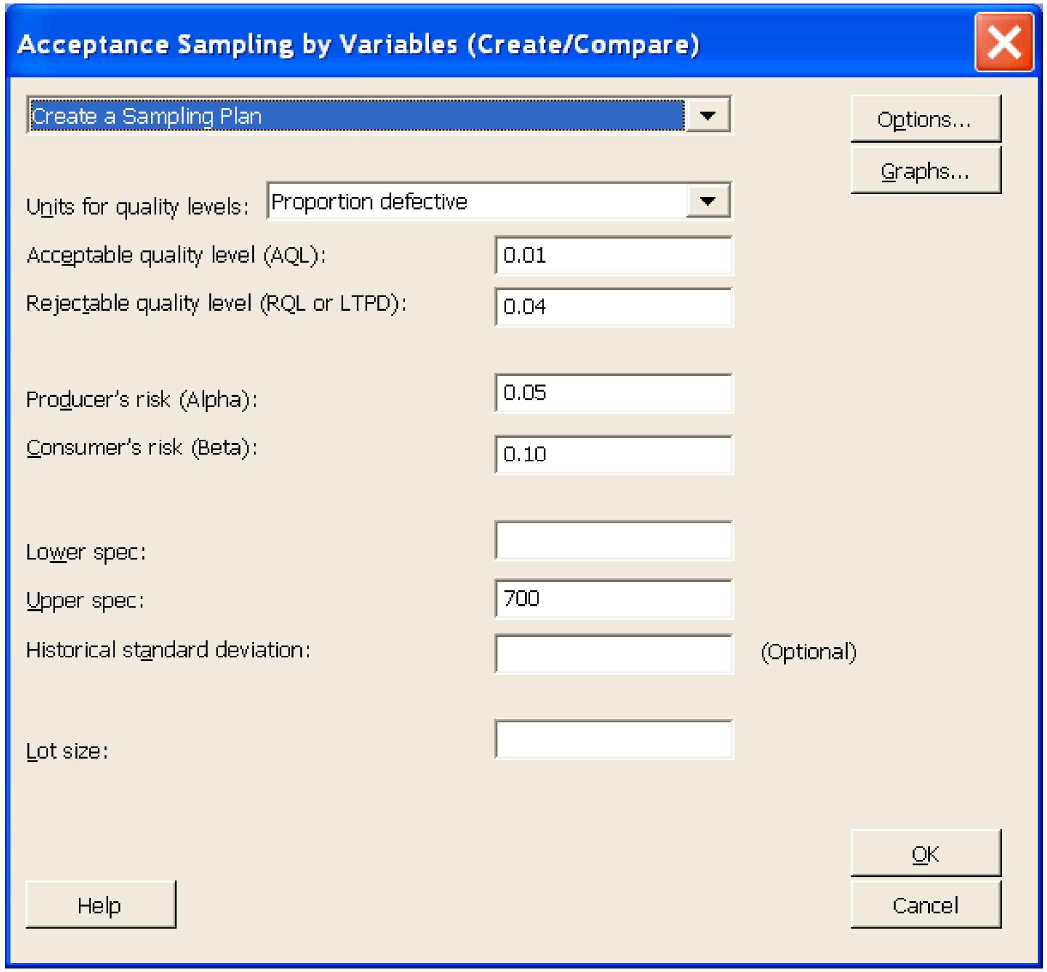
Problem: Revise the sample size for the previous example assuming that σ is unknown and will be estimated from the sample data.

$$n_{\sigma\ unknown} = n_{\sigma\ known}\left(1 + \frac{k^2}{2}\right)$$

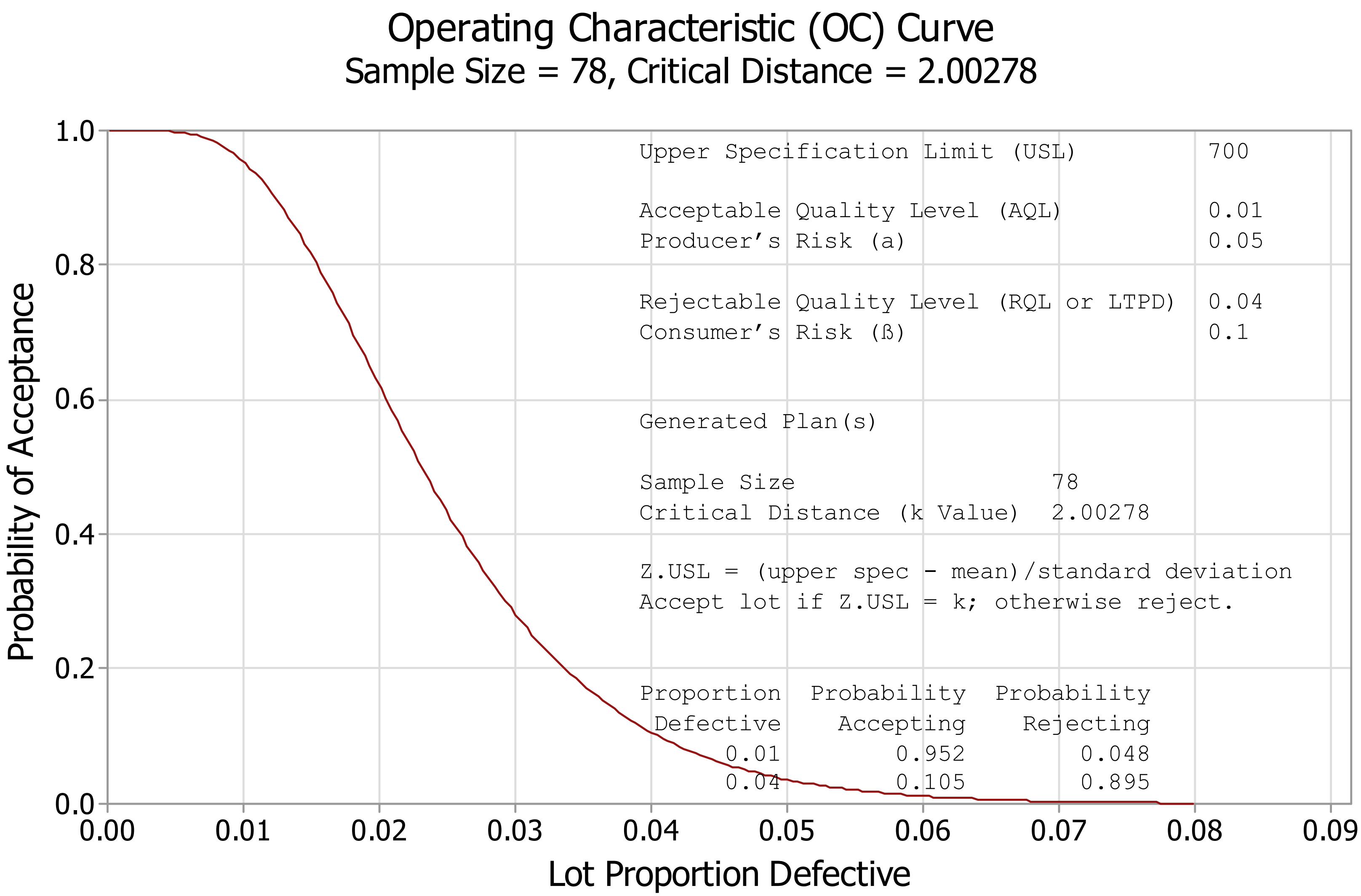
$$= 26\left(1 + \frac{2.0034^2}{2}\right)$$

$$= 79$$

Variables Sampling Plan: Example



Variables Sampling Plan: Example



Comparison of ASP to VSP Sample Sizes

Attribute and variables sampling plans can both be designed to meet the same AQL and RQL conditions. In that case the ratio of the sample sizes is given by

$$\frac{n_{attributes}}{n_{variables}} = \frac{\left(\frac{z_\alpha\sqrt{p_0(1-p_0)}+z_\beta\sqrt{p_1(1-p_1)}}{p_1-p_0}\right)^2}{\left(\frac{z_\alpha+z_\beta}{z_{p_0}-z_{p_1}}\right)^2}$$

For the special case of $\alpha = \beta$ and when p_0 and p_1 are both small, say, less than about 10%, this ratio simplifies and approximates to

$$\frac{n_{attributes}}{n_{variables}} \simeq \frac{1}{4}\left(\frac{z_{p_0} - z_{p_1}}{\sqrt{p_1} - \sqrt{p_0}}\right)^2$$

Comparison of ASP to VSP Sample Sizes

Example: Determine the sample size ratio for attributes and variables inspection plans that will accept 95% of the lots with 0.1% defectives and reject 95% of the lots with 0.4% defectives.

Solution: The two points on the OC curve are $(p_0 = 0.001, 1 - \alpha = 0.95)$ and $(p_1 = 0.004, \beta = 0.05)$. Because $\alpha = \beta = 0.05$ and both p_0 and p_1 are relatively small the ratio of the attributes- to variables-based sample sizes is approximately

$$\begin{aligned} \frac{n_{attributes}}{n_{variables}} &\simeq \frac{1}{4}\left(\frac{z_{0.001} - z_{0.004}}{\sqrt{0.004} - \sqrt{0.001}}\right)^2 \\ &\simeq \frac{1}{4}\left(\frac{3.090 - 2.652}{\sqrt{0.004} - \sqrt{0.001}}\right)^2 \\ &\simeq 48 \end{aligned}$$

Acceptance Sampling Standards

- ANSI ASQ Z1.9 Acceptance Sampling for Attributes (formerly MIL-STD-105)
- ANSI ASQ Z1.4 Acceptance Sampling for Variables (formerly MIL-STD-414)
- Dodge-Romig Rectifying Inspection by Attributes
- Squeglia, Zero Acceptance Number Sampling Plans
- MIL-HDBK-H108 Reliability Sampling

References

1. Montgomery, *Introduction to Statistical Quality Control*
2. Grant and Leavenworth, *Statistical Quality Control*
3. Mathews, *Sample Size Calculations: Practical Methods for Engineers and Scientists*