

0.1 Normal Form Games and Nash Equilibria

0.2 Bimatrix Games and Best Response Polytopes

What do I give for granted? subsection 1 on games, subsection 2 on games and polytopes?

read: unit games article on vS website

A (d -dimensional) *simplicial polytope* P is the convex hull of a set of at least $d + 1$ points v in \mathbb{R}^d in general position, that is, no $d + 1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P . A *facet* of P is the convex hull $\text{conv } F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P ; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F .

The following theorem, due to Balthasar and von Stengel [?, ?], establishes a connection between general labeled polytopes and equilibria of certain $d \times n$ bimatrix games (U, B) .

THIS FROM VvS

Theorem 1. *Consider a labeled d -dimensional simplicial polytope Q with $\mathbf{0}$ in its interior, with vertices $-e_1, \dots, -e_d, c_1, \dots, c_n$, so that $F_0 = \text{conv}\{-e_1, \dots, -e_d\}$ is a facet of Q . Let $-e_i$ have label i for $i \in [d]$, and let c_j have label $l(j) \in [d]$ for $j \in [n]$. Let (U, B) be the $d \times n$ bimatrix game with $U = [e_{l(1)} \cdots e_{l(n)}]$ and $B = [b_1 \cdots b_n]$, where $b_j = c_j / (1 + \mathbf{1}^T c_j)$ for $j \in [n]$. Then the completely labeled facets F of Q , with the exception of F_0 , are in one-to-one correspondence to the Nash equilibria (x, y) of the game (U, B) as follows: if v is the normal vector of F , then $x = (v + \mathbf{1}) / \mathbf{1}^T (v + \mathbf{1})$, and $x_i = 0$ if and only if $-e_i \in F$ for $i \in [d]$; any other label j of F , so that c_j is a vertex of F , represents a pure best reply to x . The mixed strategy y is the uniform distribution on the set of pure best replies to x .*

In the preceding theorem, any simplicial polytope can take the role of Q as long as it has one completely labeled facet F_0 . Then an affine transformation, which does not change the incidences of the facets of Q , can be used to map F_0 to the negative unit vectors $-e_1, \dots, -e_d$ as described, with Q if necessary expanded in the direction $\mathbf{1}$ so that $\mathbf{0}$ is in its interior.

A $d \times n$ bimatrix game (U, B) is a *unit vector game* if all columns of U are unit vectors. For such a game B with $B = [b_1 \cdots b_n]$, the columns b_j for $j \in [n]$ can be obtained from c_j as in Theorem ?? if $b_j > \mathbf{0}$ and $\mathbf{1}^T b_j < 1$. This is always possible via a positive-affine transformation of the payoffs in B , which does not change the game. The unit vectors $e_{l(j)}$ that constitute the columns of U define the labels of the vertices c_j . The corresponding polytope with these vertices is simplicial if the game (U, B) is nondegenerate [?], which here means that no mixed strategy x of the row player has more than $|\{i \in [d] \mid x_i > 0\}|$ pure best replies. Any game can be made nondegenerate by a suitable “lexicographic” perturbation of B , which can be implemented symbolically.

Unit vector games encode arbitrary bimatrix games: An $m \times n$ bimatrix game (A, B) with (w.l.o.g.) positive payoff matrices A, B can be symmetrized so that its Nash equilibria are in one-to-correspondence to the symmetric equilibria of the $(m+n) \times (m+n)$ symmetric game (C^T, C) where

$$C = \begin{pmatrix} 0 & B \\ A^\top & 0 \end{pmatrix}.$$

In turn, as shown by McLennan and Tourky [?], the symmetric equilibria (x, x) of any symmetric game (C^T, C) are in one-to-one correspondence to the Nash equilibria (x, y) of the “imitation game” (I, C) where I is the identity matrix; the mixed strategy y of the second player is simply the uniform distribution on the set $\{i \mid x_i > 0\}$. Clearly, I is a matrix of unit vectors, so (I, C) is a special unit vector game.