1 Introduction

2 Complexity, Games, Polytopes and Gale Strings

2.1 The Complexity Classes P and PPAD

2.2 Normal Form Games and Nash Equilibria

file: background

2.3 Bimatrix Games and Best Response Polytopes

file: polytopes

2.4 Cyclic Polytopes and Gale Strings

A special case of games is obtained by taking a particular case of best response polytope in theorem **??**.

A *cyclic polytope* P in dimension d with n vertices is the convex hull of distinct points $\mu(t_j)$, where $j \in [n]$ and μ is the *moment curve*

$$\mu \colon t \mapsto (t, t^2, \dots, t^d)^\top$$

Restricting the study of best response polytopes to the case of cyclic polytopes gives an interesting case, since cyclic polytopes can be represented as a combinatorial structure, called *Gale strings*. These are a case of *bitstrings*, that is a string of 0's and 1's.

Formally: given an integer k and a set S, we can represent the function $f_s : [k] \to S$ as the string $s = s(1)s(2)\cdots s(k)$. In the case where $S = \{0, 1\}$ we call s a bitstring.

A maximal substring of consecutive 1's in a bitstring is called a *run*.

We denote with $G(d, n)$ the set of all *Gale strings of length n and dimension d*, defined as the set of all bitstrings s of length n such that *Gale string* is a

1. exactly d bits in s are 1 and

2. s fulfills the *Gale evenness condition*:

$$01^k0 \text{ is a substring of } s \quad \Rightarrow \quad k \text{ is even.}$$

The Gale evenness condition characterises Gale strings in $G(d,n)$ as the bitstrings of length n with exactly d elements equal to 1, such that *interior* runs (that is, runs bounded on both sides by 0s) must be of even length. In general, this condition allows Gale strings to start or end with an odd-length run. When d is even, on the other hand, s starts with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d,n)$ with even d as a "loop" obtained by "glueing together" the extremes of the string to form an even run; more formally, we can see the indices of the string as equivalence classes modulo n, so that we identify $s(i+n) = s(i)$. This also implies that the set of Gale strings of even dimension is therefore invariant under a cyclic shift of the strings.

Example 2.1. We consider $G(4,6)$. We have

$$\begin{aligned} G(4,6) = \{&111100,\\ &111001,\\ &110011,\\ &100111,\\ &001111,\\ &011110,\\ &110110,\\ &101101,\\ &011011\} \end{aligned}$$

The strings 111100, 111001, 110011, 100111, 001111 and 011110 are equivalent under a cyclic shift (if considering the strings as loops, the 1's are all consecutive), as are the strings 110110, 101101 and 011011 (if considering the strings as loops, the even runs of 1's are two couples separated by a single 0).

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The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [**?**].

Theorem 1 ([**?**]). *For any positive integer n, assume that $t_1 < t_2 < \cdots < t_n$ and let P be the cyclic polytope obtained by taking t_j, where $j \in [n]$, in definition* **??**.

Then the facets of P are encoded by $G(d,n)$; that is, F is a facet of P if and only if

$$F = \mathrm{conv}\{\mu(t_i) \mid i \in 1(s)\} \qquad \textit{for some } s \in G(d,n)$$

sketch of pf if not too long and it uses relevant techniques

graphics of cyclic polytope - parallel to gale string

From this point forward, we will assume that d is even.

give something to generalise to odd case

2.5 Labeling and the Problem Another Gale

Given a set G of bitstrings of length n and a parameter d, a *labeling* is a function $l : [n] \to [d]$. A string s in $G(d,n)$ is *completely labeled* if $l(\imath(s)) = [d]$. Any $l(i) \in [d]$ is called a *label*

If $s \in G(d,n)$ is completely labeled for the labeling $l : [n] \to [d]$, then for each label $l(i)$ there is a bit $s(i) = 1$. We therefore have exactly d positions i for which $s(i) = 1$; hence, $|l(\imath(s))| = d$.

Example 2.2. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: 111100, 110110, 100111 and 101101.

123432	**12**3**43**2	**1**23**432**	**1**2**34**3**2**
111100	**11**0**11**0	**1**00**111**	**1**0**11**0**1**

Sometimes there aren't any completely labeled Gale strings that are associated with a given labeling.

Example 2.3. For $l = 121314$, there are no completely labeled Gale strings.

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graphics of labeled cyclic polytope

For this cyclic polytope P, a labeling $l : [n] \to [d]$ can be understood as a label $l(j)$ for each vertex $\mu(t_j)$ for $j \in [n]$. A completely labeled Gale string s therefore represents a facet F of P that is completely labeled.

Special games are obtained by using cyclic polytopes in Theorem **??**, suitably affinely transformed with a completely labeled facet F_0. When Q is a cyclic polytope in dimension d with $d+n$ vertices, then the string of labels $l(1)\cdots l(n)$ in Theorem **??** defines a labeling $l' : [d+n] \to [d]$ where $l'(i) = i$ for $i \in [d]$ and $l'(d+j) = l(j)$ for $j \in [n]$. In other words, the string of labels $l(1)\cdots l(n)$ is just prefixed with the string $1\,2\cdots d$ to give l'. Then l' has a trivial completely labeled Gale string $1^d 0^n$ which defines the facet F_0. Then the problem Another Gale defines exactly the problem of finding a Nash equilibrium of the unit vector game (I, B). Note again that B is here not a general matrix (which would define a general game) but obtained from the last n of $d+n$ vertices of a cyclic polytope in dimension d.

Another Gale	
input :	A labeling $l : [n] \to [d]$, where d is even and $d < n$, and an associated completely labeled Gale string s in $G(d, n)$.
output :	A completely labeled Gale string s' in $G(d, n)$ associated with l, such that $s' \neq s$.

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