

# 1 Complexity, Games, Polytopes and Gale Strings

## 1.1 Some Complexity Classes

A *computational problem* is given by the combination of an *input* and a related *output*. A specific input gives an *instance* of the problem.

Computational problems can be classified according to the form of their output: for instance, the output of *decision problems* is either “YES” or “NO”. A *function problem*, on the other hand, has a more generic output  $y$  related to the input  $x$  by a  $R(x, y)$  given by the problem.

An example of decision problem could be “(input) given a graph, (question) is it possible to find an *Euler tour*, that is, a path through the input graph that starts and end at the same vertex that traverses each edge exactly once?” On the other hand, a function problem could be “(input) given a graph, (output) return an Euler tour.”

check: search problems - def

*Search problems* return either an output  $y$  satisfying a given relation  $R(x, y)$ , where  $x$  is the input of the problem, or “NO”, if it’s not possible to find any such  $y$ . If  $y$  is guaranteed to exist, the problem is called a *total function problem*. *Counting problems*, finally, return the *number* of  $y$ ’s that satisfy  $R(x, y)$ ; given a problem  $R$  we denote the associated counting problem  $\#R$ .

Computational problems are also classified according to their *computational complexity*, given by the *reducibility* from each other.

Turing machines: here

Let  $P_1$  be a computational problem. For an instance  $x$  of  $P_1$ , let  $|x|$  be the the number of bits needed to encode  $x$ .  $P_1$  *reduces to the problem*  $P_2$  *in polynomial time*, denoted  $P_1 \leq_P P_2$ , if there exists a *polynomial-time reduction*, that is, a function  $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$  and a *Turing machine*  $\mathcal{M}$  such that for all  $x \in \{0, 1\}^*$

$$\bullet \quad x \in P_1 \quad \Longleftrightarrow \quad f(x) \in P_2$$

- $\mathcal{M}$  computes  $f(x)$
- $\mathcal{M}$  stops after  $p(|x|)$  steps, where  $p$  is a polynomial

The complexity class P contains all the polynomially decidable problems, that is, all problems  $P$  such that there exists a Turing machine  $\mathcal{M}$  that outputs either “YES” or “NO” for all inputs  $x \in \{0,1\}^*$  of  $P$  after  $p(|x|)$  steps, where  $p$  is a polynomial. problems in  $P$  are often described as *efficient*. The class FP of all the function problems that can be solved in polynomial time is analogously defined.

NP - finding certificates  
counting certificates of NP: # P  
FNP / TFNP  $\Rightarrow$  PPA(D)

## 1.2 Normal Form Games and Nash Equilibria

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## 1.3 Best Response Polytopes

file: polytopes-subsection

## 1.4 Cyclic Polytopes and Gale Strings

## 1.5 The Problem ANOTHER GALE

file: gale-def-subsection  
merge in one section "Gale strings" or "CP and GS"?

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