

## 1 Introduction

## 2 Complexity, Games, Polytopes and Gale Strings

### 2.1 Some Complexity Classes

### 2.2 Normal Form Games and Nash Equilibria

file: background-subsection

### 2.3 Some Geometrical Notation

### 2.4 Bimatrix Games, Labels and Polytopes

file: polytopes-subsection

$$P = \{x \in \mathbb{R}^m | x \geq \mathbf{0}, B^\top x \leq \mathbf{1}\}, \quad Q = \{y \in \mathbb{R}^n | y \geq \mathbf{0}, Ay \leq \mathbf{1}\}, \quad (1)$$

a point in  $P$  has label  $k$  iff  $x_k = 0$  for  $k \in \{1, \dots, m\}$  or  $(B^\top x)_{k-m} = 0$  for  $k \in \{m+1, \dots, m+n\}$ ; analogously, a point in  $Q$  has label  $k$  if and only if either  $y_{k-m} = 0$  for  $k \in \{m+1, \dots, m+n\}$  or  $(Ay)_k = 0$  for  $k \in \{1, \dots, m\}$ .

**proposition 1.** *Let  $(A, B)$  be a bimatrix game and  $(x, y)$  be one of its Nash equilibria. Then  $(z, z)$ , where  $z = (x, y)$ , is a Nash equilibrium of the symmetric game  $(C, C^\top)$ , where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}.$$

**Theorem 1.** [15] *Let  $l : [n] \rightarrow [m]$ , and let  $(U, B)$  be the unit vector game where  $U = (e_{l(1)} \cdots e_{l(n)})$ . Consider the polytopes  $P^l$  and  $Q^l$  where*

$$P^l = \{x \in \mathbb{R}^m | x \geq \mathbf{0}, B^\top x \leq \mathbf{1}\} \quad (2)$$

$$Q^l = \{y \in \mathbb{R}^n | y \geq \mathbf{0}, \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\} \quad (3)$$

where  $N_i = \{j \in [n] | l(j) = i\}$  for  $i \in [m]$ .

Give a labeling  $l_f$  of the facets of  $P^l$  according to the inequality defining it, as follows:

$$x_i \geq 0 \text{ has label } i \text{ for } i \in [m] \quad (4)$$

$$B^\top x)_j \leq 1 \text{ has label } l(j) \text{ for } j \in [n] \quad (5)$$

Then  $x \in P^l$  is a completely labeled point of  $P^l \setminus \{\mathbf{0}\}$  if and only if there is some  $y \in Q^l$  such that, after scaling, the pair  $(x, y)$  is a Nash equilibrium of  $(U, B)$

**Theorem 2.** [1] Let  $Q$  be a labeled  $m$ -dimensional simplicial polytope with  $\mathbf{0}$  in its interior, with vertices  $e_1, \dots, e_m, c_1, \dots, c_n$ , so that  $F_0 = \text{conv}\{e_i \mid i \in [m]\}$  is a facet of  $Q$ .

Let  $l : [n] \rightarrow [m]$ , and let  $(U, B)$  be the unit vector game with  $U = (e_{l(1)} \cdots e_{l(n)})$  and  $B = (b_1 \cdots b_n)$ , where  $b_j = \frac{c_j}{1 + \mathbf{1}^\top c_j}$  for  $j \in [n]$ .

Label the vertices of  $Q$  as follows:

$$l_v(-e_i) = i \text{ for } i \in [m] \quad (6)$$

$$l_v(c_j) = l(j) \text{ for } j \in [n] \quad (7)$$

Then a facet  $F \neq F_0$  of  $Q$  with normal vector  $v$  is completely labeled if and only if  $(x, y)$  is a Nash equilibrium of  $(U, B)$ , where  $x = \frac{v + \mathbf{1}}{\mathbf{1}^\top (v + \mathbf{1})}$ , and  $x_i = 0$  if and only if  $e_i \in F$  for  $i \in [m]$ . Any  $j$  so that  $c_j$  is a vertex of  $F$  represents a pure best reply to  $x$ ; the mixed strategy  $y$  is the uniform distribution on the set of the pure best replies to  $x$ .

## 2.5 Cyclic Polytopes and Gale Strings

## 2.6 Labeling and the Problem ANOTHER GALE

file: gale-def-subsection

# 3 Algorithmic and Complexity Results

### 3.1 Lemke Paths and the Lemke-Howson for Gale Algorithm

We have NEs  $\Leftrightarrow$  completely labeled things (facets, vertices, GS) We give now different versions of fundam algorithm to deal with labeling looking for compl.label. - in particular in these cases  
 first in version on simple polytopes with labeled facets, (name: Lemke-Howson; Lemke-Howson 1964, Shapley 1974 beautiful exposition)  
 then in its corresponding version on Gale strings (name: Lemke-Howson for Gale, where???).

We will also mention dual version on simplicial polytopes with labeled vertices (name: exchange algorithm, Edmonds - Sanità). (This case: index more problematic - see )

Sth about why (exp lemke paths; conjecture)

Consider a labeling  $l : [n] \rightarrow [m]$ , for a set  $X$  with  $|X| = n$ . Then  $x = (x_1, \dots, x_m) \in X^m$  is *almost completely labeled* if  $|\{j \in [n] \mid x_i = j \text{ for some } i \in [m]\}| = [m] \setminus \{k\}$  for exactly one  $k \in [m]$ . This mean that all labels appear once in  $x$ , except for the *missing label*  $k$ , and a *duplicate label*  $\bar{k} \in [m]$  that appears twice.

now we see in poly with labeled facets, cl vertices

Let  $P$  be a simple polytope in dimension  $m$  with  $n$  facets. We define the operation of *pivoting on vertices* as moving from a vertex  $x$  of  $P$  to another vertex  $y$  such that there is an edge between  $x$  and  $y$ . Note that, since  $P$  is simple, there are exactly  $m$  possible choices for  $y$ .

Now let  $l_f : [n] \rightarrow [m]$  be a labeling of the facets of  $P$  such that there is at least one completely labeled vertex  $x_0$  of  $P$ . Note that if we pivot from vertex  $v$  we “leave behind” a facet  $F$ , that has label  $k$ ; we call this *dropping label*  $k$ . We will then reach a vertex  $w$  that shares with  $v$  all facets except  $F$  (that contains  $v$  but not  $w$ ) and another facet  $G$  (that contains  $w$  but not  $v$ ) that has label  $j$ ; we will call this *picking up label*  $j$ . We give the *Lemke-Howson algorithm* as follows:

reference!

The steps of the Lemke-Howson algorithm result in a *Lemke path* that

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**Algorithm 1:** Lemke-Howson algorithm

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**input** : A simple  $m$ -polytope  $P$  with  $n$  facets. A labeling  $l_f : [n] \rightarrow [m]$  of the facets of  $P$ . A vertex  $x_0$  of  $P$ , completely labeled for  $l$ .

**output:** A completely labeled vertex  $y \neq x_0$  of  $P$ .

```
1 choose a label  $k \in [n]$ 
2 pivot from  $x_0$  to  $y$  dropping label  $k$ 
3 while  $y$  is not completely labeled do
4   | pivot from  $y$  to  $y'$  dropping duplicate label  $j$ , moving away from
   |    $x_0$ 
5   | rename  $x_0 = y$ ,  $y = y'$ 
6 return  $y$ 
```

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connects two completely labeled vertices through  $k$ -almost complementary vertices and edges, that is, almost completely labeled vertices and edges where the only missing label is  $k$ . It remains to show that  $y \neq x_0$ . This comes from the fact that the Lemke paths are *simple paths*, that is, there are no “loops” where a vertex is visited more than once. This is not possible because at each vertex there are only two edges corresponding to the missing label  $k$ , since  $P$  is not degenerate; one is the edge that is traversed to get to the vertex, one is the one that is traversed to leave it in the next step. This proves the following.

**Theorem 3.** *The Lemke-Howson algorithm returns a solution to the problem ANOTHER COMPLETELY LABELED VERTEX.*

In the context of finding the Nash equilibrium of a bimatrix game  $(A, B)$ , there are two equivalent implementations of the Lemke-Howson algorithm.

We can consider the game  $C$  as in proposition 1, and the associated polytope  $S = \{z \in \mathbb{R}^{m+n} \mid z \geq \mathbf{0}, Cz \leq \mathbf{1}\}$ , labeling the  $2(m+n)$  inequalities defining the facets of  $S$  as  $1, \dots, m+n, 1, \dots, m+n$ . Then applying the

Lemke-Howson algorithm starting from vertex  $\mathbf{0}$  returns a Nash equilibrium  $(z, z)$  of  $C$  and a corresponding  $(x, y) = z$  a Nash equilibrium of  $(A, B)$ .

We can also follow the “traditional” version of the Lemke-Howson algorithm; a very clear exposition of this can be found in Shapley [16]. Let  $P$  and  $Q$  be the best response polytopes of  $(A, B)$  as in 1. We then move alternately on  $P$  and  $Q$ , starting from the couple of vertices  $(\mathbf{0}, \mathbf{0})$ . Since we move in  $\mathbb{R}^m$  and  $\mathbb{R}^n$  instead of  $\mathbb{R}^{m+n}$ , this version is more practical to visualize, as shown in the following example.

ex Savani - von Stengel, pag. 11; fig 8 are Schegel diagrams of BR polytopes.

*Example 3.1.*

Theorem 3 has a straightforward dual version. Let  $P$  be a simplicial polytope in dimension  $m$  with  $n$  vertices. We *pivot on facets* by moving from facet  $F$  to facet  $G$  that shares an edge with  $F$ . Since  $P$  is simplicial, there are exactly  $m$  possible choices for  $G$ . Let  $l_v : [n] \rightarrow [m]$  be a labeling of the vertices of  $P$  such that there is at least one completely labeled facet  $F_0$ . We *drop label  $k$*  and *pick up label  $j$*  when pivoting from a facet  $F$  to a facet  $G$  that shares with  $F$  all vertices except a vertex  $v$  with label  $k$  that belongs to  $F$  but not  $G$ , and another vertex  $w$  with label  $j$  that belongs to  $G$  but not  $F$ . The Lemke-Howson algorithm then becomes the *exchange algorithm* as follows:

reference!

Considering the dual of Lemke paths on (almost) completely labeled facets, we get the dual result to theorem 3.

**Theorem 4.** *The exchange algorithm returns a solution to the problem AN-OTHER COMPLETELY LABELED FACET.*

If the

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**Algorithm 2:** Exchange algorithm

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**input** : A simplicial  $m$ -polytope  $P$  with  $n$  vertices. A labeling  $l_v : [n] \rightarrow [m]$  of the vertices of  $P$ . A vertex  $F_0$  of  $P$ , completely labeled for  $l$ .

**output:** A completely labeled facet  $F \neq F_0$  of  $P$ .

```
1 choose a label  $k \in [n]$ 
2 pivot from  $F_0$  to  $F$  dropping label  $k$ 
3 while  $F$  is not completely labeled do
4   pivot from  $F$  to  $F'$  dropping duplicate label  $j$ , moving away from
    $F_0$ 
5   rename  $F_0 = F$ ,  $F = F'$ 
6 return  $y$ 
```

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sp case cycl poly leads to gs

6 in thm 2

then gs:

first we note a result on generic unit vector games (SvS 15)...

(here? before? - it's on all games! - nowhere?) thm Savani '06 - pg 12

SvS-15: not only LH paths on  $P \times Q$  simple path, but projection to simple paths on both  $P$  and  $Q$ .

In the case of a unit vector game  $(U, B)$ , where  $U = (e_{l(1)} \cdots e_{l(n)})$  for a labeling  $l : [n] \rightarrow [m]$ , let as in 2; theorem 1 then shows that  $P^l$  encodes all the Nash equilibria of  $(U, B)$  as completely labeled vertices. We can apply the Lemke-Howson algorithm to both  $P^l$  and to the product of the best response polytopes  $P$  and  $Q$ . The projection of a Lemke path for a missing label  $i \in [m]$  on  $P \times Q$  to  $P$  defines a Lemke path in  $P^l$ . However,  $P \times Q$  has  $m + n$  labels, therefore there could be Lemke paths for a missing label  $m + j$  with  $j \in [n]$  on  $P \times Q$  that get lost in the projection on  $P^l$ . The following theorem, proved in Savani and von Stengel [15], shows that there

how?

is no loss of generality in studying Lemke paths on  $P^l$ .

**Theorem 5.** *Let  $(U, B)$  be a unit vector game, with  $U = (e_{l(1)} \cdots e_{l(n)})$  for a labeling  $l : [n] \rightarrow [m]$ ; let  $P = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, B^\top x \leq \mathbf{1}\}$  and  $Q = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0}, Ay \leq \mathbf{1}\}$ , as in 1; and let  $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, B^\top x \leq \mathbf{1}\}$  as in 2. Then the Lemke path on  $P \times Q$  for the missing label  $k$  projects to a path on  $P$  that is the Lemke path on  $P^l$  for missing label  $k$  if  $k \in [m]$ , and for missing label  $l(j)$  if  $k = m + j$  with  $j \in [n]$ .*

so if poly dual cyclic, when cons from pow gale strings we could take the str of labels  $l(n+j) \cdots l(n+m)$  instead of  $l(1) \cdots l(n+m)$ , that is, we could cut the “artificial” first labels  $12 \dots n$ .

After all, in main we’re studying ANOTHER GALE in general, not nec starting from  $12 \dots n$ ; and we’re interested in finding *one* eq that’s not the one we started from (and is at other end of LPath, since index and so on), *not all equilibria*; but the eq we started from is not nec the artificial one - actually, if we go with this we can take any NE to start looking for another, and we’re sure to find a “non-artificial” one. Note: if we were looking for all NE, LH doesn’t work anyway - see ex by Wilson in Shapley, where “disconnected” paths between equilibria.

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complexity considerations, PPAD

Needed for why result interesting; it can be done at the end, with Morris paths



### 3.2 The Complexity of GALE and ANOTHER GALE

file: main-result-subsection

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