

# 1 Complexity Classes, Games, Polytopes and Gale Strings

## 1.1 The Complexity Classes P and PPAD

## 1.2 Normal Form Games and Nash Equilibria

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## 1.3 Bimatrix Games and Best Response Polytopes

read: unit games article on vS website

Nash equilibria can be analysed from a geometrical point of view, using polytopes.

In the following, we will consider vectors  $u, v \in \mathbb{R}^d$  as column vectors, so  $u^\top v$  is their scalar product. A vector in  $\mathbb{R}^d$  for which all components are 0's will be denoted as  $\mathbf{0}$ ; similarly, a vectors for which all components are 1's will be denoted as  $\mathbf{1}$ . The  $i$ -th component of the *unit vector*  $e_i$  is 1, whereas all the other components are 0's. An inequality of the form  $u \geq v$  (and analogous) holds for every component; that is,  $u_i \geq v_i$  for all  $i \in [d]$ .

An *affine combination* of points in an Euclidean space  $z_1, \dots, z_k$  is

$$\sum_{i=1}^k \lambda_i z_i \quad \text{where } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^k \lambda_i = 1$$

If furthermore  $\lambda_i \geq 0$  for all  $i \in [k]$

An affine combination of points  $z_1, \dots, z_k$  in some Euclidean space is of the form  $\sum_{i=1}^k \lambda_i z_i$  where  $\lambda_1, \dots, \lambda_k$  are reals with  $\sum_{i=1}^k \lambda_i = 1$ . It is called a convex combination if  $\lambda_i \geq 0$  for all  $i$ . A set of points is convex if it is closed under forming convex combinations. Given points are affinely independent if none of these points is an affine combination of the others. A convex set has dimension  $d$  if and only if it has  $d + 1$ , but no more, affinely independent points.

A ( $d$ -dimensional) *simplicial polytope*  $P$  is the convex hull of a set of at least  $d + 1$  points  $v$  in  $\mathbb{R}^d$  in general position, that is, no  $d + 1$  of them are on a common hyperplane.

If a point  $v$  cannot be omitted from these points without changing  $P$  then  $v$  is called a *vertex* of  $P$ . A *facet* of  $P$  is the convex hull  $\text{conv } F$  of a set  $F$  of  $d$  vertices of  $P$  that lie on a hyperplane  $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$  so that  $a^T u < a_0$  for all other vertices  $u$  of  $P$ ; the vector  $a$  (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices  $F$ .

The following theorem, due to Balthasar and von Stengel [?, ?], establishes a connection between general labeled polytopes and equilibria of certain  $d \times n$  bimatrix games  $(U, B)$ .

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**Theorem 1.** *Consider a labeled  $d$ -dimensional simplicial polytope  $Q$  with  $\mathbf{0}$  in its interior, with vertices  $-e_1, \dots, -e_d, c_1, \dots, c_n$ , so that  $F_0 = \text{conv}\{-e_1, \dots, -e_d\}$  is a facet of  $Q$ . Let  $-e_i$  have label  $i$  for  $i \in [d]$ , and let  $c_j$  have label  $l(j) \in [d]$  for  $j \in [n]$ . Let  $(U, B)$  be the  $d \times n$  bimatrix game with  $U = [e_{l(1)} \cdots e_{l(n)}]$  and  $B = [b_1 \cdots b_n]$ , where  $b_j = c_j / (1 + \mathbf{1}^T c_j)$  for  $j \in [n]$ . Then the completely labeled facets  $F$  of  $Q$ , with the exception of  $F_0$ , are in one-to-one correspondence to the Nash equilibria  $(x, y)$  of the game  $(U, B)$  as follows: if  $v$  is the normal vector of  $F$ , then  $x = (v + \mathbf{1})\mathbf{1}^T(v + \mathbf{1})$ , and  $x_i = 0$  if and only if  $-e_i \in F$  for  $i \in [d]$ ; any other label  $j$  of  $F$ , so that  $c_j$  is a vertex of  $F$ , represents a pure best reply to  $x$ . The mixed strategy  $y$  is the uniform distribution on the set of pure best replies to  $x$ .*

In the preceding theorem, any simplicial polytope can take the role of  $Q$  as long as it has one completely labeled facet  $F_0$ . Then an affine transformation, which does not change the incidences of the facets of  $Q$ , can be used to map  $F_0$  to the negative unit vectors  $-e_1, \dots, -e_d$  as described, with  $Q$  if necessary expanded in the direction  $\mathbf{1}$  so that  $\mathbf{0}$  is in its interior.

A  $d \times n$  bimatrix game  $(U, B)$  is a *unit vector game* if all columns of

$U$  are unit vectors. For such a game  $B$  with  $B = [b_1 \cdots b_n]$ , the columns  $b_j$  for  $j \in [n]$  can be obtained from  $c_j$  as in Theorem 1 if  $b_j > \mathbf{0}$  and  $\mathbf{1}^T b_j < 1$ . This is always possible via a positive-affine transformation of the payoffs in  $B$ , which does not change the game. The unit vectors  $e_{l(j)}$  that constitute the columns of  $U$  define the labels of the vertices  $c_j$ . The corresponding polytope with these vertices is simplicial if the game  $(U, B)$  is nondegenerate [?], which here means that no mixed strategy  $x$  of the row player has more than  $|\{i \in [d] \mid x_i > 0\}|$  pure best replies. Any game can be made nondegenerate by a suitable “lexicographic” perturbation of  $B$ , which can be implemented symbolically.

Unit vector games encode arbitrary bimatrix games: An  $m \times n$  bimatrix game  $(A, B)$  with (w.l.o.g.) positive payoff matrices  $A, B$  can be symmetrized so that its Nash equilibria are in one-to-correspondence to the symmetric equilibria of the  $(m+n) \times (m+n)$  symmetric game  $(C^T, C)$  where

$$C = \begin{pmatrix} 0 & B \\ A^T & 0 \end{pmatrix}.$$

In turn, as shown by McLennan and Tourky [?], the symmetric equilibria  $(x, x)$  of any symmetric game  $(C^T, C)$  are in one-to-one correspondence to the Nash equilibria  $(x, y)$  of the “imitation game”  $(I, C)$  where  $I$  is the identity matrix; the mixed strategy  $y$  of the second player is simply the uniform distribution on the set  $\{i \mid x_i > 0\}$ . Clearly,  $I$  is a matrix of unit vectors, so  $(I, C)$  is a special unit vector game.

BEARSTEARD Blue

Nina Watson

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## 1.4 Cyclic Polytopes and Gale Strings

## 1.5 Labeling and the Problem ANOTHER GALE

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## References

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