1 Introduction

2 Complexity, Games, Polytopes and Gale Strings

2.1 Some Complexity Classes

2.2 Normal Form Games and Nash Equilibria

file: background-subsection

2.3 Some Geometrical Notation

2.4 Bimatrix Games, Labels and Polytopes

file: polytopes-subsection

$$P = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}, \quad Q = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}, \tag{1}$$

a point in P has label k iff $x_k = 0$ for $k \in \{1, \ldots, m\}$ or $(B^\top x)_{k-m} = 0$ for $k \in \{m+1, \ldots, m+n\}$; analogously, a point in Q has label k if and only if either $y_{k-m} = 0$ for $k \in \{m+1, \ldots, m+n\}$ or $(Ay)_k$ for $k \in \{m+1, \ldots, m+n\}$.

proposition 1. *Let (A, B) be a bimatrix game and (x, y) be one of its Nash equilibria. Then (z, z), where $z = (x, y)$, is a Nash equilibrium of the symmetric game $(C, C^\top)$, where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}.$$

Theorem 1. *[15] Let $l : [n] \to [m]$, and let (U, B) be the unit vector game where $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. Consider the polytopes P^l and Q^l where*

$$P^l = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\} \tag{2}$$

$$Q^l = \{y \in \mathbb{R}^n | y \geq \mathbf{0}, \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\} \tag{3}$$

where $N_i = \{j \in [n] | l(j) = i\}$ *for* $i \in [m]$.

Label every facet of P^l *according to the inequality defining it, as follows:*

- $x_i \geq 0$ *has label* i*, for* $i \in [m]$
- $(B^\top x)_j \leq 1$ *has label* $l(j)$*, for* $j \in [n]$

Then $x \in P^l$ *is a completely labeled point of* $P^l \setminus \{\mathbf{0}\}$ *if and only if there is some* $y \in Q^l$ *such that, after scaling, the pair* (x, y) *is a Nash equilibrium of* (U, B)

2.5 Cyclic Polytopes and Gale Strings

2.6 Labeling and the Problem Another Gale

file: gale-def-subsection

3 Algorithmic and Complexity Results

3.1 Lemke Paths and the Lemke-Howson for Gale Algorithm

We have NEs $\Leftrightarrow$ completely labeled things (facets, vertices, GS) We give now different versions of fundam algorithm to deal with labeling looking for compl.label. - in particular in these cases
first in version on simple polytopes with labeled facets, (name: Lemke-Howson; Lemke-Howson 1964, Shapley 1974 beautiful exposition)
then in its corresponding version on Gale strings (name: Lemke-Howson for Gale, where???).
We will also mention dual version on simplicial polytopes with labeled vertices (name: exchange algorithm, Edmonds - Sanità).
Sth about why (exp lemke paths; conjecture)

Consider a labeling $l : [n] \to [m]$, where $n \geq m$; then $x = \{l(i) \in [n] \mid i \in [m]\}$ is *almost completely labeled* if $x = [m] \setminus \{k\}$ for exactly one *missing label* $k \in [m]$. Since $|x| = m$, this mean that all other labels appear once in x except for one *duplicate label* j that appears twice.

Let P be a simple polytope in dimension m with n facets. We define the operation of *pivoting on vertices* as moving from a vertex x of P to another vertex y such that there is an edge between x and y. Note that, since P is simple, there are exactly m possible choices for y.

Now let $l : [n] \to [m]$ be a labeling of the facets of P such that there is at least one completely labeled vertex x_0 of P. Note that if we pivot from a vertex we "leave behind" a facet, that has label k; we call this *dropping label* k. We will then reach a vertex that lies on a new facet, that has label j; we will call this *picking up label* j We give the *Lemke-Howson algorithm* as follows: reference!

Algorithm 1: Lemke-Howson algorithm

input : A simple m-polytope P with n facets. A labeling $l : [n] \to [m]$ of the facets of P. A vertex x of P, completely labeled for l.

output: A completely labeled vertex $y \neq x_0$ of P.

choose a label $k \in [n]$
pivot from x to y dropping label k
while *y is not completely labeled* **do**
    let j be the duplicate label of y
    pivot from y to y' dropping label j *on the facet shared with x*
    rename $y = y'$
return y

no cycles (ref? from Lemke? LH? Of course well explained in Shapley...) which is condition for the alg to effectively return $y \neq x$
thm: LH returns sol of ANOTHER CL VERTEX
complexity considerations, PPAD (check def PPAD: do we need next step in P-time?)
simple paths - Lemke paths: def so by Morris 94
later: extend term "Lemke paths" to paths of every LH-style algorithms we see. ... In this case, Lemke paths "translate" as / correspond to...

In the context of finding the Nash equilibrium of a bimatrix game (A, B), there are two equivalent implementations of the Lemke-Howson algorithm.

We can consider the game C as in proposition 1, and the associated polytope $S = \{z \in \mathbb{R}^{m+n} \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1}\}$, labeling the $2(m+n)$ inequalities defining the facets of of S as $1, \ldots, m+n, 1, \ldots, m+n$. Then applying the Lemke-Howson algorithm starting from vertex $\mathbf{0}$ returns a Nash equilibrium (z, z) of C and a corresponding $(x, y) = z$ a Nash equilibrium of (A, B).

We can also follow the "traditional" version of the Lemke-Howson algorithm; a very clear exposition of this can be found in Shapley [16]. Let P and Q be the best response polytopes of (A, B) as in 1. We then move alternately on P and Q, starting from the couple of vertices $(\mathbf{0}, \mathbf{0})$. Since we move in $\mathbb{R}^m$ and $\mathbb{R}^n$ instead of $\mathbb{R}^{m+n}$, this version is more practical to visualize, as shown in the following example.

Example 3.1. ex Savani - von Stengel, pag. 11; fig 8 are Schegel diagrams of BR polytopes.

In the case of unit vector games (U, B),

(?) thm Savani: not only simple path, but projection to simple paths.
note, thm SvS-15: P^l, dim m, is enough to study all. (we could take the str of labels for the gale pow $l(n+j)\cdots l(n+m)$ instead of $l(1)\cdots l(n+m)$), that is, we could cut the "artificial" first labels 12...n.
Do we go straight for this, or? YES!
After all, in main we're studying ANOTHER GALE in general, not nec starting from 12...n; and we're interested in finding *one* eq that's not the one we started from (and is at other end of LPath, since index and so on), *not all equilibria*; but the eq we started from is not nec the artificial one - actually, if we go with this we can take any NE to start looking for another, and we're sure to find a "non-artificial" one. Note: if we were looking for all NE, LH doesn't work anyway - see ex by Wilson in Shapley, where "disconnected" paths between equilibria.

3.2 The Complexity of Gale and Another Gale

file: main-result-subsection

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