1 Complexity, Games, Polytopes and Gale Strings

1.1 Some Complexity Classes

references - cite Papadimitriou (book) for general; Papadimitriou 1994 for PPAD

A *computational problem* is given by the combination of an *input* and a related *output*. A specific input gives an *instance* of the problem.

Computational problems can be classified according to the form of their output. A *function problem* P returns for an intance x an output y that satisfies a given binary relation $R(x, y)$. In the case of a *decision problems*, y is either "YES" or "NO". The *complement* of a decision problem P is the problem $\bar{P}$ that returns "NO" for each instance of P that returns "YES", and vice versa.

Search problems are function problems that return either an output y such that $R(x, y)$, or "NO", if it's not possible to find any such y. If y is guaranteed to exist, the problem is called a *total function problem*. *Counting problems* return the *number* of y's that satisfy $R(x, y)$; given a problem R we denote the associated counting problem $\#R$.

An example of decision problem is: "(input) given a graph, (question) is it possible to find an Euler tour of the graph?" Its complement is "(input) given a graph, (question) is it possible that there isn't any Euler of the graph?" A search problem is: "(input) given a graph, (output) return one Euler tour of the graph, or "NO" if no such tour exists." A total function problem is: "(input) given an Euler graph, (output) return one of its Euler tours." Finally, a counting problem is "(input) given a graph, (output) return the number of its Euler tours."

Computational problems are also classified according to their *computational complexity*, given by the *reducibility* from each other.

Turing machines: here - not that in the following deterministic TM

Let P_1 be a computational problem. For an instance x of P_1, let $|x|$

be the the number of bits needed to encode x. P_1 *reduces to the problem* P_2 *in polynomial time*, denoted $P_1 \leq_P P_2$, if there exists a *polynomial-time reduction*, that is, a function $f : \{0,1\}^* \to \{0,1\}^*$ and a Turing machine $\mathcal{M}$ such that for all $x \in \{0,1\}^*$

1. $x \in P_1 \quad \iff \quad f(x) \in P_2$;
2. $\mathcal{M}$ computes $f(x)$;
3. $\mathcal{M}$ stops after $p(|x|)$ steps, where p is a polynomial.

For any class C of decision problems, the class of all complements of the problems in C is the *complement class* $\text{co} - \text{C}$. A problem P is *hard* for a class C if for every problem P_C in C there is a polynomial-time reduction to P; that is, if P is hard to solve at least as every problem in C. A $\text{C} - hard$ problem in C is *complete* for C.

The complexity class **P** contains all the *polynomially decidable problems*, that is, all problems P such that there exists a Turing machine $\mathcal{M}$ that outputs either "YES" or "NO" for all inputs $x \in \{0,1\}^*$ of P after $p(|x|)$ steps, where p is a polynomial. Intuitively, a decision problem is in **P** if the answer to its question can be found in a number of steps that is polynomial in the input of the problem.

A problem P belongs to the class **NP**, *non-deterministic polynomial-time problems*, if there exists a Turing machine $\mathcal{M}$ and polynomials p_1, p_2 such that

1. for all $x \in P$ there exists a *certificate* $y \in \{0,1\}^*$ which satisfies $|y| \leq p_1(|x|)$;
2. $\mathcal{M}$ accepts the combined input xy, stopping after at most $p_2(|x| + |y|)$ steps;
3. for all $x \notin P$ there does not exist $y \in \{0,1\}^*$ such that $\mathcal{M}$ accepts the combined input xy.

This means that a decision problem is in **NP** if it takes polynomial time to verify whether the "certificate solution" y is, indeed, a correct answer to the question posed by the problem. The class #**P** is the class of all problems that output the number of possible certificates for a problem in **NP**.

check formal def of # P (?)

In [7], Megiddo and Papadimitriou introduce the classes **FNP**, *function non-deterministic polynomial*, and **TFNP**, *total function non-deterministic polynomial*. The former is defined as the class of binary relations $R(x, y)$ such that there is a polynomial-time algorithm that decides whether $R(x, y)$ holds for given x, y satisfying $|y| \leq p(|x|)$, where p is a polynomial. The latter is the class of all such problems for which y is guaranteed to exist. Intuitively, **FNP** and **TFNP** are similar to **NP**, but they allow for problems of (respectively) function and total function form.

In [7], Megiddo and Papadimitriou also prove that, unless $\mathbf{NP} = \mathbf{co} - \mathbf{NP}$, it's impossible to find a **TFNP**-complete problem. To circumvent this limitation of **TFNP**, Papadimitriou ([9]) focused on the problems for which the existence of a solution is proved by a "parity argument", introducing the classes **PPA** (*Proof by Parity Argument*) and **PPAD** (*Proof by Parity Argument, Directed version*).

definition of PPA(D): one in Papadimitriou 1994 and one in DGP - the second w END OF THE LINE, use that one.

1.2 Normal Form Games and Nash Equilibria

n-NASH in TFNP and typical problem pushing the def of PPAD ([9]); in fact, PPAD-complete ([3] for $n \geq 3$, [2] for $n = 2$).

until here

1.3 Best Response Polytopes

file: polytopes

1.4 Cyclic Polytopes and Gale Strings

1.5 The Problem Another Gale

file: gale-def

merge in one section "Gale strings" or "CP and GS"?

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