Abstract

This thesis presents a report on original research, published as joint work with Merschen and von Stengel in *Electronic Notes in Discrete Mathematics* (2010). Our result shows a polynomial time algorithm to find a Nash equilibrium for a particular class of games, which was previously used by Savani and von Stengel (2006) as an example of exponential time for the classical Lemke-Howson algorithm for bimatrix games (1964).

It was conjectured that solving these games via the Lemke-Howson algorithm was complete in the class PPAD (Proof by Parity Argument, Directed version). A major motivation for the definition of this class by Papadimitriou (1994) was, in turn, to capture the pivoting technique of many results related to the Nash equilibrium, including the Lemke-Howson algorithm. A PPAD-completeness proof of the games we consider would have provided a traceable proof of the Daskalakis, Goldberg and Papaditrimiou (2005) and Chen and Deng (2009) results about the PPAD-completeness of every normal form game. Our result of polynomial-time solvability, on the other hand, indicates the existence of a special class of games, unless PPAD = P.

Our proof exploits two results. The first one is the representation of the Nash equilibria of these games as a string of labels and an associated string of 0s and 1s satisfying some conditions, called *Gale conditions*, as seen in Savani and von Stengel (2006). The second one is the polynomial-time solvability of the problem of finding a perfect matching in a graph, solved by Edmonds (1965).

Further results by Merschen (2012) and Végh and von Stengel (2014) solved the open problem of the *sign* of the equilibrium found in polynomial time.

1 Introduction

2 Bimatrix Games and Polytopes

A (d-dimensional) *simplicial polytope* P is the convex hull of a set of at least $d+1$ points v in $\mathbb{R}^d$ in general position, that is, no $d+1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P. A *facet* of P is the convex hull $\operatorname{conv} F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F.

The following theorem, due to Balthasar and von Stengel [?, ?], establishes a connection between general labeled polytopes and equilibria of certain $d \times n$ bimatrix games (U, B).

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Theorem 2.1 *Consider a labeled d-dimensional simplicial polytope Q with $\mathbf{0}$ in its interior, with vertices $-e_1, \ldots, -e_d, c_1, \ldots, c_n$, so that $F_0 = \operatorname{conv}\{-e_1, \ldots, -e_d\}$ is a facet of Q. Let $-e_i$ have label i for $i \in [d]$, and let c_j have label $l(j) \in [d]$ for $j \in [n]$. Let (U, B) be the $d \times n$ bimatrix game with $U = [e_{l(1)} \cdots e_{l(n)}]$ and $B = [b_1 \cdots b_n]$, where $b_j = c_j/(1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Then the completely labeled facets F of Q, with the exception of F_0, are in one-to-one correspondence to the Nash equilibria (x, y) of the game (U, B) as follows: if v is the normal vector of F, then $x = (v + \mathbf{1})/\mathbf{1}^\top(v + \mathbf{1})$, and $x_i = 0$ if and only if $-e_i \in F$ for $i \in [d]$; any other label j of F, so that c_j is a vertex of F, represents a pure best reply to x. The mixed strategy y is the uniform distribution on the set of pure best replies to x.*

In the preceding theorem, any simplicial polytope can take the role of Q as long as it has one completely labeled facet F_0. Then an affine transformation, which does not change the incidences of the facets of Q, can be used to map F_0 to the negative unit vectors $-e_1, \ldots, -e_d$ as described, with Q if necessary expanded in the direction $\mathbf{1}$ so that $\mathbf{0}$ is in its interior.

A $d \times n$ bimatrix game (U, B) is a *unit vector game* if all columns of U are unit vectors. For such a game B with $B = [b_1 \cdots b_n]$, the columns b_j for $j \in [n]$ can be obtained from c_j as in Theorem 2.1 if $b_j > \mathbf{0}$ and $\mathbf{1}^T b_j < 1$. This is always possible via a positive-affine transformation of the payoffs in B, which does not change the game. The unit vectors $e_{l(j)}$ that constitute the columns of U define the labels of the vertices c_j. The corresponding polytope with these vertices is simplicial if the game (U, B) is nondegenerate [?], which here means that no mixed strategy x of the row player has more than $|\{i \in [d] \mid x_i > 0\}|$ pure best replies. Any game can be made nondegenerate by a suitable "lexicographic" perturbation of B, which can be implemented symbolically.

Unit vector games encode arbitrary bimatrix games: An $m \times n$ bimatrix game (A, B) with (w.l.o.g.) positive payoff matrices A, B can be symmetrized so that its Nash equilibria are in one-to-correspondence to the symmetric equilibria of the $(m+n) \times (m+n)$ symmetric game (C^T, C) where

$$C = \begin{pmatrix} 0 & B \\ A^\top & 0 \end{pmatrix}.$$

In turn, as shown by McLennan and Tourky [?], the symmetric equilibria (x, x) of any symmetric game (C^T, C) are in one-to-one correspondence to the Nash equilibria (x, y) of the "imitation game" (I, C) where I is the identity matrix; the mixed strategy y of the second player is simply the uniform distribution on the set $\{i \mid x_i > 0\}$. Clearly, I is a matrix of unit vectors, so (I, C) is a special unit vector game.

3 Cyclic Polytopes and Gale Strings

A *cyclic polytope* P in dimension d with n vertices is the convex hull of n points $\mu(t_j)$ on the *moment curve*

$$\mu : t \mapsto (t, t^2, \ldots, t^d)^\top \text{ for } j \in [n]$$

Suppose that $t_1 < t_2 < \cdots < t_n$. Then the facets of P are encoded by $G(d, n)$, that is F is a facet of P if and only if

$$F = \operatorname{conv}\{\mu(t_i) \mid i \in 1(s)\} \text{ for some } s \in G(d, n)$$

as shown by Gale [?].

Let $[k] = \{1, \ldots, k\}$ for any positive integer k. For any set S, we can represent the function $s : [k] \to S$ as the string $s = s(1)s(2)\cdots s(k)$. Let $A \subset [k]$, and $S = \{0, 1\}$. We will then have a correspondence between the function $s : A \to \{0, 1\}$ and a *bitstring* s, that in turn corresponds to the indicator function of the set

$$\begin{aligned} 1(s) &= s^{-1}(1) \\ &= \{j \in [k] \mid s(j) = 1\} \end{aligned}$$

For a bitstring s, a maximal substring of s of consecutive 1's is called a *run*.

Definition $G(d,n)$ is the set of all bitstrings s of length n such that

1. exactly d bits in s are 1 and
2. s fulfills the *Gale evenness condition*:

$$01^k0 \text{ is a substring of } s \quad \Rightarrow \quad k \text{ is even.}$$

An element of of $G(d,n)$ is called a *Gale string of dimension d and length n*.

Definition 3 characterises Gale strings as bitstrings of length n with exactly d elements equal to 1, such that interior runs (that is, runs bounded on both sides by 0s) must be of even length. Note that this condition allows Gale strings to start or end with an odd-length run. If d is even, then any s in $G(d,n)$ that starts with an odd run also ends with an odd run; we can then consider the Gale string as a "loop" by "glueing together" the extremes of the string to form an even run. The set of Gale strings of even dimension is therefore invariant under a cyclic shift of the strings.

As an example, we can consider $G(4,6)$. We have

$$G(4,6) = \{111100, 111001, 110011, 100111, 001111, 110110, 101101, 011011\}$$

The strings 111100, 111001, 110011, 100111 and 001111 are equivalent under a cyclic shift, as are the strings 110110, 101101 and 011011.

From this point forward, we will assume that d is even.

Given a set G of bit strings of length n and a parameter d, a *labeling* is a function $l : [n] \to [d]$. Given a labeling, a string s in G is called *completely labeled* if $l(1(s)) = [d]$, that is, if every label in $[d]$ appears as $l(i)$ for at least one bit $s(i)$ so that $s(i) = 1$. Clearly, if s is completely labeled, then s has at least d bits that are 1, and if exactly d bits in s are 1, then every label in $[d]$ occurs exactly once.

For this cyclic polytope P, a labeling $l : [n] \to [d]$ can be understood as a label $l(j)$ for each vertex $\mu(t_j)$ for $j \in [n]$. A completely labeled Gale string s therefore represents a facet F of P that is completely labeled.

Special games are obtained by using cyclic polytopes in Theorem 2.1, suitably affinely transformed with a completely labeled facet F_0. When Q is a cyclic polytope in dimension d with $d+n$ vertices, then the string of labels $l(1)\cdots l(n)$ in Theorem 2.1 defines a labeling $l' : [d+n] \to [d]$ where $l'(i) = i$ for $i \in [d]$ and $l'(d+j) = l(j)$ for $j \in [n]$. In other words, the string of labels $l(1)\cdots l(n)$ is just prefixed with the string $1\,2\cdots d$ to give l'. Then l' has a trivial completely labeled Gale string 1^d0^n which defines the facet F_0. Then the problem ANOTHER COMPLETELY LABELED GALE STRING defines exactly the problem of finding a Nash equilibrium of the unit vector game (I,B). Note again that B is here not a general matrix (which would define a general game) but obtained from the last n of $d+n$ vertices of a cyclic polytope in dimension d.

3.1 The Lemke-Howson Algorithm and the Class PPAD

4 The Complexity of Completely Labeled Gale Stringand Another Completely Labeled Gale String

We consider the following decision problem.

Completely Labeled Gale String

Input: A labeling $l : [n] \to [d]$, where d is even and $d < n$.

Question: Is there a completely labeled Gale string s in $G(d,n)$?

For example, given the string of labels $l = 123432$, we see that there are four associated completely labeled Gale strings: 111100, 110110, 100111 and 101101. For 121314, there are no completely labeled Gale strings.

123432	**12**3**43**2	**1**23**432**	**1**2**34**3**2**
111100	110110	100111	101101

The set $G(d,n)$ of Gale strings has a combinatorial structure that allows the use of a "parity argument", which we consider in detail later, to show the following known property; it holds for odd d as well but we assume throughout that d is even.

Theorem 4.1 *For any labeling $l : [n] \to [d]$, where d is even and $d < n$, the number of completely labeled Gale strings in $G(d,n)$ is even.*

Theorem 4.1 implies that if there is one completely labeled Gale string, there is also a second one. The following function problem asks to compute a completely labeled Gale string if one such string is already given.

Another completely labeled Gale string

Input: A labeling $l : [n] \to [d]$, where d is even and $d < n$, and a completely labeled Gale string s in $G(d,n)$.

Output: A completely labeled Gale string s' in $G(d,n)$ where $s' \neq s$.

We now show that both Completely Labeled Gale Stringand Another Completely Labeled Gale Stringcan be solved in polynomial time.

The proof uses a reduction to the following problem, which was first shown to be solvable in polynomial time by Edmonds [**?**].

PERFECT MATCHING

Input:	Graph $G = (V, E)$.
Question:	Is there a set $M \subseteq E$ of pairwise non-adjacent edges so that every vertex $v \in V$ is incident to exactly one edge in M?

Theorem 4.2 *The problems* COMPLETELY LABELED GALE STRING *and* ALMOST COMPLETELY LABELED GALE STRING *can be solved in polynomial time.*

Proof. We give a rather simple reduction to PERFECT MATCHING. Given the labeling $l : [n] \to [d]$, construct the (multi-)graph G with vertex set $V = [d]$ and up to n (possibly parallel) edges with endpoints $l(i), l(i+1)$ for $i \in [n]$ whenever these endpoints are distinct (so G has no loops); here we let $n+1 = 1$ ("modulo n") so that $n, n+1$ is to be understood as $n, 1$. Then a completely labeled Gale string s in $G(d,n)$ splits into a number of runs which are uniquely split into $d/2$ pairs $i, i+1$ so that the labels $l(i)$ and $l(i+1)$ are distinct, and all labels $1, \ldots, n$ occur among them. So this defines a perfect matching for G.

Conversely, a perfect matching M of G defines a Gale string s where $s(i) = s(i+1) = 1$ if the edge that joins $l(i)$ and $l(i+1)$ is in M and $s(i) = 0$ otherwise, so s is completely labeled. This shows how COMPLETELY LABELED GALE STRING reduces to PERFECT MATCHING. Finding a perfect matching, or deciding that G has none, can be done in polynomial time [**?**].

The reduction for ANOTHER COMPLETELY LABELED GALE STRING is an extension of this. Consider the given completely labeled Gale string s and the matching M for it. If G has multiple edges between two nodes and one of them is in M, simply replace that edge by a parallel edge to obtain another completely labeled Gale string s'. Hence, we can assume that M has no edges that have a parallel edge. Another completely labeled Gale string s' exists by Theorem 4.1. The corresponding matching M' does not use at least one edge in M. Hence, at least one of the $d/2$ graphs G which have one of the edges of M removed has a perfect matching M', which is a perfect matching of G, and which defines a completely labeled Gale string s' different from s. The search for M' takes again polynomial time. $\square$

The significance of Theorem 4.2 is to be understood in the context of equilibrium computation for games, which we discuss next. The remainder of this paper contains only known results.

5 Further results

Appendix A: Notation

For a matrix A we denote its transpose with A^T. We treat vectors u, v in $\mathbb{R}^d$ as column vectors, so $u^T v$ is their scalar product. By $\mathbf{0}$ we denote a vector of all 0's, of suitable dimension, by $\mathbf{1}$ a vector of all 1's. A unit vector, which has a 1 in its ith component and 0 otherwise, is denoted by e_i. Inequalities like $u \geq \mathbf{0}$ hold for all components. For a set of points S we denote its convex hull by $\operatorname{conv} S$.

For $n \in \mathbb{N}$ we denote $[n] = 1, 2, \ldots, n$

Appendix B: A result about PPAD completeness of NASH

References