Complexity of the Gale String Problem for Equilibrium Computation in Games

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Declaration

I certify that this thesis I have presented for examination for the MPhil degree of the London School of Economics and Political Science is based on joint work with Julian Merschen and Bernhard von Stengel, published in [3].

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Abstract

This thesis presents a report on original research, published as joint work with Merschen and von Stengel in *Electronic Notes in Discrete Mathematics* [3]. Our result shows a polynomial time algorithm to solve two problems related to labeled Gale strings, a combinatorial structure introduced by Gale in [9] that can be used in the representation of a particular class of games.

These games were used by Savani and von Stengel [19] [20] as an example of exponential running time for the classical Lemke-Howson algorithm to find a Nash equilibrium of a bimatrix game [11]. It was conjectured that solving these games via the Lemke-Howson algorithm was complete in the class **PPAD** (Proof by Parity Argument, Directed version). A major motivation for the definition of this class by Papadimitriou [18] was, in turn, to capture the pivoting technique of many results related to the Nash equilibrium, including the Lemke-Howson algorithm.

Our result, on the contrary, sets apart this class of games as a case for which there is a polynomial-time algorithm to find a Nash equilibrium. Since Daskalakis, Goldberg and Papaditrimiou [5] and Chen and Deng [4] proved the **PPAD**-completeness of finding a Nash equilibrium in general normal-form games, we have a special class of games, unless **PPAD** = **P**.

Our proof exploits two results. The first one is the representation of the Nash equilibria of these games as Gale strings, as seen in Savani and von Stengel [19] [20]. The second one is the polynomial-time solvability of the problem of finding a perfect matching in a graph, proven by Edmonds [6].

Merschen [14] and Végh and von Stengel [22] expanded our technique to prove further interesting results about index of a Nash equilibrium, as in Shapley [21], and orientation of oiks, first introduced by Edmonds [7] and Edmonds and Sanità [8].

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Introduction

Savani and von Stengel [19] [20] have used this labeling to build Gale games that are "hard to solve" The Gale game immediately associated with the labeling gives an example of exponential running time for the Lemke-Howson algorithm, but it is still quite tractable by support enumeration; a more complex construction given in [19] and later simplified in [20]

The topic of this thesis is a problem in the field of *algorithmic game theory*, that is, the study of game-theoretic problems from the point of view of computer science. In particular, we focus on the computational complexity of a particular class of games. These

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Savani and von Stengel (2006) showed that the LH algorithm may take exponentially many steps. Their construction uses âĂIJdual cyclic polytopesâĂİ which have a well-known vertex structure for any dimension and number of linear inequalities. Morris (1994) used similarly labeled dual cyclic polytopes where all âĂIJLemke pathsâĂİ are exponentially long. A Lemke path is related to the path computed by the LH algorithm, but is defined on a single polytope that does not have a product structure corresponding to a bimatrix game. The completely labeled vertex found by a Lemke path can be interpreted as a symmetric equilibrium of a symmetric bimatrix game. However, as in the example in Figure 4 below, such a symmetric game may also have

nonsymmetric equilibria which here are easy to compute, so that the result by Morris (1994) seemed not suitable to describe games that are hard to solve with the LH algorithm.

The âĂIJimitiation gamesâĂİ defined by McLennan and Tourky (2010) changed this picture. In an imitation game, the payoff matrix of one of the players is the identity matrix. The mixed strategy of that player in any Nash equilibrium of the imitation game corresponds exactly to a symmetric equilibrium of the symmetric game defined by the payoff matrix of the other player. In that way, an algorithm that finds a Nash equilibrium of a bimatrix game can be used to find a symmetric Nash equilibrium of a symmetric game.

In one sense the two-polytope construction of Savani and von Stengel (2006) was overly com- plicated: the imitation games by McLennan and Tourky (2010) provide a simple and elegant way to turn the single-polytope construction of Morris (1994) into exponentially-long LH paths for bimatrix games. In another sense, the construction of Savani and von Stengel was not redun- dant, since it provided examples that are simultaneously bad for the LH algorithm and âĂIJsupport enumerationâĂİ, which is another natural and simple algorithm for finding equilibria. The support of a mixed strategy is the set of pure strategies that are played with positive probability. Given a pair of supports of equal size, the mixed strategy probabilities are found by equating all payoffs for the other playerâĂŹs support, which then have to be compared with payoffs outside the support to establish the equilibrium property (see Dickhaut and Kaplan, 1991).

In this paper, we extend the idea of imitation games to games where one payoff matrix is arbi- trary and the other is a set of unit vectors. We call these unit vector games.

we use them to extend MorrisâĂŹs con- struction to give bimatrix games that use only one dual cyclic polytope, rather than the two used by Savani and von Stengel, and for which both the LH algorithm and support enumeration

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are simultaneously bad. This result (Theorem 11) was first described by Savani (2006, Section 3.8).

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We begin this section with the definition of almost complete labeling; we then move on to the classic version of the Lemke-Howson algorithm for the problem ANOTHER COMPLETELY LABELED VERTEX, as given in the beautiful exposition by Shapley [21], and its dual version for ANOTHER COMPLETELY LABELED FACET. Finally, we present the Lemke-Howson for Gale algorithm. In the next session we will tackle the issue of the computational complexity of these algorithms: ANOTHER COMPLETELY LABELED FACET and ANOTHER COMPLETELY LABELED VERTEX are **PPA**, NASH is **PPAD**, as first shown in Papadimitriou [18]; furhermore, as shown by Morris [15] and by Savani and von Stengel [19], there are cases of exponential running time. This had led us to conjecture that these problems could be exploited for a proof of **PPAD** completeness, also considering that finding a completely labeled facet (or vertex, or the existence of a Nash equilibrium) is **NP** in the case of a general labeled polytope, as proven by von Stengel [25] . In the last section we will finally present our original result, that goes in the opposite direction: the problem ANOTHER GALEcan be solved in polynomial time, that is, it is a problem in **TFP**. Unit vector games with dual cyclic best response polytope present therefore a case apart, as expected, but not because they are harder than others, but because they are easier.

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Chapter 1

Preliminary definitions

This chapter consists of some background definitions and notation that will be used throughout the thesis. The first section will cover polytopes (see Ziegler [26] for further details); the second section will deal with basic game theory. We will finally focus on computational complexity; after the standard definitions (see Papadimitriou [17] for an introduction) we will move on to the more recently defined classes **TFNP** and **PPAD**, first introduced in [13] and [18], respectively. The latter, in particular, is a key concept in the study of the problems that are the focus of this thesis.

We will often need to refer to sets of natural numbers; we follow the notation $[n] = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$.

1.1   Vectors and Polytopes

We denote the transpose of a matrix A as $A^\top$. Vectors $u, v \in \mathbb{R}^d$ will be considered as column vectors, so $u^\top v$ is their scalar product. A vector in $\mathbb{R}^d$ for which all components are 0's will be denoted as $\mathbf{0}$; a vectors for which all components are 1's will be denoted as $\mathbf{1}$; the *i-th unit vector*, for which all the components are 0 except for the i-th component equal to 1, will be denoted as e_i. An inequality of the form $u \geq v$, is intended to hold for every component.

An *affine combination* of points in an Euclidean space $\{z_1, \ldots, z_m\} \subset \mathbb{R}^n$ is $\sum_{i \in [m]} \lambda_i z_i$ where $\lambda_i \in \mathbb{R}$ such that $\sum_{i \in [m]} \lambda_i = 1$; the points $z_1, \ldots, z_m$ are *affinely independent* if none of them is an affine combination of the others. The *convex hull* of the points $z_1, \ldots, z_m$ is an affine combination with $\lambda_i \geq 0$ for all $i \in [n]$; we denote it as $\text{conv}\{z_1, \ldots, z_m\} = \{\sum_i \lambda_i z_i \mid \lambda_i \geq 0, \ \sum_i \lambda_i = 1\}$. A set of points $Z = \{z_1, \ldots, z_m\}$ is *convex* if $Z = \text{conv}(Z)$, and it has *dimension* d if Z has exactly $d+1$ affinely independent points. A convex set of $d+1$ points of dimension d is called a *d-simplex*. The *standard d-simplex* is $\Delta_d = \text{conv}\{e_1, \ldots, e_{d+1}\}$.

A *polytope* is the convex hull of a finite set of points, not necessarily affinely independent, $\{z_1, \ldots, z_m\} \subset \mathbb{R}^n$; the *dimension* of the polytope is the dimension of its convex hull. A polyhedron is the intersection of finitely many closed halfspaces $\{x \in \mathbb{R}^d \mid a^T x \leq a_0\}$; notice that a bounded polyhedron is a polytope. A *vertex* of a d-dimensional polytope $P = \text{conv}(Z)$ is a point $z \in Z$ such that $\text{conv}(Z \setminus \{z\}) \neq P$; an *edge* of P is a 1-dimensional line segment that has two vertices as endpoints. A *facet* of P is the convex hull of a set of d vertices $F = \{z_1, \ldots, z_d\}$ that lie on a hyperplane of the form $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P; the vector a is the *normal vector* of the facet.

A *d-dimensional simplicial polytope* P is the convex hull of a set of at least $d+1$ points $v \in \mathbb{R}^d$ such that no $d+1$ of them are on a common hyperplane; this is equivalent to requiring that every facet of P is a d-simplex. A d-dimensional polytope P is *simple* if every point of P lies on at most than d facets; notice that the points lying on exactly d facets are exactly the vertices.

The *polar* of the polytope $P = \{x \in \mathbb{R}^d \mid x^\top c_i \leq 1, \text{ for } c_i \in \mathbb{R}^d, \ i \in [k]\}$ is $P^\Delta = \text{conv}\{c_i, i \in [k]\}$. If P is simplicial or it is simple and it contains the origin $\mathbf{0}$, then $P^{\Delta\Delta} = P$; furthermore, if P is simplicial P^Δ is simple, and vice versa.

1.2 Normal Form Games and Nash Equilibria

A *game*, first defined by von Neumann in [23], is a model of strategic inter-action. A *finite normal form game* is $\Gamma = (P, S, u)$ where both P and S are finite. The former is the set of *players*; $S = \times_{p \in P}$ is the set of *pure strategy profiles* and S_p is the set of *pure strategies* of player p; we will use the notation $S_{-p} = \times_{q \neq p} S_p$. The purpose of each player $p \in P$ is to maximize their *payoff function* $u^p : S \to \mathbb{R}$ and $u = \times_{p \in P} u^p$. In the following pages, by "game" we will always mean "finite normal form game." A *mixed strategy* of player p is a probability distribution on S_p; it can be described as a point on the $(|S_p| - 1)$-dimensional *mixed strategy simplex* $\Delta_p = \{x \in \mathbb{R}^{|S_p|} \mid x \geq \mathbf{0}, \ \mathbf{1}^\top x = 1\}$. The set of *mixed strategy profiles* is the simplicial polytope $\Delta = \times_{p \in P} \Delta_p$; we extend the payoff functions to $u^p : \Delta \to \mathbb{R}$ linearly. A *Nash equilibrium* of a game is a strategy profile in which each player cannot improve their expected payoff by unilaterally changing their strategy; such a strategy is called a *best response*. Note that applying an affine transformation to all the payoffs does not change the Nash equilibria of the game.

Proposition 1.1. *A mixed strategy $x \in \Delta_p$ is a best response against some mixed strategy profile $y \in \Delta_{-p}$ of the other players if and only if every pure strategy $s_i \in S_p$ chosen with positive probability in x is a best response to y.*

The existence of a Nash equilibrium is guaranteed by the fundamental theorem by Nash ([16]). Note that there might be more than one equilibrium.

Theorem 1. (Nash [16]) *Every finite game in normal form has a Nash equilibrium.*

Bimatrix games are games with only two players; we will refer to player 1 using feminine pronouns and to player 2 using masculine ones. Bimatrix games can be characterized by the $m \times n$ payoff matrices A and B, where a_{ij} and b_{ij} are the payoffs of respectively player 1 and of player 2 when the former plays

her ith pure strategy and the latter plays his jth pure strategy. A bimatrix game is *zero-sum* if $B = -A$, and *symmetric* if $B = A^\top$.

We give two classic examples of bimatrix games: the prisoners' dilemma and the coordination game.

Example 1.1. In the symmetric non zero-sum *prisoners' dilemma* of Table 1.1, each player must decide whether to "help" the other one or to "betray" them. If both players help each other, they will get a small reward; if both betray, they will pay a small penalty; if one betrays and the other cooperate the former will get a large reward and the latter will pay a large penalty. The only equilibrium is the profile in which both players betray. If player 2 betrays, the best response of player 1 is to betray, since it gives her payoff 1 instead of 0; if player 2 helps, her payoff for betraying is 3 and her payoff for helping is 2, so betraying is again the best response. The same holds for player 2, so at the equilibrium both players will betray.

	betray	help
betray	1 ⟍ 1	0 ⟍ 3
help	3 ⟍ 0	2 ⟍ 2

Table 1.1   The prisoners' dilemma.

Table 1.2 shows a *coordination* game. Both players drive on a mountain road; they lose if drive on the same side of the road and win if they avoid each other, regardless of which side they take. The pure strategy Nash equilibria are (mountain,valley) and (valley,mountain); there is also a symmetric equilibrium in mixed strategies at $((1/2, 1/2), (1/2, 1/2))$.

2 / 1	mountain	valley
mountain	0 / 0	1 / 1
valley	1 / 1	0 / 0

Table 1.2 A coordination game.

1.3 Some Complexity Classes

A *Turing machine* $\mathcal{M}$ is a representation of an *algorithm* that takes an *input*, runs a *program* manipulating the input, and either does not come to a *halting state h* or it returns an *output*; the latter can be the *accepting state* YES (the Turing machine *accepts* the input), the *rejecting state* NO (the Turing machine *rejects* the input), or a string $\mathcal{M}(x)$. Formally, $\mathcal{M} = (K, \Sigma, \delta, s)$. The finite set K is the set of *states*; Σ is a finite set of *symbols* (the *alphabet* of $\mathcal{M}$) such that $\Sigma \cap K = \varnothing$, and Σ always contains the symbols $\sqcup$ (*blank*) and $\triangleright$ (*first symbol*). The *transition function* δ is

$$\delta : K \times \Sigma \longrightarrow (K \cup \{h, \text{ YES, NO}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}.$$

where $\{\leftarrow, \rightarrow, -\} \nsubseteq (K \cup \Sigma)$ correspond to the *cursor directions* "left," "right" and "stay."

A *language* is a set of strings of symbols $L \subseteq (\Sigma \setminus \{\sqcup\})^*$; a *class* is a set of languages. A Turing machine $\mathcal{M}$ *decides* L if for every $x \in (\Sigma \setminus \{\sqcup\})^*$ either $\mathcal{M}(x) = \text{YES}$ if $x \in L$ or $\mathcal{M}(x) = \text{NO}$ if $x \notin L$. If for every $x \in (\Sigma \setminus \{\sqcup\})^*$ either $\mathcal{M}(x) = \text{YES}$, if and only if $x \in L$, or $\mathcal{M}(x)$ does not halt, if and only if $x \notin L$, the Turing machine $\mathcal{M}$ *accepts* L. Finally, $\mathcal{M}$ *computes* a function $f : (\Sigma \setminus \{\sqcup\})^* \to \Sigma^*$ if $\mathcal{M}(x) = f(x)$ for every $x \in (\Sigma \setminus \{\sqcup\})^*$.

An *instance* is the specific input of a problem. A problem P that outputs either YES or NO is a *decision problem*; its *complement* is the problem $\overline{P}$ that outputs "NO" for each instance of P that outputs "YES", and vice versa. A

function problem outputs a string y more generic than YES or NO; a *search problem* either outputs a string y rejects the input if it's not possible to find any such y. If y is guaranteed to exist, the problem is a *total function problem.*

An example of decision problem is: "(input) given a graph, (question) is it possible to find an Euler tour of the graph?" Its complement is "(input) given a graph, (question) is it impossible to find any Euler of the graph?" A search problem is: "(input) given a graph, (output) return one Euler tour of the graph, or "NO" if no such tour exists." A total function problem is: "(input) given an Euler graph, (output) return one of its Euler tours."

Let P_1 be a problem and let x be an instance of P_1 that is encoded in $|x|$ bits. P_1 *reduces to the problem* P_2 *in polynomial time* if there exists a *polynomial-time reduction*, that is, a function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ and a Turing machine $\mathcal{M}$ such that for all $x \in \{0,1\}^*$

1. $x \in P_1 \iff f(x) \in P_2$;

2. $\mathcal{M}$ computes $f(x)$;

3. $\mathcal{M}$ stops after $p(|x|)$ steps, where p is a polynomial.

For any class C of decision problems, the class of all complements of the problems in C is the *complement class* co − C. A problem P is *hard* for a class C if every problem in C is polynomial-time reducible to P; that is, if P is hard to solve at least as every problem in C. A *complete* for the class C is a C−*hard* problem that is also in C. Intuitively, if P_1 is polynomial-time reducible to P_2, it takes polynomial time to "translate" P_1 to P_2, and then to "translate back" a solution of P_2 as a solution of P_1. This is particularly useful if P_2 is hard; then the problem P_1 is at least as "difficult"; if P_2 is also complete, it can be used as a test of belonging to its class.

The complexity class **P** contains all the *polynomially decidable problems*; that is, all problems P such that there exists a Turing machine $\mathcal{M}$ that outputs either YES or NO for all inputs $x \in \{0,1\}^*$ of P after $p(|x|)$ steps, where p

is a polynomial. A problem P belongs to the class **NP**, *non-deterministic polynomial-time problems*, if there exists a Turing machine $\mathcal{M}$ and polynomials p_1, p_2 such that

1. for all $x \in P$ there exists a *certificate* $y \in \{0,1\}^*$ which satisfies $|y| \leq p_1(|x|)$;

2. $\mathcal{M}$ accepts the combined input xy, stopping after at most $p_2(|x| + |y|)$ steps;

3. for all $x \notin P$ there does not exist $y \in \{0,1\}^*$ such that $\mathcal{M}$ accepts the combined input xy.

The class $\#\mathbf{P}$ is the class of all problems that output the number of possible certificates for a problem in **NP**. Informally, a decision problem is in **P** if the answer to its question can be found in a number of steps that is polynomial in the input of the problem, and a decision problem is in **NP** if it takes polynomial time to verify whether the "certificate solution" y is, indeed, a correct answer to the question posed by the problem.

In [13], Megiddo and Papadimitriou introduced the classes **FNP**, *function non-deterministic polynomial*, and **TFNP**, *total function non-deterministic polynomial*. The former is defined as the class of binary relations $R(x, y)$ such that there is a polynomial-time algorithm that decides $R(x, y)$ for x, y such that $|y| \leq p(|x|)$, where p is a polynomial. The latter is the class of all such problems for which y is guaranteed to exist. **FNP** and **TFNP** can be seen as the equivalent of **NP** for (respectively) function and total function problems. Also in [13], Megiddo and Papadimitriou proved that **TFNP** is a *semantic* class, that is, a class without complete problems, unless $\mathbf{NP} = \mathbf{co} - \mathbf{NP}$, which is to this day an open problem. To circumvent this limitation, Papadimitriou [18] focused on the problems for which the existence of a solution is proved by a specific argument, introducing the classes **PPA** (*Proof by Parity Argument*) and **PPAD** (*Proof by Parity Argument, Directed version*).

The existence of a solution for a problem in **PPA** can be proved using the argument "in any undirected graph with one odd-degree node there must be another odd-degree node." It is interesting to notice that **PPA**-complete problems are yet to be found. Problems in **PPAD**, analogously, are guaranteed to have a solution by a proof employing the argument "in any directed graph in which all vertices have indegree and outdegree at most one where there is a *source* (a node with indegree zero) there must be a *sink* (a node with outdegree zero)." Formally, we can define **PPAD** as the class of problems reducible to the problem END OF THE LINE in Table 1.3. This is the definition given in Daskalakis, Goldberg and Papadimitriou [5]; the original definition in Papadimitriou [18] is given in terms of Turing machines. A *circuit* with n *input bits* and m *output bits* is a function $C : \{0,1\}^n \to \{0,1\}^m$.

END OF THE LINE

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input : Two circuits S and P with n input bits and n output bits such that $P(0^n) = 0^n \neq S(0^n)$.

output: An input $x \in \{0,1\}^n$ such that $P(S(x) \neq x)$ or $S(P(x)) \neq x \neq 0^n$

Table 1.3   The problem END OF THE LINE.

The problems in **PPAD** can be seen as a circuit S ("successor"), and a circuit P ("predecessor") that are used to build a directed graph with an edge (x, y) if and only if $S(x) = y$ and $P(y) = x$; furthermore, a *standard source* 0^n is given, guaranteeing the existence of the output, which consists in either a sink or a non-standard source. Figure 1.1 presents an example of a graph implicit in a **PPAD** problem; a graph for a **PPA** problem is analogous, but it is undirected and instead of sources and sinks there are generic endpoints.
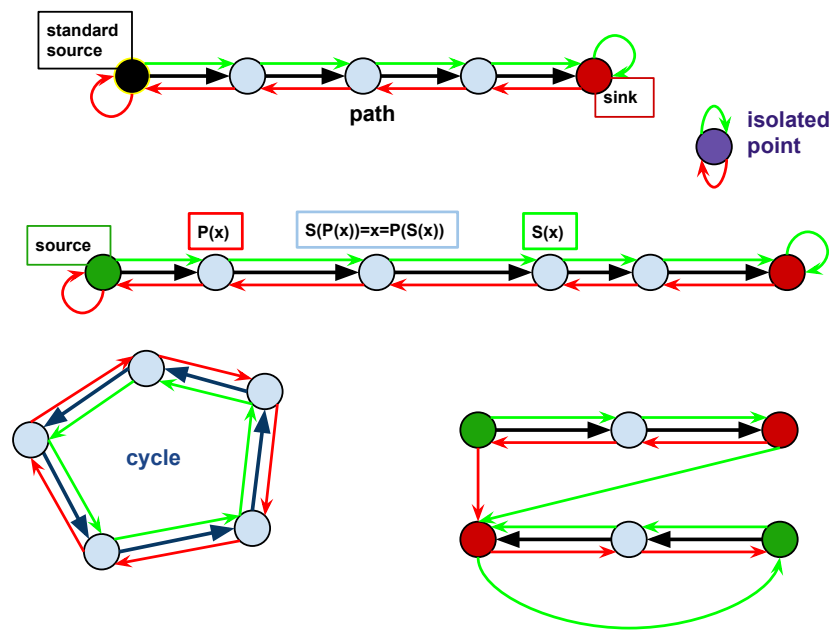
Figure 1.1 The graph in a **PPAD** problem (paths in black, cycles in blue, isolated points in purple). The input is given by the circuits S (in green) and P (in red) and the standard source (the black node). The output can be either a sink (a red node) or a nonstandard source (a green node).

n-NASH	
input : A n-player game.	
output: A Nash equilibrium of the game.	

Table 1.4 The problem n-NASH.

By theorem 1, the problem n-NASH of Table 1.4 is a total function problem. Megiddo and Papadimitriou ([13]) proved that n-NASH is in **TFNP**. Daskalakis, Goldberg and Papadimitriou [5] and Chen and Deng [4] have later proven its **PPAD**-completeness, the former for $n \geq 3$ and the latter for $n \geq 2$. A small amendment of the proof in [5] can be found in Casetti [2].

Theorem 2. (Daskalakis, Goldberg and Papadimitriou [5]; Chen and Deng [4]) *For $n \geq 2$, the problem n-NASH is **PPAD**-complete.*

We will see more problems in **PPA** and **PPAD** in chapter 3; in fact, our main result can be seen as a negative result on the **PPAD** complexity of a case of 2-NASH.

Chapter 2

Labels, Polytopes and Gale Strings

In the remainder of this thesis we will focus bimatrix games. We will identify the game with its payoff matrices (A, B), and we will assume that both A and B are non-negative, and that neither A nor $B^\top$ has a zero column; this can be done without loss of generality, applying an affine transformation to A and B if necessary.

We will start by showing a construction to translate a bimatrix game to two simplices subdivided in regions which are labeled so that the Nash equilibria of the game correspond to the "completely labeled" points of the simplices. This idea is due to Lemke and Howson [11]; we will follow the very clear approach given by Shapley [21]. From labeled regions we will then move on to an equivalent formulation in terms of polytopes; a result by McLennan and Tourky [12] on a special case of games, called imitation games, and its extension to the more general unit vector games, due to Balthasar [1], will allow us to simplify the labeling construction on polytopes; a result by Balthasar [1] gives a condition on facets of a simplicial polytope, whereas Savani and von Stengel [20] give an equivalent condition on vertices of a simple polytope. The problem 2-Nash will then be shown to have a polynomial-time reduction to a problem

on facets or vertices of a labeled polytope, under certain simple conditions.

In the second section of the chapter we will restrict our scope to Gale games, a case of unit vector games for which the facets of the polytopes in the previous section can be represented as an elegant combinatorial structure, called Gale strings. We will show this representation, then we will define a labeling for Gale strings that will lead to a reduction from 2-NASH to a problem on Gale strings.

2.1   Bimatrix Games and Labels

Let $n, m \in \mathbb{N}$ with $m \leq n$. A *labeling* of a set X is a function $l : X \to [m]$; an n-uple $x = (x_1, \ldots, x_n) \in X^n$ is *completely labeled* if each label $j \in [m]$ appears once and only once in $(l(x_1), \ldots, l(x_n))$; formally, if $\{j \in [m] \mid l(x_i) = j \text{ for some } i \in [n]\} = [m]$.

Let (A, B) be bimatrix game, and let X and Y be the mixed strategy simplices of respectively player 1 and 2, that is

$$X = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, \ \mathbf{1}^\top x = 1\}; \quad Y = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0}, \ \mathbf{1}^\top y = 1\}. \quad (2.1)$$

A *labeling* of the game is then given as follows:

1. the m pure strategies of player 1 are identified by $1, \ldots, m$;

2. the n pure strategies of player 2 are identified by $m + 1, \ldots, m + n$;

3. each mixed strategy $x \in X$ of player 1 has

    - label i for each $i \in [m]$ such that $x_i = 0$, that is if in x player 1 does not play her i-th pure strategy,

    - label $m + j$ for each $j \in [n]$ such that the j-th pure strategy of player 2 is a best response to x;

4. each mixed strategy $y \in Y$ of player 2 has

- label $m + j$ for each $j \in [n]$ such that $y_j = 0$, that is if in y player 2 does not play his j-th pure strategy,

- label i for each $i \in [m]$ such that the i-th pure strategy of player 1 is a best response to y.

The labeling of mixed strategy profiles can be used to characterize the Nash equilibria of the game.

Theorem 3. (Shapley [21]) *Let $(x, y) \in X \times Y$; then (x, y) is a Nash equilibrium of the bimatrix game (A, B) if and only if (x, y) is completely labeled.*

Proof. The mixed strategy $x \in X$ has label $m + j$ for some $j \in [n]$ if and only if the j-th pure strategy of player 2 is a best response to x; by Proposition 1.1 is a necessary and sufficient condition for player 2 to play his j-th strategy at every equilibrium where player 1 chooses x. The analogous holds for the strategies $y \in Y$ and player 1 playing her i-th strategy in response to player 2 choosing y. Therefore, at an equilibrium (x, y) all labels $m + j$, with $j \in [n]$, will appear either as labels of x or of y. $\qquad\square$

An useful graphical representation of labels on the simplices X and Y is done by labeling the outside of each simplex according to the player's own pure strategies that are *not* played, and by subdividing its interior in closed polyhedral sets corresponding to the other player's pure best response strategies, called *best response regions*. We give an example of this construction.

Example 2.1. Consider the 3×3 game (A, B) with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.2}$$

Figure 2.1 shows the mixed strategy simplices X and Y: the exterior facets are labeled with the pure strategy on the opposite vertex, where only that pure strategy is played; the interior is covered by the best response regions,

labeled by to the other player's pure best response strategies. For example, the best-response region in Y with label 1 is the set of those (y_1, y_2, y_3) so that $y_1 \geq y_2$ and $y_1 \geq y_3$. There is only one pair (x, y) that is completely labeled, namely $x = (\frac{1}{3}, \frac{2}{3}, 0)$ with labels $3, 4, 5$ and $y = (\frac{1}{2}, \frac{1}{2}, 0)$ with labels $1, 2, 6$, so this is the only Nash equilibrium of the game.
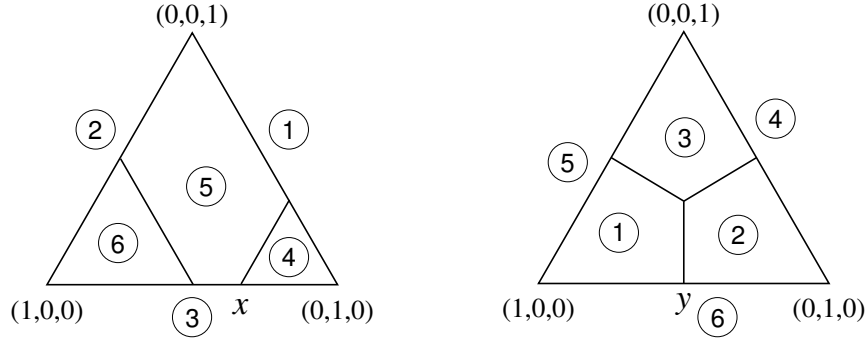
Figure 2.1   The mixed strategy sets X and Y of game (2.2) and their labeled best response regions.

The representation of the game and its Nash equilibria in terms of best response regions can be translated to an equivalent construction on polytopes. The first step is to notice that the best-response regions can be obtained as projections on X and Y of the *best-response facets* of the polyhedra

$$\overline{P} = \{(x, v) \in X \times \mathbb{R} \mid B^\top x \leq \mathbf{1}v\};$$

$$\overline{Q} = \{(y, u) \in Y \times \mathbb{R} \mid Ay \leq \mathbf{1}u\}. \tag{2.3}$$

In $\overline{P}$, these facets are the points $(x, v) \in X \times \mathbb{R}$ such that $(B^\top x)_j = v$, which in turn correspond to the strategies $x \in X$ of player 1 that give exactly payoff v to player 2 when he plays strategy j; the projection of the facet defined by $(B^\top x)_j = v$ to X has then label j. Analogously, the facet of $\overline{Q}$ given by the points $(y, u) \in Y \times \mathbb{R}$ such that $(Ay)_i = u$ projects to the best-response region of Y with label i.

Example 2.2. In Example 2.1, the inequalities $B^\top x \leq \mathbf{1}v$ are

$$3x_2 \leq v;$$
$$2x_1 + 2x_2 + 2x_3 \leq v$$
$$4x_1 \leq v.$$

Figure 2.2 shows the best-response facets of $\overline{P}$ and their projection to X by ignoring the payoff variable v, which gives the subdivision of X into best-response regions of Figure 2.1.
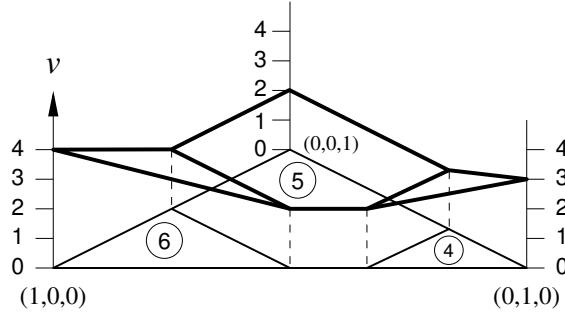
Figure 2.2   The best response polyhedron $\overline{P}$ of game (2.2).

Given the assumptions on non-negativity of A and $B^\top$, we can change coordinates to x_i/v and y_j/u and replace $\overline{P}$ and $\overline{Q}$ with the *best-response polytopes*

$$P = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, \ B^\top x \leq \mathbf{1}\};$$
$$Q = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0}, \ Ay \leq \mathbf{1}\}. \tag{2.4}$$

The polytope P is the intersection the $m + n$ half spaces corresponding to either player 1 avoiding her i-th pure strategy or a best response of player 2 that gives non-zero probability to his j-th strategy, where $i \in [m]$ and $j \in [n]$. Formally, a point $x \in P$ has label k if and only if either $x_k = 0$ for $k \in [m]$ or $(B^\top x)_k = 0$ for $k \in [n]$, and a point in Q has label k if and only if either $y_k = 0$ for $k \in [n]$ or $(Ay)_k$ for $k \in [m]$. Then a point $(x, y) \in P \times Q$ is completely

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labeled if and only if it satisfies the *complementarity condition*

$$x_i = 0 \quad \text{or} \quad (Ay)_i = 1 \qquad \text{for all } i \in [m];$$

$$y_j = 0 \quad \text{or} \quad (B^\top x)_j \qquad \text{for all } j \in [n]. \tag{2.5}$$

Therefore, if $(x, y) \in P \times Q$ is completely labeled, either the corresponding point in $\overline{P} \times \overline{Q}$ is a Nash equilibrium, or $(x, y) = (\mathbf{0}, \mathbf{0})$; we will refer to the latter case as *artificial equilibrium*.

Example 2.3. Keeping on with Example 2.1, the best response polyhedron $\overline{P}$ of Figure 2.2 becomes the best response polytope of Figure 2.3. Notice that the vertex $(\mathbf{0}, \mathbf{0})$ is completely labeled, since it is labeled by the labels $x_k = 0$ with $k = 1, 2, 3$ in P and $k = 4, 5, 6$ in Q.
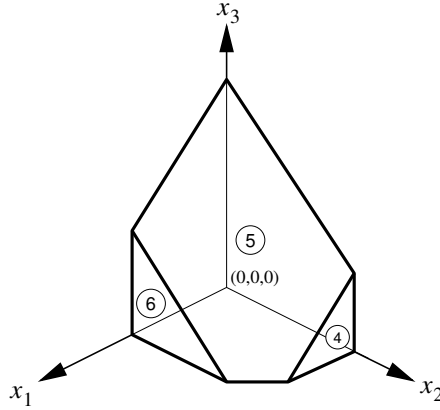
Figure 2.3   The best response polytope P of game (2.2).

We will now see some special cases of games connected by polynomial-time reductions. First of all, we note that any bimatrix game can be "symmetrized"; the result is due to Gale, Kuhn and Tucker [10] for zero-sum games and its extension to non-zero-sum games is a folklore result.

Proposition 2.1. *Let (A, B) be a bimatrix game and let (x, y) be one of its Nash equilibria. Then (z, z), where $z = (x, y)$, is a Nash equilibrium of the*

symmetric game $(C, C^\top)$, *where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}. \tag{2.6}$$

McLennan and Tourky [12] have proven a result in the opposite direction of Proposition 2.1: any symmetric game can be translated into a bimatrix game (I, B); games of this kind are called *imitiation games*. Since it takes polynomial time in the size of a matrix to calculate its transpose, we have a polynomial-time reduction from 2-NASH to IMITATION NASH.

Theorem 4. (McLennan and Tourky [12]) *The pair (x, x) is a symmetric Nash equilibrium of the symmetric bimatrix game $(C, C^\top)$ if and only if there is some y such that (x, y) is a Nash equilibrium of the imitation game (I, B) with $B = C^\top$.*

In any Nash equilibrium of (I, B), the mixed strategy x of player 1 corresponds exactly to the symmetric equilibrium (x, x) in the symmetric game defined by the payoff matrix of player 2. Note, though, that Theorem 4 applies to the symmetric equilibria of the symmetric game but not to all its Nash equilibria; there could be non-symmetric equilibria of $(C, C^\top)$ that are not found through the imitation game, as shown in the following example.

Example 2.4. As an example, consider the symmetric game $(C, C^\top)$ with

$$C = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \qquad C^\top = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \tag{2.7}$$

so that the corresponding imitation game is $(I, C^\top) = (A, B)$, of Example 2.1. Figure 2.4 shows the labeled mixed-strategy simplices X and Y for the game (2.7); since the game is symmetric, only the labels are different. In addition to the symmetric equilibrium (x, x) where $x = (\frac{1}{3}, \frac{2}{3}, 0)$, the game has two non-symmetric equilibria in (a, b) and (b, a) with $a = (\frac{1}{2}, \frac{1}{2}, 0)$ and $b = (0, \frac{2}{3}, \frac{1}{3})$; the imitation game (A, B) has only one equilibrium (x, y), with $y = (\frac{1}{2}, \frac{1}{2}, 0)$, corresponding to (x, x).
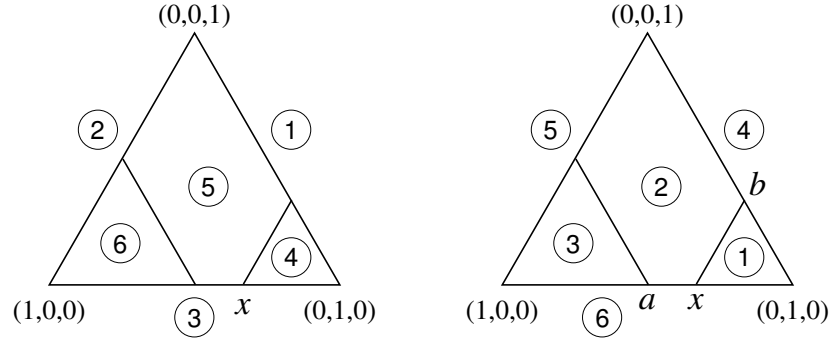
Figure 2.4 Best response regions of the symmetric game (2.7).

The characterization of Nash equilibria as completely labeled pairs (x, y) holds for arbitrary bimatrix games. From now on, we will impose a further condition: all points in P will have at most m labels, and all points in Q have at most n labels. These games are called *nondegenerate*; since any game can be made nondegenerate by lexicographic perturbation (see von Stengel [24]), we can impose the nondegeneracy condition without loss of generality. In an equilibrium (x, y) of a nondegenerate game each label appears exactly once; this also means that the number of pure best response strategies against a mixed strategy is never larger than the size of the support of that mixed strategy.

Geometrically, this means that no point of the best response polytope P lies on more than m facets and no point of the best response polytope Q lies on more than n facets, so both P and Q are simple. Furthermore, a point of P has exactly m labels if and only if it is a vertex, and a point of Q has exactly n labels if and only if it is a vertex; therefore, all completely labeled points (x, y) are vertices of the best response polytopes, and Nash equilibria are isolated points.

Example 2.5. An example of degenerate game is given by $(C, C^\top)$ where

$$C = \begin{pmatrix} 0 & 4 & 0 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \qquad C^\top = \begin{pmatrix} 0 & 2 & 4 \\ 4 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \qquad (2.8)$$

As it is shown in Figure 2.5 left, the mixed strategy $x = (\frac{1}{2}, \frac{1}{2}, 0)$, that also defines the unique symmetric equilibrium (x, x) of the game, has three pure best responses. Notice that the Nash equilibria (x, y) of the imitation game $(I, C^\top)$ are not unique, since any convex combination of $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ can be chosen for y, as shown in Figure 2.5 right.
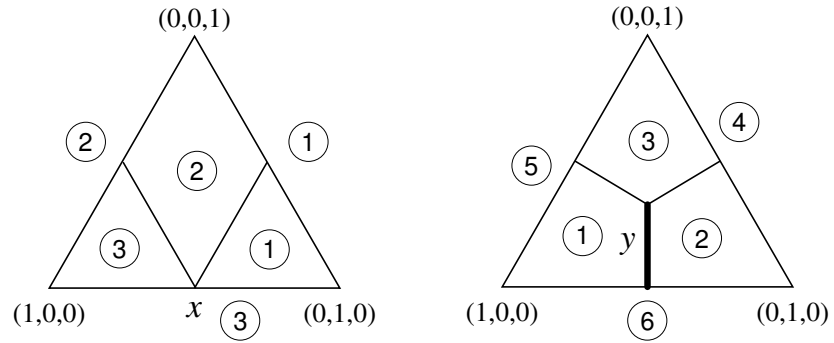
Figure 2.5   Left: The best-response regions on the mixed strategy simplices of the degenerate symmetric game (2.8).

Right: The labeled mixed-strategy sets on the mixed strategy simplex Y of the imitation game $(I, C^\top)$.

A generalization of imitation games is the class of *unit vector games*, introduced by Balthasar [1]; they are defined as bimatrix games of the form (U, B) where the columns of the matrix U are unit vectors. By the results above, finding a Nash equilibrium of a bimatrix game is at least as hard as finding a Nash equilibrium of a unit vector game; that is, 2-NASH reduces to UNIT VECTOR NASH. Savani and von Stengel [20] have shown that in unit vector games the problem of finding a completely labeled vertex of the product of the best response polytopes $P \times Q$ can be simplified to the problem of finding a completely labeled vertex of one polytope that encodes all the game.

Theorem 5. (Savani and von Stengel [20]) *Let $l : [n] \to [m]$ be a labeling, and let (U, B) be the unit vector game with $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. Let $N_i = \{j \in [n] \ | \ l(j) = i\}$ for $i \in [m]$, and consider the polytopes P^l and Q^l*

$$P^l = \{x \in \mathbb{R}^m \ | \ x \geq \mathbf{0}, \ B^\top x \leq \mathbf{1}\}; \tag{2.9}$$
$$Q^l = \{y \in \mathbb{R}^n \ | \ y \geq \mathbf{0}, \ \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\}.$$

Let l_f be the labeling of the facets of P^l defined as follows:

$$\begin{aligned} x_i \geq 0 \qquad &\textit{has label } i \qquad \textit{for } i \in [m]; \\ (B^\top x)_j \leq 1 \quad &\textit{has label } l(j) \quad \textit{for } j \in [n]. \end{aligned} \tag{2.10}$$

Then $x \in P^l$ is a completely labeled vertex of $P^l \setminus \{\mathbf{0}\}$ if and only if there is some $y \in Q^l$ such that, after scaling, the pair (x, y) is a Nash equilibrium of (U, B)

Proof. Let P and Q be the best response polytopes of (U, B) as in (2.4), and let $(x, y) \in P \times Q \setminus \{(\mathbf{0}, \mathbf{0})\}$ be a Nash equilibrium of (U, B); then (x, y) is completely labeled in $[m + n]$. If $x_i = 0$, then x has label $i \in m$; if $x_i > 0$, then y has label i, so $(Uy)_i = 1$; therefore for some $j \in [n]$ we have $y_j > 0$ and $U_j = e_i$; that is, we have $y_j > 0$ and $l(j) = i$ for some $j \in [n]$. Since $y_j > 0$,

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$x \in P$ has label $m + j$; then, $(B^\top x)_j = 1$; therefore $x \in P^l$ has label $l(j) = i$. Hence, x is a completely labeled vertex of P^l.

Conversely, let $x \in P^l \setminus \{\mathbf{0}\}$ be completely labeled. If $x_i > 0$, then there is $j \in [n]$ such that $(B^\top x) = j$ and $l(j) = i$; that is, $j \in N_i$. For all $i \in [m]$ such that $x_i > 0$, define y as follows: $y_j = 1$; $y_h = 0$ for all $h \in N_i \setminus \{j\}$. Then $(x, y) \in P \times Q$ is completely labeled. $\qquad\square$

Example 2.6. The game in Example 2.1 is a unit vector game with $l(i) = i$. In the polytope P^l of Figure 2.6 the labels 4, 5 and 6 of the best response polytope P of Figure 2.3 are replaced by 1, 2 and 3, since the corresponding columns of A are the unit vectors e_1, e_2, e_3. The only completely labeled point of P^l are the origin $\mathbf{0}$, corresponding to the "artificial" equilibrium, and x, corresponding to the unique Nash equilibrium of the unit vector game (2.2).
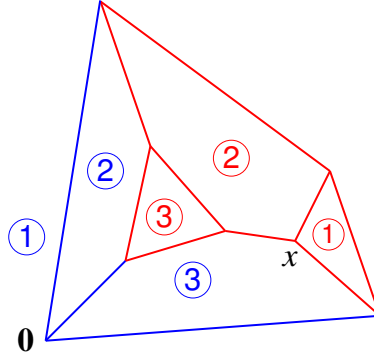
Figure 2.6 The polytope P^l of the unit vector game (2.2). Label 1 refers to the hidden facet.

We will now move on the dual version of Theorem 5, as given by Balthasar [1]. We can traslate the polytope P^l of (2.9) to $P = \{x - \mathbf{1} \mid x \in P^l\}$, multiplying all payoffs in B by a constant, if necessary, so that $\mathbf{0}$ is in the interior of P. We have that

$$\widetilde{P} = \{x + \mathbf{1} \geq \mathbf{0}, \ (x + \mathbf{1})^\top B \leq \mathbf{1}\} =$$
$$= \{x \in \mathbb{R}^m \mid -x_i \leq 1 \text{ for } i \in [m], \ x^\top (b_j/(1 - \mathbf{1}^\top b_j)) \leq 1 \text{ for } j \in [n]\}.$$

The polar of $\widetilde{P}$ is then

$$Q = \widetilde{P}^\Delta = \mathrm{conv}(\{e_i \mid i \in [m]\} \ \cup \ \{c_j \mid j \in [n]\}) \tag{2.11}$$

where $c_j = b_j/(1 - \mathbf{1}^\top b_j)$. Since $\widetilde{P}$ is a simple polytope with $\mathbf{0}$ in its interior, $Q^\Delta = \widetilde{P}^{\Delta\Delta} = \widetilde{P}$. Furthermore, Q is simplicial and its facets correspond to the vertices of P and vice versa. We label the vertices of Q as the corresponding facets in P^l, so the completely labeled facets of Q correspond to the completely labeled vertices of P^l. In particular, the facet corresponding to $\mathbf{0}$ is

$$\begin{aligned}
F_0 &= \{x \in \widetilde{P}^\Delta \mid -\mathbf{1}^\top x = 1\} \\
&= \mathrm{conv}\{e_i \mid i \in [m]\}. \tag{2.12}
\end{aligned}$$

Theorem 5 then translates into the following.

Theorem 6. (Balthasar [1]) *Let Q be a labeled m-dimensional simplicial polytope with $\mathbf{0}$ in its interior and vertices $e_1, \ldots, e_m, c_1, \ldots, c_n$ such that (2.12) is a facet of Q. Let (U, B) be a unit vector game, with $U = (e_{l(1)} \ \cdots \ e_{l(n)})$ for a labeling $l : [n] \to [m]$ and $B = (b_1 \ \cdots \ b_n)$, where $b_j = c_j/(1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Let l_v be the labeling of the vertices of Q given by*

$$\begin{aligned}
l_v(-e_i) &= i && \text{for } i \in [m]; \\
l_v(c_j) &= l(j) && \text{for } j \in [n]. \tag{2.13}
\end{aligned}$$

Then a facet $F \neq F_0$ of Q with normal vector v is completely labeled if and only if (x, y) is a Nash equilibrium of (U, B), where $x = (v + \mathbf{1})/(\mathbf{1}^\top(v + \mathbf{1}))$, so that $x_i = 0$ if and only if $e_i \in F$ for $i \in [m]$ and the mixed strategy y is the uniform distribution on the set of the pure best replies to x, which in turn correspond to all $j \in [n]$ such that c_j is a vertex of F.

Theorem 5 gives a correspondence between completely labeled vertices of P^l and equilibria of the unit vector game (U, B) with the "artificial" equilibrium corresponding to the vertex $\mathbf{0}$; Theorem 6 gives a correspondence between

completely labeled facets of Q and equilibria of (U, B) with the "artificial" equilibrium corresponding to the facet F_0 in (2.12).

Given a bimatrix game (A, B), it takes polynomial time to write and solve the linear equations defining its best response polyhedra $\overline{P}, \overline{Q}$ and its best response polytopes P, Q. It also take polynomial time to label $\overline{P}, \overline{Q}$ and P, Q. Analogously, given a unit vector game (U, B), it takes polynomial time to construct and label the polytope P^l. Therefore, Theorem 5 implies a polynomial time reduction from the problem 2-Nash to the problem Another Completely Labeled Vertex.

Another Completely Labeled Vertex

input : A m-dimensional simple polytope P with $m + n$ facets; a labeling $l_f : [m + n] \to [n]$; a vertex v_0 of P that is completely labeled by l_f.

output: A vertex $v \neq v_0$ of P that is completely labeled by l_s.

Table 2.1 The problem Another Completely Labeled Vertex.

Proposition 2.2. 2-Nash *reduces in polynomial time to* Another Completely Labeled Vertex.

Theorem 6 gives the dual of Proposition 2.2; since the construction of the polar polytope from the original one is also polynomial, the reduction is to the problem Another Completely Labeled Facet.

Proposition 2.3. 2-Nash *reduces in polynomial time to* Another Completely Labeled Facet.

2.2   Cyclic Polytopes and Gale Strings

We now apply the results of the previous section to unit vector games for which the best response polytope satisfies a further condition: being the dual

input : A simplicial m-dimensional polytope Q with $m + n$ vertices; a labeling $l_v : [m + n] \to [n]$; a facet F_0 of Q that is completely labeled by l_v.

output: A facet $F \neq F_0$ of Q that is completely labeled by l_v.

Table 2.2   The problem ANOTHER COMPLETELY LABELED FACET.

of a cyclic polytope. These can be represented in a synthetic way using a combinatorial structure called Gale strings. We will first give the definition of cyclic polytope, then of Gale string, then the theorem by Gale [9] that shows the equivalence of the two representations.

The *moment curve* in dimension d is defined as

$$\mu_d : \mathbb{R} \longrightarrow \mathbb{R}^d \qquad \mu_d : t \longmapsto (t, t^2, \ldots, t^d)^\top. \qquad (2.14)$$

The *cyclic polytope* in dimension d with n vertices, where $n > d$, is

$$C_d(n) = \mathrm{conv}\{\mu_d(t_i) \mid t_1 < \ldots < t_n \text{ affinely independent }\}. \qquad (2.15)$$

Example 2.7. Figure 2.7 shows the cyclic polytope in dimension 3 with 6 facets.
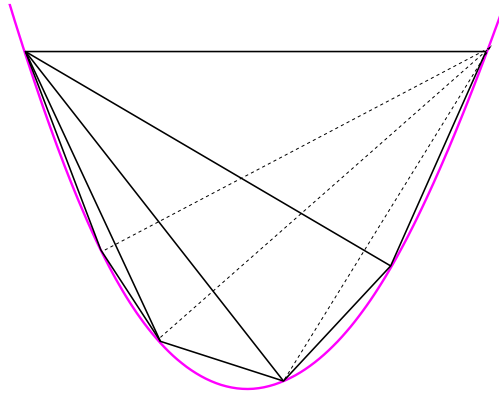
Figure 2.7   The cyclic polytope $C_3(6)$.

Given $k \in \mathbb{N}$ and a set S, we can represent the function $f : [k] \to S$ as the string $s = s(1)s(2) \cdots s(k)$; we have a *bitstring* if $S = \{0, 1\}$. A maximal

substring of consecutive 1's in a bitstring is called a *run*; an *interior run* is bounded on both sides by 0's. We will use the notation $\mathbf{1}^k$ for a run of length k, and 0^k for a string of 0's of length k A *Gale string of length n and dimension d*, where $n > d$, is a bitstring $s \in G(d, n)$ satisfying the following conditions:

1. exactly d bits of s are equal to $\mathbf{1}$;

2. (*Gale evenness condition*)

$$01^k0 \text{ is a substring of } s \implies k \text{ is even.} \tag{2.16}$$

In general, the Gale evenness conditions allows for Gale strings that start or end with an odd-length run; if d is even then s can start with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d, n)$ with even d as "loops" obtained by "glueing together" the endpoints of the strings; then all runs on the loops are even. Formally, we can see the indices of a Gale string $s \in G(d, n)$ with d even as equivalence classes modulo n.

Example 2.8. As an example of even d, we have

$$G(4, 6) = \{\mathbf{1111}00, \mathbf{111}00\mathbf{1}, \mathbf{11}00\mathbf{11}, \mathbf{1}00\mathbf{111}, 00\mathbf{1111},$$

$$0\mathbf{1111}0, \mathbf{11}0\mathbf{11}0, \mathbf{1}0\mathbf{11}0\mathbf{1}, 0\mathbf{11}0\mathbf{11}\}$$

The strings $\mathbf{1111}00, \mathbf{111}00\mathbf{1}, \mathbf{11}00\mathbf{11}, \mathbf{1}00\mathbf{111}, 00\mathbf{1111}$ and $0\mathbf{1111}0$ are equivalent under a cyclic shift (if considering the strings as "loops", the $\mathbf{1}$'s are all consecutive), as are the strings $\mathbf{11}0\mathbf{11}0, \mathbf{1}0\mathbf{11}0\mathbf{1}$ and $0\mathbf{11}0\mathbf{11}$ (two runs of two $\mathbf{1}$'s separated by a single 0).

As an example of odd d, we have

$$G(3, 5) = \{\mathbf{111}00, \mathbf{1}0\mathbf{11}0, \mathbf{1}00\mathbf{11}, \mathbf{11}00\mathbf{1}, 0\mathbf{11}0\mathbf{1}, 00\mathbf{111}\}$$

Notice that $0\mathbf{1}0\mathbf{11}$ is a shift of $\mathbf{1}0\mathbf{11}0$ but it is not a Gale string.

The relation between cyclic polytopes and Gale strings was given by Gale [9].

Theorem 7. (Gale [9]) *For any $d, n \in \mathbb{N}$, where $n > d$,*

$$F \text{ is a facet of } C_d(n)$$

$$\Longleftrightarrow$$

$$F = \text{conv}\{\mu(t_j) \mid s(j) = 1 \quad \text{for } s \in G(d, n)\}. \quad\quad (2.17)$$

Proof. By definition, $C_d(n) = \text{conv}\{\mu_d(t_j) \mid t_1 < \ldots < t_n \text{ affinely independent}\}$. Let $\overline{t_1} < \ldots < \overline{t_d}$ be a choice of some of the t_j's in the definition of $C_d(n)$. Then the $\mu_d(\overline{t_i})$'s define an hyperplane H that crosses the moment curve $\mu_d(t)$ at all and only the $\overline{t_i}$'s. Given the definition of moment curve, the hyperplane H is never tangent to the moment curve, and every crossing gives a "change of sign"; that is, if $t, t' \notin \{\overline{t_i}\}$ and $t < \overline{t_i} < t'$ for one and only one $\overline{t_i}$ then $\mu_d(t)$ and $\mu_d(t')$ have opposite sign. A facet F of the cyclic polytope $C_d(n)$ then corresponds to a choice of $\overline{t_i}$'s with $i \in [d]$, such that for all the t_k's with $k \in [n]$ and $t_k \notin \{\overline{t_i} \mid i \in [d]\}$ all the $\mu_d(t_k)$'s have the same sign. This can happen only if for every couple of t_k's the moment curve has an even number of changes of sign between them; therefore there is an even number of $\overline{t_i}$'s between any two t_k's. Let s be the bitstring in which the **1**'s correspond to the t_i's and and the 0's correspond to the other t_k's; then the condition for being a facet in (**??**) becomes the Gale evenness condition. $\qquad\square$

Notice that the moment curve has exactly d zeroes implies that the each facet of $C_d(n)$ is a d-simplex, so $C_d(n)$ is simplicial and the choice of the t_j's in the proof is irrelevant.

Example 2.9. Consider the facet F of the cyclic polytope $C_3(6)$ marked in blue in Figure 2.8. If we label the vertices on the moment curve as t_i with $i \in [n]$ and we set $s(i) = 1$ if t_i is a vertex of F and $s(i) = 0$ otherwise, we see that the corresponding Gale string $s \in G(3, 6)$ is $s = \mathbf{1}00\mathbf{11}0$. Figure 2.9 shows the moment curve and its intersections with the hyperplane H in $\mathbb{R}^3$ defined by the t_i such that $s(i) = 1$.
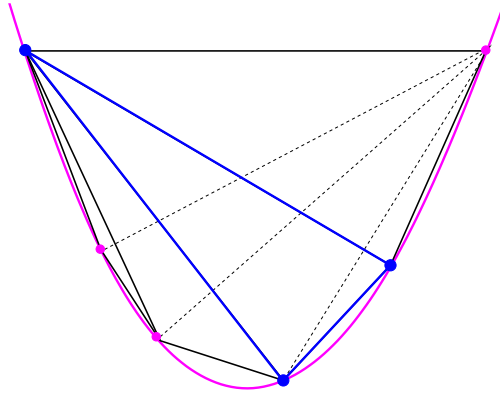
Figure 2.8 The facet of the cyclic polytope $C_3(6)$ corresponding to the Gale string $s = \mathbf{1}00\mathbf{11}0 \in G(3,6)$.
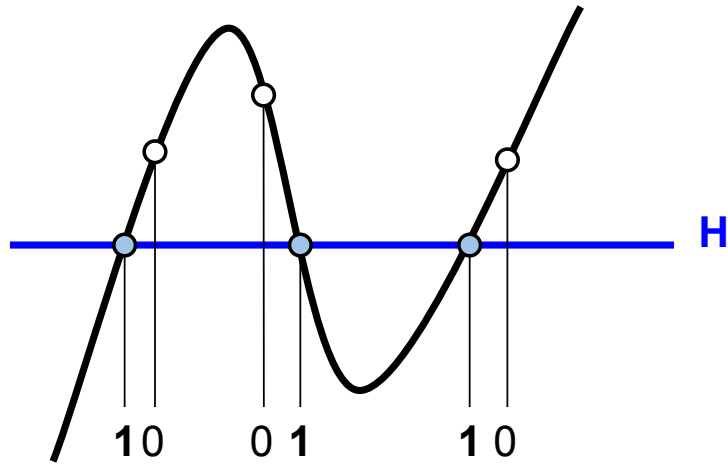
Figure 2.9 The facet of the cyclic polytope $C_3(6)$ given by the intersection of the moment curve and the hyperplane where $\mu_3(t_1) = \mu_3(t_4) = \mu_3(t_5) = 0$ corresponds to the Gale string $s = \mathbf{1}00\mathbf{11}0 \in G(3,6)$.

Example 2.10. Figure 2.10 shows the cyclic polytope $C_4(6)$, with the exterior facet corresponding to the Gale string $s = \mathbf{1111}00$. Figure 2.11 shows $s = \mathbf{1111}00$ as intersection with the hyperplane H.
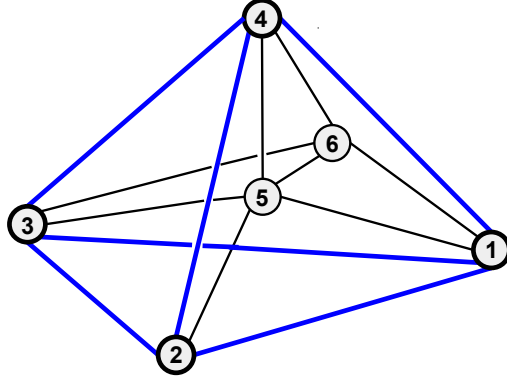
Figure 2.10   The cyclic polytope $C_4(6)$. The thin lines represent the edges inside the exterior facet, in bold lines, corresponding to the Gale string $s = \mathbf{1111}0 \in G(4,6)$.
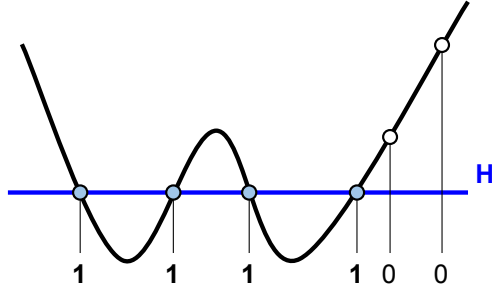

Figure 2.11   The facet of the cyclic polytope $C_4(6)$ given by the intersection of the moment curve and the hyperplane where $\mu_4(t_1) = \mu_4(t_2) = \mu_4(t_3) = \mu_4(t_4) = 0$ corresponds to the Gale string $s = \mathbf{1111}00 \in G(4,6)$.

Example 2.11. As a counterexample, consider Figure 2.12. The points $t = t_4$ and $t' = t_6$ lie on the moment curve but are on opposite sides of the hyperplane H. The corresponding bitstring is $s = \mathbf{111}010$, which is not a Gale string; the violation of the Gale evenness condition correspond to the change of sign between the t_j's. An analogous case is shown in Figure 2.13; the corresponding bitstring should be $s = \mathbf{101010}$.
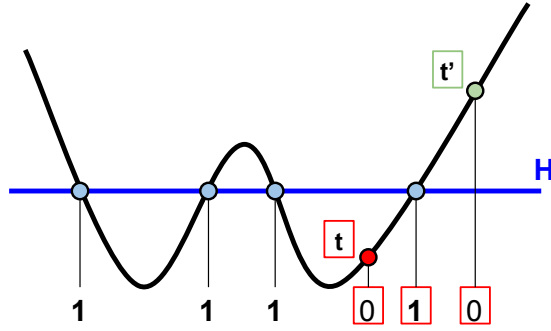
Figure 2.12   There is a change of sign between two t_j's; the bitstring $s = \mathbf{111}010$ does not satisfy the Gale evenness condition, and the t_j's do not correspond to a facet of $C_4(6)$.
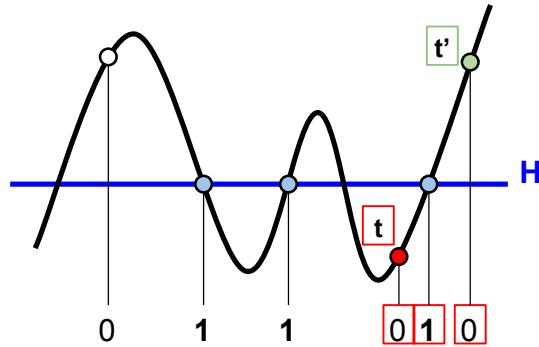
Figure 2.13   There is a change of sign between two t_j's; the bitstring $s = 0\mathbf{11}010$ does not satisfy the Gale evenness condition, and the t_j's do not correspond to a facet of $C_3(6)$.

We now apply Theorem 11 to the study of bimatrix games. Proposition 2.3 states that 2-Nash can be reduced to Another Completely Labeled Facet. If the polytope Q in Theorem 6 is cyclic and we define a labeling for Gale strings such that a completely labeled Gale string corresponds to a completely labeled facet of Q, we can study unit vector games and their dual cyclic best response polytope as Gale strings. We say that $s \in G(d, n)$ is a *completely labeled Gale string* for some labeling function $l_s : [n] \to [d]$ if $\{l_s(i) \mid s(i) = \mathbf{1} \text{ for } i \in [n]\} = [d]$. Since $s \in G(d, n)$ has exacty d bits equal to $\mathbf{1}$, this means that for each $j \in [d]$ there is exactly one $i \in [n]$ such that $s(i) = \mathbf{1}$ and $l_s(i) = j$. Note that it is not always possible to find a completely labeled Gale string.

Example 2.12. For $l_s = 121314$, there are no completely labeled Gale strings. The labels $l_s(i) = 2, 3, 4$ appear only once in l_s, as $l_s(2), l_s(4), l_s(6)$ respectively; therefore we must have $s(2) = s(4) = s(6) = 1$. For every other $i \in [n]$ we have $l(i) = 1$, so we have $l_s(i) = 1$ for exactly one $i = 1, 3, 5$. The candidate strings are then $s = \mathbf{1}\mathbf{1}0\mathbf{1}0\mathbf{1}$, $s' = 0\mathbf{1}\mathbf{1}\mathbf{1}0\mathbf{1}$, $s'' = 0\mathbf{1}0\mathbf{1}\mathbf{1}\mathbf{1}$; but none of these satisfies the Gale evenness condition.

A Gale game is a unit vector game (U, B) where $U = (e_{l(1)},...,e_{l(d)})$ for some labeling $l : [n] \to [d]$ and for which the dual of the best response polytope is a cyclic polytope $Q = \text{conv}\{e_1, \ldots, e_d, c_1, \ldots, c_n\}$. Theorem 6 gives the labeling (2.13) for the $d + n$ vertices of Q as

$$l_v(-e_i) = i \qquad \text{for } i \in [m];$$
$$l_v(c_j) = l(j) \qquad \text{for } j \in [n].$$

We define the labeling $l_s : [d + n] \to [d]$ of $G(d, n)$ as

$$l_s(i) = i \qquad \text{for } i \in [d];$$
$$l_s(d + j) = l(j) \qquad \text{for } j \in [n]. \tag{2.18}$$

Then the Gale strings $s \in G(d, d + n)$ that are completely labeled for l_s corre-

spond exactly to the completely labeled facets of Q, with the facet F_0 corresponding to the "trivial" completely labeled string $\mathbf{1}^d 0$.

From this point forward, we will assume that d is even. We will also assume that the labeling $l_s : [n] \to [d]$ is such that $l_s(i) \neq l_s(i+1)$; this can be done without loss of generality, given the following consideration. Suppose that $l_s(i) = l_s(i+1)$ for some index i, and let s be a completely labeled Gale string for l_s. Then only one of $s(i)$ and $s(i+1)$ can be equal to $\mathbf{1}$ (notice that it's possible that both are 0s). So $s(i)s(i+1)$ will never be a run of even length that "interferes" with the Gale Evenness Condition, so we can identify the indices i and $i+1$.

Example 2.13. Given the string of labels $l_s = 123432$, there are four associated completely labeled Gale strings in $G(4,6)$: $s_A = \mathbf{111100}$, $s_B = \mathbf{110110}$, $s_C = \mathbf{100111}$ and $s_D = \mathbf{101101}$. These correspond to the completely labeled facets for the labeling shown in Figure 2.14.

facet	1 2 3 4 3 2
A	1 1 1 1 0 0
B	1 1 0 1 1 0
C	1 0 0 1 1 1
D	1 0 1 1 0 1
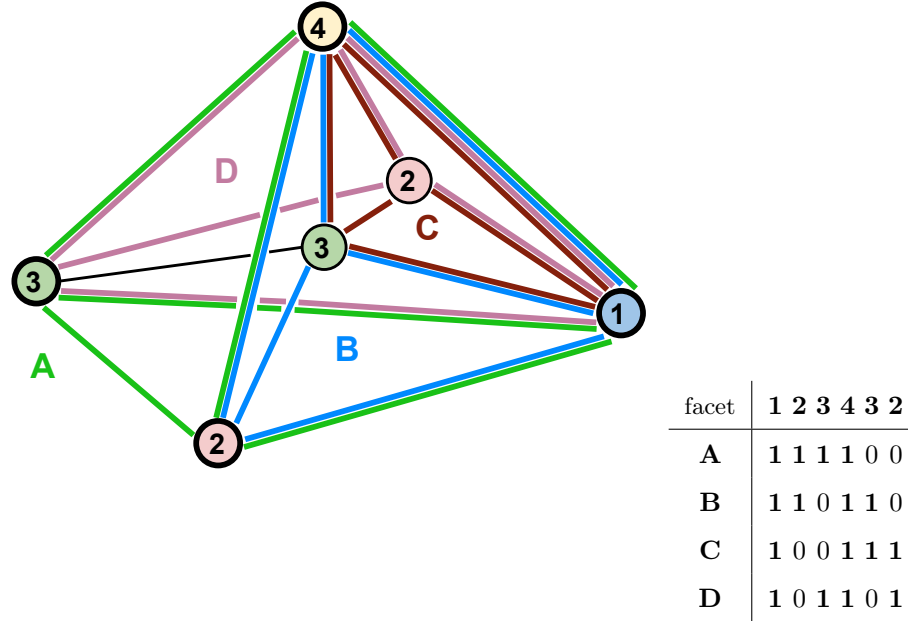
Figure 2.14 The cyclic polytope $C_4(6)$; the labeling of the vertices correspond to the labeling of $G(4,6)$ given by $l_s = 123432$.

We can now define the problem ANOTHER GALE as in Table 2.3.

ANOTHER GALE

input : A labeling $l : [n] \to [d]$, where d is even and $d < n$. A Gale string $s \in G(d, n)$, completely labeled by l.

output: A Gale string $s' \in G(d, n)$, completely labeled by l, such that $s' \neq s$.

Table 2.3 The problem ANOTHER GALE.

It takes polynomial time to translate the facets of the cyclic polytope $C_d(n)$ into the corresponding Gale strings in $G(d, n)$, following the proof of Theorem 11. Furthermore, defining the labeling l_s from the labeling l_v also takes polynomial time: for the labels $i \in [d]$ it is immediate, for the labels $d + j$, where $j \in [n]$, we have to check the $d \times n$ matrix U of the imitation game. Therefore, by Proposition 2.3, we have a reduction from GALE NASH to ANOTHER GALE.

Proposition 2.4. GALE NASH *is polynomial-time reducible to* ANOTHER GALE.

Chapter 3

Algorithmic and Complexity Results

In the previous chapter we have defined some problems of the form "find another completely labeled..." for vertices, facets and Gale strings. In this chapter we will finally study the complexity of these problems. The first section will present different versions of the standard algorithm introduced by Lemke and Howson [11]. We will first apply it to the problem ANOTHER COMPLETELY LABELED VERTEX, in particular as a tool to find a solution for 2-NASH, its original motivation; we will then touch on its dual version for the problem ANOTHER COMPLETELY LABELED FACET; finally, we will describe the Lemke-Howson for Gale Algorithm, which returns a solution to ANOTHER GALE. We will follow each one of these algorithms with a very straightforward proof of the **PPA** complexity of ANOTHER COMPLETELY LABELED VERTEX and ANOTHER COMPLETELY LABELED FACET; in the case of ANOTHER GALEwe will give a result of **PPAD** complexity. We will close the section with an example of labeling, due to Morris [15], for which the Lemke-Howson for Gale algorithm presents exponential running time; this is the labeling that Savani and von Stengel [19] [20] have used to build Gale games that are "hard to solve".

The second section presents our original result: a polynomial time algorithm for ANOTHER GALE, therefore proving that GALE NASH is in **FP**. Unless **PPAD**=**P**, this goes in the opposite direction of our first conjecture of PPAD-completeness suggested by the "hard to solve" games. Our proof relies on a theorem by Edmonds [6] that gives a polyomial-time algorithm to find a perfect matching of a graph, or deciding that it is not possible to find one. The key of the proof is the construction of a graph from any string of labels; the perfect matchings of the graph will correspond to the completely labeled Gale strings for the labels. We will first prove the **FP** complexity of finding one of these completely labeled Gale strings; we will then extend the proof to find the second string required by ANOTHER GALE.

3.1   The Lemke-Howson Algorithm

Let P be a simple d-polytope with n facets. We *pivot on the vertices* of P by moving from a vertex x to another vertex y connected to x by an edge. Note that, since P is simple, there are exactly d possible choices for y. Analogously, we *pivot on the facets* of a simplicial polytope Q in dimension d by moving from a facet F to a facet G that share all vertices but one; since Q is simplicial, there are d possible choices for G.

Suppose now that there is a labeling $l_f : [n] \to [d]$ of the facets of the simple polytope P. If we pivot from a vertex x to a vertex x' we "leave behind" a facet F with label k; so, if x has labels $(l_1, \ldots, k, \ldots, l_d)$, then x' has labels $(l_1, \ldots, h, \ldots, l_d)$, where h is the label of the facet F' that does not have x as its vertex. We call this *dropping label k and picking up label h*, or *pivoting on label k*; see Figure 3.1. Analogously, if there is a labeling $l_v : [n] \to [d]$ of the vertices of the simplicial poytope Q and we pivot from a facet F with labels $(l_1, \ldots, k, \ldots, l_d)$ to a facet F' with labels $(l_1, \ldots, h, \ldots, l_d)$, we say that we *drop label k and pick up label h*, or that we *pivot on label k*; see Figure 3.2.
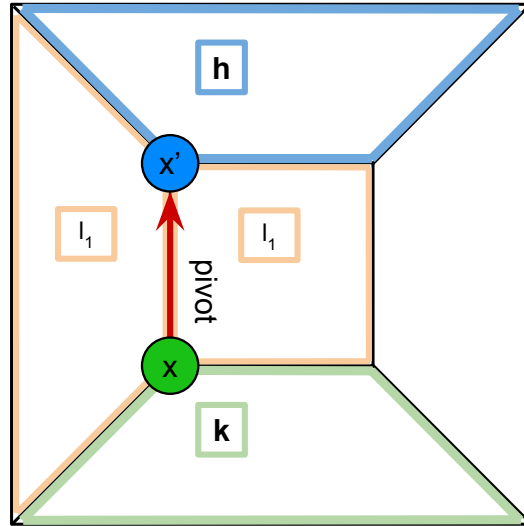
Figure 3.1 A pivot on the vertices of the cube.
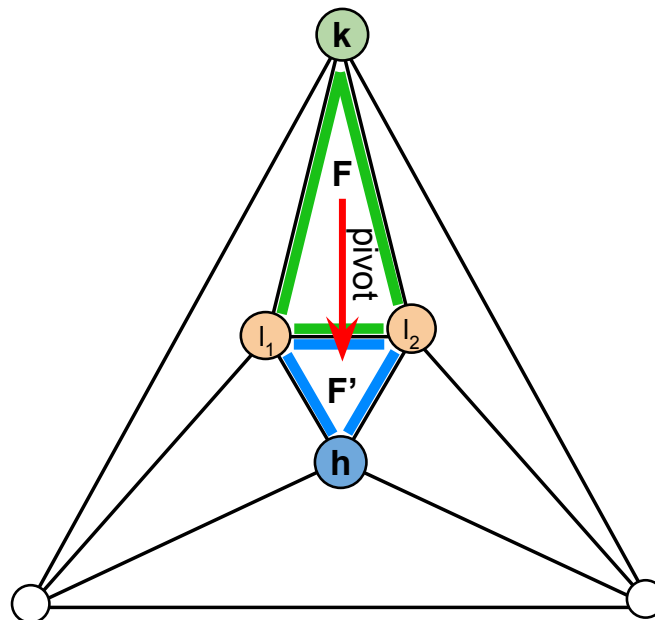
Figure 3.2 A pivot on the facets of the octahedron.

Let $m, n \in \mathbb{N}$ with $m \leq n$; consider a set X and a labeling $l : X \to [m]$. The n-uple $x = (x_1, \ldots, x_n) \in X^n$ is *almost completely labeled* if there is exactly one $k \in [m]$ such that $\{j \in [m] \mid x_i = j \text{ for some } i \in [n]\} = [m] \setminus \{k\}$. That is, an n-uple x is almost completely labeled if all labels appear once in x except for the *missing label $k \in [m]$* and the *duplicate label $h \in [m]$* that appears twice. It's easy to see that if we pivot from an almost completely labeled facet (or vertex) on the duplicate label, or from a completely labeled facet (or vertex) on any label, we reach either an almost completely labeled or a completely labeled facet (or vertex).

Algorithm 1: Lemke-Howson

    input : A simple d-polytope P with n facets. A labeling $l_f : [n] \to [d]$ of the facets of P. A vertex x_0 of P, completely labeled for l.

    output: A completely labeled vertex $x \neq x_0$ of P.

1 choose any label $k \in [d]$

2 pivot on label k from x_0 to x

3 **while** *x is not completely labeled* **do**

4   pivot on the duplicate label h from x to $x' \neq x_0$

5   set $x_0 = x$, $x = x'$

6 **return** x

The classic Lemke-Howson Algorithm 1, first introduced by Lemke and Howson [11], employs pivoting on the vertices of a simple polytope, moving through a succession of almost completely labeled vertices with missing label k; these vertices can be seen as steps of a path, called *Lemke path*. We will use Lemke paths to prove some fundamental properties of both the Lemke-Howson Algorithm and the problem ANOTHER COMPLETELY LABELED VERTEX.

Proposition 3.1. *The Lemke-Howson Algorithm 1 returns a solution to the* **PPA** *problem* ANOTHER COMPLETELY LABELED VERTEX.

Furthermore, the number of completely labeled vertices in a simple polytope

with labeled facets is even.

Proof. We first show that the Lemke-Howson Algorithm works. Let x' be an almost completely labeled vertex of the Lemke path with duplicate label h. There are only two facets with duplicate label h that contain x; since P is simple, each one of which corresponds to exactly an edge. One of these edges has been traversed to get to x', the other edge will be traversed to leave x' in the next step. Therefore, there are no "loops" where a vertex is visited more than once; the Lemke paths are *simple*.

The parity is proven by the following argument: each Lemke path is uniquely determined by its missing label and its starting point, so the Lemke path from the endpoint with the same missing label will lead back to the starting point. Since the endpoint and the starting point are different, the Lemke paths must connect an even number of points.

Finally, for each label $k \in [d]$ chosen in line 1 of Algorithm 1, the Lemke paths are disjoint paths connecting all the completely labeled vertices of P, with a standard endpoint x_0. The problem ANOTHER COMPLETELY LABELED VERTEX correspond to finding a non-standard endpoints of this graph, which is a **PPA** problem. $\qquad\square$

Applying the parity result of Proposition 3.1 to the case of a bimatrix game (not necessarily a unit vector game), and remembering that the point $(\mathbf{0}, \mathbf{0})$ corresponds to the "artificial" equilibrium, we have the following result, due to Lemke and Howson [11].

Theorem 8. (Lemke-Howson [11]) *Every non-degenerate bimatrix game has an odd number of Nash equilibria.*

There are two ways of using the Lemke-Howson Algorithm to find a Nash equilibrium of a bimatrix game (A, B). The first one is to "symmetrize" the game as in Proposition 2.1. Let $S = \{z \in \mathbb{R}^{m+n} \mid z \geq \mathbf{0}, \; Cz \leq \mathbf{1}\}$ be the polytope associated to the game $(C, C^\top)$, where $C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}$. The facets

of C correspond to $2(m + n)$ inequalities; we label both the i-th and the $(m + n + i)$-th inequality as $i \in [m + n]$ and we apply the Lemke-Howson Algorithm starting from the vertex $\mathbf{0}$. This returns a Nash equilibrium (z, z) of C, which corresponds to a Nash equilibrium $(x, y) = z$ of (A, B). We can also follow the "traditional" version of the Lemke-Howson Algorithm alternating a move on the best response polytopes P and a move on the best response polytope Q of (2.4). Since the polytopes P and Q are in $\mathbb{R}^m$ and $\mathbb{R}^n$, whereas S is a polytope in $\mathbb{R}^{m+n}$, this second version is much easier to visualize.

Example 3.1. Consider the 3×3 game (A, B) of Example 2.1.

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}.
$$

The best response polytopes can be represented as the best response regions of Figure 2.1 extended to the origin $\mathbf{0}$, as in Figure 3.3. The path starts in $(\mathbf{0}, \mathbf{0})$; we drop the label 2 and we move on the polytope P. The label 6 is duplicate; so we drop it and we make the next move on the polytope Q, and so on until we reach the point x, that is the only Nash equilibrium of (A, B).
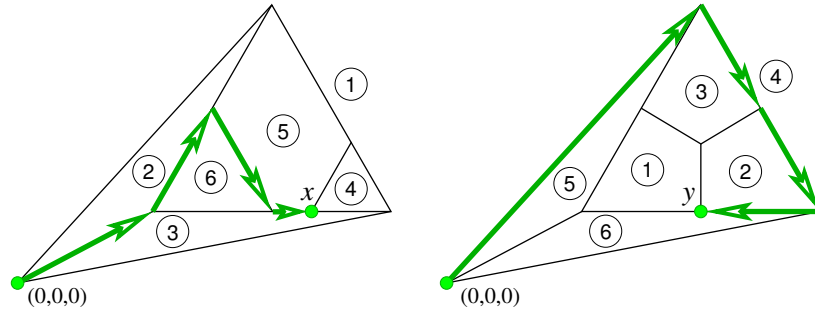
Figure 3.3 Lemke path for missing label 2 on the best response polytopes P (left) and Q (right) of game (2.2).

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It is possible to have an equilibrium that cannot be reached applying the Lemke-Howson Algorithm from the artificial equilibrium, or even from the endpoint of a Lemke path from the artificial equilibrium. This is can be seen in the next example, due to Wilson, given in Shapley [21]. Notice that, by the parity result in Proposition 3.1, there must be at least another equilibrium that is "disconnected" from the artificial one in this way.

Example 3.2.

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

The equilibria of (A, B) are $(x_1, y_1) = ((0, 0, 1), (0, 0, 1))$, $(x_2, y_2) = ((0, 1/2, 1/2), (0, 3/4, 1/4))$ and $(x_3, y_3) = ((0, 1, 0), (0, 1, 0))$. All Lemke paths from the artificial equilibrium $(0, 0)$ end at (x_1, y_1), and consequently all other Lemke paths connect (x_2, y_2) and (x_3, y_3); see Figure 3.4.
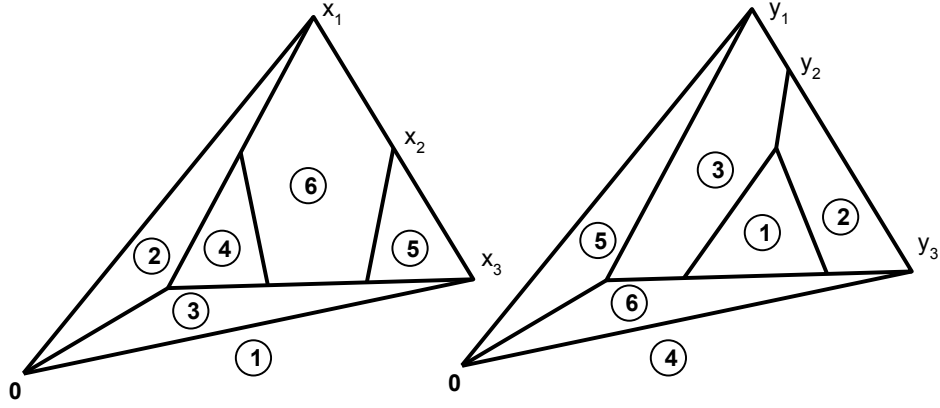
Figure 3.4 Best response polytopes of game 3.1.

The dual version of the Lemke-Howson Algorithm 1 and of proposition 3.1 is quite straightforward.

Algorithm 2: Dual Lemke-Howson

    input : A simplicial m-polytope Q with n vertices. A labeling
        $l_v : [n] \to [d]$ of the vertices of P. A vertex F_0 of Q, completely
        labeled for l.

    output: A completely labeled facet $F \neq F_0$ of Q.

1 choose any label $k \in [d]$

2 pivot on label k from F_0 to F

3 **while** *x is not completely labeled* **do**

4   pivot on the duplicate label h from F to $F' \neq x_0$

5   set $F_0 = x$, $F = F'$

6 **return** x

Proposition 3.2. *The Dual Lemke-Howson Algorithm 2 returns a solution to the* **PPA** *problem* ANOTHER COMPLETELY LABELED FACET.

Furthermore, the number of completely labeled facets in a simplicial polytope with labeled vertices is even.

By Theorem 5 and Theorem 6, in the case of unit vector games it is enough to apply the Lemke-Howson Algorithm to the polytope $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, \ B^\top x \leq \mathbf{1}\}$ in (2.9), or the Dual Lemke-Howson Algorithm to the polytope $Q = \mathrm{conv}(\{e_1, \ldots, e_m\}) \ \cup \ \{c_1, \ldots, c_n\})$ in (2.11). The following theorem by Savani and von Stengel [20] guarantees that not only this yield a Nash equilibrium, but no potential solutions are "lost" considering the polytope P^l with m labels instead of the product of polytopes $P \times Q$ with $m + n$ labels; an analogous result holds for the dual case.

Theorem 9. *Let (U, B) be a unit vector game, with $U = (e_{l(1)} \cdots e_{l(n)})$ for a labeling $l : [n] \to [m]$. Let $P = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ and $Q = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}$ as in (2.4); let $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ as in (2.9). Then for the missing label $k \in [m]$ the Lemke path on $P \times Q$ projects to a path on P that corresponds to the Lemke path on P^l for the missing label k; for the missing label $k = m + j$, where $j \in [n]$, the Lemke path on $P \times Q$ projects to a path on Q that corresponds to the Lemke path on P^l for the missing label $l(j)$.*

We finally focus on the case of unit vector games where the simplicial polytope Q is cyclic, which we can study from the point of view of Gale strings. We will consider $s \in G(d, n)$, with d even, as "loops". Let $s(i) = 1$ for an index $i \in [n]$; then, by Gale evenness condition, there is an odd run of $\mathbf{1}$'s either on the left or on the right of position i in s; let j be the first index after this run. A *pivot on s* is given by setting $s(i) = 0$ and $s(j) = 1$. If there is a labeling $l_s : [n] \to [m]$, we say that we *pivot on label $l_s(i)$, dropping label $l_s(i)$* and *picking up label $l_s(j)$*. The *Lemke Howson for Gale Algorithm* is given as Algorithm 3.

Algorithm 3: Lemke-Howson for Gale

    input : A labeling $l_s : [n] \to [d]$, where d is even, such that there is a completely labeled Gale string $s_0 \in G(d, n)$.

    output: A completely labeled Gale string $s \in G(d, n)$ such that $s \neq s_0$.

1 choose a label $k \in [d]$

2 pivot on label k from s_0 to s

3 **while** *s is not completely labeled* **do**

4   pivot on the duplicate label h from s to $s' \neq s_0$

5   rename $s_0 = s$, $s = s'$

6 **return** s

The next example illustrates the correspondence between the Lemke-Howson Algorithm and the Lemke-Howson for Gale Algorithm.

Example 3.3. Figure 3.5 shows the cyclic polytope $C_4(6)$ with the labeling

$$
\begin{aligned}
l_v(i) &= \quad i \qquad \text{for } i \in [4]; \\
l_v(5) &= \quad 3; \\
l_v(6) &= \quad 2.
\end{aligned}
$$

This corresponds to the labeling l_s for $G(4,6)$ given in Example 2.13, which gives the four completely labeled Gale strings $s_A = \mathbf{111100}$, $s_B = \mathbf{110110}$, $s_C = \mathbf{100111}$ and $s_D = \mathbf{101101}$. These correspond to the facets A, B, C and D of $C_4(6)$. Pivoting from $s_A = \mathbf{111100}$ dropping label 3 returns $s_B = \mathbf{110110}$, analogously, pivoting from facet A dropping label 3 returns facet B.

facet	**1 2 3 4 3 2**
A	**1 1 $\underline{\mathbf{1}}$ 1 0 0**
B	**1 1 0 1 $\bar{\mathbf{1}}$ 0**
facet	**1 2 3 4 3 2**
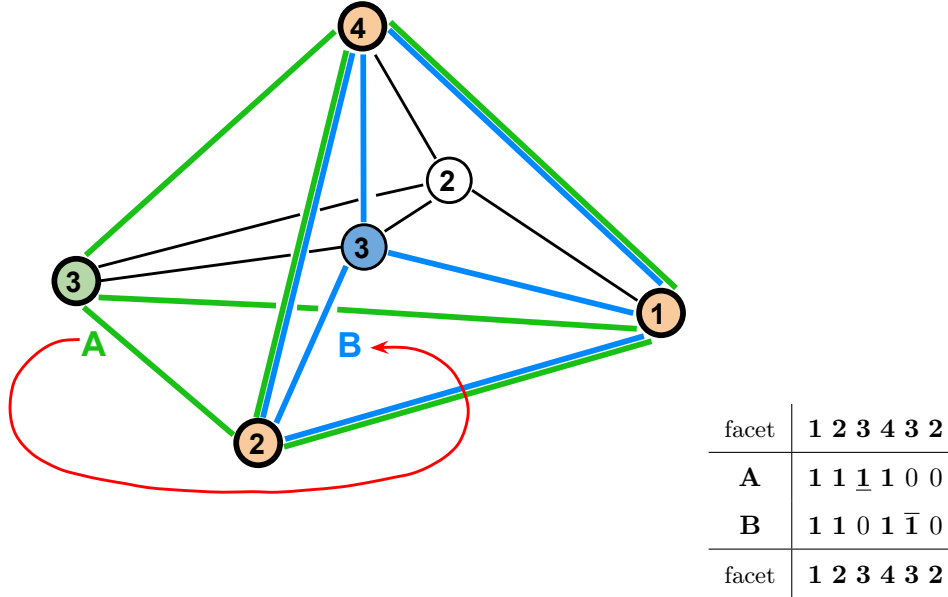
Figure 3.5 The pivoting to $s_A = \mathbf{111100}$ to $s_B = \mathbf{110110}$ in the Lemke-Howson for Gale Algorithm corresponds to the pivoting from facet A to facet B in the Dual Lemke-Howson Algorithm.

We will now show a construction that we will allow us to extend the analogous of Proposition 3.1 for the Lemke-Howson for Gale Algorithm to prove the **PPAD** complexity of ANOTHER GALE. A *permutation* of the elements of an ordered set S is a sequence without repetition of elements of S; a *transposition* is a permutation of exactly two elements of S. The *sign of a permutation σ* is $\mathrm{sign}(\sigma) = (-1)^m$, where m is the number of transpositions needed to get $\sigma_0 = 1 \ldots n$ from σ. It is immediate to see that any two permutations that differ in only one transposit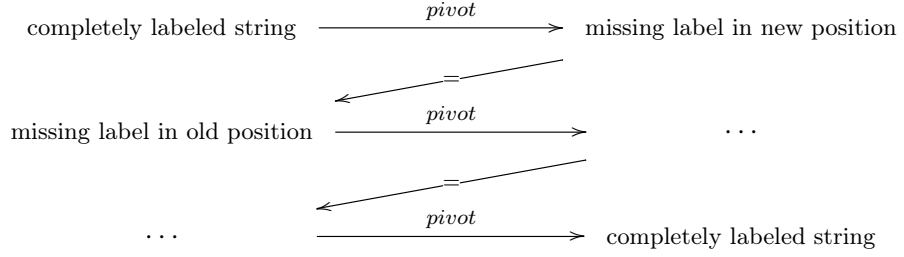ion have opposite sign. The *sign of a completely labeled Gale string $s \in G(d,n)$* is defined as follows: let $l : [n] \to [d]$ be the relative labeling of $G(d,n)$, and let l_0 be the string of labels $l(i)$ such that $s(i) = 1$ and that two labels corresponding to a run in l are adjacent in l_0. Then we define $\mathrm{sign}(s) = \mathrm{sign}(l_0)$. Notice that if $l(i) = i$ for $i \in [d]$ then the sign of the completely labeled Gale string $1^d 0^{(n-d)}$ is always positive. The *sign of an almost completely labeled Gale string $s \in G(d,n)$* with missing label k and duplicate label h is defined on two different strings. Let i_1 be the index of h reached by the last pivot (the "new" position of the **1**) and let i_2 be the index of h such that $s(i_2) = 1$ before the last pivot (the "old" position of the **1**). Let l_1 be the string obtained as l_0 substituting k to h at index i_1, and let l_2 be the string obtained as l_0 substituting k to h at index i_2. Notice that $\mathrm{sign}(l_1) = -\mathrm{sign}(l_2)$, since they can be obtained from each other applying the transposition (kh).

Consider now the steps of the Lemke paths in the Lemke-Howson for Gale Algorithm; we assume that $\mathrm{sign}(s_0) = +1$ without loss of generality, since the negative case is the same just with opposite signs. If the first pivot returns another completely labeled Gale string s, this must have negative sign because it has been obtained "jumping" over an odd number of **1**'s. For the same reason, if the pivoting returns an almost completely labeled Gale string, we have that $\mathrm{sign}(l_1) = -1$, therefore $\mathrm{sign}(l_2) = +1$. The next pivoting step drops the label h from index i_2, so again we change sign. Running the Lemke-Howson

for Gale Algorithm will therefore result in the sign of the completely labeled Gale strings swinging as in Table 3.1. Notice that all this construction can be done in polynomial time; orienting all Lemke paths from positive to negative reduces the problem ANOTHER GALE to END OF THE LINE.

completely labeled string ————————$pivot$————————→ missing label in new position

←——————$=$——————

missing label in old position ————————$pivot$————————→ $\cdots$

←——————$=$——————

$\cdots$ ————————$pivot$————————→ completely labeled string

Table 3.1 Sign switching on the Lemke-Howson for Gale Algorithm.

Proposition 3.3. *The Lemke-Howson for Gale Algorithm 3 returns a solution to the* **PPAD** *problem* ANOTHER GALE.

Furthermore, the number of completely labeled Gale strings $s \in G(d, n)$, where d is even, is even.

A result similar to Proposition 3.3, but more general, is given in Shapley [21]: two Nash equilibria at the ends of a Lemke path have opposite *index*, which is analogous to the sign but defined using determinants on the payoff matrices for the equilibrium support; the artificial equilibrium is assigned index $+1$. This is used to prove that a nondegenerate game with n Nash equilibria with index $+1$ has $n + 1$ Nash equilibria with index -1.

Example 3.4. Let $l_s = 123432$; consider the Lemke path from the completely labeled Gale string $s = \mathbf{1111}00$ dropping label 1. Figure 3.6 shows the graph of Table 3.1. Notice that $\text{sign}(\mathbf{101101}) = \text{sign}(l(6)l(1)l(3)l(4)) = \text{sign}((2134))$, since $s(6) = s(1) = 1$ and therefore the indices 6 and 1 are consecutive in the same run.
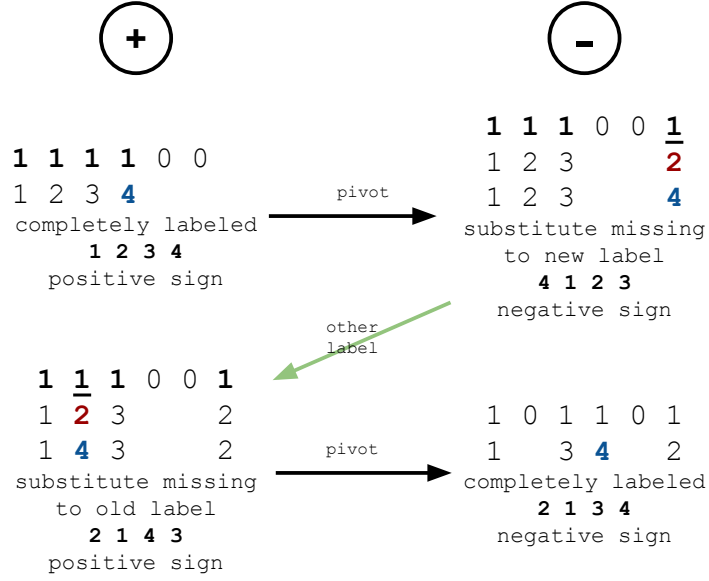
Figure 3.6 Pivoting with sign on 123432.

An example of labeling $l : [2d] \to [d]$ where the length of the Lemke paths on the cyclic polytope $C_d(2d)$, therefore on $G(d, 2d)$, grows exponentially in d, regardless of the label that is dropped, is given by Morris [15]. This labeling is given for d both even and odd as follows:

$$l(k) = k \qquad \text{for } k \in [d];$$
$$l(d + 1) = d;$$
$$l(d + k) = d - k \qquad \text{for } 2 \le k < d, \; k \text{ even ;}$$
$$l(d + k) = d - k + 2 \qquad \text{for } 2 \le k \le d, \; k \text{ odd ;}$$
$$l(2d) = 1 \qquad \text{for } k \text{ even.}$$

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Avoiding repetitions and restricting to the case of d even, as we assumed, the Morris' labeling is simplified to $l : [2d - 2] \to [d]$ as follows:

$$l(k) = k \qquad \text{for } k \in [d];$$
$$l(d + k) = d - k + 1 \qquad \text{for } k \in [d], \ k \text{ even };$$
$$l(d + k) = d - k - 1 \qquad \text{for } k \in [d], \ k \text{ odd }.$$

Example 3.5. Consider the labeling $l = 1234564523$ for $G(6, 10)$. The only two completely labeled Gale string are $s = \mathbf{111111}00$ and $s' = \mathbf{1}0000\mathbf{011111}$. Figure 3.7 shows the Lemke path for label 1.

$$
\begin{array}{cccccccccc}
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{4} & \mathbf{5} & \mathbf{2} & \mathbf{3} \\
\hline
\underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & \mathbf{1} & \underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & \overline{\mathbf{1}} & 0 & 0 & 0 \\
0 & \mathbf{1} & \mathbf{1} & 0 & \underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & \overline{\mathbf{1}} & 0 & 0 \\
0 & \underline{\mathbf{1}} & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \overline{\mathbf{1}} & 0 \\
0 & 0 & \mathbf{1} & \overline{\mathbf{1}} & 0 & \mathbf{1} & \underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & \overline{\mathbf{1}} & \mathbf{1} & 0 & \underline{\mathbf{1}} & \mathbf{1} & 0 \\
0 & 0 & \underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1} & \overline{\mathbf{1}} \\
0 & 0 & 0 & \underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & \overline{\mathbf{1}} & 0 & \mathbf{1} & \mathbf{1} \\
0 & 0 & 0 & 0 & \underline{\mathbf{1}} & \mathbf{1} & \mathbf{1} & \overline{\mathbf{1}} & \mathbf{1} & \mathbf{1} \\
\overline{\mathbf{1}} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\hline
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{4} & \mathbf{5} & \mathbf{2} & \mathbf{3} \\
\end{array}
$$

Figure 3.7 The Morris path for $C(6, 10)$.

Savani and von Stengel [19] [20] relied on Morris' example to build different "hard to solve" games; these results give a strong motivation to study the compelxity of ANOTHER GALE. Our main result, in the next section, will give a **FP** algorithm that circumvents the problem of any exponential running time of the Lemke-Howson for Gale Algorithm.

3.2 The Complexity of ANOTHER GALE

A *matching* of a graph $G = (V, E)$ is a set $M \subseteq E$ such that every vertex $v \in V$ is the endpoint of at most one edge $m \in M$. A *perfect matching* is a matching such that there is an edge $m \in M$ incident to every vertex $v \in V$. Edmonds [6] proved the **FP** complexity of finding a perfect matching. The result can be easily extended to multigraphs.

PERFECT MATCHING

---

input : A multigraph $G = (V, E)$.

output: A perfect matching for G, or NO if there is no possible perfect matching for G.

Table 3.2 The problem PERFECT MATCHING.

Theorem 10. (Edmonds [6]) *It takes polynomial time to find the perfect matching of a graph or decide that no such matching exists.*

Proposition 3.4. PERFECT MATCHING *is in* **PF**.

To prove our main result on ANOTHER GALE, we will prove that the accessory problem GALE can be solved in polynomial time.

GALE

---

input : A labeling $l : [n] \to [d]$, where d is even and $d < n$.

output: A Gale string $s \in G(d, n)$ that is completely labeled by l

---

Table 3.3 The problem GALE.

Theorem 11. GALE *is in* **FP**.

Proof. Consider the multigraph $G = (V, E)$ with $V = [d]$, so that the vertices of G correspond to the labels $l(i) \in [d]$, and $E = \{(l(i), l(i + 1)) \text{ for } i \in [n]\}$,

so that there is an edge between two vertices if and only if the corresponding labels are next to each other at some index i.

Let $s \in G(d, n)$ be a completely labeled Gale string. By the Gale evenness condition, every run of length k in s corresponds uniquely to $k/2$ pairs of indices $(i, i+1)$ with $s(i) = s(i+1) = 1$. Since s is completely labeled, all labels $l(i) \in [d]$ occur at exactly one of these indices and the edges $(l(i), l(i+1))$ form a perfect matching of G.

Conversely, let M be a perfect matching for $G = (V, E)$, defined as above. Let s be a bitstring with $s(i) = s(i+1)$ for every $(l(i), l(i+1)) \in M$ and $s(i) = 0$ otherwise. Since M is a matching, all the $(l(i), l(i+1)) \in M$ are disjoint, so every run of s is of even length; therefore s satisfies the Gale evenness condition. Furthermore, since M is perfect, s is completely labeled.

This proves that GALE reduces to PERFECT MATCHING in polynomial time; therefore, by Proposition 3.4, Algorithm 4 takes polynomial time to find a solution of an instance of GALE or deciding that no such solution exists. □

Algorithm 4: Completely Labeled Gale String Finder

    input : A labeling $l : [n] \to [d]$.

    output: A Gale string $s \in G(d, n)$ completely labeled by l.

1 Set $G = (V, E)$, with $V = [d]$ and $E = \varnothing$

2 Set $s = 0^n$

3 for $i \in [n - 1]$ **do**

4 $E = E \cup \{(l(i), l(i + 1))\}$

5 $E = E \cup \{(l(n), l(1))\}$

6 Run the Edmonds' Algorithm on G

7 if *The Edmonds' Algorithm returns a perfect matching M of G* **then**

8   **for** $(l(i), l(j)) \in M$ **do**

9 $s(i) = s(j) = \mathbf{1}$

10   Return s

11 else

12   Return No

We give an example of the construction, using the Morris' labeling of Example 3.5.

Example 3.6. Figure 3.8 shows the graph for the Morris labeling $l_s = 1234564523$, and its two matchings $M = \{e_1, e_3, e_5\}$ and $M' = \{e_8, e_6, e_{10}\}$. These, in turn, correspond to the completely labeled Gale strings $s = \mathbf{1111110000}$ and $s = \mathbf{1000011111}$.
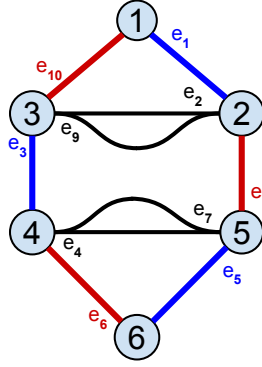
Figure 3.8 The graph G and its matchings for the Morris labeling $l = 1234564523$.

Notice that the existence of a completely labeled Gale string is not guaranteed.

Example 3.7. Consider the labeling $l_s = 121314$. Figure 3.9 shows the corresponding graph, that has no possible perfect matching. Analogously, as we have seen in Example 2.12, there isn't any possible completely labeled Gale string for l_s.
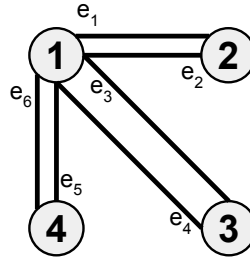
Figure 3.9 The graph for the labeling $l = 121314$

We finally extend the proof of Theorem 11 to the complexity of Another Gale.

Theorem 12. Another Gale *is in* **FP**.

Proof. Let $G = (V, E)$ be the graph corresponding to the labeling $l : [n] \to [d]$ as in the proof of Theorem 11 and let M_0 be the perfect matching of G corresponding to the completely labeled Gale string $s_0 \in G(d, n)$.

Theorem **??** guarantees the existence of another completely labeled Gale string $s \neq s_0$, therefore of the corresponding perfect matching $M \neq M_0$. Since $M \neq M_0$, there is at least one edge $e \in M_0$ such that $e \notin M$. Consider the $d/2$ graphs $G_i = (V, E_i)$, where $E_i = E \setminus \{e_i\}$ for $e_i \in M_0$. Since $V(G) = V(G_i)$ and $E(G_i) \subset E(G)$, if G_i has a perfect matching M_i, then this is a perfect matching for G as well. Since $e_i \notin E_i$, then $e_i \notin M_i$, and $M_i \neq M$. Since the number of G_i is exactly $d/2$, by Proposition 10 it takes at most polynomial time to find a perfect matching $M \neq M_0$ in one of the G_i's, and therefore the corresponding completely labeled Gale string $s \neq s_0$, applying Algorithm 5. $\qquad\square$

Algorithm 5: Another Gale Finder

---

input : A labeling $l : [n] \to [d]$ and a Gale string $s_0 \in G(d, n)$
completely labeled by l.

output: A Gale string $s \in G(d, n)$ completely labeled by l, such that
$s \neq s_0$.

---

1 Set $G = (V, E)$, with $V = [d]$ and $E = M_0 = \varnothing$

2 Set $s = 0^n$

3 **for** $i \in [n - 1]$ **do**

4 $\quad$ $E = E \cup \{(l(i), l(i + 1))\}$

5 $\quad$ **if** $s_0(i) = s_0(i + 1) = 1$ **then**

6 $\quad\quad$ $M_0 = M_0 \cup \{(l(i), l(i + 1))\}$

7 $E = E \cup \{(l(n), l(1))\}$

8 **if** $s_0(n) = s_0(1) = 1$ *and* $(s_0(1), s_0(2)) \notin M_0$ **then**

9 $\quad$ $M_0 = M_0 \cup \{(l(n), l(1))\}$

10 **for** $m \in M_0$ **do**

11 $\quad$ $E_m = E \setminus \{m\}$

12 $\quad$ Run the Edmonds' Algorithm on $G_m = (V, E_m)$

13 $\quad$ **if** *The Edmonds' Algorithm returns a perfect matching M_m of G_m*
then

14 $\quad\quad$ **for** $(l(i), l(j)) \in M_m$ **do**

15 $\quad\quad\quad$ $s(i) = s(j) = \mathbf{1}$

16 $\quad\quad$ Return s

We give two examples of the construction of theorem 12.

Example 3.8. The labeling $l = 123432$ gives the graph G in Figure 3.10. Suppose that Edmonds' algorithm returns the matching $M_0 = \{e_1, e_3\}$ of Figure 3.10 left, corresponding to the completely labeled Gale string $s_0 = \mathbf{1111}00$; a second perfect matching $M = \{e_3, e_6\}$, corresponding to the Gale string $s = \mathbf{1}0\mathbf{11}0\mathbf{1}$, can be found in the graph $G_1 = (V(G), E(G) \setminus \{e_1\})$, see Figure 3.10 right.
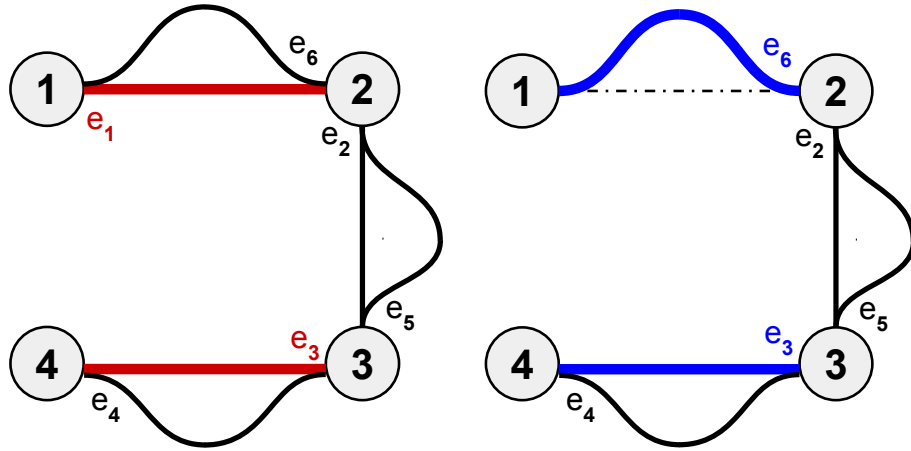
Figure 3.10   Left: The graph for the labeling $l = 123432$ with the perfect matchings corresponding to the completely labeled Gale strings $s_0 = \mathbf{1111}00$.

Right: The graph $G_1 = (V(G), E(G) \setminus \{e_1\})$ and the perfect matching corresponding to the Gale string $s = \mathbf{1}0\mathbf{11}0\mathbf{1}$.

Example 3.9. Consider the Morris graph of Example 3.6; suppose that Edmonds' algorithm returns the perfect matching $M_0 = \{e_1, e_3, e_5\}$, as in Figure 3.11 left, corresponding to the completely labeled Gale string $s_0 = \mathbf{111111}0000$. We can then delete the edge e_1 to obtain the graph G_1, as in Figure 3.11 center. The graph G_1 has a perfect matching $M = \{e_6, e_8, e_{10}\}$, shown in Figure 3.11 right; this is also a perfect matching of G, corresponding to the only other completely labeled Gale string $s = \mathbf{1}000\mathbf{11111}$.
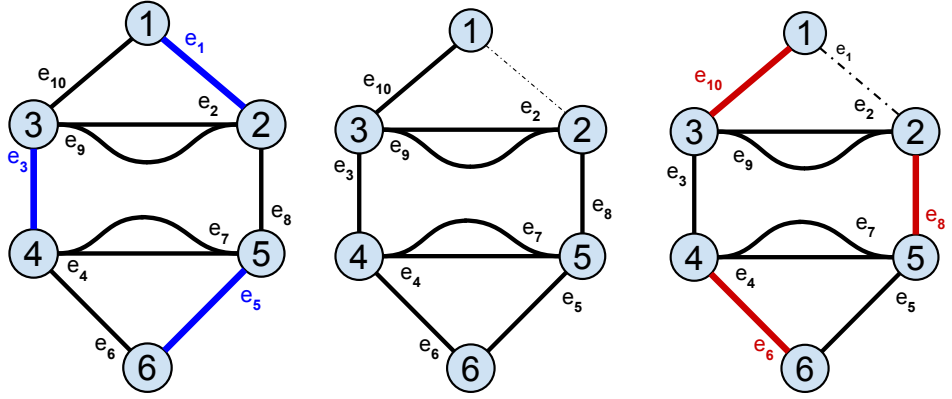
Figure 3.11   Left: The Morris graph $G = (V, E)$ with the matching $M = \{e_1, e_3, e_5\}$.

Centre: The graph $G_1 = (V(G), E(G) \setminus \{e_1\})$.

Right: The set $M' = \{e_6, e_8, e_{10}\}$ is a perfect matching of bot G_1 and G.

Applying Theorem 2.4 to Theorem 12 we have our main result: GALE NASH is in **FP**.

Theorem 13. *Finding a Nash equilibrium of a Gale game takes polynomial time.*

Notice that the Nash equilibrium that is found applying the construction in Theorem 12 is not necessarily the same Nash equilibrium that is found applying the Lemke-Howson for Gale Algorithm.

Chapter 4

Further results

P complexity of finding all (?)

from SvS-15

Ve ÌĄgh and von Stengel (2014, Thm. 12) give a near-linear time algorithm that finds such a second perfect matching that, in addition, has opposite sign, which corresponds to a Nash equilibrium of positive index as it would be found by a Lemke path (which, however, can be exponentially long). So this combinatorial problem is simpler than the problem of finding a Nash equilibrium of a bimatrix game, even though it gives rise to games that are hard to solve by the standard methods considered in Theorem 11.

from VvS

This paper presents three main contributions in this context. First, we define an abstract framework called pivoting systems that describes âĂIJcomplementary pivoting with directionâĂİ in a canonical manner. Similar abstract pivoting systems have been proposed by Todd (1976) and Lemke and Grotzinger (1976); we compare these with our approach in Section 5. Second, using this framework, we extend the concept of orientation to oiks and show that room partitions at the two ends of a pivoting path have opposite sign, provided the underlying oik is oriented. For two-dimensional oiks, which are Euler graphs, room partitions are perfect matchings. Their orientation is the

sign of a perfect matching as defined for Pfaffian orientations of graphs. Our third result is a polynomial-time algorithm for the following problem: Given a graph 2 with an Eulerian orientation and a perfect matching, find another perfect matching of opposite sign. The complementary pivoting algorithm that achieves this may take exponential time.

We conclude with open questions on the computational complexity of pivoting sys- tems. Consider a labeled oriented pivoting system whose components (in particular the pivoting operation) are specified as polynomial-time computable functions. Assume one CL state is given. The problem of finding a second CL state belongs to the com- plexity class PPAD (Papadimitriou, 1994). This problem is also PPAD-complete, because finding a Nash equilibrium of a bimatrix game is PPAD-complete (Chen and Deng, 2006), which is a special case of an oriented pivoting system by Proposition 1. However, there should be a much simpler proof of this fact because pivoting sys- tems are already rather general, so that it should be possible to encode an instance of the PPAD-complete problem âĂIJEnd of the LineâĂİ (see Daskalakis, Goldberg, and Papadimitriou, 2009) directly into a pivoting system. Finding a Nash equilibrium of a bimatrix game is PPAD-complete, and LemkeâĂŞ Howson paths may be exponentially long. Savani and von Stengel (2006) showed this with games defined by dual cyclic polytopes for the payoff matrices of both players, and a simpler way to do this is to use the Lemke paths by Morris (1994). One motivation for the study of Casetti, Merschen, and von Stengel (2010) was the question if finding a second completely labeled Gale string is PPAD-complete. This is unlikely because this problem can be solved in polynomial time with a matching algorithm. For the complexity class PPADS, where one looks for a second CL state of opposite sign (Daskalakis, Goldberg, and Papadimitriou, 2009), this problem is also solvable in polynomial time with our algorithm of Theorem 12. However, for room partitions of 3-oiks, already manifolds, finding a second room partition is likely to be more complicated. Is this problem

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PPAD-complete? We leave these questions for further research.

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appendix/acknowledgments

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