1 Introduction

2 Complexity, Games, Polytopes and Gale Strings

2.1 Some Complexity Classes

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2.3 Some Geometrical Notation

2.4 Bimatrix Games, Labels and Polytopes

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$$P = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}, \quad Q = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}, \tag{1}$$

a point in P has label k iff $x_k = 0$ for $k \in \{1, \ldots, m\}$ or $(B^\top x)_{k-m} = 0$ for $k \in \{m+1, \ldots, m+n\}$; analogously, a point in Q has label k if and only if either $y_{k-m} = 0$ for $k \in \{m+1, \ldots, m+n\}$ or $(Ay)_k$ for $k \in \{m+1, \ldots, m+n\}$.

proposition 1. *Let (A, B) be a bimatrix game and (x, y) be one of its Nash equilibria. Then (z, z), where $z = (x, y)$, is a Nash equilibrium of the symmetric game $(C, C^\top)$, where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}.$$

Theorem 1. *[15] Let $l : [n] \to [m]$, and let (U, B) be the unit vector game where $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. Consider the polytopes P^l and Q^l where*

$$P^l = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\} \tag{2}$$

$$Q^l = \{y \in \mathbb{R}^n | y \geq \mathbf{0}, \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\} \tag{3}$$

where $N_i = \{j \in [n] | l(j) = i\}$ *for* $i \in [m]$.

Give a labeling l_f *of the facets of* P^l *according to the inequality defining it, as follows:*

$$x_i \geq 0 \text{ has label } i \text{ for } i \in [m] \tag{4}$$

$$B^\top x)_j \leq 1 \text{ has label } l(j) \text{ for } j \in [n] \tag{5}$$

Then $x \in P^l$ *is a completely labeled point of* $P^l \setminus \{\mathbf{0}\}$ *if and only if there is some* $y \in Q^l$ *such that, after scaling, the pair* (x, y) *is a Nash equilibrium of* (U, B)

Theorem 2. *[1] Let* Q *be a labeled* m*-dimensional simplicial polytope with* $\mathbf{0}$ *in its interior, with vertices* $e_1, \ldots, e_m, c_1, \ldots, c_n$*, so that* $F_0 = \mathrm{conv}\{e_i \mid i \in [m]\}$ *is a facet of* Q*.*

Let $l : [n] \to [m]$*, and let* (U, B) *be the unit vector game with* $U = (e_{l(1)} \ \cdots \ e_{l(n)})$ *and* $B = (b_1 \ \cdots \ b_n)$*, where* $b_j = \frac{c_j}{1+\mathbf{1}^\top c_j}$ *for* $j \in [n]$*.*

Label the vertices of Q *as follows:*

$$l_v(-e_i) = i \text{ for } i \in [m] \tag{6}$$

$$l_v(c_j) = l(j) \text{ for } j \in [n] \tag{7}$$

Then a facet $F \neq F_0$ *of* Q *with normal vector* v *is completely labeled if and only if* (x, y) *is a Nash equilibrium of* (U, B)*, where* $x = \frac{v+\mathbf{1}}{\mathbf{1}^\top(v+\mathbf{1})}$*, and* $x_i = 0$ *if and only if* $e_i \in F$ *for* $i \in [m]$*. Any* j *so that* c_j *is a vertex of* F *represents a pure best reply to* x*; the mixed strategy* y *is the uniform distribution on the set of the pure best replies to* x*.*

2.5 Cyclic Polytopes and Gale Strings

2.6 Labeling and the Problem Another Gale

file: gale-def-subsection

$$l_s(i) = i \text{ for } i \in [d] \tag{8}$$

$$l_s(d+j) = l(j) \text{ for } j \in [n]. \tag{9}$$

3 Algorithmic and Complexity Results

3.1 Lemke Paths and the Lemke-Howson for Gale Algorithm

We have NEs ⇔ completely labeled things (facets, vertices, GS) We give now different versions of fundam algorithm to deal with labeling looking for compl.label. - in particular in these cases
first in version on simple polytopes with labeled facets, (name: Lemke-Howson; Lemke-Howson 1964, Shapley 1974 beautiful exposition)
then dual case with labeled vertices (name???? exchange? cite from???)
then in special case Gale strings (name: Lemke-Howson for Gale, cite from???).
Mention general version - or leave it further results? (maybe better) (name: exchange algorithm, Edmonds - Sanità). (This case: index more problematic - see)

Consider a labeling $l : [n] \to [m]$, for a set X with $|X| = n$. Then $x = (x_1, \ldots, x_m) \in X^m$ is *almost completely labeled* if $|\{j \in [n] \mid x_i = j \text{ for some } i \in [m]\}| = [m] \setminus \{k\}$ for exactly one $k \in [m]$. This mean that all labels appear once in x, except for the *missing label* k, and a *duplicate label* $\bar{k} \in [m]$ that appears twice.

now we see in poly with labeled facets, cl vertices

Let P be a simple polytope in dimension m with n facets. We define the operation of *pivoting on vertices* as moving from a vertex x of P to another vertex y such that there is an edge between x and y. Note that, since P is simple, there are exactly m possible choices for y.

Now let $l_f : [n] \to [m]$ be a labeling of the facets of P such that there is at least one completely labeled vertex x_0 of P. Note that if we pivot from vertex v we "leave behind" a facet F, that has label k; we call this *dropping label k*. We will then reach a vertex w that shares with v all facets except F (that contains v but not w) and another facet G (that contains w but not v) that has label j; we will call this *picking up label j*. We give the

Lemke-Howson algorithm as in 1. reference!

Algorithm 1: Lemke-Howson algorithm

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input : A simple m-polytope P with n facets. A labeling $l_f : [n] \to [m]$ of the facets of P. A vertex x_0 of P, completely labeled for l.

output: A completely labeled vertex $y \neq x_0$ of P.

```
choose a label k ∈ [n]
pivot from x_0 to y dropping label k
while y is not completely labeled do
    pivot from y to y' dropping duplicate label j, moving away from x_0
    rename x_0 = y, y = y'
return y
```

The steps of the Lemke-Howson algorithm result in a *Lemke path* that connects two completely labeled vertices through *k-almost complementary* vertices and edges, that is, almost completely labeled vertices and edges where the only missing label is k. It remains to show that $y \neq x_0$. This comes from the fact that the Lemke paths are *simple paths*, that is, there are no "loops" where a vertex is visited more than once. This is not possible because at each vertex there are only two edges corresponding to the missing label k, since P is not degenerate; one is the edge that is traversed to get to the vertex, one is the one that is traversed to leave it in the next step. This proves the following.

Theorem 3. *The Lemke-Howson algorithm 1 returns a solution to the problem* ANOTHER COMPLETELY LABELED VERTEX*.*

In the context of finding the Nash equilibrium of a bimatrix game (A, B), there are two equivalent implementations of the Lemke-Howson algorithm.

We can consider the game C as in proposition 1, and the associated polytope $S = \{z \in \mathbb{R}^{m+n} \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1}\}$, labeling the $2(m+n)$ inequalities defining the facets of of S as $1, \ldots, m+n, 1, \ldots, m+n$. Then applying the Lemke-Howson algorithm starting from vertex $\mathbf{0}$ returns a Nash equilibrium (z, z) of C and a corresponding $(x, y) = z$ a Nash equilibrium of (A, B).

We can also follow the "traditional" version of the Lemke-Howson algorithm; a very clear exposition of this can be found in Shapley [16]. Let P and Q be the best response polytopes of (A, B) as in 1. We then move alternately on P and Q, starting from the couple of vertices $(\mathbf{0}, \mathbf{0})$. Since we move in $\mathbb{R}^m$ and $\mathbb{R}^n$ instead of $\mathbb{R}^{m+n}$, this version is more practical to visualize, as shown in the following example.

Example 3.1. ex Savani - von Stengel, pag. 11; fig 8 are Schegel diagrams of BR polytopes.

Theorem 3 has a straightforward dual version. Let Q be a simplicial polytope in dimension m with n vertices. We *pivot on facets* by moving from facet F to facet G that shares an edge with F. Since P is simplicial, there are exactly m possible choices for G. Let $l_v : [n] \to [m]$ be a labeling of the vertices of P such that there is at least one completely labeled facet F_0. We *drop label* k and *pick up label* j when pivoting from a facet F to a facet G that shares with F all vertices except a vertex v with label k that belongs to F but not G, and another vertex w with label j that belongs to G but not F. The Lemke-Howson algorithm then becomes the theorem in 2. reference!

Considering the dual of Lemke paths on (almost) completely labeled facets, we get the dual result to theorem 3.

Theorem 4. *The algorithm 2 returns a solution to the problem* ANOTHER COMPLETELY LABELED FACET.

To find a Nash equilibrium of a unit vector game (U, B), where $U = (e_{l(1)} \cdots e_{l(n)})$ for a labeling $l : [n] \to [m]$, we can apply theorem 1 and algorithm 1, or we can apply the dual theorem 2 and algorithm 2. The first

Algorithm 2: Lemke-Howson algorithm on facets

input : A simplicial m-polytope Q with n vertices. A labeling $l_v : [n] \to [m]$ of the vertices of P. A vertex F_0 of Q, completely labeled for l.

output: A completely labeled facet $F \neq F_0$ of Q.

```
choose a label k ∈ [n]
pivot from F_0 to F dropping label k
while F is not completely labeled do
    pivot from F to F' dropping duplicate label j, moving away from F_0
    rename F_0 = F, F = F'
return F
```

case relies on the polytope $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ defined in 2; theorem 1 shows that P^l encodes all the Nash equilibria of (U, B) as completely labeled vertices, with an "artificial" equilibrium corresponding to the vertex $\mathbf{0}$. The second case relies on the polytope Q, defined as the convex hull of vertices $-e_i$ for $i \in [m]$ and $c_j = \frac{b_j}{1-\mathbf{1}^\top b_j}$ for $j \in [n]$; analogously, theorem 2 shows that Q encodes all the Nash equilibria of (U, B) as completely labeled facets, with the "artificial" equilibrium corresponding to the facet $F_0 = \mathrm{conv}(-e_1, \ldots, -e_m)$.

On the other hand, we can consider (U, B) as any bimatrix game, and apply algorithm 1 to the product of the best response polytopes P and Q. The projection of a Lemke path for a missing label $i \in [m]$ on $P \times Q$ to P defines a Lemke path in P^l. However, $P \times Q$ has $m + n$ labels, therefore how? there could be Lemke paths for a missing label $m + j$ with $j \in [n]$ on $P \times Q$ that get lost in the projection on P^l. The following theorem, proved by Savani and von Stengel in [15], shows that there is no loss of generality in studying Lemke paths on P^l; an analogous result holds for the dual case.

Theorem 5. *Let* (U, B) *be a unit vector game, with* $U = (e_{l(1)} \cdots e_{l(n)})$ *for a labeling* $l : [n] \to [m]$*; let* $P = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ *and* $Q = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}$*, as in 1; and let* $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ *as in 2. Then the Lemke path on* $P \times Q$ *for the missing label* k *projects to a path on* P *that is the Lemke path on* P^l *for missing label* k *if* $k \in [m]$*, and for missing label* $l(j)$ *if* $k = m + j$ *with* $j \in [n]$*.*

As before, we now consider the case of unit vector games where the simplicial polytope Q is cyclic; that is, the case that we can study from the point of view of Gale strings.

sp case cycl poly leads to gs
thanks to dual of theorem SvS-15 5, when doing labeling as in 8 we can take the str of labels $l(n+j)\cdots l(n+m)$ instead of $l(1)\cdots l(n+m)$, that is, we could cut the "artificial" first labels $12...n$.
After all, in main we're studying ANOTHER GALE in general, not nec starting from $12...n$; and we're interested in finding *one* eq that's not the one we started from (and is at other end of LPath, since index and so on), *not all equilibria*; but the eq we started from is not nec the artificial one - actually, if we go with this we can take any NE to start looking for another, and we're sure to find a "non-artificial" one. Note: if we were looking for all NE, LH doesn't work anyway - see ex by Wilson in Shapley, where "disconnected" paths between equilibria.
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complexity considerations, PPAD
Needed for why result interesting; it can be done at the end, with Morris paths

3.2 The Complexity of Gale and Another Gale

file: main-result-subsection

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