# 1 Introduction

# 2 Complexity, Games, Polytopes and Gale Strings

## 2.1 The Complexity Classes P and PPAD

## 2.2 Normal Form Games and Nash Equilibria

file: background

## 2.3 Some Geometrical Notation

## 2.4 Bimatrix Games, Labels and Polytopes

**Theorem 1.** *Let $l : [n] \to [m]$, and let $(U, B)$ be the unit vector game where $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. Consider the polytopes $P^l$ and $Q^l$ where*

$$P^l = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\} \tag{1}$$

$$Q^l = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\} \tag{2}$$

*where $N_i = \{j \in [n] | l(j) = i\}$ for $i \in [m]$.*
*Label every facet of $P^l$ according to the inequality defining it, as follows:*

- *$x_i \geq 0$ has label $i$, for $i \in [m]$*
- *$(B^\top x)_j \leq 1$ has label $l(j)$, for $j \in [n]$*

*Then $x \in P^l$ is a completely labeled point of $P^l \setminus \{\mathbf{0}\}$ if and only if there is some $y \in Q^l$ such that, after scaling, the pair $(x, y)$ is a Nash equilibrium of $(U, B)$*

file: polytopes

## 2.5 Cyclic Polytopes and Gale Strings

We have now built a correspondence between labeled best response polytopes and bimatrix games. We now consider a special case of games, where the polytope in theorem 1 is the dual of a particular kind of polytope that can be represented as a combinatorial structure. We first give the definition of these particular polytopes, called *cyclic polytopes*, then we define the combinatorial structure that we will use to study them, the *Gale strings*.

A *cyclic polytope* $P$ in dimension $d$ with $n$ vertices is the convex hull of $n$ distinct points $\mu_d(t_j)$ on the *moment curve* $\mu_d : \ t \mapsto (t, t^2, \ldots, t^d)^\top$ for $j \in [n]$ such that $t_1 < t_2 < \cdots < t_n$.

example(s) with graphics! There's one in Ziegler

A *bitstring* is a string of labels that are either 0s or 1s. Formally: given an integer $k$ and a set $S$, we can represent the function $f_s : [k] \to S$ as the string $s = s(1)s(2) \cdots s(k)$; we have a bitstring in the case where $S = \{0, 1\}$. A maximal substring of consecutive 1's in a bitstring is called a *run*.

A *Gale string of length $n$ and dimension $d$* is a bitstring of length $n$, denoted as $s \in G(d, n)$, such that

1. exactly $d$ bits in $s$ are 1 and
2. (*Gale evenness condition*)

$$01^k0 \text{ is a substring of } s \quad \Longrightarrow \quad k \text{ is even.} \tag{3}$$

The Gale evenness condition characterises Gale strings in $G(d, n)$ as the bitstrings of length $n$ with exactly $d$ elements equal to 1, such that *interior* runs (that is, runs bounded on both sides by 0s) must be of even length. In general, this condition allows Gale strings to start or end with an odd-length run; but when $d$ is even this means that $s$ starts with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d, n)$ with even $d$ as a "loop" obtained by "glueing together" the extremes of the string to form an even run. Formally, we can see the indices of a Gale string $s \in G(d,n)$ with $d$ even as equivalence classes modulo $n$, identifying $s(i+n) = s(i)$. This also shows that the set of Gale strings of even dimension is invariant under a cyclic shift of the strings.

*Example* 2.1. As an example of $d$ even, we have

$$G(4,6) = \{\mathbf{1111}00, \mathbf{111}00\mathbf{1}, \mathbf{11}00\mathbf{11}, \mathbf{1}00\mathbf{111}, 00\mathbf{1111}, 0\mathbf{1111}0, \mathbf{11}0\mathbf{11}0, \mathbf{1}0\mathbf{11}0\mathbf{1}, 0\mathbf{11}0\mathbf{11}\}$$

The strings $\mathbf{1111}00$, $\mathbf{111}00\mathbf{1}$, $\mathbf{11}00\mathbf{11}$, $\mathbf{1}00\mathbf{111}$, $00\mathbf{1111}$ and $0\mathbf{1111}0$ are equivalent under a cyclic shift (if considering the strings as "loops", the $\mathbf{1}$'s are all consecutive), as are the strings $\mathbf{11}0\mathbf{11}0$, $\mathbf{1}0\mathbf{11}0\mathbf{1}$ and $0\mathbf{11}0\mathbf{11}$ (if considering the strings as "loops", the even runs of $\mathbf{1}$'s are two couples separated by a single 0).

As an example for $d$ odd, we have

$$G(3,5) = \{\mathbf{111}00, \mathbf{1}0\mathbf{11}0, \mathbf{1}00\mathbf{11}, \mathbf{11}00\mathbf{1}, 0\mathbf{11}0\mathbf{1}, 00\mathbf{111}\}$$

Note how $0\mathbf{1}0\mathbf{11}$ is a cyclic shift of $\mathbf{1}0\mathbf{11}0$, but it is not a Gale string.

The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [6].

**Theorem 2** ([6])**.** *For any positive integers $d, n$ let $P$ be the cyclic polytope in dimension $d$ with $n$ vertices. Then the facets of $P$ are encoded by $G(d,n)$; that is,*

$$F \text{ is a facet of } P$$

$$\Longleftrightarrow$$

$$F = \mathrm{conv}\{\mu(t_j) \mid s(j) = 1 \textit{ for some } j \in [n] \textit{ and } s \in G(d,n)\}$$

pf - see Ziegler

Essentially, this holds because any set $S \subset [n]$ the moment curve defines a unique hyperplane which is crossed (and not just touched) by the moment curve; if the bitstring $s$ that encodes $F$ as $1(s)$ has a substring $01^k0$

example of cyclic polytope + equivalent gale string - pg 35 JM has nice one

We now give define a *labeling* for Gale strings, corresponding to the labeling of best-response polytopes.

A string $s$ is *completely labeled* for some labeling function $l : [n] \to [d]$ if $\{\bar{l} \in [d] | s(i) = 1 \text{ and } l(i) = \bar{l} \text{ for some } i \in [n]\} = [d]$. If $s \in G(d, n)$, this implies that for every $\bar{l} \in [d]$ there is exactly one $s(i)$ such that $s(i) = 1$ and $l(i) = \bar{l}$, since there are exactly $d$ positions such that $s(i) = 1$.

*Example* 2.2. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: 111100, 110110, 100111 and 101101.

| **1** | **2** | **3** | **4** | **3** | **2** |
|---|---|---|---|---|---|
| **1** | **1** | **1** | **1** | · | · |
| **1** | **1** | · | **1** | **1** | · |
| **1** | · | · | **1** | **1** | **1** |
| **1** | · | **1** | **1** | · | **1** |

Sometimes it is not possible to find a completely labeled Gale string.

*Example* 2.3. For $l = 121314$, there are no completely labeled Gale strings.

The labels $l(i) = 2, 3, 4$ appear only once in $l$, as $l(2), l(4), l(6)$ respectively; therefore we must have $s(2) = s(4) = s(6) = 1$. For every other $i \in [n]$ we have $l(i) = 1$, so we have $l(i) = 1$ for exactly one $i = 1, 3, 5$. The candidate strings are then **110101**, 0**11101**, 0**10111**; but none of these satisfies the Gale evenness condition.

From now on, we will always consider labeling $l$ such that $l(i) \neq l(i+1)$; we now show that we can do it without loss of generality. In the case when $l(i) = \ldots = l(i+k)$, by Gale Evenness Condition we have $s(i+j) = 0$ for $j \in [k-1]$; we can therefore "simplify" by deleting all indices $k$. Suppose now that $l(i-1) \neq l(i) = l(i+1) \neq l(i+2)$ for some index, and let $s$ be a completely labeled Gale string for $l$. Then only one of $s(i)$ and $s(i+1)$ can be equal to **1** (note that it's possible that both are **0**s). So $s(i)s(i+1)$ will never be a run of even length that "interferes" with the Gale Evenness Condition, so we can "simplify" by deleting the index $i+1$.

geometrically? projection?

how labeled GS are corresponding to br cyclic polytopes

NB: we have given DUAL version (SvS 15) of Balthasar!!! facets –¿ vertices (we must suppose a facet $F_0$ that maps to $\mathbf{0}$)

graphics

ANOTHER GALE

By theorem 2, the Gale strings $s \in G(d,n)$ encode the facets of a cyclic polytope in dimension $d$ with $n$ vertices. We want to exploit this to study

We now give a correspondence between the labeling of the facets of a cyclic polytope chosen as best-response polytope $Q$ for a unit vector game $(U,B)$ as in theorem 1 and a Gale string $s \in G(d,d+n)$.

We look for: labeling $l : [d+n] \to [d]$ for each vertex $\mu(t_j)$ of $P^\Delta$, where $j \in [n]$,

so we can look for a CL facet $F$ of $P^\Delta$

When $P^\Delta$ is a cyclic polytope in dimension $d$ with $d+n$ vertices, then the string of labels $l(1)\cdots l(n)$ in Theorem *balthasar*

defines a labeling $l' : [d+n] \to [d]$ where $l'(i) = i$ for $i \in [d]$ and $l'(d+j) = l(j)$ for $j \in [n]$. In other words, the string of labels $l(1)\cdots l(n)$ is just prefixed with the string $1\,2\cdots d$ to give $l'$. Then $l'$ has a trivial completely labeled Gale string $1^d0^n$ which defines the facet $F_0$.

From this point forward, we will assume that $d$ is even.

Then the problem ANOTHER GALE defines exactly the problem of finding a Nash equilibrium of the unit vector game $(I,B)$.

| ANOTHER GALE | | |
|---|---|---|
| **input** | **:** | A labeling $l : [n] \to [d]$, where $d$ is even and $d < n$, and an associated completely labeled Gale string $s$ in $G(d,n)$. |
| **output** | **:** | A completely labeled Gale string $s'$ in $G(d,n)$ associated with $l$, such that $s' \neq s$. |

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