1 Introduction

2 Complexity, Games, Polytopes and Gale Strings

2.1 The Complexity Classes P and PPAD

2.2 Normal Form Games and Nash Equilibria

2.3 Bimatrix Games and Best Response Polytopes

2.4 Cyclic Polytopes and Gale Strings

A special case of games is obtained by taking a particular case of best response polytope in theorem **??**.

Definition 1. A *cyclic polytope* P in dimension d with n vertices is the convex hull of distinct points $\mu(t_j)$, where $j \in [n]$ and μ is the *moment curve*

$$\mu\colon t \mapsto (t, t^2, \dots, t^d)^\top$$

Cyclic polytopes can be represented through a combinatorial structure, the *Gale strings*. This makes their study particularly interesting, and, as we will see, it can be used to obtain very elegant proofs.

Definition 2. For any integer k and any set S, we can represent the function $f_s : [k] \to S$ as the string $s = s(1)s(2)\cdots s(k)$. If $S = \{0,1\}$ we denote

$$\begin{aligned} \mathbb{1}(s) &= s^{-1}(1) \\ &= \{j \in [k] \mid s(j) = 1\} \end{aligned}$$

The indicator function of $\mathbb{1}(s)$ will then correspond to a *bitstring* s, a sequence of 0's and 1's.

A maximal substring of consecutive 1's in a bitstring is called a *run*.

Example 2.1. Let $k = 6$, and let $f_s(j) = 0$ if j is even and $f_s(j) = 1$ if j is odd. Then $s = 101010$ and $1(s) = 1, 3, 5$.

We can now give the definition of *Gale string*.

Definition 3. We denote as $G(d,n)$ the set of all bitstrings s of length n such that

1. exactly d bits in s are 1 and
2. s fulfills the *Gale evenness condition*:

$$01^k0 \text{ is a substring of } s \quad \Rightarrow \quad k \text{ is even.}$$

An element of of $G(d,n)$ is called a *Gale string of dimension d and length n*.

Definition 3 characterises Gale strings as bitstrings of length n with exactly d elements equal to 1, such that *interior* runs (that is, runs bounded on both sides by 0s) must be of even length. Note that this condition allows Gale strings to start or end with an odd-length run.

This leads to an important consequence when d is even.

Property 2.1. *Let d be even, and let s in $G(d,n)$. Then if s starts with an odd run it will also end with an odd run, and if s starts with an even run it will end with an even run.*

use "modulo" - even more than "cyclic shift"

That is, the set of Gale strings of even dimension is therefore invariant under a cyclic shift of the strings.

We can then consider the Gale strings in $G(d,n)$ with even d as a "loop" obtained by "glueing together" the extremes of the string to form an even run.

Example 2.2. We consider $G(4,6)$. We have

$$\begin{aligned} G(4,6) = \{&111100,\\ &111001,\\ &110011,\\ &100111,\\ &001111,\\ &011110,\\ &110110,\\ &101101,\\ &011011\} \end{aligned}$$

change mention of "cyclic shift" if not used before

The strings 111100, 111001, 110011, 100111, 001111 and 011110 are equivalent under a cyclic shift, as are the strings 110110, 101101 and 011011.

here to end subsect: polytopes

The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [**?**].

Theorem 1 ([**?**]). *For any positive integer n, assume that $t_1 < t_2 < \cdots < t_n$ and let P be the cyclic polytope obtained by taking t_j, where $j \in [n]$, in definition 1.*

Then the facets of P are encoded by $G(d,n)$; that is, F is a facet of P if and only if

$$F = \mathrm{conv}\{\mu(t_i) \mid i \in 1(s)\} \qquad \textit{for some } s \in G(d,n)$$

sketch of pf if not too long and it uses relevant techniques

graphics of cyclic polytope - cfr vS articles and talks

From this point forward, we will assume that d is even.

give something to generalise to odd case

2.5 Labeling and the Problem Another Gale

Definition 4. Given a set G of bitstrings of length n and a parameter d, a *labeling* is a function $l : [n] \to [d]$. A string s in $G(d, n)$ is *completely labeled* if $l(\imath(s)) = [d]$. Any $l(i) \in [d]$ is called a *label*

If $s \in G(d, n)$ is completely labeled for the labeling $l : [n] \to [d]$, then for each label $l(i)$ there is a bit $s(i) = 1$. We therefore have exactly d positions i for which $s(i) = 1$; hence, $|l(\imath(s))| = d$.

Example 2.3. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: 111100, 110110, 100111 and 101101.

123432	**12**3**43**2	**1**2**3432**	**1**2**34**3**2**
111100	**11**0**11**0	**1**00**111**	**1**0**11**0**1**

Sometimes there aren't any completely labeled Gale strings that are associated with a given labeling.

Example 2.4. For $l = 121314$, there are no completely labeled Gale strings.

here to end subsect: polytopes

graphics of labeled cyclic polytope

For this cyclic polytope P, a labeling $l : [n] \to [d]$ can be understood as a label $l(j)$ for each vertex $\mu(t_j)$ for $j \in [n]$. A completely labeled Gale string s therefore represents a facet F of P that is completely labeled.

Special games are obtained by using cyclic polytopes in Theorem **??**, suitably affinely transformed with a completely labeled facet F_0. When Q is a cyclic polytope in dimension d with $d+n$ vertices, then the string of labels $l(1)\cdots l(n)$ in Theorem **??** defines a labeling $l' : [d+n] \to [d]$ where $l'(i) = i$ for $i \in [d]$ and $l'(d+j) = l(j)$ for $j \in [n]$. In other words, the string of labels $l(1)\cdots l(n)$ is just prefixed with the string $1\,2\cdots d$ to give l'. Then l' has a trivial completely labeled Gale string $1^d 0^n$ which defines the facet F_0. Then the problem Another Gale defines exactly the problem of finding a Nash equilibrium of the unit vector game (I, B). Note again that B is here not a general matrix (which would define a general game) but obtained from the last n of $d+n$ vertices of a cyclic polytope in dimension d.

Another Gale	
input :	A labeling $l : [n] \to [d]$, where d is even and $d < n$, and an associated completely labeled Gale string s in $G(d, n)$.
output :	A completely labeled Gale string s' in $G(d, n)$ associated with l, such that $s' \neq s$.

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