1 Complexity, Games, Labels, Polytopes, Strings

1.1 Some Complexity Classes

1.2 Normal Form Games and Nash Equilibria

file: background

1.3 Some Geometrical Notation

so results in labels section - after this - don't get lost in boredom, and each section in background is about 150-200 lines (see: getting lost). Maybe turn this into an appendix? It would make sense if something more about proof of Nash

We denote the transpose of a matrix A as $A^\top$. We consider vectors $u, v \in \mathbb{R}^d$ as column vectors, so $u^\top v$ is their scalar product. A vector in $\mathbb{R}^d$ for which all components are 0's will be denoted as $\mathbf{0}$; similarly, a vectors for which all components are 1's will be denoted as $\mathbf{1}$. The *unit vector* e_i is the vector that has i-th component $e_{ii} = 1$ and $e_{ij} = 0$ for all other components. When writing an inequality of the form $u \geq v$ (and analogous), we mean that it holds for every component; that is, $u_i \geq v_i$ for all $i \in [d]$.

An *affine combination* of points in an Euclidean space $z_1, \ldots, z_n$ is

$$\sum_{i=1}^{n} \lambda_i z_i \quad \text{where } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} \lambda_i = 1$$

The points $z_1, \ldots, z_n$ are *affinely independent* if none of them is an affine combination of the others.

A *convex combination* of points $z_1, \ldots, z_n$ is an affine combination where $\lambda_i \geq 0$ for all $i \in [n]$. Note that such λ_i's can be seen as a probability distribution over the z_i's.

def simplex: here?

A set of point Z is *convex* if it is closed under forming convex combinations, that is, if $\bar{z} = \sum_{i=1}^{n} \lambda_i z_i$, where $z_i \in Z$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, then $\bar{z} \in Z$. A convex set has *dimension* d if it has exactly $d+1$ affinely independent points.

def simplex: here?

convex hull (needed for def cyclic poly);

pow hyperplanes;

polyhedron, polytopes

polar: $Q = \{x \in \mathbb{R}^d \mid x^\top c_i \leq 1,\ i \in [k]\}$

with $c_i \in \mathbb{R}^d$. Then the polar (Ziegler, 1995) of Q is given by

$Q^\Delta = \mathrm{conv}\{c_i, i \in [k]\}$

from here: notes - copy-paste

A (d-dimensional) *simplicial polytope* P is the convex hull of a set of at least $d+1$ points v in $\mathbb{R}^d$ in general position, that is, no $d+1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P. A *facet* of P is the convex hull $\mathrm{conv}\, F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F.

1.4 Bimatrix Games, Labels and Polytopes

In the rest of this thesis we will focus on two-player normal-form games. For sake of readability, we will use feminine pronouns when referring to player 1 and masculine pronouns when referring to player 2.

Two-player normal-form games are also called *bimatrix games*, since they can be characterized by the $m \times n$ payoff matrices A and B, where a_{ij} and b_{ij} are the payoffs of player 1 and 2 when she plays her ith pure strategy and he plays his jth pure strategy. We will assume that (A, B) are non-negative, and that A and $B^\top$ have no zero column. This can be easily obtained without loss of generality via an affine transformation that will not affect the equilibria of the game.

The Nash equilibria of bimatrix games can be analysed from a combinatorial point of view using *labels*. This method is due to Shapley [16], in a study building on ideas introduced in a paper by Lemke and Howson [8].

Let (A, B) be bimatrix game. The mixed-strategy simplices of player 1 and 2 are, respectively

$$X = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ \mathbf{1}^\top x = 1\}, \quad Y = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ \mathbf{1}^\top y = 1\} \tag{1}$$

A *labeling* of the game is then given as follows:

1. the m pure strategies of player 1 are identified by $1, \ldots, m$;
2. the n pure strategies of player 2 are identified by $m+1, \ldots, m+n$;
3. each mixed strategy $x \in X$ of player 1 has
   - label i for each $i \in [m]$ such that $x_i = 0$, that is if in x player 1 does not play her ith pure strategy;
   - label $m+j$ for each $j \in [n]$ such that the jth pure strategy of player 2 is a best response to x;
4. each mixed strategy $y \in Y$ of player 2 has

- label $m+j$ for each $j \in [n]$ such that $y_j = 0$, that is if in y player 2 does not play his jth pure strategy;
- label i for each $i \in [m]$ such that the ith pure strategy of player 1 is a best response to y;

A strategy profile $(x, y) \in X \times Y$ is *completely labeled* if every label $1, \ldots, m+n$ is a label of either x or y. We have the following theorem (Theorem 1 in [16]):

Theorem 1. *Let $(x, y) \in X \times Y$; then (x, y) is a Nash equilibrium of the bimatrix game (A, B) if and only if (x, y) is completely labeled.*

Proof. The mixed strategy $x \in X$ has label $m+j$ for some $j \in [n]$ if and only if the jth pure strategy of player 2 is a best response to x; this, in turn, is a necessary and sufficient condition for player 2 to play his jth strategy at an equilibrium against x. Therefore, at an equilibrium (x, y) all labels $m+1, \ldots, m+n$ will appear either as labels of x or of y. The analogous holds for the labels $i \in [n]$. □

An useful geometrical representation of labels can be given on the mixed strategies simplices X and Y. The outside of each simplex is labeled according to the player's own pure strategies that are *not* played; so, for instance, the outside of X will have labels $1, \ldots, n$. The interior of each simplex is subdivided in closed polyhedral sets, called *best-response regions.* These are labeled according to the other player's pure strategy that is a best response in that set; so, for instance, the inside of X will have labels $m+1, \ldots, m+n$.

We give an example of this construction.

Example 1.1. page 3–4 of Savani, von Stengel, Unit Vector Games. With graphics.

We will now give a description of labeling on polytopes equivalent to the construction based on best-response regions.

We begin by noticing that the best-response regions can be obtained as projections on X and Y of the *best-response facets* of the polyhedra

$$\bar{P} = \{(x, v) \in X \times \mathbb{R} | B^\top x \leq \mathbf{1}v\}, \quad \bar{Q} = \{(y, u) \in Y \times \mathbb{R} | Ay \leq \mathbf{1}u\}. \quad (2)$$

These facets in $\bar{P}$ are defined as the points $(x, v) \in X \times \mathbb{R}$ such that $(B^\top x)_j = v$. These points represent the strategies $x \in X$ of player 1 that give exactly payoff v to player 2 when he plays strategy j. The projection of the facet defined by $(B^\top x)_j = v$ to X will have label j. Analogously, in $\bar{Q}$, the facets are the points $(y, u) \in Y \times \mathbb{R}$ such that $a_i y = u$, and their projection to Y will be the best-response region with label i.

Example 1.2. cont of ex above, page 4–5, image on page 5 left

Given the assumptions on non-negativity of A and $B^\top$, we can give a change coordinates to x_i/v and y_j/u and replace $\bar{P}$ and $\bar{Q}$ with the *best-response polytopes*

$$P = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}, \quad Q = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}, \quad (3)$$

Each one of these polytope is defined by half spaces corresponding to either the player's own strategy that is not being played or the other player's best response; each one of the facets of the polytope is labeled by the strategy corresponding to the relative half-space.

This means that a point in P has label k if and only if either $x_k = 0$ for $k \in \{1, \ldots, m\}$ or $(B^\top x)_{k-m} = 0$ for $k \in \{m+1, \ldots, m+n\}$; analogously, a point in Q has label k if and only if either $y_{k-m} = 0$ for $k \in \{m+1, \ldots, m+n\}$ or $(Ay)_k$ for $k \in \{m+1, \ldots, m+n\}$. A point $(x, y) \in P \times Q$ is *completely labeled* if every $k \in [m+n]$ is a label of x or y. Note that the point $(\mathbf{0}, \mathbf{0})$ is completely labeled. Rescaling back to $\bar{P}$ and $\bar{Q}$, all the non-zero completely labeled points give exactly all the equilibria of (A, B). In this construction, we will call $(\mathbf{0}, \mathbf{0})$ *artificial equilibrium.*

Example 1.3. ex in Savani, von Stengel, image on page 5 right

A characterization of the completely labeled pairs in $P \times Q$ can be given as follows.

proposition 1. *The pair $(x, y) \in P \times Q$ is completely labeled if and only if one of the following condition holds:*

- (Complementarity condition)

$$x_i = 0 \text{ or } (Ay)_i = 1 \text{ for all } i \in [m], \quad y_j = 0 \text{ or } (B^\top x)_j \text{ for all } j \in [n] \tag{4}$$

- (Orthogonality condition)

$$x^\top(\mathbf{1} - Ay) = 0, \quad y^\top(\mathbf{1} - B^\top x) = 0 \tag{5}$$

Proposition 1 can be used to prove a useful property: *symmetric games*, that is, games that have payoff matrix of the form $(C, C^\top)$ for some matrix C, can be used to study generic bimatrix games without loss of generality. The result is due to Gale, Kuhn and Tucker [7] for zero-sum games; its extension to non-zero-sum games is a folklore result.

proposition 2. *Let (A, B) be a bimatrix game and (x, y) be one of its Nash equilibria. Then (z, z), where $z = (x, y)$, is a Nash equilibrium of the symmetric game $(C, C^\top)$, where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}.$$

The converse has been proved by McLennan and Tourky [9] in their study of *imitiation games*, that is, bimatrix games of the form (I, B).

proposition 3. *The pair (x, x) is a symmetric Nash equilibrium of the symmetric bimatrix game $(C, C^\top)$ if and only if there is some y such that (x, y) is a Nash equilibrium of the imitation game $(I, C^\top)$.*

Example 1.4. Consider the symmetric game $(C, C^\top)$, where $C^\top = B$ in the previous examples.

ex Savani, von Stengel, pg 8

Savani and von Stengel [15] extended proposition 3 to *unit vector games*, that is, games of the form (U, B), where the columns of the matrix U are unit vectors. Their study relies on a theorem, first proved in dual form by Balthasar [1], that exploits the techinque of labeling.

Theorem 2. *[15] Let $l : [n] \to [m]$, and let (U, B) be the unit vector game where $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. Consider the polytopes P^l and Q^l where*

$$P^l = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\} \tag{6}$$

$$Q^l = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\} \tag{7}$$

where $N_i = \{j \in [n] | l(j) = i\}$ for $i \in [m]$.

Label every facet of P^l according to the inequality defining it, as follows:

- *$x_i \geq 0$ has label i, for $i \in [m]$*
- *$(B^\top x)_j \leq 1$ has label $l(j)$, for $j \in [n]$*

Then $x \in P^l$ is a completely labeled point of $P^l \setminus \{\mathbf{0}\}$ if and only if there is some $y \in Q^l$ such that, after scaling, the pair (x, y) is a Nash equilibrium of (U, B)

Proof. Let P, Q be the polytopes associated to the game (U, B) as before.

Let $(x, y) \in P \times Q \setminus \{\mathbf{0}, \mathbf{0}\}$ be a Nash equilibrium of (U, B), therefore completely labeled in $[m + n]$. Then, if $x_i = 0$, then x has label $i \in m$. If

$x_i > 0$ instead, then y has label i, therefore $(Uy)_i = 1$, therefore for some $j \in [n]$ we have $y_j > 0$ and $U_j = e_i$, so $l(j) = i$. Since $y_j > 0$ and (x, y) is completely labeled, $x \in P$ has label $m + j$, that is, $(B^\top x)_j = 1$, therefore $x \in P^l$ has label $l(j) = i$. Hence, x is a completely labeled point of P^l.

Conversely, let $x \in P^l \setminus \{\mathbf{0}\}$ be completely labeled. If $x_i > 0$, then there is $j \in [m]$ such that $(B^\top x) = j$ and $l(j) = i$, that is, $j \in N_i$. For all i such that $x_i > 0$, define y as follows: $y_h = 0$ for all $h \in N_i \setminus \{j\}$, $y_j = 1$. Then $(x, y) \in P \times Q$ is completely labeled. □

The dual version of theorem 2 is based on the following construction (see [1]).

We translate the polytope P^l in theorem 2 to $P = \{x - \mathbf{1} \mid x \in P^l\}$. Multiplying all payoffs in B by a constant if necessary (an operation that does not change the game), we can have $\mathbf{1}$ is in the interior of P^l and $\mathbf{0}$ in the interior of P. We have $x \in P$ if and only if $x + \mathbf{1} \geq \mathbf{0}$ and $B^\top(x + \mathbf{1}) = (x + \mathbf{1})^\top B \leq \mathbf{1}$; that is, if and only if $-x_i \leq 1$ for $i \in [m]$ and $x^\top \frac{b_j}{\mathbf{1}\mathbf{1}^\top b_j} \leq 1$ for $j \in [n]$.

The polar of P is then

$$P^\Delta = \operatorname{conv}(\{e_i \mid i \in [m]\} \cup \{\frac{b_j}{\mathbf{1}\mathbf{1}^\top b_j}\}) \tag{8}$$

P is simple (why? show it before), so P^Δ is simplicial

Since $\mathbf{0}$ is in the interior of P, we have that $P^{\Delta\Delta} = P$, and the facets of P^Δ correspond to the vertices of P and vice versa. We can then label the vertices of $P\Delta$ with the labels of the corresponding facets in P^l, so the completely labeled facets of $P\Delta$ will correspond to the completely labeled vertices of P^l.

In particular, the facet corresponding to $\mathbf{0}$ is

$$F_0 = \{x \in P^\Delta \mid -\mathbf{1}^\top x = 1\} = \operatorname{conv}\{e_i \mid i \in [m]\}. \tag{9}$$

We therefore have the dual of theorem 2, as enunciated in [1].

Theorem 3. *[1] Let Q be a labeled m-dimensional simplicial polytope with $\mathbf{0}$ in its interior, with vertices $e_1, \ldots, e_m, c_1, \ldots, c_n$, so that $F_0 = \mathrm{conv}\{e_i \mid i \in [m]\}$ is a facet of Q.*

Let $l : [n] \to [m]$, and let (U, B) be the unit vector game with $U = (e_{l(1)} \ \cdots \ e_{l(n)})$ and $B = (b_1 \ \cdots \ b_n)$, where $b_j = \frac{c_j}{1+\mathbf{1}^\top c_j}$ for $j \in [n]$.

Label the vertices of Q as follows:

- *$-e_i$ has label $i \in [m]$*
- *c_j has label $l(j)$ for $j \in [n]$*

Then a facet $F \neq F_0$ of Q with normal vector v is completely labeled if and only if (x, y) is a Nash equilibrium of (U, B), where $x = \frac{v+\mathbf{1}}{\mathbf{1}^\top(v+\mathbf{1})}$, and $x_i = 0$ if and only if $e_i \in F$ for $i \in [m]$. Any j so that c_j is a vertex of F represents a pure best reply to x; the mixed strategy y is the uniform distribution on the set of the pure best replies to x.

F_0 correspond to "artificial" equilibrium, as $(\mathbf{0})$ before.
see as comp problems: ANOTHER CL FACET / ANOTHER CL VERTEX sp case of 2-NASH
nondegeneracy; made nondegenerate by "lexicographic" perurbation (what does it mean?);
ex pg 9; odd no eq, mention homotopy method (find ref) (tie with Nash, again?)

1.5 Cyclic Polytopes and Gale Strings

file: gale-def

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