

`\subsection{Cyclic Polytopes and Gale Strings}\label{gs-ssect}`

`\begin{example}\label{no-clgs}`

For  $l = 121314$ , there are no completely labeled Gale strings. ■

`\end{example}`

`\begin{theorem}\label{even-number-gale}`

For any labeling  $l: [n] \rightarrow [d]$ , where  $d$  is even and  $d < n$ , the number of completely labeled Gale strings associated with  $l$  is even. ■

`\end{theorem}`

`\bibitem{edm}` J. Edmonds (1965).

Paths, trees, and flowers.

`\emph{Canad. J. Math.}` 17, pp. 449--467.

## 0.1 The Complexity of GALE and ANOTHER GALE

We will now give our main result: ANOTHER GALE can be solved in polynomial time. Therefore, it takes polynomial time to find a Nash Equilibrium of a bimatrix game for which the best response polytope is cyclic.

Our proof will be based on a simple graph construction.

**Definition 1.** A *perfect matching* for a graph  $G = (V, E)$  is a set  $M \subseteq E$  of pairwise non-adjacent edges so that every vertex  $v \in V$  is incident to exactly one edge in  $M$ .

We define the problem PERFECT MATCHING as follows:

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PERFECT MATCHING

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**input** : A graph  $G = (V, E)$ .

**question** : Is there a perfect matching for  $G$ ?

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The complexity of PERFECT MATCHING has been proven to be in P by Edmonds [?].

**Theorem 1** ([?]). *The problem PERFECT MATCHING is solvable in polynomial time.*

We will first consider the accessory problem GALE, and we will show that it is solvable in polynomial time by using theorem 1.

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GALE

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**input** : A labeling  $l : [n] \rightarrow [d]$ , where  $d$  is even and  $d < n$ .

**question** : Does there exists a completely labeled Gale string  $s$  in  $G(d, n)$  associated with  $l$ ?

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**Theorem 2.** *The problem GALE is solvable in polynomial time.*

*Proof.* We give a reduction of GALE to PERFECT MATCHING.

In the following, we will consider every Gale string as a “loop,” as seen in section ??, so  $n + 1 = 1$ .

Given the labeling  $l : [n] \rightarrow [d]$ , let  $V = [d]$ , let  $E = \{(l(i), l(i+1)) \text{ for } i \in [n] \text{ for every } i \text{ such that } l(j) \neq l(i+1)\}$ , and consider the multigraph  $G = (V, E)$ .

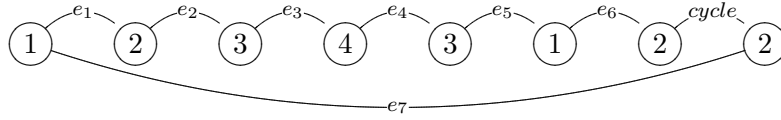
Let  $s \in G(d, n)$  be a completely labeled Gale string. Then every run of  $s$  splits uniquely into  $d/2$  pairs  $(i, i+1)$  such that the labels  $l(i)$  satisfy the condition  $l(i) \neq l(i+1)$ , and all the labels  $l(i) \in [d]$  occur. Then the labels will correspond to all the vertices of  $G$ , and the pairs will correspond to the edges of a perfect matching for  $G$ .

Conversely, let  $l : [n] \rightarrow [d]$  be a labeling, and let  $M$  be a perfect matching for  $G$  as above. We can construct a string  $s$  such that  $s(i) = s(i+1)$  for every  $(l(i), l(i+1)) \in M$  and  $s(i) = 0$  otherwise. Since  $M$  is a matching, all the  $(l(i), l(i+1)) \in M$  are disjoint, so, considering  $s$  as a “loop,” every run is of even length. Furthermore, since  $M$  is a perfect matching, every vertex  $v \in [d]$  is the endpoint of an edge  $(l(i), l(i+1))$ , so  $s$  is completely labeled.

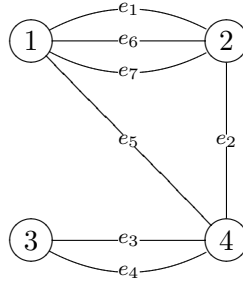
We have a reduction from GALE to the problem PERFECT MATCHING, which is polynomial-time solvable by theorem 1. Finding a Gale string for a given labeling, or deciding that there isn't one, can therefore be done in polynomial time.  $\square$

We give two examples of the construction used in theorem 2.

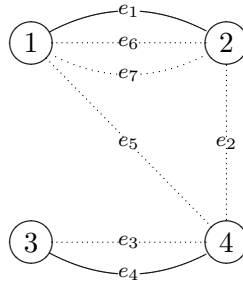
*Example 0.1.* Let  $l = 12343122$  be a string of labels. Then the edges  $e_i$  of the graph  $G$  obtained from the construction in the proof of theorem 2 will be as follow:



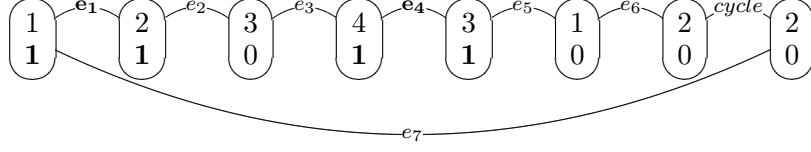
Given the vertices  $v \in [4]$ , the graph  $G$  will be:



A perfect matching for  $G$  is given by  $M = \{e_1, e_4\}$ .

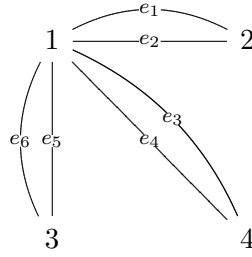


In turn, this corresponds to the completely labeled Gale string 11011000.



A perfect matching for a graph, and therefore a Gale string for a labeling, is not always possible, as shown in the next example.

*Example 0.2.* Let us consider the labeling  $l = 121314$ . The associated graph  $G$  will be



Since there aren't any disjoint edges, it's not possible to find a perfect matching for  $G$ . Analogously, we have seen in example ?? that there isn't any possible completely labeled Gale string for the labeling  $l$ .

We finally extend the proof of theorem 2 to show that ANOTHER GALE is polynomial-time solvable.

**Theorem 3.** *The problem ANOTHER GALE is solvable in polynomial time.*

*Proof.* Let  $l : [n] \rightarrow [d]$  be a labeling, and let  $s \in G(d, n)$  be a completely labeled Gale string for  $l$ . Let  $G = (V, E)$  be the graph constructed from  $l$  as in the proof of theorem 2, and let  $M$  be its perfect matching for  $G$  corresponding to  $s$ .

If there is an edge  $e = (l(i), l(i + 1)) \in M$  and there is an edge  $e' \neq e$  in  $G$  such that  $e' = (l(i), l(i + 1))$  (recall that  $G$  can be a multigraph), we simply consider the matching  $M' = M \setminus \{e\} \cup \{e'\}$ . Let  $s'$  be the completely labeled Gale string corresponding to  $M'$ ; the 1's corresponding to the labels

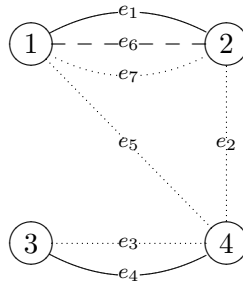
$l(i), l(i+1)$ , that in  $s$  were in the position given by the edge  $e$ , for  $s'$  are in the position given by  $e' \neq e$ . Therefore, we have a completely labeled Gale string that is different from the one in the input of the problem.

We now assume that all the edges in every perfect matching  $M$  for  $G$  don't have a parallel edge. Note that this condition is only on the edges in the matching;  $G$  can still be a multigraph.

Theorem ?? guarantees the existence of a completely labeled Gale string  $s' \neq s$ ; since the two strings are different, the perfect matching  $M' \neq M$  corresponding to one of these  $s'$  does not use at least one edge  $e \in M$ . There are  $d/2$  possible graphs  $G'_i = (V, E'_i)$ , where  $E'_i = E \setminus \{e_i\}$  for each  $e_i \in M$ ; since  $V(G) = V(G')$  and  $E(G) \subset E(G')$ , every perfect matching for  $G'$  is a perfect matching for  $G$  as well. The existence of  $s'$  implies that there is at least one graph  $G'$  with a perfect matching  $M' \neq M$ . With a brute force approach, the time to find this  $G'$  and the corresponding  $M'$  will be given by the time to find a perfect matching multiplied by a factor  $O(d)$ . Therefore, searching for a completely labeled Gale string  $s' \neq s$  takes again polynomial time.  $\square$

We give two examples of the construction of theorem 3.

*Example 0.3.* We consider the labeling the string of labels  $l = 1234312$ . We have found in example 0.1 the completely labeled Gale string 1101100, corresponding to the perfect matching  $M = \{e_1, e_4\}$  in the graph  $G$ .



If instead of  $e_1$  we take the parallel edge  $e_6$ , the resulting matching is still perfect.

A case in which all the edges in every perfect matching don't have a parallel edge is the following; note that  $G$  is a multigraph.

*Example 0.4.* We consider the labeling  $l = 123142$ . There are only two possible perfect matchings for the corresponding graph:  $M = \{e_2, e_4\}$ , that corresponds to the completely labeled Gale string  $s = 011110$ , and  $M' = \{e_3, e_5\}$ , that corresponds to  $s' = 001111$ .

