1 Introduction

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file: background

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Theorem 1. *[1] Let Q be a labeled m-dimensional simplicial polytope with $\mathbf{0}$ in its interior, with vertices $e_1, \ldots, e_m, c_1, \ldots, c_n$, so that $F_0 = \mathrm{conv}\{e_i \mid i \in [m]\}$ is a facet of Q.*

Let $l : [n] \to [m]$, and let (U, B) be the unit vector game with $U = (e_{l(1)} \ \cdots \ e_{l(n)})$ and $B = (b_1 \ \cdots \ b_n)$, where $b_j = \frac{c_j}{1+\mathbf{1}^\top c_j}$ for $j \in [n]$.

Label the vertices of Q as follows:

$$l_v(-e_i) = i \text{ for } i \in [m] \tag{1}$$

$$l_v(c_j) = l(j) \text{ for } j \in [n] \tag{2}$$

Then a facet $F \neq F_0$ of Q with normal vector v is completely labeled if and only if (x, y) is a Nash equilibrium of (U, B), where $x = \frac{v+\mathbf{1}}{\mathbf{1}^\top(v+\mathbf{1})}$, and $x_i = 0$ if and only if $e_i \in F$ for $i \in [m]$. Any j so that c_j is a vertex of F represents a pure best reply to x; the mixed strategy y is the uniform distribution on the set of the pure best replies to x.

Proposition 1. *The problem 2-*Nash *for unit vector games is polynomial-time reducible to the problems* Another Completely Labeled Vertex *and its dual* Another Completely Labeled Facet, *where*

do we have to prove "polynomial"?

Another Completely Labeled Vertex	
input :	*A simple m-dimensional polytope S with $m+n$ facets; a labeling $l : [m+n] \to [n]$; a facet F_0 of S, completely labeled by l.*
output :	*A facet $F \neq F_0$ of S, completely labeled by l.*

Another Completely Labeled Vertex	
input :	*A simplicial m-dimensional polytope S with $m+n$ vertices; a labeling $l : [m+n] \to [n]$; a vertex v_0 of S, completely labeled by l.*
output :	*A vertex $v \neq v_0$ of S, completely labeled by l.*

2.5 Cyclic Polytopes and Gale Strings

In the last section we have built a correspondence between labeling of best response polytopes and bimatrix games. We now focus on a special case of games, where the polytope in theorem 1 is a *cyclic polytope*, a particular kind of polytope that can be represented as a combinatorial structure called *Gale string*. We give first the definition of cyclic polytope, then the definition of Gale string, and we show their relation as proven by Gale in [6].

A *cyclic polytope* P in dimension d with n vertices is the convex hull of n distinct points $\mu_d(t_j)$ on the *moment curve* $\mu_d : \ t \mapsto (t, t^2, \ldots, t^d)^\top$ for $j \in [n]$ such that $t_1 < t_2 < \cdots < t_n$.

example(s) with graphics! There's one in Ziegler

A *bitstring* is a string of labels that are either 0s or 1s. Formally: given an integer k and a set S, we can represent the function $f_s : [k] \to S$ as the string $s = s(1)s(2)\cdots s(k)$; we have a bitstring in the case where $S = \{0, 1\}$. A maximal substring of consecutive 1's in a bitstring is called a *run*.

A *Gale string of length n and dimension d* is a bitstring of length n, denoted as $s \in G(d, n)$, such that

1. exactly d bits in s are 1 and
2. (*Gale evenness condition*)

$$01^k0 \text{ is a substring of } s \quad \Longrightarrow \quad k \text{ is even.} \tag{3}$$

The Gale evenness condition characterises Gale strings in $G(d, n)$ as the bitstrings of length n with exactly d elements equal to 1, such that *interior* runs (that is, runs bounded on both sides by 0s) must be of even length. In general, this condition allows Gale strings to start or end with an odd-length run; but when d is even this means that s starts with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d, n)$ with even d as a "loop" obtained by "glueing together" the extremes of the string to form an even run. Formally, we can see the indices of a Gale string $s \in G(d,n)$ with d even as equivalence classes modulo n, identifying $s(i+n) = s(i)$. This also shows that the set of Gale strings of even dimension is invariant under a cyclic shift of the strings.

Example 2.1. As an example of d even, we have

$$G(4,6) = \{\mathbf{1111}00, \mathbf{111}00\mathbf{1}, \mathbf{11}00\mathbf{11}, \mathbf{1}00\mathbf{111}, 00\mathbf{1111}, \\ 0\mathbf{1111}0, \mathbf{11}0\mathbf{11}0, \mathbf{1}0\mathbf{11}0\mathbf{1}, 0\mathbf{11}0\mathbf{11}\}$$

The strings $\mathbf{1111}00$, $\mathbf{111}00\mathbf{1}$, $\mathbf{11}00\mathbf{11}$, $\mathbf{1}00\mathbf{111}$, $00\mathbf{1111}$ and $0\mathbf{1111}0$ are equivalent under a cyclic shift (if considering the strings as "loops", the $\mathbf{1}$'s are all consecutive), as are the strings $\mathbf{11}0\mathbf{11}0$, $\mathbf{1}0\mathbf{11}0\mathbf{1}$ and $0\mathbf{11}0\mathbf{11}$ (if considering the strings as "loops", the even runs of $\mathbf{1}$'s are two couples separated by a single 0).

As an example for d odd, we have

$$G(3,5) = \{\mathbf{111}00, \mathbf{1}0\mathbf{11}0, \mathbf{1}00\mathbf{11}, \mathbf{11}00\mathbf{1}, 0\mathbf{11}0\mathbf{1}, 00\mathbf{111}\}$$

Note how $0\mathbf{1}0\mathbf{11}$ is a cyclic shift of $\mathbf{1}0\mathbf{11}0$, but it is not a Gale string.

The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [6].

Theorem 2 ([6])**.** *For any positive integers d, n let P be the cyclic polytope in dimension d with n vertices. Then the facets of P are encoded by $G(d,n)$; that is,*

$$F \text{ is a facet of } P$$
$$\Longleftrightarrow$$
$$F = \operatorname{conv}\{\mu(t_j) \mid s(j) = 1 \text{ for some } j \in [n] \text{ and } s \in G(d,n)\}$$

pf - see Ziegler

Essentially, this holds because any set $S \subset [n]$ the moment curve defines a unique hyperplane which is crossed (and not just touched) by the moment curve; if the bitstring s that encodes F as $1(s)$ has a substring 01^k0

example of cyclic polytope + equivalent gale string - pg 35 JM has nice one

We now give define a *labeling* for Gale strings, corresponding to the labeling of best-response polytopes.

A string s is *completely labeled* for some labeling function $l : [n] \to [d]$ if $\{\bar{l} \in [d] | s(i) = 1 \text{ and } l(i) = \bar{l} \text{ for some } i \in [n]\} = [d]$. If $s \in G(d, n)$, this implies that for every $\bar{l} \in [d]$ there is exactly one $s(i)$ such that $s(i) = 1$ and $l(i) = \bar{l}$, since there are exactly d positions such that $s(i) = 1$.

Example 2.2. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: **1111**00, **11**0**11**0, **1**00**111** and **1**0**11**0**1**.

1	**2**	**3**	**4**	**3**	**2**
1	**1**	**1**	**1**	·	·
1	**1**	·	**1**	**1**	·
1	·	·	**1**	**1**	**1**
1	·	**1**	**1**	·	**1**

Sometimes it is not possible to find a completely labeled Gale string.

Example 2.3. For $l = 121314$, there are no completely labeled Gale strings.

The labels $l(i) = 2, 3, 4$ appear only once in l, as $l(2), l(4), l(6)$ respectively; therefore we must have $s(2) = s(4) = s(6) = 1$. For every other $i \in [n]$ we have $l(i) = 1$, so we have $l(i) = 1$ for exactly one $i = 1, 3, 5$. The candidate strings are then **11**0**1**0**1**, 0**111**0**1**, 0**1**0**111**; but none of these satisfies the Gale evenness condition.

From this point forward, we will assume that d is even. We will also assume that the labeling $l : [n] \to [d]$ is such that $l(i) \neq l(i + 1)$; this can be done without loss of generality, given the following consideration. Suppose that $l(i) = l(i + 1)$ for some index i, and let s be a completely labeled Gale string for l. Then only one of $s(i)$ and $s(i + 1)$ can be equal to **1** (note that it's possible that both are 0s). So $s(i)s(i + 1)$ will never be a run of even length that "interferes" with the Gale Evenness Condition, so we can "simplify" by identifying the indices i and $i + 1$.

We will now focus on the problem of finding the Nash equilibria of a unit vector game (U, B) where $U = (e_{l(1)} \cdots e_{l(d)})$ for some labeling $l : [n] \to [d]$

and the best-response polytope Q is cyclic, exploiting theorems 1 and 2. To do so, we must find a labeling l such that the Gale strings $s \in G(d, d+n)$ encoding the completely labeled facets of the corresponding cyclic d-polytope Q with $d+n$ vertices are exactly the Gale strings of dimension d and length $d+n$ that are completely labeled for l.

Theorem 1 relies on a labeling of the vertices $l_v : [d+n] \to [d]$ defined in 1 such that $l_v(-e_i) = i$ for $i \in [d]$, and $l_v(c_j) = l(j)$ for the vertices $c_j \neq -e_i$, where $j \in [n]$. We define the labeling $l : [d+n] \to [d]$ as follows:

$$l_s(i) = i \text{ for } i \in [d] \tag{4}$$

$$l_s(d+j) = l(j) \text{ for } j \in [n]. \tag{5}$$

The Gale strings $s \in G(d, d+n)$ that are completely labeled for l_s correspond exactly to the completely labeled facets of Q, with the facet F_0 corresponding to the "trivial" completely labeled string $\mathbf{1}^d 0$.

Then, by proposition 1, we have the following theorem.

Theorem 3. *The problem of finding a Nash equilibrium for a unit vector game for which the best response polytope is the dual of a cyclic polytope is polynomial-time reducible to the problem* Another Gale*, where*

do we have to prove "polynomial"?

Another Gale	
input **:**	*A labeling $l : [n] \to [d]$, where d is even and $d < n$. A Gale string $s \in G(d, n)$, completely labeled by l.*
output **:**	*A Gale string $s' \in G(d, n)$, completely labeled by l, such that $s' \neq s$.*

References

[1] A. V. Balthasar (2009). "Geometry and equilibria in bimatrix games." PhD Thesis, London School of Economics and Political Science.

[2] M. M. Casetti, J. Merschen, B. von Stengel (2010). "Finding Gale Strings." *Electronic Notes in Discrete Mathematics* 36, pp. 1065–1082.

[3] X. Chen, X. Deng (2006). "Settling the Complexity of 2-Player Nash Equilibrium." *Proc. 47th FOCS*, pp. 261–272.

[4] C. Daskalakis, P. W. Goldberg, C. H. Papadimitriou (2006). "The Complexity of Computing a Nash Equilibrium." *SIAM Journal on Computing*, 39(1), pp. 195–259.

[5] J. Edmonds (1965). "Paths, Trees, and Flowers." *Canad. J. Math.* 17, pp. 449–467.

[6] D. Gale (1963). "Neighborly and Cyclic Polytopes." *Convexity, Proc. Symposia in Pure Math.*, Vol. 7, ed. V. Klee, American Math. Soc., Providence, Rhode Island, pp. 225–232.

[7] D. Gale, H. W. Kuhn, A. W. Tucker (1950). "On Symmetric Games." *Contributions to the Theory of Games* I, eds. H. W. Kuhn and A. W. Tucker, *Annals of Mathematics Studies* 24, Princeton University Press, Princeton, pp. 81–87.

[8] C. E. Lemke, J. T. Howson, Jr. (1964). "Equilibrium Points of Bimatrix Games." *J. Soc. Indust. Appl. Mathematics* 12, pp. 413–423.

[9] A. McLennan, R. Tourky (2010). "Imitation Games and Computation." *Games and Economic Behavior* 70, pp. 4–11.

[10] N. Megiddo, C. H. Papadimitriou (1991). "On Total Functions, Existence Theorems and Computational Complexity." *Theoretical Computer Science* 81, pp. 317–324.

[11] J. Merschen (2012). "Nash Equilibria, Gale Strings, and Perfect Matchings." PhD Thesis, London School of Economics and Political Science.

[12] J. F. Nash (1951). "Noncooperative games." *Annals of Mathematics*, 54, pp. 289-295.

[13] C. H. Papadimitriou (1994). "On the Complexity of the Parity Argument and Other Inefficient Proofs of Existence." *J. Comput. System Sci.* 48, pp. 498–532.

[14] R. Savani, B. von Stengel (2006). "Hard-to-solve Bimatrix Games." *Econometrica* 74, pp. 397–429.

better other article, "Exponentially many steps for finding a NE in a bimatrix game." In the 45th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2004.?

[15] R. Savani, B. von Stengel (2015). "Unit Vector Games." arXiv:1501.02243v1 [cs.GT]

[16] L. S. Shapley (1974). "A Note on the Lemke-Howson Algorithm." *Mathematical Programming Study 1: Pivoting and Extensions*, pp. 175–189

[17] L. Végh, B. von Stengel "Oriented Euler Complexes and Signed Perfect Matchings." arXiv:1210.4694v2 [cs.DM]

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