1 Introduction

2 Complexity, Games, Polytopes and Gale Strings

2.1 The Complexity Classes P and PPAD

2.2 Normal Form Games and Nash Equilibria

file: background

2.3 Some Geometrical Notation

2.4 Bimatrix Games, Labels and Polytopes

Theorem 1. *Let $l : [n] \to [m]$, and let (U, B) be the unit vector game where $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. Consider the polytopes P^l and Q^l where*

$$Q^l = \{y \in \mathbb{R}^n | y \geq \mathbf{0}, \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\} \tag{1}$$

where $N_i = \{j \in [n] | l(j) = i\}$ for $i \in [m]$, and

$$P^l = \{x \in \mathbb{R}^m | x \geq \mathbf{0}, \ B^\top x \leq \mathbf{1}\} \tag{2}$$

Label the $m + n$ inequalities in the definition of P^l as follows, and label every facet of P^l according to the inequality defining it.

- *$x_i \geq 0$ has label i, for $i \in [m]$*
- *$(B^\top x)_j \leq 1$ has label $l(j)$, for $j \in [n]$*

Then $x \in P^l$ is a completely labeled point of $P^l \setminus \{\mathbf{0}\}$ if and only if there is some $y \in Q^l$ such that, after scaling, the pair (x, y) is a Nash equilibrium of (U, B)

file: polytopes

2.5 Cyclic Polytopes and Gale Strings

We have now built a correspondence between labeled best response polytopes and bimatrix games. We now consider a special case of games, where the polytope in theorem 1 can be represented as a combinatorial structure. We first give the definition of these particular polytopes, called *cyclic polytopes*, then we define the combinatorial structure that we will use to study them, the *Gale strings*.

A *cyclic polytope* P in dimension d with n vertices is the convex hull of n distinct points $\mu_d(t_j)$ on the *moment curve* $\mu_d : t \mapsto (t, t^2, \dots, t^d)^\top$ for $j \in [n]$ such that $t_1 < t_2 < \dots < t_n$.

example(s) with graphics!

A *bitstring* is a string of labels that are either 0s or 1s. Formally: given an integer k and a set S, we can represent the function $f_s : [k] \to S$ as the string $s = s(1)s(2)\cdots s(k)$; we have a bitstring in the case where $S = \{0, 1\}$. A maximal substring of consecutive 1's in a bitstring is called a *run*.

A *Gale string of length n and dimension d* is a bitstring of length n, denoted as $s \in G(d, n)$, such that

1. exactly d bits in s are 1 and
2. (*Gale evenness condition*)

$$01^k0 \text{ is a substring of } s \implies k \text{ is even.} \tag{3}$$

The Gale evenness condition characterises Gale strings in $G(d, n)$ as the bitstrings of length n with exactly d elements equal to 1, such that *interior* runs (that is, runs bounded on both sides by 0s) must be of even length. In general, this condition allows Gale strings to start or end with an odd-length run; but when d is even this means that s starts with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d, n)$ with even d as a "loop" obtained by "glueing together" the extremes of the string to form an even run. Formally, we can see the indices of a Gale string $s \in G(d,n)$ with d even as equivalence classes modulo n, identifying $s(i+n) = s(i)$. This also shows that the set of Gale strings of even dimension is invariant under a cyclic shift of the strings.

Example 2.1. For instance, as a case where d is even, we have

$$\begin{aligned} G(4,6) = \{&\mathbf{1111}00, \\ &\mathbf{111}00\mathbf{1}, \\ &\mathbf{11}00\mathbf{11}, \\ &\mathbf{1}00\mathbf{111}, \\ &00\mathbf{1111}, \\ &0\mathbf{1111}0, \\ &\mathbf{11}0\mathbf{11}0, \\ &\mathbf{1}0\mathbf{11}0\mathbf{1}, \\ &0\mathbf{11}0\mathbf{11}\} \end{aligned}$$

The strings 111100, 111001, 110011, 100111, 001111 and 011110 are equivalent under a cyclic shift (if considering the strings as "loops", the 1's are all consecutive), as are the strings 110110, 101101 and 011011 (if considering the strings as "loops", the even runs of 1's are two couples separated by a single 0).

Example 2.2. As a case where d is odd, we consider

$$\begin{aligned} G(3,5) = \{&\mathbf{111}00, \\ &\mathbf{1}0\mathbf{11}0, \\ &\mathbf{1}00\mathbf{11}, \\ &\mathbf{11}00\mathbf{1}, \\ &0\mathbf{11}0\mathbf{1}, \\ &00\mathbf{111}\} \end{aligned}$$

Note how 01011 is a cyclic shift of 10110, but it is not a Gale string.

The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [6].

Theorem 2 ([6]). *For any positive integers d, n let P be the cyclic polytope in dimension d with n vertices. Then the facets of P are encoded by $G(d, n)$; that is,*

$$\begin{aligned} &F \text{ is a facet of } P \\ &\Longleftrightarrow \\ &F = \operatorname{conv}\{\mu(t_i) \mid i \in 1(s)\} \text{ for some } s \in G(d, n) \end{aligned} \tag{4}$$

sketch of pf if not too long and it uses relevant techniques
example of cyclic polytope + equivalent gale string - pg 35 JM has nice one

From this point forward, we will assume that d is even.

give something to generalise to odd case

The labeling of a cyclic polytope has a straigtforward equivalent in its representation as a Gale string. A string s is *completely labeled* for some labeling function $l : [n] \to [d]$ if $\{\bar{l} \in [d] | s(i) = \bar{l} \text{ for some } i \in [n]\} = [d]$. If $s \in G(d, n)$, this implies that for every $\bar{l} \in [d]$ there is exactly one $s(i)$ such that $s(i) = 1$ and $l(i) = \bar{l}$, since there are exactly d positions such that $s(i) = 1$.

Example 2.3. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: 111100, 110110, 100111 and 101101.

1	**2**	**3**	**4**	**3**	**2**
1	**1**	**1**	**1**	·	·
1	**1**	·	**1**	**1**	·
1	·	·	**1**	**1**	**1**
1	·	**1**	**1**	·	**1**

Sometimes there aren't any completely labeled Gale strings that are associated with a given labeling.

Example 2.4. For $l = 121314$, there are no completely labeled Gale strings.

The labels $l(i) = 2, 3, 4$ appear only once in l, as $l(2), l(4), l(6)$ respectively; therefore we must have $s(2) = s(4) = s(6) = 1$. For every other $i \in [n]$ we have $l(i) = 1$, so we have $l(i) = 1$ for exactly one $i = 1, 3, 5$. The candidate strings are then **11** 0**1** 0**1**, 0**111** 0**1**, 0**1** 0**111**; but none of these satisfies the Gale evenness condition.

from here: labeled cyclic polytopes and their correspondence to labeled GS

For this cyclic polytope P, a labeling $l : [n] \to [d]$ can be understood as a label $l(j)$ for each vertex $\mu(t_j)$ for $j \in [n]$. A completely labeled Gale string s therefore represents a facet F of P that is completely labeled.

graphics of labeled cyclic polytope

Special games are obtained by using cyclic polytopes in Theorem 1, suitably affinely transformed with a completely labeled facet F_0. When Q is a cyclic polytope in dimension d with $d+n$ vertices, then the string of labels $l(1)\cdots l(n)$ in Theorem 1 defines a labeling $l' : [d+n] \to [d]$ where $l'(i) = i$ for $i \in [d]$ and $l'(d+j) = l(j)$ for $j \in [n]$. In other words, the string of labels $l(1)\cdots l(n)$ is just prefixed with the string $1\,2\cdots d$ to give l'. Then l' has a trivial completely labeled Gale string $1^d 0^n$ which defines the facet F_0. Then the problem Another Gale defines exactly the problem of finding a Nash equilibrium of the unit vector game (I, B). Note again that B is here not a general matrix (which would define a general game) but obtained from the last n of $d+n$ vertices of a cyclic polytope in dimension d.

Another Gale	
input :	A labeling $l : [n] \to [d]$, where d is even and $d < n$, and an associated completely labeled Gale string s in $G(d, n)$.
output :	A completely labeled Gale string s' in $G(d, n)$ associated with l, such that $s' \neq s$.

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