

Complexity of the Gale String Problem for Equilibrium Computation in Games

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Declaration

I certify that this thesis I have presented for examination for the MPhil degree of the London School of Economics and Political Science is based on joint work with Julian Merschen and Bernhard von Stengel, published in [3].

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Abstract

This thesis presents a report on original research, published as joint work with Merschen and von Stengel in *Electronic Notes in Discrete Mathematics* [3]. Our result shows a polynomial time algorithm to solve two problems related to labeled Gale strings, a combinatorial structure introduced by Gale in [9] that can be used in the representation of a particular class of games.

These games were used by Savani and von Stengel [19] as an example of exponential running time for the classical Lemke-Howson algorithm to find a Nash equilibrium of a bimatrix game [11]. It was conjectured that solving these games via the Lemke-Howson algorithm was complete in the class **PPAD** (Proof by Parity Argument, Directed version). A major motivation for the definition of this class by Papadimitriou [18] was, in turn, to capture the pivoting technique of many results related to the Nash equilibrium, including the Lemke-Howson algorithm.

Our result, on the contrary, sets apart this class of games as a case for which there is a polynomial-time algorithm to find a Nash equilibrium. Since Daskalakis, Goldberg and Papaditrimiou [5] and Chen and Deng [4] proved the **PPAD**-completeness of finding a Nash equilibrium in general normal-form games, we have a special class of games, unless **PPAD** = **P**.

Our proof exploits two results. The first one is the representation of the Nash equilibria of these games as Gale strings, as seen in Savani and von Stengel [19]. The second one is the polynomial-time solvability of the problem of finding a perfect matching in a graph, proven by Edmonds [6].

Merschen [14] and Végh and von Stengel [22] expanded our technique to prove further interesting results about index of a Nash equilibrium (see Shapley [21]) and orientation of an oik (see Edmonds [7] and Edmonds and Sanità [8]).

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Introduction

The topic of this thesis is a problem in the field of *algorithmic game theory*, that is, the study of game-theoretic problems from the point of view of computer science. In particular, we focus on the computational complexity of a particular class of games. These

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Savani and von Stengel (2006) showed that the LH algorithm may take exponentially many steps. Their construction uses “dual cyclic polytopes” which have a well-known vertex structure for any dimension and number of linear inequalities. Morris (1994) used similarly labeled dual cyclic polytopes where all “Lemke paths” are exponentially long. A Lemke path is related to the path computed by the LH algorithm, but is defined on a single polytope that does not have a product structure corresponding to a bimatrix game. The completely labeled vertex found by a Lemke path can be interpreted as a symmetric equilibrium of a symmetric bimatrix game. However, as in the example in Figure 4 below, such a symmetric game may also have nonsymmetric equilibria which here are easy to compute, so that the result by Morris (1994) seemed not suitable to describe games that are hard to solve with the LH algorithm.

The “imitation games” defined by McLennan and Tourky (2010) changed this picture. In an imitation game, the payoff matrix of one of the

players is the identity matrix. The mixed strategy of that player in any Nash equilibrium of the imitation game corresponds exactly to a symmetric equilibrium of the symmetric game defined by the payoff matrix of the other player. In that way, an algorithm that finds a Nash equilibrium of a bimatrix game can be used to find a symmetric Nash equilibrium of a symmetric game.

In one sense the two-polytope construction of Savani and von Stengel (2006) was overly complicated: the imitation games by McLennan and Tourky (2010) provide a simple and elegant way to turn the single-polytope construction of Morris (1994) into exponentially-long LH paths for bimatrix games. In another sense, the construction of Savani and von Stengel was not redundant, since it provided examples that are simultaneously bad for the LH algorithm and support enumeration, which is another natural and simple algorithm for finding equilibria. The support of a mixed strategy is the set of pure strategies that are played with positive probability. Given a pair of supports of equal size, the mixed strategy probabilities are found by equating all payoffs for the other player's support, which then have to be compared with payoffs outside the support to establish the equilibrium property (see Dickhaut and Kaplan, 1991).

In this paper, we extend the idea of imitation games to games where one payoff matrix is arbitrary and the other is a set of unit vectors. We call these unit vector games.

we use them to extend Morris's construction to give bimatrix games that use only one dual cyclic polytope, rather than the two used by Savani and von Stengel, and for which both the LH algorithm and support enumeration are simultaneously bad. This result (Theorem 11) was first described by Savani (2006, Section 3.8).

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We begin this section with the definition of almost complete labeling; we then move on to the classic version of the Lemke-Howson algorithm for the

problem ANOTHER COMPLETELY LABELED VERTEX, as given in the beautiful exposition by Shapley [21], and its dual version for ANOTHER COMPLETELY LABELED FACET. Finally, we present the Lemke-Howson for Gale algorithm. In the next session we will tackle the issue of the computational complexity of these algorithms: ANOTHER COMPLETELY LABELED FACET and ANOTHER COMPLETELY LABELED VERTEX are **PPA**, NASH is **PPAD**, as first shown in Papadimitriou [18]; furthermore, as shown by Morris [15] and by Savani and von Stengel [19], there are cases of exponential running time. This had led us to conjecture that these problems could be exploited for a proof of **PPAD** completeness, also considering that finding a completely labeled facet (or vertex, or the existence of a Nash equilibrium) is **NP** in the case of a general labeled polytope, as proven by von Stengel [25]. In the last section we will finally present our original result, that goes in the opposite direction: the problem ANOTHER GALE can be solved in polynomial time, that is, it is a problem in **TFP**. Unit vector games with dual cyclic best response polytope present therefore a case apart, as expected, but not because they are harder than others, but because they are easier.

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Chapter 1

Preliminary definitions

We begin by giving some background definitions and notation that will be used throughout this thesis. The first section will cover polytopes (see Ziegler [26] for further details); the second section will deal with basic game theory. We will finally focus on computational complexity; after the standard definitions (see Papadimitriou [17] for an introduction) we will move on to the more recently defined classes **TFNP** and **PPAD**, first introduced in [13] and [18], respectively. The latter, in particular, is a key concept in the study of the problems that are the focus of this thesis.

We will often need to refer to subsets of \mathbb{N} ; we follow the notation $[n] = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$.

1.1 Vectors and Polytopes

We denote the transpose of a matrix A as A^\top . Vectors $u, v \in \mathbb{R}^d$ will be considered as column vectors, so $u^\top v$ is their scalar product. A vector in \mathbb{R}^d for which all components are 0's will be denoted as $\mathbf{0}$; a vectors for which all components are 1's will be denoted as $\mathbf{1}$; the *i-th unit vector*, for which all the components are 0 except for the *i*-th component equal to 1, will be denoted as e_i . An inequality of the form $u \geq v$, is intended to hold for every component.

An *affine combination* of points in an Euclidean space $\{z_1, \dots, z_m\} \subset \mathbb{R}^n$ is $\sum_{i \in [m]} \lambda_i z_i$ where $\lambda_i \in \mathbb{R}$ such that $\sum_{i \in [m]} \lambda_i = 1$; the points z_1, \dots, z_m are *affinely independent* if none of them is an affine combination of the others. The *convex hull* of the points z_1, \dots, z_m is an affine combination with $\lambda_i \geq 0$ for all $i \in [m]$; we denote it as $\text{conv}\{z_1, \dots, z_m\} = \{\sum_i \lambda_i z_i \mid \lambda_i \geq 0, \sum_i \lambda_i = 1\}$. A set of points $Z = \{z_1, \dots, z_m\}$ is *convex* if $Z = \text{conv}(Z)$, and it has *dimension* d if Z has exactly $d+1$ affinely independent points. A convex set of dimension d is called a *d-simplex*. The *standard d-simplex* is $\Delta_d = \text{conv}\{e_1, \dots, e_{d+1}\}$.

A *polytope* is the convex hull of a finite set of points $\{z_1, \dots, z_m\} \subset \mathbb{R}^n$, not necessarily affinely independent; the *dimension* of the polytope is the dimension of its convex hull. A *polyhedron* is the intersection of finitely many closed halfspaces $\{x \in \mathbb{R}^d \mid a^T x \leq a_0\}$; note that a bounded polyhedron is a polytope. A *vertex* of a d -dimensional polytope $P = \text{conv}(Z)$ is a point $z \in Z$ such that $\text{conv}(Z \setminus \{z\}) \neq P$; an *edge* of P is a 1-dimensional line segment that has two vertices as endpoints. A *facet* of P is the convex hull of a set of d vertices $F = \{z_1, \dots, z_d\}$ that lie on a hyperplane of the form $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P ; the vector a (taken as unique up to a scalar multiple) is called the *normal vector* of the facet.

A *d-dimensional simplicial polytope* P is the convex hull of a set of at least $d+1$ points $v \in \mathbb{R}^d$ such that no $d+1$ of them are on a common hyperplane; this is equivalent to requiring that every facet of P is a d -simplex. A d -dimensional polytope P is *simple* if every point of P lies on at most d facets; note that the points lying on exactly d facets are exactly the vertices.

The *polar* of the polytope $P = \{x \in \mathbb{R}^d \mid x^\top c_i \leq 1, i \in [k]\}$ with $c_i \in \mathbb{R}^d$ is $P^\Delta = \text{conv}\{c_i, i \in [k]\}$. If P is simplicial or it is simple and it contains the origin $\mathbf{0}$, then $P^{\Delta\Delta} = P$; furthermore, if P is simplicial P^Δ is simple, and vice versa.

1.2 Normal Form Games and Nash Equilibria

A *game*, first defined by von Neumann in [23], is a model of strategic interaction. A *finite normal form game* is $\Gamma = (P, S, u)$ where both P and S are finite. The former is the set of *players*; $S = \times_{p \in P} S_p$ is the set of *pure strategy profiles* and S_p is the set of *pure strategies* of player p ; we will use the notation $S_{-p} = \times_{q \neq p} S_q$. The purpose of each player $p \in P$ is to maximize their *payoff function* $u^p : S \rightarrow \mathbb{R}$, where $u = \times_{p \in P} u^p$. In the following pages, by “game” we will always mean “finite normal form game.” If there are only two players, we will refer to player 1 using feminine pronouns and to player 2 using masculine ones; such games are called *bimatrix games* since they can be characterized by the $m \times n$ payoff matrices A and B , where a_{ij} and b_{ij} are the payoffs of respectively player 1 and of player 2 when the former plays her i th pure strategy and the latter plays his j th pure strategy. A bimatrix game is *zero-sum* if $B = -A$, and *symmetric* if $B = A^\top$.

A *mixed strategy* of player p is a probability distribution on S_p ; it can be described as a point on the $(|S_p| - 1)$ -dimensional *mixed strategy simplex* $\Delta_p = \{x \in \mathbb{R}^{|S_p|} \mid x \geq \mathbf{0}, \mathbf{1}^\top x = 1\}$. The set of *mixed strategy profiles* is the simplicial polytope $\Delta = \times_{p \in P} \Delta_p$; we extend the payoff functions to $u^p : \Delta \rightarrow \mathbb{R}$ linearly.

A *Nash equilibrium* of a game is a strategy profile in which each player cannot improve their expected payoff by unilaterally changing their strategy; such a strategy is called a *best response*. Note that applying an affine transformation to all the payoffs does not change the Nash equilibria of the game.

Proposition 1.1. *A mixed strategy $x \in \Delta_p$ is a best response against some mixed strategy profile $y \in \Delta_{-p}$ of the other players if and only if every pure strategy $s_i \in S_p$ chosen with positive probability in x is a best response to y .*

The existence of a Nash equilibrium is guaranteed by the fundamental theorem by Nash ([16]). Note that there might be more than one equilibrium.

Theorem 1. (Nash [16]) *Every finite game in normal form has a Nash equilibrium.*

We give two classic examples of games: the prisoners' dilemma and a coordination game.

Example 1.1. In the symmetric non zero-sum *prisoners' dilemma* of Figure 1.1, each player must decide whether to “help” the other one or to “betray” them. If both players help each other, they will get a small reward; if both betray, they will pay a small penalty; if one betrays and the other cooperate the former will get a large reward and the latter will pay a large penalty.

		2	
		betray	help
1	betray	1 0	
	help	3 2	

Figure 1.1 The prisoners' dilemma.

The only equilibrium is the profile in which both players betray. If player 2 betrays, the best response of player 1 is to betray, since it gives her payoff 1 instead of 0; if player 2 helps, her payoff for betraying is 3 and her payoff for helping is 2, so betraying is again the best response. The same holds for player 2, so at the equilibrium both players will betray.

Figure 1.2 shows a *coordination* game. Both players drive on a mountain road; they lose if drive on the same side of the road and win if they avoid each other, regardless of which side they take.

The pure strategy Nash equilibria are (mountain, valley) and (valley, mountain); there is also a symmetric equilibrium in mixed strategies at $((1/2, 1/2), (1/2, 1/2))$.

		2	
		mountain	valley
1	mountain	0	1
	valley	1	0

Figure 1.2 A coordination game.

1.3 Some Complexity Classes

A *Turing machine* \mathcal{M} is a representation of an *algorithm* that takes an *input*, runs a *program* manipulating the input, and either does not come to a *halting state* or it returns an *output*; the latter can be YES (in which case the Turing machine *accepts* the input), NO (the Turing machine *rejects* the input), or a string $\mathcal{M}(x)$. Formally, $\mathcal{M} = (K, \Sigma, \delta, s)$. The finite set K is the set of *states*; Σ is a finite set of *symbols* (the *alphabet* of \mathcal{M}) such that $\Sigma \cap K = \emptyset$, and Σ always contains the symbols \sqcup (*blank*) and \triangleright (*first symbol*). The *transition function* δ is

$$\delta : K \times \Sigma \longrightarrow (K \cup \{h, \text{YES}, \text{NO}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$$

where h is the *halting state*, YES is the *accepting state*, NO is the *rejecting state*, and $\{\leftarrow, \rightarrow, -\} \not\subseteq (K \cup \Sigma)$ correspond to the *cursor directions* “left,” “right” and “stay.”

A *language* is a set of strings of symbols $L \subseteq (\Sigma \setminus \{\sqcup\})^*$; a Turing machine \mathcal{M} *decides* L if for every $x \in (\Sigma \setminus \{\sqcup\})^*$ either $\mathcal{M}(x) = \text{YES}$ if $x \in L$ or $\mathcal{M}(x) = \text{NO}$ if $x \notin L$. We say that \mathcal{M} *accepts* L if for every $x \in (\Sigma \setminus \{\sqcup\})^*$ we have that $\mathcal{M}(x) = \text{YES}$ if $x \in L$ and $\mathcal{M}(x)$ does not halt if $x \notin L$. Given a function $f : (\Sigma \setminus \{\sqcup\})^* \rightarrow \Sigma^*$, we say that \mathcal{M} *computes* f if $\mathcal{M}(x) = f(x)$ for every $x \in (\Sigma \setminus \{\sqcup\})^*$. A *class* is a set of languages.

A specific input of a problem is called an *instance*. If a problem P outputs either YES or NO, P is a *decision problem*; its *complement* is the problem

\overline{P} that outputs “NO” for each instance of P that outputs “YES”, and vice versa. A *function problem* outputs a string y , more generic than YES or NO, that satisfies a binary relation $R(x, y)$, where x is the instance of P ; a *search problem* either outputs y as above or it rejects the input if it’s not possible to find any such y . If y is guaranteed to exist, the problem is called a *total function problem*.

An example of decision problem is: “(input) given a graph, (question) is it possible to find an Euler tour of the graph?” Its complement is “(input) given a graph, (question) is it possible that there isn’t any Euler of the graph?” A search problem is: “(input) given a graph, (output) return one Euler tour of the graph, or “NO” if no such tour exists.” A total function problem is: “(input) given an Euler graph, (output) return one of its Euler tours.”

Let P_1 be a problem and let x be an instance of P_1 that is encoded in $|x|$ bits. P_1 *reduces to the problem P_2 in polynomial time*, denoted $P_1 \leq_P P_2$, if there exists a *polynomial-time reduction*, that is, a function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and a Turing machine \mathcal{M} such that for all $x \in \{0, 1\}^*$

1. $x \in P_1 \iff f(x) \in P_2$;
2. \mathcal{M} computes $f(x)$;
3. \mathcal{M} stops after $p(|x|)$ steps, where p is a polynomial.

Intuitively, if P_1 is polynomial-time reducible to P_2 , it takes polynomial time to “translate” P_1 to P_2 , and then to “translate back” a solution of P_2 as a solution of P_1 . This is particularly useful if P_2 is “difficult to solve”; then the problem P_1 is at least as “difficult.”

For any class C of decision problems, the class of all complements of the problems in C is the *complement class* $\text{co} - C$. A problem P is *hard* for a class C if every problem in C is polynomial-time reducible to P ; that is, if P is hard to solve at least as every problem in C . A *complete* for the class C is a C –*hard* problem that is also in C .

The complexity class **P** contains all the *polynomially decidable problems*; that is, all problems P such that there exists a Turing machine \mathcal{M} that outputs either YES or NO for all inputs $x \in \{0,1\}^*$ of P after $p(|x|)$ steps, where p is a polynomial. Intuitively, a decision problem is in **P** if the answer to its question can be found in a number of steps that is polynomial in the input of the problem. A problem P belongs to the class **NP**, *non-deterministic polynomial-time problems*, if there exists a Turing machine \mathcal{M} and polynomials p_1, p_2 such that

1. for all $x \in P$ there exists a *certificate* $y \in \{0,1\}^*$ which satisfies $|y| \leq p_1(|x|)$;
2. \mathcal{M} accepts the combined input xy , stopping after at most $p_2(|x| + |y|)$ steps;
3. for all $x \notin P$ there does not exist $y \in \{0,1\}^*$ such that \mathcal{M} accepts the combined input xy .

Intuitively, a decision problem is in **NP** if it takes polynomial time to verify whether the “certificate solution” y is, indeed, a correct answer to the question posed by the problem. The class $\#\mathbf{P}$ is the class of all problems that output the number of possible certificates for a problem in **NP**.

In [13], Megiddo and Papadimitriou introduced the classes **FNP**, *function non-deterministic polynomial*, and **TFNP**, *total function non-deterministic polynomial*. The former is defined as the class of binary relations $R(x, y)$ such that there is a polynomial-time algorithm that decides $R(x, y)$ for x, y such that $|y| \leq p(|x|)$, where p is a polynomial. The latter is the class of all such problems for which y is guaranteed to exist. **FNP** and **TFNP** can be seen as the equivalent of **NP** for (respectively) function and total function problems. Also in [13], Megiddo and Papadimitriou proved that **TFNP** is a *semantic* class, that is, a class without complete problems, unless $\mathbf{NP} = \mathbf{co-NP}$. To circumvent this limitation, Papadimitriou [18] focused on the problems for which the existence of a solution is proved by a specific argument, introducing

the classes **PPA** (*Proof by Parity Argument*) and **PPAD** (*Proof by Parity Argument, Directed version*).

The existence of a solution for a problem in **PPA** can be proved using the argument “in any undirected graph with one odd-degree node there must be another odd-degree node.” It is interesting to note that **PPA**-complete problems are yet to be found. Problems in **PPAD**, on the other hand, are guaranteed to have a solution by a proof employing the argument “in any directed graph with one unbalanced node (that is, with outdegree different from its indegree) there must be another unbalanced node.” This can be simplified to the argument “in any directed graph in which all vertices have indegree and outdegree at most one, if there is a *source* (a node with indegree zero), then there must be a *sink* (a node with outdegree zero).” Formally, we can define **PPAD** as the class of problems reducible to the problem END OF THE LINE. This is the definition given in Daskalakis, Goldberg and Papadimitriou [5]; the original definition in Papadimitriou [18] is given in terms of Turing machines.

END OF THE LINE

input : Two circuits S and P with n input bits and n output bits such that $P(0^n) = 0^n \neq S(0^n)$.

output: An input $x \in \{0, 1\}^n$ such that $P(S(x) \neq x)$ or $S(P(x)) \neq x \neq 0^n$

A *circuit* is formally defined as a directed acyclic graph with n vertices with indegree 0 called *input nodes*, m vertices with outdegree 0 called *output nodes*, and *internal nodes* with indegree 1 or 2; when each input node receives an input in $\{0, 1\}$, the internal nodes with indegree 2 compute the Boolean functions *and* or *or*, the internal nodes with indegree 1 compute the Boolean function *not* and each output node returns a value in $\{0, 1\}$ accordingly. The problems in **PPAD** can be seen as a circuit S (“successor”), and a circuit P (“predecessor”) that are used to build a directed graph with an edge (x, y) if and only if $S(x) = y$ and $P(y) = x$, with a *standard source* 0^n ; the output

is either a sink or a non-standard source. Figure 1.3 presents an example of graph of a **PPAD** problem; a graph for a **PPA** problem is analogous, but the resulting graph is not directed and instead of sources and sinks there are generic endpoints.

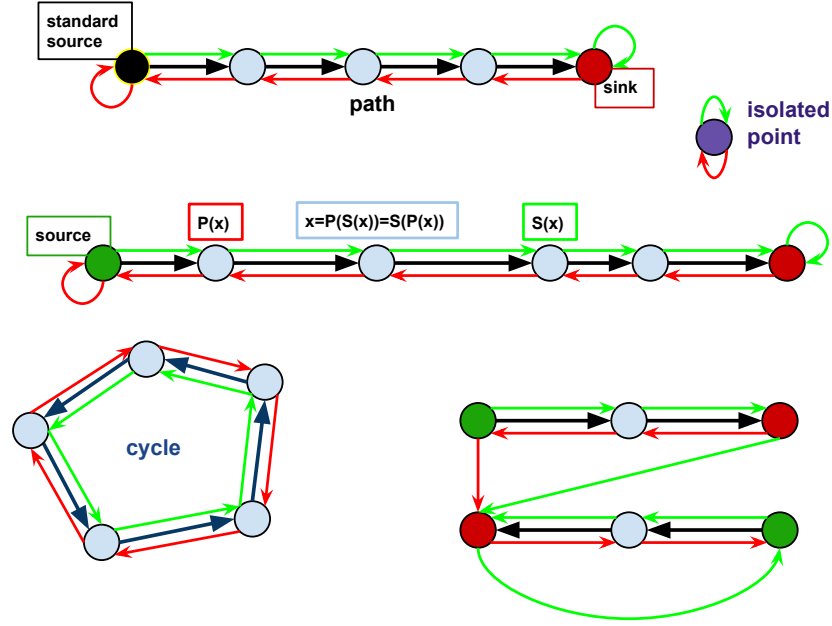


Figure 1.3 In green, the circuit S ; in red, the circuit P . The standard source is the black node; the green nodes are the other sources; the red nodes are the sinks. The graph can include paths, cycles and isolated points.

By theorem 1, the problem n -NASH, defined as follows, is a total function problem.

n -NASH

input : A n -player game.

output: A Nash equilibrium of the game.

Megiddo and Papadimitriou ([13]) proved that n -NASH is in **TFNP**. Daskalakis, Goldberg and Papadimitriou [5] and Chen and Deng [4] have later proven its

PPAD-completeness, the former for $n \geq 3$ and the latter for $n \geq 2$. A small amendment of the proof in [5] can be found in Casetti [2].

Theorem 2. (Daskalakis, Goldberg and Papadimitriou [5]; Chen and Deng [4]) *For $n \geq 2$, the problem n -NASH is **PPAD**-complete.*

We will see more problems in **PPA** and **PPAD** in chapter ??; in fact, our main result can be seen as a negative result on the **PPAD** complexity of a case of 2-Nash.

Chapter 2

Labels, Polytopes and Gale Strings

In the rest of this thesis we will focus bimatrix games, analyzing their Nash equilibria with the use of labels and combinatorial structures. This idea is due to Lemke and Howson [11]; we will follow the approach given by Shapley [21]. We will first relate the labeling construction that allows to identify Nash equilibria with “completely labeled” vertices or facets of certain polytopes; we will then move on to unit vector games, a special class of games that allows to simplify the labeling construction; see Balthasar [1] and Savani and von Stengel [20]. In the second section of the chapter we will restrict our scope to Gale games, a further case of unit vector games. The labeling problems for these games are equivalent to an elegant combinatorial problem on a case of bitstrings satisfying some simple conditions, called Gale strings. We will define Gale strings, see their correspondence to the structures given in the first section, and finally get a reduction from 2-NASH to the Gale string problem ANOTHER GALE.

We will identify the bimatrix game with its payoff matrices (A, B) , and we will assume that both A and B are non-negative, and that neither A nor B^\top has a zero column; this can be done without loss of generality, applying an

affine transformation to A and B if necessary.

2.1 Bimatrix Games and Labels

Let $n, m \in \mathbb{N}$ with $m \leq n$, and consider a set X with $|X| = n$. A *labeling* of X is a function $l : X \rightarrow [m]$; an m -uple $x = (x_1, \dots, x_m) \in X^m$ is *completely labeled* if each label $j \in [m]$ appears once and only once in $(l(x_1), \dots, l(x_m))$; formally, if $\{i \in [m] \mid l(x_i) = j \text{ for some } i \in [m]\} = [m]$.

Let (A, B) be bimatrix game, and let X and Y be the mixed strategy simplices of respectively player 1 and 2; that is

$$X = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, \mathbf{1}^\top x = 1\}; \quad Y = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0}, \mathbf{1}^\top y = 1\}. \quad (2.1)$$

A *labeling* of the game is then given as follows:

1. the m pure strategies of player 1 are identified by $1, \dots, m$;
2. the n pure strategies of player 2 are identified by $m + 1, \dots, m + n$;
3. each mixed strategy $x \in X$ of player 1 has
 - label i for each $i \in [m]$ such that $x_i = 0$, that is if in x player 1 does not play her i -th pure strategy,
 - label $m + j$ for each $j \in [n]$ such that the j -th pure strategy of player 2 is a best response to x ;
4. each mixed strategy $y \in Y$ of player 2 has
 - label $m + j$ for each $j \in [n]$ such that $y_j = 0$, that is if in y player 2 does not play his j -th pure strategy,
 - label i for each $i \in [m]$ such that the i -th pure strategy of player 1 is a best response to y .

The labeling of mixed strategy profiles can be used to characterize the Nash equilibria of the game.

Theorem 3. (Shapley [21]) *Let $(x, y) \in X \times Y$; then (x, y) is a Nash equilibrium of the bimatrix game (A, B) if and only if (x, y) is completely labeled.*

Proof. The mixed strategy $x \in X$ has label $m + j$ for some $j \in [n]$ if and only if the j -th pure strategy of player 2 is a best response to x ; this, in turn, is a necessary and sufficient condition for player 2 to play his j -th strategy at an equilibrium against x . Therefore, at an equilibrium (x, y) all labels $m + j$, with $j \in [n]$, will appear either as labels of x or of y , and analogously for the strategies $y \in Y$. \square

An useful graphical representation of labels on the simplices X and Y is done by labeling the outside of each simplex according to the player's own pure strategies that are *not* played, and by subdividing its interior in closed polyhedral sets corresponding to the other player's pure best response strategies, called *best response regions*. We give an example of this construction.

Example 2.1. Consider the 3×3 game (A, B) with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \quad (2.2)$$

The pure strategies of player 1 are labeled as 1, 2, 3; the pure strategies of player 2 are labeled as 4, 5, 6. In the following figures the labels of the strategies will be represented as circled numbers. Figure 2.1 shows X and Y : the exterior facets are labeled with the pure strategy on the opposite vertex, where only that pure strategy is played; the interior is covered by the best response regions, by to the other player's pure best response strategies. For example, the best-response region in Y with label 1 is the set of those (y_1, y_2, y_3) so that $y_1 \geq y_2$ and $y_1 \geq y_3$. There is only one pair (x, y) that is completely labeled, namely $x = (\frac{1}{3}, \frac{2}{3}, 0)$ with labels 3, 4, 5 and $y = (\frac{1}{2}, \frac{1}{2}, 0)$ with labels 1, 2, 6, so this is the only Nash equilibrium of the game.

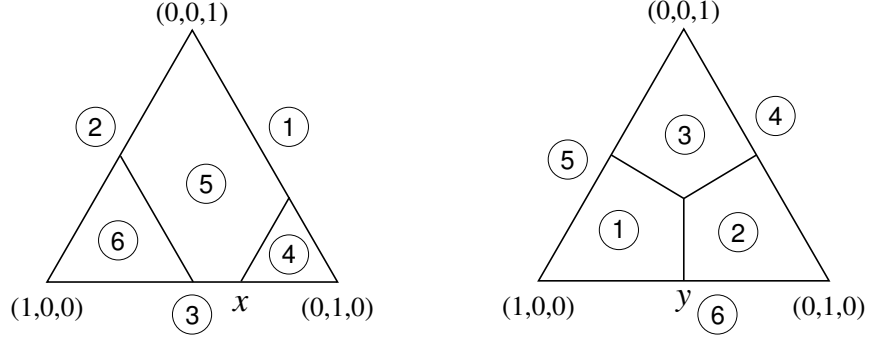


Figure 2.1 Mixed strategy sets X and Y of game (2.2) and their labeled best response regions.

The point of view of best response regions can be translated to an equivalent construction on polytopes. The first step is to notice that the best-response regions can be obtained as projections on X and Y of the *best-response facets* of the polyhedra

$$\bar{P} = \{(x, v) \in X \times \mathbb{R} \mid B^\top x \leq \mathbf{1}v\}; \quad \bar{Q} = \{(y, u) \in Y \times \mathbb{R} \mid Ay \leq \mathbf{1}u\}. \quad (2.3)$$

In \bar{P} , these facets are the points $(x, v) \in X \times \mathbb{R}$ such that $(B^\top x)_j = v$, which in turn correspond to the strategies $x \in X$ of player 1 that give exactly payoff v to player 2 when he plays strategy j ; the projection of the facet defined by $(B^\top x)_j = v$ to X has then label j . Analogously, the facet of \bar{Q} given by the points $(y, u) \in Y \times \mathbb{R}$ such that $(Ay)_i = u$ will project to the best-response region of Y with label i .

Example 2.2. In Example 2.1, the inequalities $B^\top x \leq \mathbf{1}v$ are

$$\begin{aligned} 3x_2 &\leq v; \\ 2x_1 + 2x_2 + 2x_3 &\leq v \\ 4x_1 &\leq v. \end{aligned}$$

Figure 2.2 shows the best-response facets of \bar{P} and their projection to X by ignoring the payoff variable v , which gives the subdivision of X into best-response regions of Figure 2.1.

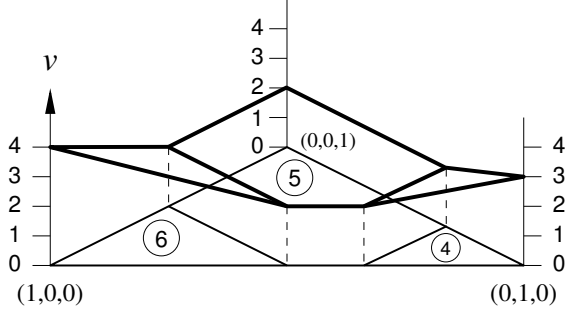


Figure 2.2 Best response polyhedron \bar{P} of game (2.2).

Given the assumptions on non-negativity of A and B^\top , we can change coordinates to x_i/v and y_j/u and replace \bar{P} and \bar{Q} with the *best-response polytopes*

$$P = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, B^\top x \leq \mathbf{1}\}; \quad Q = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0}, Ay \leq \mathbf{1}\}. \quad (2.4)$$

The polytope P is the intersection the $m + n$ half spaces corresponding to either player 1 avoiding her i -th pure strategy, with $i \in [m]$, or to a best response of player 2 that gives non-zero probability to his j -th strategy, with $j \in [n]$. Formally, a point $x \in P$ has label k if and only if either $x_k = 0$ for $k \in [m]$ or $(B^\top x)_k = 0$ for $k \in [n]$, and a point in Q has label k if and only if either $y_k = 0$ for $k \in [n]$ or $(Ay)_k$ for $k \in [m]$. Then a point $(x, y) \in P \times Q$ is completely labeled if and only if it satisfies the *complementarity condition*

$$\begin{aligned} x_i = 0 \text{ or } (Ay)_i = 1 & \quad \text{for all } i \in [m]; \\ y_j = 0 \text{ or } (B^\top x)_j & \quad \text{for all } j \in [n]. \end{aligned} \quad (2.5)$$

Therefore, if $(x, y) \in P \times Q$ is completely labeled, either the corresponding point in $\bar{P} \times \bar{Q}$ is a Nash equilibrium, or $(x, y) = (\mathbf{0}, \mathbf{0})$; we will refer to the latter case as *artificial equilibrium*.

Example 2.3. Keeping on with Example 2.1, the best response polyhedron \bar{P} of Figure 2.2 becomes the best response polytope of Figure 2.3. Note that the vertex $(\mathbf{0}, \mathbf{0})$ is completely labeled, since it has labels 1, 2, 3 in P and labels 4, 5, 6 in Q .

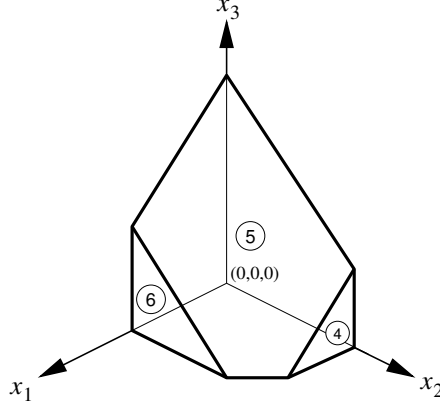


Figure 2.3 Best response polytope of game (2.2).

We will now see some special cases of games connected by polynomial-time reductions; our goal is to find a class of games that captures the complexity of 2-NASH. First of all, we note that any bimatrix game can be “symmetrized”; the result is due to Gale, Kuhn and Tucker [10] for zero-sum games and its extension to non-zero-sum games is a folklore result.

Proposition 2.1. *Let (A, B) be a bimatrix game and (x, y) be one of its Nash equilibria. Then (z, z) , where $z = (x, y)$, is a Nash equilibrium of the symmetric game (C, C^\top) , where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}. \quad (2.6)$$

McLennan and Tourky [12] have proven a result in the opposite direction of proposition 2.1: any symmetric game can be translated into a bimatrix game of the form (I, B) , called *imitation game*. Since it takes polynomial time in the size of a matrix to calculate its transpose, we have a polynomial-time reduction from 2-NASH to IMITATION NASH.

Theorem 4. (McLennan and Tourky [12]) *The pair (x, x) is a symmetric Nash equilibrium of the symmetric bimatrix game (C, C^\top) if and only if there is some y such that (x, y) is a Nash equilibrium of the imitation game (I, B) with $B = C^\top$.*

In any Nash equilibrium of (I, B) , the mixed strategy x of player 1 corresponds exactly to the symmetric equilibrium (x, x) in the symmetric game defined by the payoff matrix of player 2. Note, though, that Theorem 4 applies to the symmetric equilibria of the symmetric game, but not to all its equilibria; there could be non-symmetric equilibria of (C, C^\top) that are not found through the imitation game. We see an example illustrating this.

Example 2.4. As an example, consider the symmetric game (C, C^\top) with

$$C = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \quad C^\top = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad (2.7)$$

so that the corresponding imitation game is $(I, C^\top) = (A, B)$, as in (2.2). Figure 2.4 shows the labeled mixed-strategy simplices X and Y for the game 2.7; since the game is symmetric, only the labels are different. In addition to the symmetric equilibrium (x, x) where $x = (\frac{1}{3}, \frac{2}{3}, 0)$, the game has two non-symmetric equilibria in (a, b) and (b, a) with $a = (\frac{1}{2}, \frac{1}{2}, 0)$ and $b = (0, \frac{2}{3}, \frac{1}{3})$; the corresponding imitation game (A, B) has only one equilibrium (x, y) , corresponding to (x, x) .

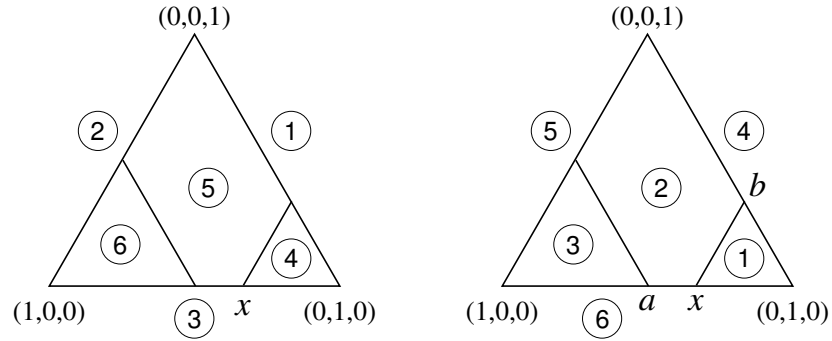


Figure 2.4 Best response regions of the symmetric game (2.7).

The characterization of Nash equilibria as completely labeled pairs (x, y) holds for arbitrary bimatrix games. From now on, we will also impose a further

condition: that all points in P have at most m labels, and all points in Q have at most n labels. These games are called *nondegenerate*; since any game can be made nondegenerate by lexicographic perturbation (see von Stengel [24]), we can impose the nondegeneracy condition without loss of generality. In an equilibrium (x, y) of a nondegenerate game each label appears exactly once; this also means that the number of pure best response strategies against a mixed strategy is never larger than the size of the support of that mixed strategy. Geometrically, this means that no point of the best response polytope P lies on more than m facets and no point of the best response polytope Q lies on more than n facets, so both P and Q are simple. Furthermore, a point of P has exactly m labels if and only if it is a vertex, and a point of Q has exactly n labels if and only if it is a vertex; therefore, all completely labeled points (x, y) are vertices, and Nash equilibria are isolated points.

Example 2.5. An example of degenerate game is given by (C, C^\top) where

$$C = \begin{pmatrix} 0 & 4 & 0 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \quad C^\top = \begin{pmatrix} 0 & 2 & 4 \\ 4 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \quad (2.8)$$

As it is shown in Figure 2.5, the mixed strategy $x = (\frac{1}{2}, \frac{1}{2}, 0)$, that also defines the unique symmetric equilibrium (x, x) of the game, has three pure best responses. Note that the Nash equilibria (x, y) of the imitation game (I, C^\top) are not unique, since any convex combination of $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ can be chosen for y , as shown in Figure 2.6.

A generalization of imitation games is the class of *unit vector games*, introduced by Balthasar [1]; they are defined as bimatrix games of the form (U, B) where the columns of the matrix U are unit vectors. By the results above, finding a Nash equilibrium of a bimatrix game is at least as hard as finding a Nash equilibrium of a unit vector game; that is, 2-NASH reduces to UNIT VECTOR NASH. In unit vector games, the problem of finding a completely labeled vertex of the polytope $P \times Q$ can be translated to the problem

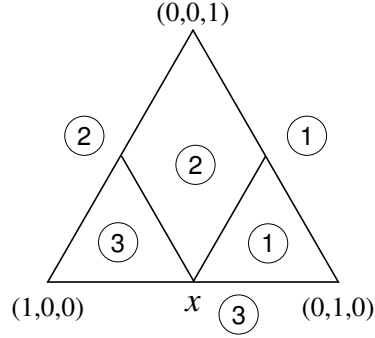


Figure 2.5 Best-response regions of the degenerate symmetric game (2.8). Label 3 appears twice at the unique symmetric equilibrium.

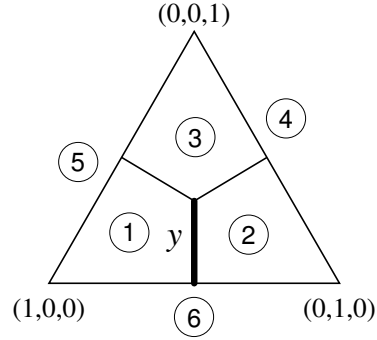


Figure 2.6 Labeled mixed-strategy sets for the imitation game (I, C^\top) . The equilibria (x, y) corresponding to the unique symmetric equilibrium of the symmetric game (2.8) are not unique.

of finding a completely labeled vertex of one simple polytope that encodes all the game. We first give this result as in Savani and von Stengel [20]; we will later see the version in Balthasar [1].

Theorem 5. (Savani and von Stengel [20]) *Let $l : [n] \rightarrow [m]$, and let (U, B) be the unit vector game with $U = (e_{l(1)} \cdots e_{l(n)})$. Let $N_i = \{j \in [n] \mid l(j) = i\}$ for $i \in [m]$, and consider the polytopes P^l and Q^l*

$$P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0}, B^\top x \leq \mathbf{1}\}; \quad (2.9)$$

$$Q^l = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0}, \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\}.$$

Let l_f be the labeling of the facets of P^l defined as follows:

$$\begin{aligned} x_i \geq 0 \text{ has label } i \text{ for } i \in [m]; \\ (B^\top x)_j \leq 1 \text{ has label } l(j) \text{ for } j \in [n]. \end{aligned} \tag{2.10}$$

Then $x \in P^l$ is a completely labeled vertex of $P^l \setminus \{\mathbf{0}\}$ if and only if there is some $y \in Q^l$ such that, after scaling, the pair (x, y) is a Nash equilibrium of (U, B)

Proof. Let P, Q be the best response polytopes of (U, B) as in 2.4, and let $(x, y) \in P \times Q \setminus \{\mathbf{0}, \mathbf{0}\}$ be a Nash equilibrium of (U, B) . Then (x, y) is completely labeled in $[m + n]$; so if $x_i = 0$, then x has label $i \in m$. If $x_i > 0$, then y has label i , so $(Uy)_i = 1$; then for some $j \in [n]$ we have $y_j > 0$ and $U_j = e_i$; that is, we have $y_j > 0$ and $l(j) = i$ for some $j \in [n]$. Since $y_j > 0$, $x \in P$ has label $m + j$; then, $(B^\top x)_j = 1$; therefore $x \in P^l$ has label $l(j) = i$. Hence, x is a completely labeled vertex of P^l .

Conversely, let $x \in P^l \setminus \{\mathbf{0}\}$ be completely labeled. If $x_i > 0$, then there is $j \in [m]$ such that $(B^\top x)_j = 1$ and $l(j) = i$; that is, $j \in N_i$. For all i such that $x_i > 0$, define y as follows: $y_j = 1$; $y_h = 0$ for all $h \in N_i \setminus \{j\}$. Then $(x, y) \in P \times Q$ is completely labeled. \square

Example 2.6. The game in Example 2.1 is a unit vector game, with $l(i) = i$. In the polytope P^l of Figure 2.7 the labels 4, 5 and 6 of the best response polytope are replaced by 1, 2 and 3, since the corresponding columns of A are the unit vectors e_1, e_2, e_3 . Apart from the origin $\mathbf{0}$, the only completely labeled point of P^l is x , corresponding to the unique equilibrium of game (2.2).

We will now move on the dual version of Theorem 5, as given in Balthasar [1]. We translate the polytope P^l of (2.9) to $P = \{x - \mathbf{1} \mid x \in P^l\}$, multiplying all payoffs in B by a constant, if necessary, so that $\mathbf{0}$ is in the interior of P .

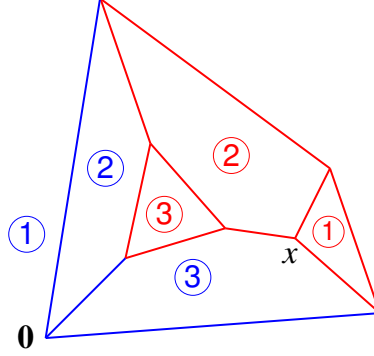


Figure 2.7 The polytope P^l of the unit vector game (2.2). Label 1 refers to the hidden facet.

We have that

$$\begin{aligned} P &= \{x + \mathbf{1} \geq \mathbf{0}, (x + \mathbf{1})^\top B \leq \mathbf{1}\} = \\ &= \{x \in \mathbb{R}^m \mid -x_i \leq 1 \text{ for } i \in [m], x^\top (b_j / (1 - \mathbf{1}^\top b_j)) \leq 1 \text{ for } j \in [n]\}. \end{aligned}$$

The polar of P is then

$$P^\Delta = \text{conv}(\{e_i \mid i \in [m]\} \cup \{c_j \mid j \in [n]\}) \quad (2.11)$$

where $c_j = b_j / (1 - \mathbf{1}^\top b_j)$. Since P is a simple polytope with $\mathbf{0}$ in its interior, $P^{\Delta\Delta} = P$. Furthermore, P^Δ is simplicial, therefore the facets of P^Δ correspond to the vertices of P and vice versa. We label the vertices of P^Δ as the corresponding facets in P^l , so the completely labeled facets of P^Δ correspond to the completely labeled vertices of P^l . In particular, the facet corresponding to $\mathbf{0}$ is

$$F_0 = \{x \in P^\Delta \mid -\mathbf{1}^\top x = 1\} = \text{conv}\{e_i \mid i \in [m]\}. \quad (2.12)$$

Theorem 5 then translates into the following.

Theorem 6. (Balthasar [1]) *Let Q be a labeled m -dimensional simplicial polytope with $\mathbf{0}$ in its interior and vertices $e_1, \dots, e_m, c_1, \dots, c_n$ such that (2.12) is a facet of Q . Let (U, B) be a unit vector game, with $U = (e_{l(1)} \cdots e_{l(n)})$ for*

a labeling $l : [n] \rightarrow [m]$ and $B = (b_1 \cdots b_n)$, where $b_j = c_j / (1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Let l_v be the labeling of the vertices of Q given by

$$\begin{aligned} l_v(-e_i) &= i \text{ for } i \in [m]; \\ l_v(c_j) &= l(j) \text{ for } j \in [n]. \end{aligned} \tag{2.13}$$

Then a facet $F \neq F_0$ of Q with normal vector v is completely labeled if and only if (x, y) is a Nash equilibrium of (U, B) , where $x = (v + \mathbf{1}) / (\mathbf{1}^\top (v + \mathbf{1}))$, so $x_i = 0$ if and only if $e_i \in F$ for $i \in [m]$ and the mixed strategy y is the uniform distribution on the set of the pure best replies to x , which in turn correspond to $j \in [n]$ such that c_j is a vertex of F .

As in Theorem 5 we have a correspondence between completely labeled vertices of P^l and equilibria of the unit vector game (U, B) with the “artificial” equilibrium corresponding to the vertex $\mathbf{0}$, in Theorem 6 we have a correspondence between the completely labeled facets of the polytope Q and equilibria of (U, B) with the “artificial” equilibrium corresponding to the facet F_0 in (2.12).

Given a bimatrix game (A, B) , it takes polynomial time to write and solve the linear equations defining its best response polyhedra $\overline{P}, \overline{Q}$ and its best response polytopes P, Q . It also take polynomial time to label $\overline{P}, \overline{Q}$ and P, Q . Analogously, given a unit vector game (U, B) , it takes polynomial time to construct and label the polytope P^l . Therefore, Theorem 5 implies a polynomial time reduction from the problem 2-NASH to the problem ANOTHER COMPLETELY LABELED VERTEX.

ANOTHER COMPLETELY LABELED VERTEX

input : A simple m -dimensional polytope P with $m + n$ facets; a labeling

$l_f : [m + n] \rightarrow [n]$; a facet F_0 of P , completely labeled by l_f .

output: A facet $F \neq F_0$ of S , completely labeled by l .

Proposition 2.2. 2-NASH reduces in polynomial time to ANOTHER COMPLETELY LABELED VERTEX.

Theorem 6 gives the dual of Proposition 2.2; since the construction of the polar polytope from the original one is also polynomial, the reduction is to the problem ANOTHER COMPLETELY LABELED FACET.

ANOTHER COMPLETELY LABELED FACET

input : A simplicial m -dimensional polytope Q with $m + n$ vertices; a labeling $l_v : [m + n] \rightarrow [n]$; a facet F_0 of Q , completely labeled by l_v .

output: A facet $F \neq F_0$ of S completely labeled by l_v .

Proposition 2.3. *2-NASH reduces in polynomial time to ANOTHER COMPLETELY LABELED FACET.*

Chapter 3

Further results

P complexity of finding all (?)

from SvS-15

Ve Āgh and von Stengel (2014, Thm. 12) give a near-linear time algorithm that finds such a second perfect matching that, in addition, has opposite sign, which corresponds to a Nash equilibrium of positive index as it would be found by a Lemke path (which, however, can be exponentially long). So this combinatorial problem is simpler than the problem of finding a Nash equilibrium of a bimatrix game, even though it gives rise to games that are hard to solve by the standard methods considered in Theorem 11.

from VvS

This paper presents three main contributions in this context. First, we define an abstract framework called pivoting systems that describes ĀIJcomplementary pivoting with directionĀĀ in a canonical manner. Similar abstract pivoting systems have been proposed by Todd (1976) and Lemke and Grotzinger (1976); we compare these with our approach in Section 5. Second, using this framework, we extend the concept of orientation to oiks and show that room partitions at the two ends of a pivoting path have opposite sign, provided the underlying oik is oriented. For two-dimensional oiks, which are Euler graphs, room partitions are perfect matchings. Their orientation is the

sign of a perfect matching as defined for Pfaffian orientations of graphs. Our third result is a polynomial-time algorithm for the following problem: Given a graph G with an Eulerian orientation and a perfect matching, find another perfect matching of opposite sign. The complementary pivoting algorithm that achieves this may take exponential time.

We conclude with open questions on the computational complexity of pivoting systems. Consider a labeled oriented pivoting system whose components (in particular the pivoting operation) are specified as polynomial-time computable functions. Assume one CL state is given. The problem of finding a second CL state belongs to the complexity class PPAD (Papadimitriou, 1994). This problem is also PPAD-complete, because finding a Nash equilibrium of a bimatrix game is PPAD-complete (Chen and Deng, 2006), which is a special case of an oriented pivoting system by Proposition 1. However, there should be a much simpler proof of this fact because pivoting systems are already rather general, so that it should be possible to encode an instance of the PPAD-complete problem “End of the Line” (see Daskalakis, Goldberg, and Papadimitriou, 2009) directly into a pivoting system. Finding a Nash equilibrium of a bimatrix game is PPAD-complete, and Lemke’s Howson paths may be exponentially long. Savani and von Stengel (2006) showed this with games defined by dual cyclic polytopes for the payoff matrices of both players, and a simpler way to do this is to use the Lemke paths by Morris (1994). One motivation for the study of Casetti, Merschen, and von Stengel (2010) was the question if finding a second completely labeled Gale string is PPAD-complete. This is unlikely because this problem can be solved in polynomial time with a matching algorithm. For the complexity class PPADS, where one looks for a second CL state of opposite sign (Daskalakis, Goldberg, and Papadimitriou, 2009), this problem is also solvable in polynomial time with our algorithm of Theorem 12. However, for room partitions of 3-oids, already manifolds, finding a second room partition is likely to be more complicated. Is this problem

PPAD-complete? We leave these questions for further research.

Acknowledgements

appendix/acknowledgments

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