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\subsection{Cyclic Polytopes and Gale Strings}\label{gs-ssect}

\begin{example}\label{no-clgs}
For $l = 121314$, there are no completely labeled Gale strings.
\end{example}

\begin{theorem}\label{even-number-gale}
For any labeling $l:[n]\to[d]$, where $d$ is even and $d<n$,
the number of completely labeled Gale strings associated with $l$ is even.
\end{theorem}

\bibitem{edm} J. Edmonds (1965).
Paths, trees, and flowers.
\emph{Canad. J. Math.} 17, pp. 449--467.
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0.1 The Complexity of GALE and ANOTHER GALE

We will now give our main result: ANOTHER GALE can be solved in polynomial time. Therefore, it takes polynomial time to find a Nash Equilibrium of a bimatrix game for which the best response polytope is cyclic.

Our proof will be based on a simple graph construction.

Definition 1. A *perfect matching* for a graph $G = (V, E)$ is a set $M \subseteq E$ of pairwise non-adjacent edges so that every vertex $v \in V$ is incident to exactly one edge in M.

We define the problem PERFECT MATCHING as follows:

PERFECT MATCHING	
input :	A graph $G = (V, E)$.
output:	A perfect matching for G.

The complexity of Perfect Matching has been proven to be in P by Edmonds [4].

Theorem 1 ([4])**.** *The problem* Perfect Matching *is solvable in polynomial time.*

We will first consider the accessory problem Gale, and we will show that it is solvable in polynomial time by using theorem 1.

Gale	
input :	A labeling $l : [n] \to [d]$, where d is even and $d < n$.
output:	A completely labeled Gale string s in $G(d, n)$ associated with l.

Theorem 2. *The problem* Gale *is solvable in polynomial time.*

Proof. We give a reduction of Gale to Perfect Matching.

In the following, we will consider every Gale string as a "loop," as seen in section **??**, so $n + 1 = 1$.

Given the labeling $l : [n] \to [d]$, let $V = [d]$, let $E = \{(l(i), l(i+1))$ for $i \in [n]$ for every i such that $l(j) \neq l(i + 1)\}$, and consider the multigraph $G = (V, E)$.

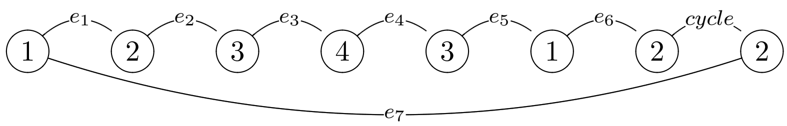
Let $s \in G(d, n)$ be a completely labeled Gale string. Then every run of s splits uniquely into $d/2$ pairs $(i, i + 1)$ such that the labels $l(i)$ satisfy the condition $l(i) \neq l(i + 1)$, and all the labels $l(i) \in [d]$ occur. Then the labels will correspond to all the vertices of G, and the pairs will correspond to the edges of a perfect matching for G.

Conversely, let $l : [n] \to [d]$ be a labeling, and let M be a perfect matching for G as above. We can construct a string s such that $s(i) = s(i + 1)$ for every $(l(i), l(i + 1)) \in M$ and $s(i) = 0$ otherwise. Since M is a matching, all the $(l(i), l(i + 1)) \in M$ are disjoint, so, considering s as a "loop," every run is of even length. Furthermore, since M is a perfect matching, every vertex $v \in [d]$ is the endpoint of an edge $(l(i), l(i + 1))$, so s is completely labeled.

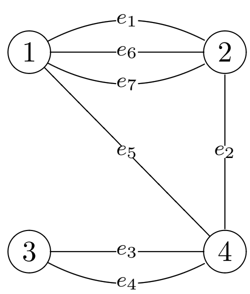
We have a reduction from GALE to the problem PERFECT MATCHING, which is polynomial-time solvable by theorem 1. Finding a Gale string for a given labeling, or deciding that there isn't one, can therefore be done in polynomial time. □

We give two examples of the construction used in theorem 2.

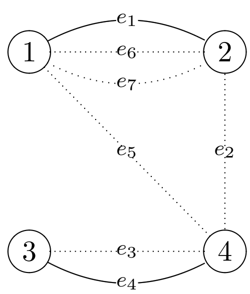
Example 0.1. Let $l = 12343122$ be a string of labels. Then the egdes e_i of the graph G obtained from the construction in the proof of theorem 2 will be as follow:



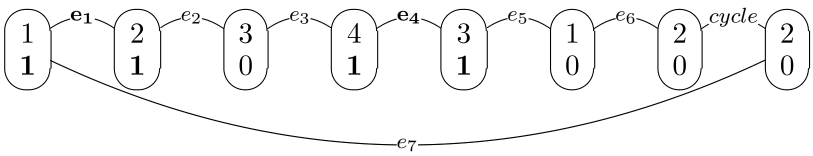
Given the vertices $v \in [4]$, the graph G will be:



A perfect matching for G is given by $M = \{e_1, e_4\}$.

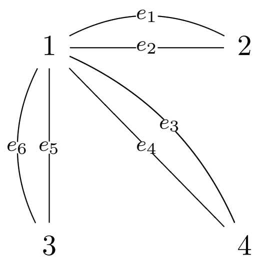


In turn, this corresponds to the completely labeled Gale string 11011000.



A perfect matching for a graph, and therefore a Gale string for a labeling, is not always possible, as shown in the next example.

Example 0.2. Let us consider the labeling $l = 121314$. The associated graph G will be



Since there aren't any disjoint edges, it's not possible to find a perfect matching for G. Analogously, we have seen in example **??** that there isn't any possible completely labeled Gale string for the labeling l.

We finally extend the proof of theorem 2 to show that ANOTHER GALE is polynomial-time solvable.

Theorem 3. *The problem* ANOTHER GALE *is solvable in polynomial time.*

Proof. Let $l : [n] \to [d]$ be a labeling, and let $s \in G(d, n)$ be a completely labeled Gale string for l. Let $G = (V, E)$ be the graph constructed from l as in the proof of theorem 2, and let M be its perfect matching for G corresponding to s.

If there is an edge $e = (l(i), l(i+1)) \in M$ and there is an edge $e' \neq e$ in G such that $e' = (l(i), l(i+1))$ (recall that G can be a multigraph), we simply consider the matching $M' = M \setminus \{e\} \cup \{e'\}$. Let s' be the completely labeled Gale string corresponding to M'; the 1's corresponding to the labels

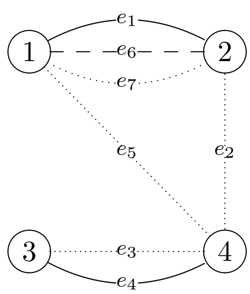
$l(i), l(i+1)$, that in s were in the position given by the edge e, for s' are in the position given by $e' \neq e$. Therefore, we have a completely labeled Gale string that is different from the one in the input of the problem.

We now assume that all the edges in every perfect matching M for G don't have a parallel edge. Note that this condition is only on the edges in the matching; G can still be a multigraph.

Theorem **??** guarantees the existence of a completely labeled Gale string $s' \neq s$; since the two strings are different, the perfect matching $M' \neq M$ corresponding to one of these s' does not use at least one edge $e \in M$. There are $d/2$ possible graphs $G'_i = (V, E'_i)$, where $E'_i = E \setminus \{e_i\}$ for each $e_i \in M$; since $V(G) = V(G')$ and $E(G) \subset E(G')$, every perfect matching for G' is a perfect matching for G as well. The existence of s' implies that there is at least one graph G' with a perfect matching $M' \neq M$. With a brute force approach, the time to find this G' and the corresponding M' will be given by the time to find a perfect matching multiplied by a factor $O(d)$. Therefore, searching for a completely labeled Gale string $s' \neq s$ takes again polynomial time. □

We give two examples of the construction of theorem 3.

Example 0.3. We consider the labeling the string of labels $l = 1234312$. We have found in example 0.1 the completely labeled Gale string 1101100, corresponding to the perfect matching $M = \{e_1, e_4\}$ in the graph G.



If instead of e_1 we take the parallel edge e_6, the resulting matching is still perfect.

A case in which all the edges in every perfect matching don't have a parallel edge is the following; note that G is a multigraph.

Example 0.4. We consider the labeling $l = 123142$. There are only two possible perfect matchings for the corresponding graph: $M = \{e_2, e_4\}$, that corresponds to the completely labeled Gale string $s = 011110$, and $M' = \{e_3, e_5\}$, that corresponds to $s' = 001111$.