

1 Introduction

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file: background

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Theorem 1. Let Q be a labeled m -dimensional simplicial polytope with $\mathbf{0}$ in its interior, with vertices $e_1, \dots, e_m, c_1, \dots, c_n$, so that $F_0 = \text{conv}\{e_i \mid i \in [m]\}$ is a facet of Q .

Let $l : [n] \rightarrow [m]$, and let (U, B) be the unit vector game with $U = (e_{l(1)} \cdots e_{l(n)})$ and $B = (b_1 \cdots b_n)$, where $b_j = \frac{c_j}{\mathbf{1}^\top c_j}$ for $j \in [n]$.

Label the vertices of Q as follows:

- $-e_i$ has label $i \in [m]$
- c_j has label $l(j)$ for $j \in [n]$

Then a facet $F \neq F_0$ of Q with normal vector v is completely labeled if and only if (x, y) is a Nash equilibrium of (U, B) , where $x = \frac{v+1}{\mathbf{1}^\top (v+1)}$, and $x_i = 0$ if and only if $e_i \in F$ for $i \in [m]$. Any j so that c_j is a vertex of F represents a pure best reply to x ; the mixed strategy y is the uniform distribution on the set of the pure best replies to x .

2.5 Cyclic Polytopes and Gale Strings

In the last section we have built a correspondence between labeling of best response polytopes and bimatrix games. We now focus on a special case of games, where the polytope in theorem 1 is a *cyclic polytope*, a particular kind of polytope that can be represented as a combinatorial structure called *Gale string*. We give first the definition of cyclic polytope, then the definition of Gale string, and we show their relation as proven by Gale in [6].

A *cyclic polytope* P in dimension d with n vertices is the convex hull of n distinct points $\mu_d(t_j)$ on the *moment curve* $\mu_d : t \mapsto (t, t^2, \dots, t^d)^\top$ for $j \in [n]$ such that $t_1 < t_2 < \dots < t_n$.

example(s) with graphics! There's one in Ziegler

A *bitstring* is a string of labels that are either 0s or 1s. Formally: given an integer k and a set S , we can represent the function $f_s : [k] \rightarrow S$ as the string $s = s(1)s(2)\dots s(k)$; we have a bitstring in the case where $S = \{0, 1\}$. A maximal substring of consecutive 1's in a bitstring is called a *run*.

A *Gale string of length n and dimension d* is a bitstring of length n , denoted as $s \in G(d, n)$, such that

1. exactly d bits in s are 1 and
2. (*Gale evenness condition*)

$$01^k0 \text{ is a substring of } s \implies k \text{ is even.} \quad (1)$$

The Gale evenness condition characterises Gale strings in $G(d, n)$ as the bitstrings of length n with exactly d elements equal to 1, such that *interior* runs (that is, runs bounded on both sides by 0s) must be of even length. In general, this condition allows Gale strings to start or end with an odd-length run; but when d is even this means that s starts with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d, n)$ with even d as a “loop” obtained by “glueing together” the extremes of the string to form an even run. Formally, we can see the indices of a Gale

string $s \in G(d, n)$ with d even as equivalence classes modulo n , identifying $s(i+n) = s(i)$. This also shows that the set of Gale strings of even dimension is invariant under a cyclic shift of the strings.

Example 2.1. As an example of d even, we have

$$G(4, 6) = \{\mathbf{111100}, \mathbf{111001}, \mathbf{110011}, \mathbf{100111}, \mathbf{001111}, \\ \mathbf{011110}, \mathbf{110110}, \mathbf{101101}, \mathbf{011011}\}$$

The strings $\mathbf{111100}$, $\mathbf{111001}$, $\mathbf{110011}$, $\mathbf{100111}$, $\mathbf{001111}$ and $\mathbf{011110}$ are equivalent under a cyclic shift (if considering the strings as “loops”, the **1**’s are all consecutive), as are the strings $\mathbf{110110}$, $\mathbf{101101}$ and $\mathbf{011011}$ (if considering the strings as “loops”, the even runs of **1**’s are two couples separated by a single 0).

As an example for d odd, we have

$$G(3, 5) = \{\mathbf{11100}, \mathbf{10110}, \mathbf{10011}, \mathbf{11001}, \mathbf{01101}, \mathbf{00111}\}$$

Note how $\mathbf{01011}$ is a cyclic shift of $\mathbf{10110}$, but it is not a Gale string.

The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [6].

Theorem 2 ([6]). *For any positive integers d, n let P be the cyclic polytope in dimension d with n vertices. Then the facets of P are encoded by $G(d, n)$; that is,*

$$F \text{ is a facet of } P$$

$$\iff$$

$$F = \text{conv}\{\mu(t_j) \mid s(j) = 1 \text{ for some } j \in [n] \text{ and } s \in G(d, n)\}$$

pf - see Ziegler

Essentially, this holds because any set $S \subset [n]$ the moment curve defines a unique hyperplane which is crossed (and not just touched) by the moment curve; if the bitstring s that encodes F as $1(s)$ has a substring 01^k0 example of cyclic polytope + equivalent gale string - pg 35 JM has nice one

We now give define a *labeling* for Gale strings, corresponding to the labeling of best-response polytopes.

A string s is *completely labeled* for some labeling function $l : [n] \rightarrow [d]$ if $\{\bar{l} \in [d] | s(i) = 1 \text{ and } l(i) = \bar{l} \text{ for some } i \in [n]\} = [d]$. If $s \in G(d, n)$, this implies that for every $\bar{l} \in [d]$ there is exactly one $s(i)$ such that $s(i) = 1$ and $l(i) = \bar{l}$, since there are exactly d positions such that $s(i) = 1$.

Example 2.2. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: **111100**, **110110**, **100111** and **101101**.

$$\begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{3} & \mathbf{2} \\ \hline \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdot & \cdot \\ \mathbf{1} & \mathbf{1} & \cdot & \mathbf{1} & \mathbf{1} & \cdot \\ \mathbf{1} & \cdot & \cdot & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \cdot & \mathbf{1} & \mathbf{1} & \cdot & \mathbf{1} \end{array}$$

Sometimes it is not possible to find a completely labeled Gale string.

Example 2.3. For $l = 121314$, there are no completely labeled Gale strings.

The labels $l(i) = 2, 3, 4$ appear only once in l , as $l(2), l(4), l(6)$ respectively; therefore we must have $s(2) = s(4) = s(6) = 1$. For every other $i \in [n]$ we have $l(i) = 1$, so we have $l(i) = 1$ for exactly one $i = 1, 3, 5$. The candidate strings are then **110101**, **011101**, **010111**; but none of these satisfies the Gale evenness condition.

From this point forward, we will assume that d is even. We will also assume that the labeling $l : [n] \rightarrow [d]$ is such that $l(i) \neq l(i+1)$; this can be done without loss of generality, given the following consideration. Suppose that $l(i) = l(i+1)$ for some index i , and let s be a completely labeled Gale string for l . Then only one of $s(i)$ and $s(i+1)$ can be equal to **1** (note that it's possible that both are 0s). So $s(i)s(i+1)$ will never be a run of even length that "interferes" with the Gale Evenness Condition, so we can "simplify" by identifying the indices i and $i+1$.

We will now focus on the problem of finding the Nash equilibria of a unit vector game (U, B) where the best-response polytope Q is cyclic, ex-

not generic,
via Savani-
von Sten-
gel [15]?

ploiting theorems 1 and 2. To do so, we must find a labeling l such that the Gale strings $s \in G(d, d+n)$ encoding the completely labeled facets of the corresponding cyclic d -polytope Q with $d+n$ vertices are exactly the Gale strings of dimension d and length $d+n$ that are completely labeled for l .

Theorem 1 relies on a labeling of the vertices $l_Q : [d+n] \rightarrow [d]$ defined as

- $l_Q(-e_i) = i$ for $i \in [d]$
- $l_Q(c_j) = l_U(j)$ for the vertices $c_j \neq -e_i$, for $j \in [n]$

where $l_U : [n] \rightarrow [d]$ is the the labeling associated to the unit vector game (U, B) . We define the labeling $l : [d+n] \rightarrow [d]$ as follows:

- $l(i) = i$ for $i \in [d]$
- $l(d+j) = l_U(j)$ for $j \in [n]$

The Gale strings $s \in G(d, d+n)$ that are completely labeled for l correspond exactly to the completely labeled facets of Q , with the facet F_0 corresponding to the “trivial” completely labeled string $\mathbf{1}^d 0$.

Then the problem of finding a Nash equilibrium for a unit vector game (U, B) is equivalent to the problem ANOTHER GALE, defined as follows.

ANOTHER GALE

input : A labeling $l : [n] \rightarrow [d]$, where d is even and $d < n$, and an associated completely labeled Gale string s in $G(d, n)$.

output : A completely labeled Gale string s' in $G(d, n)$ associated with l , such that $s' \neq s$.

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