1 Complexity Classes, Games, Polytopes and Gale Strings

1.1 The Complexity Classes P and PPAD

1.2 Normal Form Games and Nash Equilibria

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1.3 Bimatrix Games and Best Response Polytopes

read: unit games article on vS website

Nash equilibria can be analysed from a geometrical point of view, using polytopes.

In the following, we will consider vectors $u, v \in \mathbb{R}^d$ as column vectors, so $u^\top v$ is their scalar product. A vector in $\mathbb{R}^d$ for which all components are 0's will be denoted as $\mathbf{0}$; similarly, a vectors for which all components are 1's will be denoted as $\mathbf{1}$. The i-th component of the *unit vector* e_i is 1, whereas all the other components are 0's. An inequality of the form $u \geq v$ (and analogous) holds for every component; that is, $u_i \geq v_i$ for all $i \in [d]$.

An *affine combination* of points in an Euclidean space $z_1, \ldots, z_n$ is

$$\sum_{i=1}^{n} \lambda_i z_i \quad \text{where } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} \lambda_i = 1$$

If furthermore $\lambda_i \geq 0$ for all $i \in [n]$, the points z_i form a *convex combination*. Note that such λ_i's can be seen as a probability distribution over the z_i's.

The points $z_1, \ldots, z_n$ are *affinely independent* if none of them is an affine combination of the others. The set Z is *convex* if it is closed under forming convex combinations, that is,

$$\bar{z} = \sum_{i=1}^{n} \lambda_i z_i \text{ for some } z_i \in Z \text{ and } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0$$

$$\Rightarrow$$

$$\bar{z} \in Z \quad (1)$$

A convex set has *dimension* d if it has exactly $d+1$ affinely independent points.

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A (d-dimensional) *simplicial polytope* P is the convex hull of a set of at least $d+1$ points v in $\mathbb{R}^d$ in general position, that is, no $d+1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P. A *facet* of P is the convex hull $\operatorname{conv} F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F.

The following theorem, due to Balthasar and von Stengel [**?**, **?**], establishes a connection between general labeled polytopes and equilibria of certain $d \times n$ bimatrix games (U, B).

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Theorem 1. *Consider a labeled d-dimensional simplicial polytope Q with $\mathbf{0}$ in its interior, with vertices $-e_1, \ldots, -e_d, c_1, \ldots, c_n$, so that $F_0 = \operatorname{conv}\{-e_1, \ldots, -e_d\}$ is a facet of Q. Let $-e_i$ have label i for $i \in [d]$, and let c_j have label $l(j) \in [d]$ for $j \in [n]$. Let (U, B) be the $d \times n$ bimatrix game with $U = [e_{l(1)} \cdots e_{l(n)}]$ and $B = [b_1 \cdots b_n]$, where $b_j = c_j/(1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Then the completely labeled facets F of Q, with the exception of F_0, are in one-to-one correspondence to the Nash equilibria (x, y) of the game (U, B) as follows: if v is the normal vector of F, then $x = (v + \mathbf{1})\mathbf{1}^\top(v + \mathbf{1})$, and $x_i = 0$ if* *and only if $-e_i \in F$ for $i \in [d]$; any other label j of F, so that c_j is a vertex of F, represents a pure best reply to x. The mixed strategy y is the uniform distribution on the set of pure best replies to x.*

In the preceding theorem, any simplicial polytope can take the role of Q as long as it has one completely labeled facet F_0. Then an affine transformation, which does not change the incidences of the facets of Q, can be used to map F_0 to the negative unit vectors $-e_1, \ldots, -e_d$ as described, with Q if necessary expanded in the direction $\mathbf{1}$ so that $\mathbf{0}$ is in its interior.

A $d \times n$ bimatrix game (U, B) is a *unit vector game* if all columns of U are unit vectors. For such a game B with $B = [b_1 \cdots b_n]$, the columns b_j for $j \in [n]$ can be obtained from c_j as in Theorem 1 if $b_j > \mathbf{0}$ and $\mathbf{1}^T b_j < 1$. This is always possible via a positive-affine transformation of the payoffs in B, which does not change the game. The unit vectors $e_{l(j)}$ that constitute the columns of U define the labels of the vertices c_j. The corresponding polytope with these vertices is simplicial if the game (U, B) is nondegenerate [**?**], which here means that no mixed strategy x of the row player has more than $|\{i \in [d] \mid x_i > 0\}|$ pure best replies. Any game can be made nondegenerate by a suitable "lexicographic" perturbation of B, which can be implemented symbolically.

Unit vector games encode arbitrary bimatrix games: An $m \times n$ bimatrix game (A, B) with (w.l.o.g.) positive payoff matrices A, B can be symmetrized so that its Nash equilibria are in one-to-correspondence to the symmetric equilibria of the $(m + n) \times (m + n)$ symmetric game (C^T, C) where

$$C = \begin{pmatrix} 0 & B \\ A^\top & 0 \end{pmatrix}.$$

In turn, as shown by McLennan and Tourky [**?**], the symmetric equilibria (x, x) of any symmetric game (C^T, C) are in one-to-one correspondence to the Nash equilibria (x, y) of the "imitation game" (I, C) where I is the identity matrix; the mixed strategy y of the second player is simply the uniform distribution on the set $\{i \mid x_i > 0\}$. Clearly, I is a matrix of unit vectors, so (I, C) is a special unit vector game.

BEARSTEARD Blue

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1.4 Cyclic Polytopes and Gale Strings

1.5 Labeling and the Problem Another Gale

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