

1 Complexity Classes, Games, Polytopes and Gale Strings

1.1 The Complexity Classes P and PPAD

1.2 Normal Form Games and Nash Equilibria

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1.3 Bimatrix Games and Best Response Polytopes

read: unit games article on vS website

Nash equilibria can be analysed from a geometrical point of view, using polytopes.

In the following, we will consider vectors $u, v \in \mathbb{R}^d$ as column vectors, so $u^\top v$ is their scalar product. A vector in \mathbb{R}^d for which all components are 0's will be denoted as **0**; similarly, a vectors for which all components are 1's will be denoted as **1**. The i -th component of the *unit vector* e_i is 1, whereas all the other components are 0's. An inequality of the form $u \geq v$ (and analogous) holds for every component; that is, $u_i \geq v_i$ for all $i \in [d]$.

An *affine combination* of points in an Euclidean space z_1, \dots, z_k is

$$\sum_{i=1}^k \lambda_i z_i \quad \text{where } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^k \lambda_i = 1$$

If furthermore $\lambda_i \geq 0$ for all $i \in [k]$

An affine combination of points z_1, \dots, z_k in some Euclidean space is of the form $\sum_{i=1}^k \lambda_i z_i$ where $\lambda_i \geq 0$ for all i , $\sum_{i=1}^k \lambda_i = 1$. It is called a convex combination if $\lambda_i > 0$ for all i . A set of points is convex if it is closed under forming convex combinations. Given points are affinely independent if none of these points is an affine combination of the others. A convex set has dimension d if and only if it has $d + 1$, but no more, affinely independent points.

A (d -dimensional) *simplicial polytope* P is the convex hull of a set of at least $d + 1$ points v in \mathbb{R}^d in general position, that is, no $d + 1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P . A *facet* of P is the convex hull $\text{conv } F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P ; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F .

The following theorem, due to Balthasar and von Stengel [?, ?], establishes a connection between general labeled polytopes and equilibria of certain $d \times n$ bimatrix games (U, B) .

THIS FROM VvS

Theorem 1. Consider a labeled d -dimensional simplicial polytope Q with $\mathbf{0}$ in its interior, with vertices $-e_1, \dots, -e_d, c_1, \dots, c_n$, so that $F_0 = \text{conv}\{-e_1, \dots, -e_d\}$ is a facet of Q . Let $-e_i$ have label i for $i \in [d]$, and let c_j have label $l(j) \in [d]$ for $j \in [n]$. Let (U, B) be the $d \times n$ bimatrix game with $U = [e_{l(1)} \cdots e_{l(n)}]$ and $B = [b_1 \cdots b_n]$, where $b_j = c_j / (1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Then the completely labeled facets F of Q , with the exception of F_0 , are in one-to-one correspondence to the Nash equilibria (x, y) of the game (U, B) as follows: if v is the normal vector of F , then $x = (v + \mathbf{1})\mathbf{1}^\top(v + \mathbf{1})$, and $x_i = 0$ if and only if $-e_i \in F$ for $i \in [d]$; any other label j of F , so that c_j is a vertex of F , represents a pure best reply to x . The mixed strategy y is the uniform distribution on the set of pure best replies to x .

In the preceding theorem, any simplicial polytope can take the role of Q as long as it has one completely labeled facet F_0 . Then an affine transformation, which does not change the incidences of the facets of Q , can be used to map F_0 to the negative unit vectors $-e_1, \dots, -e_d$ as described, with Q if necessary expanded in the direction $\mathbf{1}$ so that $\mathbf{0}$ is in its interior.

A $d \times n$ bimatrix game (U, B) is a *unit vector game* if all columns of

U are unit vectors. For such a game B with $B = [b_1 \cdots b_n]$, the columns b_j for $j \in [n]$ can be obtained from c_j as in Theorem 1 if $b_j > \mathbf{0}$ and $\mathbf{1}^T b_j < 1$. This is always possible via a positive-affine transformation of the payoffs in B , which does not change the game. The unit vectors $e_{l(j)}$ that constitute the columns of U define the labels of the vertices c_j . The corresponding polytope with these vertices is simplicial if the game (U, B) is nondegenerate [?], which here means that no mixed strategy x of the row player has more than $|\{i \in [d] \mid x_i > 0\}|$ pure best replies. Any game can be made nondegenerate by a suitable “lexicographic” perturbation of B , which can be implemented symbolically.

Unit vector games encode arbitrary bimatrix games: An $m \times n$ bimatrix game (A, B) with (w.l.o.g.) positive payoff matrices A, B can be symmetrized so that its Nash equilibria are in one-to-correspondence to the symmetric equilibria of the $(m + n) \times (m + n)$ symmetric game (C^T, C) where

$$C = \begin{pmatrix} 0 & B \\ A^\top & 0 \end{pmatrix}.$$

In turn, as shown by McLennan and Tourky [?], the symmetric equilibria (x, x) of any symmetric game (C^T, C) are in one-to-one correspondence to the Nash equilibria (x, y) of the “imitation game” (I, C) where I is the identity matrix; the mixed strategy y of the second player is simply the uniform distribution on the set $\{i \mid x_i > 0\}$. Clearly, I is a matrix of unit vectors, so (I, C) is a special unit vector game.

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Nina Watson

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1.4 Cyclic Polytopes and Gale Strings

1.5 Labeling and the Problem ANOTHER GALE

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