1 Complexity, Games, Labels, Polytopes, Strings

1.1 Some Complexity Classes

1.2 Normal Form Games and Nash Equilibria

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1.3 Bimatrix Games, Labels and Polytopes

In the rest of this thesis we will focus on two-player normal-form games, also called *bimatrix games*, since they can be characterized by the payoff matrices by the two players. The Nash equilibria of these games can be analysed from a combinatorial point of view using *labels*. This method is due to Shapley [12], in a study building on ideas introduced in a paper by Lemke and Howson [6].

From here on, we will assume that the payoff matrices (A, B) of both players are non-negative, and that A and $B^\top$ have no zero column. This can be easily obtained without loss of generality via an affine transformation that will not affect the equilibria of the game.

Let (A, B) be bimatrix game. The mixed-strategy simplices of player 1 and 2 are, respectively

$$X = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ \mathbf{1}^\top x = 1\}, \quad Y = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ \mathbf{1}^\top y = 1\} \tag{1}$$

A *labeling* of the game is then given as follows:

1. the m pure strategies of player 1 are identified by $1, \ldots, m$;
2. the n pure strategies of player 2 are identified by $m+1, \ldots, m+n$;
3. each mixed strategy $x \in X$ of player 1 has
    - label i for each $i \in [m]$ such that $x_i = 0$, that is if in x player 1 does not play her ith pure strategy;

- label $m + j$ for each $j \in [n]$ such that the jth pure strategy of player 2 is a best response to x;

4. each mixed strategy $y \in Y$ of player 2 has

   - label $m+j$ for each $j \in [n]$ such that $y_j = 0$, that is if in y player 2 does not play his jth pure strategy;
   - label i for each $i \in [m]$ such that the ith pure strategy of player 1 is a best response to y;

A strategy profile $(x, y) \in X \times Y$ is *completely labeled* if every label $1, \ldots, m + n$ is a label of either x or y. We have the following theorem (Theorem 1 in [12]):

Theorem 1. *Let $(x, y) \in X \times Y$; then (x, y) is a Nash equilibrium of the bimatrix game (A, B) if and only if (x, y) is completely labeled.*

Proof. The mixed strategy $x \in X$ has label $m + j$ for some $j \in [n]$ if and only if the jth pure strategy of player 2 is a best response to x; this, in turn, is a necessary and sufficient condition for player 2 to play his jth strategy at an equilibrium against x. Therefore, at an equilibrium (x, y) all labels $m + 1, \ldots, m + n$ will appear either as labels of x or of y. The analogous holds for the labels $i \in [n]$. $\square$

An useful geometrical representation of labels can be given on the mixed strategies simplices X and Y. The outside of each simplex is labeled according to the player's own pure strategies that are *not* played; so, for instance, the outside of X will have labels $1, \ldots, n$. The interior of each simplex is subdivided in closed polyhedral sets, called *best-response regions.* These are labeled according to the other player's pure strategy that is a best response in that set; so, for instance, the inside of X will have labels $m+1, \ldots, m+n$.

We give an example of this construction.

Example 1.1. page 3–4 of Savani, von Stengel, Unit Vector Games. With graphics.

We will now give a description of labeling on polytopes equivalent to the construction based on best-response regions.

We begin by noticing that the best-response regions can be obtained as projections on X and Y of the *best-response facets* of the polyhedra

$$\bar{P} = \{(x, v) \in X \times \mathbb{R} | B^\top x \le \mathbf{1} v\}, \quad \bar{Q} = \{(y, u) \in Y \times \mathbb{R} | Ay \le \mathbf{1} u\}. \quad (2)$$

These facets are defined as the points (x, v) such that $b_i^\top x = v$ in $\bar{P}$; these correspond to the strategies $x \in X$ of player 1 that give exactly payoff v to player 2. Analogously, in $\bar{Q}$, the facets are the points (y, u) such that $a_j y = u$.

labels on these facets - that then are on projection = brreg

Example 1.2. cont of ex above, page 4–5

Given the assumptions on non-negativity of A and $B^\top$, we can give a change coordinates to x_i/v and y_j/u and replace $\bar{P}$ and $\bar{Q}$ with the *best-response polytopes*

$$P = \{x \in \mathbb{R}^m | x \ge \mathbf{0} B^\top x \le \mathbf{1}\}, \quad Q = \{y \in \mathbb{R}^n | y \ge \mathbf{0} Ay \le \mathbf{1}\}, \quad (3)$$

labels on facets, ex cont'd

compl label + 0 = equil

compl and orth conditions

symmetric games; vice versa, imitation games; ex pg 8

nondegeneracy; ex pg 9; odd no eq, mention homotopy method (find ref) (tie with Nash, again?)

1.4 Some Geometrical Notation

so results in labels section - after this - don't get lost in boredom, and each section in background is about 150-200 lines (see: getting lost). Maybe turn this into an appendix? It would make sense if something more about proof of Nash

We denote the transpose of a matrix A as $A^\top$. We consider vectors $u, v \in \mathbb{R}^d$ as column vectors, so $u^\top v$ is their scalar product. A vector in $\mathbb{R}^d$ for which all components are 0's will be denoted as $\mathbf{0}$; similarly, a vectors for which all components are 1's will be denoted as $\mathbf{1}$. The *unit vector* e_i is the vector that has i-th component $e_{ii} = 1$ and $e_{ij} = 0$ for all other components. When writing an inequality of the form $u \geq v$ (and analogous), we mean that it holds for every component; that is, $u_i \geq v_i$ for all $i \in [d]$.

An *affine combination* of points in an Euclidean space $z_1, \ldots, z_n$ is

$$\sum_{i=1}^{n} \lambda_i z_i \quad \text{where } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} \lambda_i = 1$$

The points $z_1, \ldots, z_n$ are *affinely independent* if none of them is an affine combination of the others.

A *convex combination* of points $z_1, \ldots, z_n$ is an affine combination where $\lambda_i \geq 0$ for all $i \in [n]$. Note that such λ_i's can be seen as a probability distribution over the z_i's.

def simplex: here?

A set of point Z is *convex* if it is closed under forming convex combinations, that is, if $\bar{z} = \sum_{i=1}^{n} \lambda_i z_i$, where $z_i \in Z$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, then $\bar{z} \in Z$. A convex set has *dimension* d if it has exactly $d + 1$ affinely independent points.

def simplex: here?

convex hull; pow hyperplanes; polyhedron, polyopes; needed? yes, for cyclic poly!

from here: notes - copy-paste

A (d-dimensional) *simplicial polytope* P is the convex hull of a set of at least $d+1$ points v in $\mathbb{R}^d$ in general position, that is, no $d+1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P. A *facet* of P is the convex hull $\operatorname{conv} F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F.

this from VvS

The following theorem, due to Balthasar and von Stengel [**?**, **?**], establishes a connection between general labeled polytopes and equilibria of certain $d \times n$ bimatrix games (U, B).

Theorem 2. *Consider a labeled d-dimensional simplicial polytope Q with $\mathbf{0}$ in its interior, with vertices $-e_1, \ldots, -e_d, c_1, \ldots, c_n$, so that $F_0 = \operatorname{conv}\{-e_1, \ldots, -e_d\}$ is a facet of Q. Let $-e_i$ have label i for $i \in [d]$, and let c_j have label $l(j) \in [d]$ for $j \in [n]$. Let (U, B) be the $d \times n$ bimatrix game with $U = [e_{l(1)} \cdots e_{l(n)}]$ and $B = [b_1 \cdots b_n]$, where $b_j = c_j/(1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Then the completely labeled facets F of Q, with the exception of F_0, are in one-to-one correspondence to the Nash equilibria (x, y) of the game (U, B) as follows: if v is the normal vector of F, then $x = (v + \mathbf{1})\mathbf{1}^\top(v + \mathbf{1})$, and $x_i = 0$ if and only if $-e_i \in F$ for $i \in [d]$; any other label j of F, so that c_j is a vertex of F, represents a pure best reply to x. The mixed strategy y is the uniform distribution on the set of pure best replies to x.*

In the preceding theorem, any simplicial polytope can take the role of Q as long as it has one completely labeled facet F_0. Then an affine transformation, which does not change the incidences of the facets of Q, can be used to map F_0 to the negative unit vectors $-e_1, \ldots, -e_d$ as described, with Q if necessary expanded in the direction $\mathbf{1}$ so that $\mathbf{0}$ is in its interior.

A $d \times n$ bimatrix game (U, B) is a *unit vector game* if all columns of U are unit vectors. For such a game B with $B = [b_1 \cdots b_n]$, the columns b_j for $j \in [n]$ can be obtained from c_j as in Theorem 2 if $b_j > \mathbf{0}$ and $\mathbf{1}^T b_j < 1$. This is always possible via a positive-affine transformation of the payoffs in B, which does not change the game. The unit vectors $e_{l(j)}$ that constitute the columns of U define the labels of the vertices c_j. The corresponding polytope with these vertices is simplicial if the game (U, B) is nondegenerate [**?**], which here means that no mixed strategy x of the row player has more than $|\{i \in [d] \mid x_i > 0\}|$ pure best replies. Any game can be made nondegenerate by a suitable “lexicographic” perturbation of B, which can be implemented symbolically.

Unit vector games encode arbitrary bimatrix games: An $m \times n$ bimatrix game (A, B) with (w.l.o.g.) positive payoff matrices A, B can be symmetrized so that its Nash equilibria are in one-to-correspondence to the symmetric equilibria of the $(m+n) \times (m+n)$ symmetric game (C^T, C) where

$$C = \begin{pmatrix} 0 & B \\ A^\top & 0 \end{pmatrix}.$$

In turn, as shown by McLennan and Tourky [**?**], the symmetric equilibria (x, x) of any symmetric game (C^T, C) are in one-to-one correspondence to the Nash equilibria (x, y) of the "imitation game" (I, C) where I is the identity matrix; the mixed strategy y of the second player is simply the uniform distribution on the set $\{i \mid x_i > 0\}$. Clearly, I is a matrix of unit vectors, so (I, C) is a special unit vector game.

until here

1.5 Cyclic Polytopes and Gale Strings

file: gale-def

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