Complexity of the Gale String Problem for Equilibrium Computation in Games

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Declaration

I certify that chapter 2 of this thesis I have presented for examination for the MPhil degree of the London School of Economics and Political Science is based on joint work with Julian Merschen and Bernhard von Stengel, published in [2]. The appendix was the result of previous study for the Master of Science degree I undertook at the London School of Economics and Political Science in 2008, see [3].

Abstract

This thesis presents a report on original research, published as joint work with Merschen and von Stengel in *Electronic Notes in Discrete Mathematics* [2]. Our result shows a polynomial time algorithm to solve two problems related to labeled Gale strings, a combinatorial structure consisting a string of labels and a bitstring satisfying certain conditions introduced by Gale in [7].

Gale strings can be used in the representation of a particular class of games that Savani and von Stengel [17] used as an example of exponential running time for the classical Lemke-Howson algorithm to find a Nash equilibrium of a bimatrix game [9]. It was conjectured that solving these games via the Lemke-Howson algorithm was complete in the class **PPAD** (Proof by Parity Argument, Directed version). A major motivation for the definition of this class by Papadimitriou [16] was, in turn, to capture the pivoting technique of many results related to the Nash equilibrium, including the Lemke-Howson algorithm.

Our result, on the contrary, sets apart this class of games as a case for which there is a polynomial-time algorithm to find a Nash equilibrium. Since Daskalakis, Goldberg and Papaditrimiou [5] and Chen and Deng [4] proved the **PPAD**-completeness of finding a Nash equilibrium in general normal-form games, we have a special class of games, unless **PPAD** = **P**.

Our proof exploits two results. The first one is the representation of the Nash equilibria of these games as Gale strings, as seen in Savani and von Stengel [17]. The second one is the polynomial-time solvability of the problem of finding a perfect matching in a graph, proven by Edmonds [6].

Merschen [12] and Végh and von Stengel [20] expanded our technique to prove further interesting results.

An appendix relates an amendment to the proof of the **PPAD**-completeness result by Daskalakis, Goldberg and Papaditrimiou [5].

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Introduction

The topic of this thesis is a problem in the field of *algorithmic game theory*, that is, the study of game-theoretic problems from the point of view of computer science. In particular, we focus on the computational complexity of a particular class of games. These

General refs for comp compl [15]

General refs for geometry [23]

Chapter 1

Complexity, Games, Labels and Gale Strings

1.1 Some Complexity Classes

We start by recalling some standard definitions of computational complexity theory; we then move on to the more recent classes **TFNP** and **PPAD**, first introduced in [11] and [16] respectively. The latter, in particular, is a key concept in the study of the problems that are the focus of this thesis.

A *computational problem* is given by the combination of an *input* and a related *output*. A specific input gives an *instance* of the problem. Computational problems can be classified according to the form of their output. A *function problem* P returns for an intance x an output y that satisfies a given binary relation $R(x, y)$. In the case of a *decision problems*, y answers a "Yes / No" question. The *complement* of a decision problem P is the problem $\bar{P}$ that returns "No" for each instance of P that returns "Yes", and vice versa. *Search problems* are function problems that return either an output y such that $R(x, y)$, or "No" if it's not possible to find any such y. If y is guaranteed to exist, the problem is called a *total function problem.*

An example of decision problem is: "(input) given a graph, (question) is it possible to find an Euler tour of the graph?" Its complement is "(input) given a graph, (question) is it possible that there isn't any Euler of the graph?" A search problem is: "(input) given a graph, (output) return one Euler tour of the graph, or "NO" if no such tour exists." A total function problem is: "(input) given an Euler graph, (output) return one of its Euler tours."

Computational problems are also classified according to their *computational complexity*

, given by the *reducibility* from each other.

deterministic Turing machines: here (we use only deterministic ones)

Let P_1 be a computational problem, such that its instance x is encoded by $|x|$ bits. P_1 *reduces to the problem* P_2 *in polynomial time*, denoted $P_1 \leq_P P_2$, if there exists a *polynomial-time reduction*, that is, a function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ and a Turing machine $\mathcal{M}$ such that for all $x \in \{0,1\}^*$

1. $x \in P_1 \iff f(x) \in P_2$;
2. $\mathcal{M}$ computes $f(x)$;
3. $\mathcal{M}$ stops after $p(|x|)$ steps, where p is a polynomial.

Intuitively, if P_1 is polynomial-time reducible to P_2, it takes polynomial time to "translate" P_1 to P_2, and then to "translate back" a solution of P_2 as a solution of P_1.

For any class C of decision problems, the class of all complements of the problems in C is the *complement class* $\mathrm{co} - \mathrm{C}$. A problem P is *hard* for a class C if for every problem P_C in C there is a polynomial-time reduction to P; that is, if P is hard to solve at least as every problem in C. A $\mathrm{C} - hard$ problem in C is *complete* for C.

The complexity class **P** contains all the *polynomially decidable problems*, that is, all problems P such that there exists a Turing machine $\mathcal{M}$ that outputs either "YES" or "NO" for all inputs $x \in \{0,1\}^*$ of P after $p(|x|)$ steps, where

p is a polynomial. Intuitively, a decision problem is in **P** if the answer to its question can be found in a number of steps that is polynomial in the input of the problem.

A problem P belongs to the class **NP**, *non-deterministic polynomial-time problems*, if there exists a Turing machine $\mathcal{M}$ and polynomials p_1, p_2 such that

1. for all $x \in P$ there exists a *certificate* $y \in \{0,1\}^*$ which satisfies $|y| \leq p_1(|x|)$;
2. $\mathcal{M}$ accepts the combined input xy, stopping after at most $p_2(|x| + |y|)$ steps;
3. for all $x \notin P$ there does not exist $y \in \{0,1\}^*$ such that $\mathcal{M}$ accepts the combined input xy.

This means that a decision problem is in **NP** if it takes polynomial time to verify whether the "certificate solution" y is, indeed, a correct answer to the question posed by the problem. The class **#P** is the class of all problems that output the number of possible certificates for a problem in **NP**.

check formal def of # P (?)

In [11], Megiddo and Papadimitriou introduce the classes **FNP**, *function non-deterministic polynomial*, and **TFNP**, *total function non-deterministic polynomial*. The former is defined as the class of binary relations $R(x, y)$ such that there is a polynomial-time algorithm that decides whether $R(x, y)$ holds for given x, y satisfying $|y| \leq p(|x|)$, where p is a polynomial. The latter is the class of all such problems for which y is guaranteed to exist. Intuitively, **FNP** and **TFNP** are similar to **NP**, but they allow for problems of (respectively) function and total function form.

In [11], Megiddo and Papadimitriou also prove that, unless $\mathbf{NP} = \mathbf{co-NP}$, **TFNP** is a *semantic* class, that is, a class without complete problems. To circumvent this limitation of **TFNP**, Papadimitriou ([16]) focused on the problems for which the existence of a solution is proved by a "parity argument", introducing the classes **PPA** (*Proof by Parity Argument*) and **PPAD** (*Proof by Parity Argument, Directed version*).

The formal definition of **PPA** and **PPAD**,

definition of PPA(D): one in Papadimitriou 1994 and one in DGP the second w END OF THE LINE, use that one.

as an example, SPERNER (look at Papadimitriou 1994); will reconnect to NASH in next subsection - BROUWER mention, maybe

1.2 Normal Form Games and Nash Equilibria

We now give the game-theoretic background that will be used in this thesis. A *game*, as first defined by von Neumann in [21], is a model of strategic interaction. A *finite normal form game* is $\Gamma = (P, S = \times_{p \in P} S_p, u = \times_{p \in P} u^p)$ where both the set of *players* P and the sets of *pure strategies* S_p (and therefore the set of *pure strategy profiles* S) are finite. We will use the notation $S_{-p} = \times_{q \neq p} S_p$. The purpose of each player $p \in P$ is to maximize her *payoff function* $u^p : S \to \mathbb{R}$. In the following pages, by "game" we will always mean "finite normal form game." If there are only two players, we will refer to player 1 using feminine pronouns and to player 2 using masculine ones; such games are called *bimatrix games* since they can be characterized by the $m \times n$ payoff matrices A and B, where a_{ij} and b_{ij} are the payoffs of respectively player 1 and of player 2 when the former plays her ith pure strategy and the latter plays his jth pure strategy. A bimatrix game is *zero-sum* if $B = -A$, and *symmetric* if $B = A^\top$.

A *mixed strategy* of player p is a probability distribution on S_P; it can be described as a point $x = (x_1^p, \ldots, x_{|S_p|}^p)$ on the $(|S_p| - 1)$-dimensional *mixed strategy simplex* $\Delta_p = \{x \in \mathbb{R}^{|S_p|} | x \geq \mathbf{0},\ \mathbf{1}^\top x = 1\}$; the set of *mixed strategy profiles* will be the simplicial polytope $\Delta = \times_{p \in P} \Delta_p$. We extend the payoff functions to $u^p : \Delta \to \mathbb{R}$ by linearity.

A *Nash equilibrium* of a game is a strategy profile in which each player cannot improve her expected payoff by unilaterally changing her strategy. Such a strategy is called a *best response*; a strategy that is not a best response is called *dominated*. Formally: for $s \in S_{-p}$ let $x_s = \prod_{q \neq p} x^q_{s_q}$; then a Nash equilibrium is a strategy profile x such that for every $p \in P$ and every $\sigma, \tau \in S_p$

$$\sum_{s \in S_{-p}} u^p(\sigma, s) x_s > \sum_{s \in S_{-p}} u^p(\tau, s) x_s \quad \Rightarrow \quad x^p_\tau = 0 \tag{1.1}$$

Note that applying an affine transformation to all the payoffs does not change the Nash equilibria of the game. Note also that there might be more than one equilibrium. The existence of a Nash equilibrium is guaranteed by the fundamental theorem by Nash ([14]).

Theorem 1. (Nash [14]) *Every finite game in normal form has a Nash equilibrium.*

We give three classic examples of game: matching pennies, the prisoners' dilemma and coordination.

Example 1.1. In the non-symmetric zero-sum game *matching pennies*, both players have payoff zero unless they play the same strategy. In this case, player 2 (the *evader*) pays a sum to payoff 1 (the *pursuer*).

pursuer \ evader	up	down
up	-1 1	0 0
down	0 0	-1 1

At the unique equilibrium of the game, each player follows the uniform distribution over their strategies.

In the symmetric non zero-sum *prisoners' dilemma*, each player must decide whether to "help" the other one or to "betray" them. If both players help each other, they will get a small reward; if both betray, they will pay a small penalty; if one betrays and the other cooperate the former will get a large reward and the latter will pay a large penalty.

1 \ 2	betray	help
betray	10 -2	-8 -2
help	5 -8	5 10

The only equilibrium is the profile in which both players betray. Assume that player 1 helps: then she must switch to betrayal, since she would get 10 instead of 5 if player 2 helps and -2 instead of -8 if player 2 betrays. The same applies to player 2, so both players will betray. Note that the payoff matrices can be rewritten as $A = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right)$ and $B = A^\top$.

As an example of game with more than one equilibrium is *coordination*: both players drive on a mountain road; they lose if drive on the same side of the road and win if they avoid each other, regardless of which side they take.

1 \ 2	mountain	valley
mountain	1 0	1 0
valley	0 1	0 1

Both (mountain,road) and (road,mountain) are Nash equilibria.

Consider the problem n-Nash, as follows.

---

n-Nash

---

input : A n-player game Γ.

output: A Nash equilibrium of Γ.

By theorem 1, n-Nash is a total function problem; Megiddo and Papadimitriou ([11]) proved that it is in **TFNP**. Daskalakis, Goldberg and Papadimitriou [5] and Chen and Deng [4] have proven its **PPAD**-completeness, the former for $n \geq 3$ and the latter for $n \geq 2$.

Theorem 2. (Daskalakis, Goldberg and Papadimitriou [5]; Chen and Deng [4]) *For $n \geq 2$, the problem n-*NASH *is* **PPAD**-*complete.*

1.3 Bimatrix Games and Labels

In the rest of this thesis we will focus bimatrix games. We will assume that the payoff matrices (A, B) are non-negative, and that neither A nor $B^\top$ has a zero column, if necessary applying an affine transformation, which does not affect the equilibria of the game.

The Nash equilibria of bimatrix games can be analysed from a combinatorial point of view using *labels*. This method is due to Shapley [19], in a study building on ideas introduced in a paper by Lemke and Howson [9]. Let $n, m \in \mathbb{N}$ with $m \leq n$, and consider a set X with $|X| = n$. A *labeling* of X is a function $l : X \to [m]$. An m-uple $x = (x_1, \dots, x_m) \in X^m$ is *completely labeled* if $\{i \in [m] \mid l(x_j) = i \text{ for some } j \in [m]\} = [m]$, that is, if each label $j \in [m]$ appears once and only once in $(l(x_1), \dots, l(x_m))$.

Let (A, B) be bimatrix game, and let X and Y be the mixed strategy simplices of respectively player 1 and 2; that is

$$X = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ \mathbf{1}^\top x = 1\}; \quad Y = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0},\ \mathbf{1}^\top y = 1\}. \quad (1.2)$$

A *labeling* of the game is then given as follows:

1. the m pure strategies of player 1 are identified by $1, \dots, m$;
2. the n pure strategies of player 2 are identified by $m + 1, \dots, m + n$;
3. each mixed strategy $x \in X$ of player 1 has
    - label i for each $i \in [m]$ such that $x_i = 0$, that is if in x player 1 does not play her ith pure strategy,
    - label $m+j$ for each $j \in [n]$ such that the jth pure strategy of player 2 is a best response to x;

4. each mixed strategy $y \in Y$ of player 2 has

   - label $m + j$ for each $j \in [n]$ such that $y_j = 0$, that is if in y player 2 does not play his jth pure strategy,
   - label i for each $i \in [m]$ such that the ith pure strategy of player 1 is a best response to y.

The labeling of mixed strategy profiles can be used to characterize the Nash equilibria of the game.

Theorem 3. (Shapley [19]) *Let $(x, y) \in X \times Y$; then (x, y) is a Nash equilibrium of the bimatrix game (A, B) if and only if (x, y) is completely labeled.*

Proof. The mixed strategy $x \in X$ has label $m + j$ for some $j \in [n]$ if and only if the jth pure strategy of player 2 is a best response to x; this, in turn, is a necessary and sufficient condition for player 2 to play his jth strategy at an equilibrium against x. Therefore, at an equilibrium (x, y) all labels $m+1, \ldots, m+n$ will appear either as labels of x or of y. The analogous holds for the labels $i \in [n]$. $\square$

An useful graphical representation of labels on the simplices X and Y. is done by labeling the outside of each simplex according to the player's own pure strategies that are *not* played, and by subdividing its interior in closed polyhedral sets corresponding to the other player's best response, called *best response regions*. We give an example of this construction.

Example 1.2. Consider the 3×3 game (A, B) with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{1.3}$$

The pure strategies of player 1 are labeled as $1, 2, 3$; the pure strategies of player 2 are labeled as $4, 5, 6$.

In the following figures the labels of the strategies will be represented as circled numbers. Figure 1.1 shows X and Y: the exterior is labeled with the player's own pure strategies where these are not played, so the facet labeled with a pure strategy is opposite to the vertex where only that pure strategy is played; the interior is covered by the best response regions, with labels corresponding to the other player's pure strategies that are in the best response to the player's own strategy in the region. For example, the best-response region in Y with label 1 is the set of those (y_1, y_2, y_3) so that $y_1 \geq y_2$ and $y_1 \geq y_3$. There is only one pair (x, y) that is completely labeled, namely $x = (\frac{1}{3}, \frac{2}{3}, 0)$ with labels $3, 4, 5$ and $y = (\frac{1}{2}, \frac{1}{2}, 0)$ with labels $1, 2, 6$, so this is the only Nash equilibrium of the game.

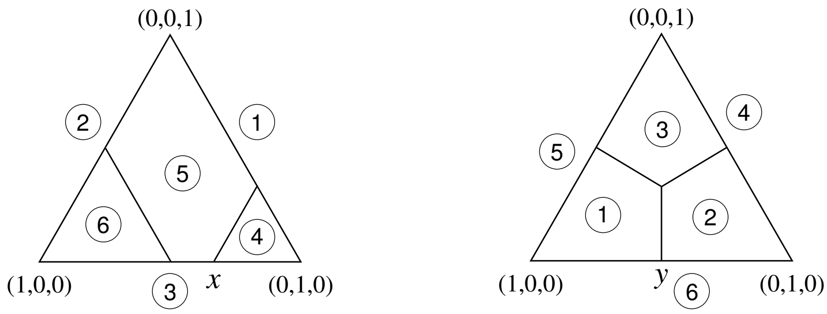

Figure 1.1 Labeled mixed strategy sets X and Y for the game (1.3).

The point of view of best response regions can be translated to an equivalent construction of *best response polytopes*. We begin by noticing that the best-response regions can be obtained as projections on X and Y of the *best-response facets* of the polyhedra

$$\bar{P} = \{(x, v) \in X \times \mathbb{R} \mid B^\top x \leq \mathbf{1}v\}; \quad \bar{Q} = \{(y, u) \in Y \times \mathbb{R} \mid Ay \leq \mathbf{1}u\}. \quad (1.4)$$

These facets in $\bar{P}$ are defined as the points $(x, v) \in X \times \mathbb{R}$ such that $(B^\top x)_j = v$, which in turn represent the strategies $x \in X$ of player 1 that give exactly payoff v to player 2 when he plays strategy j; the projection of the facet defined by $(B^\top x)_j = v$ to X has then label j. Analogously, the facet of

$\bar{Q}$ given by the points $(y, u) \in Y \times \mathbb{R}$ such that $(Ay)_i = u$ will project to the best-response region of Y with label i.

Example 1.3. In the example 1.2, the inequalities $B^\top x \leq \mathbf{1}v$ state $3x_2 \leq v$, $2x_1 + 2x_2 + 2x_3 \leq v$, $4x_1 \leq v$. Figure 1.2 shows the best-response facets of $\bar{P}$ and their projection to X by ignoring the payoff variable v, which gives the subdivision of X into best-response regions Figure 1.1.

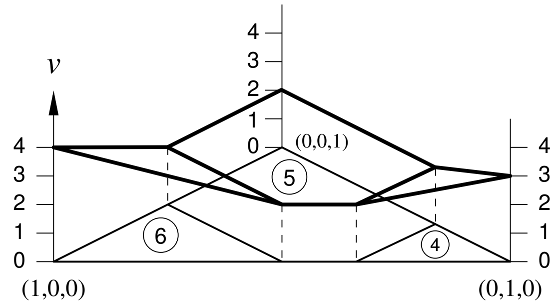

Figure 1.2 Best response facets of the polyhedron $\bar{P}$ in (1.4).

Given the assumptions on non-negativity of A and $B^\top$, we can give a change coordinates to x_i/v and y_j/u and replace $\bar{P}$ and $\bar{Q}$ with the *best-response polytopes*

$$P = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}; \quad Q = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}. \quad (1.5)$$

The polytope P is the intersection of $n + m$ half spaces, one for each inequality corresponding to either player 1 avoiding her i-th pure strategy, where $i \in [m]$, or to a best response of player 2 that gives non-zero probability to his j-th strategy, where $j \in [n]$. Formally, a point $x \in P$ has label k if and only if either $x_k = 0$ for $k \in [m]$ or $(B^\top x)_k = 0$ for $k \in [n]$. Analogously, a point in Q has label k if and only if either $y_k = 0$ for $k \in [n]$ or $(Ay)_k$ for $k \in [m]$. Then a point $(x, y) \in P \times Q$ is completely labeled if and only if it satisfies the *complementarity condition*

$$x_i = 0 \text{ or } (Ay)_i = 1 \text{ for all } i \in [m];$$
$$y_j = 0 \text{ or } (B^\top x)_j \text{ for all } j \in [n]. \quad (1.6)$$

Then either he point corresponding to (x, y) in $\bar{P} \times \bar{Q}$ is a Nash equilibrium, or $(x, y) = (\mathbf{0}, \mathbf{0})$; we will refer to the latter case as *artificial equilibrium.*

Example 1.4. Keeping on with example 1.2, the polyhedron $\bar{P}$ of Figure 1.2 becomes the polytope in Figure 1.3 1.3.

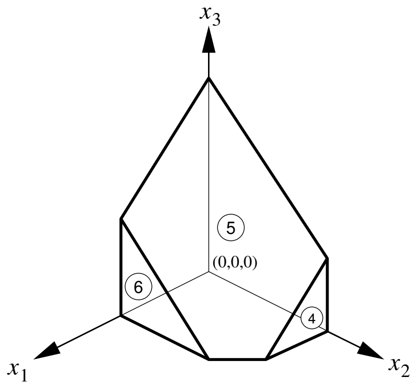

Figure 1.3 Best response polytope P in (1.5) for the game in (1.3).

Note that the vertex $(\mathbf{0}, \mathbf{0})$ has labels $1, 2, 3$ in P and, analogously, it has labels $4, 5, 6$ in Q, so it is completely labeled, and corresponds to the "artificial" equilibrium.

We will now consider a particular class of games, called *unit vector games.* These are games of the form (U, B), where the columns of the matrix U are unit vectors. We will see that a method that "solves" a unit vector game, in the sense of finding one equilibrium, or all equilibria, of the game, can be used to solve any arbitrary bimatrix game. First of all, we note that any bimatrix game can be "symmetrized." The result is due to Gale, Kuhn and Tucker [8] for zero-sum games; its extension to non-zero-sum games is a folklore result.

Proposition 1. *Let (A, B) be a bimatrix game and (x, y) be one of its Nash equilibria. Then (z, z), where $z = (x, y)$, is a Nash equilibrium of the symmetric game $(C, C^\top)$, where*

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}. \tag{1.7}$$

McLennan and Tourky [10] have proven a converse of proposition 1 for bimatrix games of the form (I, B), called *imitation games*.

Theorem 4. (McLennan and Tourky [10]) *The pair (x, x) is a symmetric Nash equilibrium of the symmetric bimatrix game $(C, C^\top)$ if and only if there is some y such that (x, y) is a Nash equilibrium of the imitation game (I, B) with $B = C^\top$.*

By theorem 4, the mixed strategy x of player 1 in any Nash equilibrium of the imitation game (I, B) corresponds exactly to the symmetric equilibrium (x, x) of the symmetric game defined by the payoff matrix of the other player. Therefore, an algorithm that finds a Nash equilibrium of a bimatrix game can be used to find a symmetric Nash equilibrium of a symmetric game.

Example 1.5. As an example, consider the symmetric game $(C, C^\top)$ with

$$C = \begin{pmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \qquad C^\top = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \tag{1.8}$$

so that $C^\top = B$ in (1.3), and the corresponding imitation game $(I, C^\top)$ is (A, B) in (1.3).

Figure 1.4 shows the labeled mixed-strategy simplices X and Y for the game 1.8; since the game is symmetric, only the labels are different. In addition to the symmetric equilibrium (x, x) where $x = (\frac{1}{3}, \frac{2}{3}, 0)$, the game has two non-symmetric equilibria in (a, b) and (b, a) with $a = (\frac{1}{2}, \frac{1}{2}, 0)$ and $b = (0, \frac{2}{3}, \frac{1}{3})$.

Note that the corresponding imitation game (A, B) has only one equilibrium in (x, y), where (x, x) is the symmetric equilibrium of $(C, C^\top)$. This shows how theorem 4 applies to the *symmetric* equilibria of the symmetric game, but not to all its equilibria; there could be non-symmetric equilibria of $(C, C^\top)$ that are not found through the imitation game.

The characterization of Nash equilibria as completely labeled pairs (x, y) holds for arbitrary bimatrix games. From now on, we will also assume that

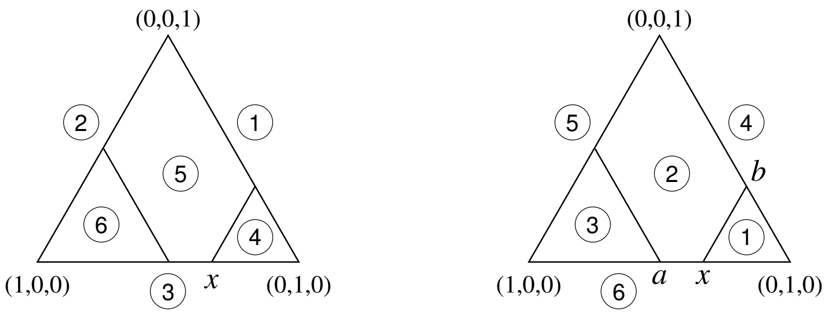

Figure 1.4 Best response regions for the symmetric game $(C, C^\top)$ in (1.8).

the games are *nondegenerate*, that is, no point in P has more than m labels, and no point in Q has more than n labels. If (x, y) is an equilibrium of a nondegenerate game, it is completely labeled and each label appears exactly once either as a label of x or as a label of y. Therefore, nondegeneracy is equivalent to the condition that the number of pure best responses against a mixed strategy is never larger than the size of the support of that mixed strategy. In turn, this implies that no point of P lies on more than m facets and no point of Q lies on more than n facets, so the best response polytopes P, Q of a nondegenerate game are simple.

"lexicographic" perurbation makes it non-degenerate (see AB).

Example 1.6. An example of degenerate game is given by $(C, C^\top)$ where

$$C = \begin{pmatrix} 0 & 4 & 0 \\ 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}, \qquad C^\top = \begin{pmatrix} 0 & 2 & 4 \\ 4 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{1.9}$$

As it is shown in Figure 1.5, the game $(C, C^\top)$ is degenerate because the mixed strategy $x = (\frac{1}{2}, \frac{1}{2}, 0)$, that also defines the unique symmetric equilibrium (x, x) of the game, has three pure best responses.

Note that the corresponding equilibria (x, y) of the imitation game $(I, C^\top)$ are not unique, because due to degeneracy any convex combination of $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ can be chosen for y, as shown in Figure 1.6.

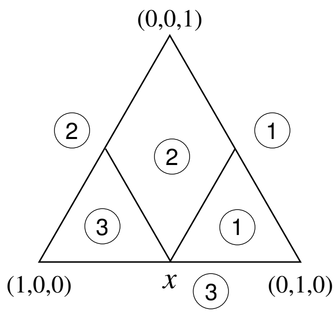

Figure 1.5 Best-response regions of the degenerate symmetric game (1.9) with a unique symmetric equilibrium.

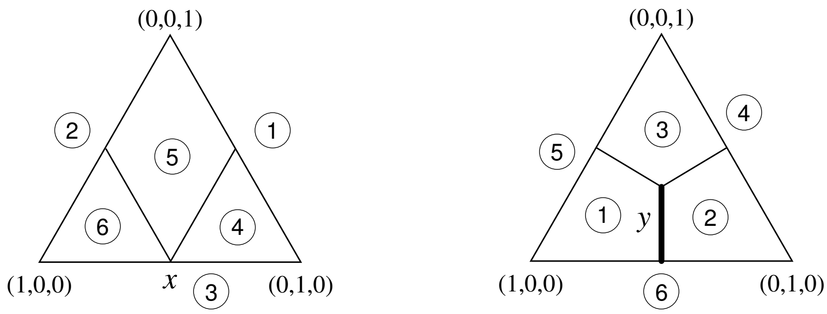

Figure 1.6 Labeled mixed-strategy sets for the imitation game $(I, C^\top)$ for the degenerate symmetric game (1.9) where the equilibria (x, y) are not unique.

This shows how the reduction between symmetric equilibria (x, x) of a symmetric game and Nash equilibria (x, y) of the corresponding imitation game as in theorem 4 does not preserve uniqueness if the game is degenerate.

nondeg: equilibria are on vertices, not simply points

The correspondence between equilibria of a unit vector game and completely labeled points of a polytope is given by the following theorem, first proved in dual form by Balthasar [1] and given here as in Savani and von Stengel [18].

also in thm: vertices, if non-deg

Theorem 5. (Savani and von Stengel [18]) *Let* $l : [n] \to [m]$, *and let* (U, B) *be the unit vector game where* $U = (e_{l(1)} \ \cdots \ e_{l(n)})$. *Let* $N_i = \{j \in [n] \mid l(j) = i\}$

for $i \in [m]$, and consider the polytopes P^l and Q^l

$$P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}; \tag{1.10}$$

$$Q^l = \{y \in \mathbb{R}^n \mid y \geq \mathbf{0},\ \sum_{\substack{j \in N_i \\ i \in [m]}} y_j \leq 1\}. \tag{1.11}$$

Let l_f be the labeling of the facets of P^l defined as follows:

$$x_i \geq 0 \text{ has label } i \text{ for } i \in [m];$$

$$(B^\top x)_j \leq 1 \text{ has label } l(j) \text{ for } j \in [n]. \tag{1.12}$$

Then $x \in P^l$ is a completely labeled point of $P^l \setminus \{\mathbf{0}\}$ if and only if there is some $y \in Q^l$ such that, after scaling, the pair (x, y) is a Nash equilibrium of (U, B) vertex

Proof. Let P, Q be the polytopes associated to the game (U, B) as before. check

Let $(x, y) \in P \times Q \setminus \{\mathbf{0}, \mathbf{0}\}$ be a Nash equilibrium of (U, B), therefore completely labeled in $[m+n]$. Then, if $x_i = 0$, then x has label $i \in m$. If $x_i > 0$ instead, then y has label i, therefore $(Uy)_i = 1$, therefore for some $j \in [n]$ we have $y_j > 0$ and $U_j = e_i$, so $l(j) = i$. Since $y_j > 0$ and (x, y) is completely labeled, $x \in P$ has label $m + j$, that is, $(B^\top x)_j = 1$, therefore $x \in P^l$ has label $l(j) = i$. Hence, x is a completely labeled point of P^l.

Conversely, let $x \in P^l \setminus \{\mathbf{0}\}$ be completely labeled. If $x_i > 0$, then there is $j \in [m]$ such that $(B^\top x) = j$ and $l(j) = i$, that is, $j \in N_i$. For all i such that $x_i > 0$, define y as follows: $y_h = 0$ for all $h \in N_i \setminus \{j\}$, $y_j = 1$. Then $(x, y) \in P \times Q$ is completely labeled. □

Example 1.7. The game in example 1.2 is a unit vector game, with $l(i) = i$. In figure 1.3 we have the labels $4, 5, 6$ on the best response facets of polytope P; the facets with labels $1, 2, 3$, that is, the facets where $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, respectively, are at back right, back left, and bottom of the polytope. In the polytope P^l the labels $4, 5, 6$ are replaced by $1, 2, 3$ because the corresponding columns of A are the unit vectors e_1, e_2, e_3. Figure 1.7 shows the polytope P^l;

the only hidden facet, with label 1, is at the bottom. Apart from the origin $\mathbf{0}$, the only completely labeled point of P^l is x, as shown in figure 1.7.

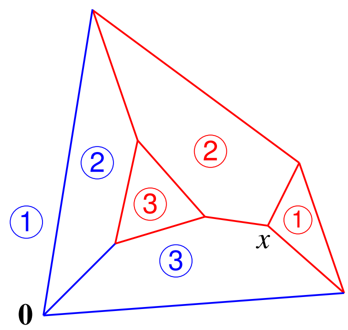

Figure 1.7 The polytope P^l for the unit vector game (1.3). The hidden facet at the back has label 1.

Theorem 5 gives a correspondence between the completely labeled vertices of the polytope P^l and the equilibria of the unit vector game (U, B), with an "artificial" equilibrium corresponding to the vertex $\mathbf{0}$. We will now construct its dual version given in [1]. We traslate the polytope P^l as in 1.10 to $P = \{x - \mathbf{1} \mid x \in P^l\}$, and multiply all payoffs in B by a constant, if necessary, so that $\mathbf{0}$ is in the interior of P. We have that

$$\begin{aligned} P &= \{x + \mathbf{1} \geq \mathbf{0},\ (x + \mathbf{1})^\top B \leq \mathbf{1}\} = \\ &= \{x \in \mathbb{R}^m \mid -x_i \leq 1 \text{ for } i \in [m],\ x^\top (b_j/(1 - \mathbf{1}^\top b_j)) \leq 1 \text{ for } j \in [n]\}. \end{aligned}$$

The polar of P is then

$$P^\Delta = \operatorname{conv}(\{e_i \mid i \in [m]\} \cup \{\frac{b_j}{1 - \mathbf{1}^\top b_j}\}) \tag{1.13}$$

Since P is a simple polytope with $\mathbf{0}$ in its interior, P^Δ is simplicial, $P^{\Delta\Delta} = P$, and the facets of P^Δ correspond to the vertices of P and vice versa. We can then label the vertices of P^Δ as the corresponding facets in P^l; the completely labeled facets of P^Δ will then correspond to the completely labeled vertices of P^l. In particular, the facet corresponding to $\mathbf{0}$ will be

$$F_0 = \{x \in P^\Delta \mid -\mathbf{1}^\top x = 1\} = \operatorname{conv}\{e_i \mid i \in [m]\}. \tag{1.14}$$

Theorem 5 then translates as in the original version by Balthasar [1].

Theorem 6. (Balthasar [1]) *Let Q be a labeled m-dimensional simplicial polytope with $\mathbf{0}$ in its interior and vertices $e_1, \ldots, e_m, c_1, \ldots, c_n$ such that $F_0 = \operatorname{conv}\{e_i \mid i \in [m]\}$ is a facet of Q. Let (U, B) be a unit vector game, with $U = (e_{l(1)} \;\cdots\; e_{l(n)})$ for a labeling $l : [n] \to [m]$ and $B = (b_1 \;\cdots\; b_n)$ with $b_j = c_j/(1 + \mathbf{1}^\top c_j)$ for $j \in [n]$. Let l_v be a labeling of the vertices of Q as follows:*

$$
\begin{aligned}
l_v(-e_i) &= i \text{ for } i \in [m]; \\
l_v(c_j) &= l(j) \text{ for } j \in [n]. \qquad (1.15)
\end{aligned}
$$

Then a facet $F \neq F_0$ of Q with normal vector v is completely labeled if and only if (x, y) is a Nash equilibrium of (U, B), where $x = (v + \mathbf{1})/(\mathbf{1}^\top(v + \mathbf{1}))$, and $x_i = 0$ if and only if $e_i \in F$ for $i \in [m]$. Any j so that c_j is a vertex of F represents a pure best reply to x; the mixed strategy y is the uniform distribution on the set of the pure best replies to x.

??? here to end thm

As in theorem 5 we have a correspondence between completely labeled vertices of P^l and equilibria of the unit vector game (U, B) with an "artificial" equilibrium corresponding to the vertex $\mathbf{0}$, in theorem 6 we have a correspondence between the completely labeled facets of the polytope Q and the equilibria of the unit vector game (U, B) with the "artificial" equilibrium corresponding to the facet F_0.

prove polynomial encoding - therefore poly reduction

edit from here

Consider now the problems

ANOTHER COMPLETELY LABELED VERTEX

input : A simple m-dimensional polytope S with $m + n$ facets; a labeling $l_f : [m + n] \to [n]$; a facet F_0 of S, completely labeled by l_f.

output: A facet $F \neq F_0$ of S, completely labeled by l.

Another Completely Labeled Facet	
input :	A simplicial m-dimensional polytope S with $m+n$ vertices; a labeling $l_v : [m+n] \to [n]$; a facet F_0 of S, completely labeled by l_v.
output:	A facet $F \neq F_0$ of S completely labeled by l_v.

Unit Vector Nash	
input :	A unit vector game Γ.
output:	A Nash equilibrium of Γ.

Proposition 2. *The problem* Unit Vector Nash *is polynomial-time reducible to the problems* Another Completely Labeled Vertex *and its dual* Another Completely Labeled Facet.

in general, 2-NASH, since we can go back to 2 polytopes - see LH

1.4 Cyclic Polytopes and Gale Strings

In theorems 5 and 6 we have built a correspondence between labeled polytopes and unit vector games, where Nash equilibria correspond to completely labeled vertices or facets. We now focus on a particular kind of simplicial polytopes, called *cyclic polytopes*, that can be represented as a combinatorial structure, the *Gale strings*. We will first give the definition of cyclic polytope, then of Gale string, then the theorem by Gale [7] about their correspondence.

The *moment curve* in dimension d is defined as

$$\mu_d : \mathbb{R} \longrightarrow \mathbb{R}^d, \quad \mu_d : t \longmapsto (t, t^2, \ldots, t^d)^\top. \tag{1.16}$$

The *cyclic polytope* in dimension d with n vertices, where $n > d$ is

$$C_d(n) = \text{conv}\{\mu_d(t_i) \text{ for } t_1 < \ldots < t_n\}. \tag{1.17}$$

Given $k \in \mathbb{N}$ and a set S, we can represent the function $f : [k] \to S$ as the string $s = s(1)s(2)\cdots s(k)$; we have a *bitstring* if $S = \{0, \mathbf{1}\}$. A maximal substring of consecutive $\mathbf{1}$'s in a bitstring is called a *run*; an *interior* run is bounded on both sides by 0's. We will use the notation $\mathbf{1}^k$ for a run of length k, and 0^k for a string of 0's of length k A *Gale string of length n and dimension d*, where $n > d$, is a bitstring $s \in G(d, n)$ satisfying the following conditions:

bitstring already def in section com-plexity?

1. exactly d bits in s are $\mathbf{1}$ and

2. (*Gale evenness condition*)

$$0\mathbf{1}^k 0 \text{ is a substring of } s \quad \Longrightarrow \quad k \text{ is even.} \tag{1.18}$$

In general, the Gale evenness conditions allows for Gale strings that start or end with an odd-length run; but if d is even then s can start with an odd run if and only if it ends with an odd run. We can then consider the Gale strings in $G(d, n)$ with even d as a the "loops" obtained by "glueing together" the extremes of the strings, so that all runs on the loops are even. Formally: we can see the indices of a Gale string $s \in G(d, n)$ with d even as equivalence classes modulo n, identifying $s(i + n) = s(i)$. This also shows that the set of Gale strings of even dimension is invariant under a cyclic shift of the strings.

Example 1.8. As an example of d even, we have

$$\begin{aligned} G(4,6) = \{&\mathbf{1111}00, \mathbf{111}00\mathbf{1}, \mathbf{11}00\mathbf{11}, \mathbf{1}00\mathbf{111}, 00\mathbf{1111}, \\ &0\mathbf{1111}0, \mathbf{11}0\mathbf{11}0, \mathbf{1}0\mathbf{11}0\mathbf{1}, 0\mathbf{11}0\mathbf{11}\} \end{aligned}$$

The strings $\mathbf{1111}00$, $\mathbf{111}00\mathbf{1}$, $\mathbf{11}00\mathbf{11}$, $\mathbf{1}00\mathbf{111}$, $00\mathbf{1111}$ and $0\mathbf{1111}0$ are equivalent under a cyclic shift (if considering the strings as "loops", the $\mathbf{1}$'s are all consecutive), as are the strings $\mathbf{11}0\mathbf{11}0$, $\mathbf{1}0\mathbf{11}0\mathbf{1}$ and $0\mathbf{11}0\mathbf{11}$ (if considering the strings as "loops", the even runs of $\mathbf{1}$'s are two couples separated by a single 0).

As an example for d odd, we have

$$G(3,5) = \{\mathbf{111}00, \mathbf{1}0\mathbf{11}0, \mathbf{1}00\mathbf{11}, \mathbf{11}00\mathbf{1}, 0\mathbf{11}0\mathbf{1}, 00\mathbf{111}\}$$

Note how **0**1**0**11 is a cyclic shift of **10110**, but it is not a Gale string.

The relation between cyclic polytopes and Gale strings is given by the following theorem by Gale [7].

Theorem 7. (Gale [7]) *For any positive integers d, n with $n > d$*

$$F \text{ is a facet of } C_d(n)$$

$$\iff$$

$$F = \operatorname{conv}\{\mu(t_j) \mid s(j) = 1 \text{ for some } j \in [n] \text{ and } s \in G(d,n)\}. \tag{1.19}$$

sketch of pf - see Ziegler - with drawing of moment curve + hyperplane

CP simplicial

CP does not depend on the choice of t_i's

Essentially, this holds because any set $S \subset [n]$ the moment curve defines a unique hyperplane which is crossed (and not just touched) by the moment curve; if the bitstring s that encodes F as $1(s)$ has a substring 01^k0

example of cyclic polytope + equivalent gale string (a simple one)

From this point forward, we will assume that d is even. We will also assume that the labeling $l : [n] \to [d]$ is such that $l(i) \neq l(i+1)$; this can be done without loss of generality, given the following consideration. Suppose that $l(i) = l(i+1)$ for some index i, and let s be a completely labeled Gale string for l. Then only one of $s(i)$ and $s(i+1)$ can be equal to **1** (note that it's possible that both are 0s). So $s(i)s(i+1)$ will never be a run of even length that "interferes" with the Gale Evenness Condition, so we can "simplify" by identifying the indices i and $i+1$.

Theorem 7 gives a correspondence between Gale strings and facets of cyclic polytopes; we have also seen that these polytopes are simplicial. On the other hand, theorem 6 gives a correspondence between completely labeled facets of a simplicial polytopes and Nash equilibria of unit vector games. To exploit these connections, we now give a definition of labeling for Gale strings that will allow us to study the Nash equilibria of a unit vector game for which the best response polytope is the dual of a cyclic polytope (recall that the polytope in theorem 5 is the best response polytope, whereas theorem 6 describes its dual version). This might seem a very specific case of bimatrix game; we will see in the next chapter that it leads to very interesting results. We therefore need a definition of "completely labeled" for Gale strings, and a labeling l_s for $G(d,n)$ such that $s \in G(d,n)$ is completely labeled if and only if the corresponding facet in $C_d(n)$ is completely labeled by l_v as given in theorem 6.

We say that $s \in G(d,n)$ is a *completely labeled Gale string* if for some labeling function $l_s : [n] \to [d]$ the set $\{i \in [n] \mid s(i) = \mathbf{1}\}$ is completely labeled by l_s. Since $s \in G(d,n)$ has exacty d bits equal to $\mathbf{1}$, this means that for each $j \in [d]$ there is exactly one $i \in [n]$ such that $s(i) = \mathbf{1}$ and $l_s(i) = j$. Note that it is not always possible to find a completely labeled Gale string.

Example 1.9. For $l = 121314$, there are no completely labeled Gale strings.

The labels $l(i) = 2, 3, 4$ appear only once in l, as $l(2), l(4), l(6)$ respectively; therefore we must have $s(2) = s(4) = s(6) = 1$. For every other $i \in [n]$ we have $l(i) = 1$, so we have $l(i) = 1$ for exactly one $i = 1, 3, 5$. The candidate strings are then **110101**, 0**11101**, 0**10111**; but none of these satisfies the Gale evenness condition.

Let (U, B), where $U = (e_{l(1),\ldots,e_{l(d)}})$ for some labeling $l : [n] \to [d]$, be a unit vector game for which the dual of the best response polytope is a cyclic polytope $Q = \text{conv}\{e_1, \ldots, e_d, c_1, \ldots, c_n\}$. Theorem 6 gives a labeling l_v of the $d + n$ vertices of Q as in 1.15:

$$l_v(-e_i) = i \text{ for } i \in [m];$$
$$l_v(c_j) = l(j) \text{ for } j \in [n].$$

Let the labeling $l_s : [d + n] \to [d]$ be defined as follows:

$$l_s(i) = i \text{ for } i \in [d];$$
$$l_s(d + j) = l(j) \text{ for } j \in [n]. \tag{1.20}$$

Then the Gale strings $s \in G(d, d+n)$ that are completely labeled for l_s correspond exactly to the completely labeled facets of Q, with the facet F_0 corresponding to the "trivial" completely labeled string $\mathbf{1}^d0$.

Example 1.10. Given the string of labels $l = 123432$, there are four associated completely labeled Gale strings: **1111**00, **11**0**11**0, **1**00**111** and **1**0**11**0**1**.

1	**2**	**3**	**4**	**3**	**2**
1	**1**	**1**	**1**	·	·
1	**1**	·	**1**	**1**	·
1	·	·	**1**	**1**	**1**
1	·	**1**	**1**	·	**1**

draw polytope

From a computational point of view, we can define the problem Another Gale and Unit Vector Cyclic Nash as follows:

Another Gale

---

input : A labeling $l : [n] \to [d]$, where d is even and $d < n$. A Gale string $s \in G(d, n)$, completely labeled by l.

output: A Gale string $s' \in G(d, n)$, completely labeled by l, such that $s' \neq s$.

Unit Vector Cyclic Nash

---

input : A unit vector game Γ with dual cyclic best response polytope.

output: A Nash equilibrium of Γ.

---

prove that encoding GS is poly

Then, by proposition 2, we have the following theorem.

Proposition 3. *The problem* Unit Vector Cyclic Nash *is polynomial-time reducible to the problem* Another Gale.

Chapter 2

Algorithmic and Complexity Results

2.1 Lemke Paths and the Lemke-Howson for Gale Algorithm

In the previous chapter we have defined different, but related, problems of the form "find another completely labeled..." In this section we will focus on different versions of a standard algorithm, first introduced by Lemke and Howson in [9], that will solve these problems through an operation called *pivoting* and introducing *almost complete* labelings next to complete ones. In the next session we will tackle the issue of the computational complexity of these algorithms: Another Completely Labeled Facet and Another Completely Labeled Vertex are **PPA**, Nash is **PPAD**, as first shown in Papadimitriou [16]; furhermore, as shown by Morris [13] and by Savani and von Stengel [17], there are cases of exponential running time. This had led us to conjecture that these problems could be exploited for a proof of **PPAD** completeness, also considering that finding a completely labeled facet (or vertex, or the existence of a Nash equilibrium) is **NP** in the case of a general labeled polytope, as proven by von Stengel [22] . In the last section we will finally present our original result, that goes in the opposite direction: the problem Another Galecan be solved in polynomial time, that is, it is a problem in **TFP**. Unit vector games with dual cyclic best response polytope present therefore a case apart, as expected, but not because they are harder than others, but because they are easier.

check proof, citation

original? main?

We begin this section with the definition of almost complete labeling; we then move on to the classic version of the Lemke-Howson algorithm for the problem Another Completely Labeled Vertex, as given in the beautiful exposition by Shapley [19], and its dual version for Another Completely Labeled Facet. Finally, we present the Lemke-Howson for Gale algorithm.

Let $m, n \in \mathbb{N}$ with $m \leq n$; consider a set X with $|X| = n$ and a labeling $l : X \to [m]$. The m-uple $x = (x_1, \ldots, x_m) \in X^m$ is *almost completely labeled* if $\{j \in [n] \mid x_i = j \text{ for some } i \in [m]\} = [m] \setminus \{k\}$ for exactly one $k \in [m]$. That is, all labels appear once in x, except for the *missing label* k and a *duplicate label* $h \in [m]$ that appears twice.

Let P be a simple polytope in dimension d with n facets. We *pivot on the vertices* of P by moving from a vertex x to another vertex y connected to x by an edge. Note that, since P is simple, there are exactly d possible choices for y. Analogously, we *pivot on the facets* of a simplicial polytope Q in dimension d by moving from a facet F to a facet G that share all vertices but one; and since Q is simplicial, there are d possible choices for G.

Suppose now that there is a labeling $l_f : [n] \to [d]$ of the facets of the simple polytope P. If we pivot from a vertex x to a vertex x' we "leave behind" a facet F with label k; so, if x has labels $(l_1, \ldots, k, \ldots, l_d)$, then x' has labels $(l_1, \ldots, h, \ldots, l_d)$, where h is the label of the facet F' that does not have x as its vertex. We call this *dropping label k and picking up label h*, or *pivoting on label k*. Analogously, if there is a labeling $l_v : [n] \to [d]$ of the vertices of the simplicial poytope Q and we pivot from a facet F with labels $(l_1, \ldots, k, \ldots, l_d)$

to a facet F' with labels $(l_1, \ldots, h, \ldots, l_d)$, we say that we *drop label k and pick up label h*, or that we *pivot on label k*.

We consider now the case of a simple polytope with labeled facets. Suppose that the labeling l_f is such that there is at least one completely labeled vertex x_0 of P. Algorithm ?? then gives the *Lemke-Howson algorithm* algorithm (see Shapley in [19]).

Algorithm 1: Lemke-Howson algorithm

---

input : A simple d-polytope P with n facets. A labeling $l_f : [n] \to [d]$ of the facets of P. A vertex x_0 of P, completely labeled for l.

output: A completely labeled vertex $x \neq x_0$ of P.

```
choose any label k ∈ [d]
pivot on label k from x_0 to x
while x is not completely labeled do
    pivot on the duplicate label h from x to x' ≠ x_0
    set x_0 = x, x = x'
return x
```

Running the Lemke-Howson algorithm defines a *Lemke path* that connects two completely labeled vertices through almost completely labeled vertices and edges where the only missing label is k. This is fundamental to show that the Lemke-Howson algorithm does, indeed, work.

Proposition 4. *The Lemke-Howson algorithm ?? returns a solution to the problem* Another Completely Labeled Vertex.

Furthermore, the number of completely labeled vertices in a simple polytope with labeled facets is even.

Proof. The fact that x is completely labeled is trivial; we must show that $x \neq x_0$. At each vertex x' of the Lemke path there are only two edges corresponding to the missing label k, since P is simple; one is the edge that has been traversed to get to x', the other one will be traversed to leave it in the next step. Therefore, there are no "loops" where a vertex is visited more than once; Lemke paths are *simple paths*.

Suppose now that there are an odd number of completely labeled vertices; that is, that there is an odd number of endpoints of Lemke paths.

pf parity via Lemke paths

□

Applying the parity result in proposition 4 to the case of a bimatrix game (not necessarily a unit vector game) , and remembering that the point $(\mathbf{0},\mathbf{0})$ corresponds to an "artificial" equilibrium, we have the following result, due to Lemke and Howson [9].

note: two polytopes, see todo at end section labels

Theorem 8. (Lemke-Howson [9]) *Every non-degenerate bimatrix game has an odd number of Nash equilibria.*

Example 2.1. We now give an example of implementing the Lemke-Howson algorithm to find a Nash equilibrium of a bimatrix game.

We can consider the game C as in proposition **??**, and the associated polytope $S = \{z \in \mathbb{R}^{m+n} \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1}\}$, labeling the $2(m+n)$ inequalities defining the facets of of S as $1,\dots,m+n,1,\dots,m+n$. Then we will apply the Lemke-Howson algorithm starting from vertex $\mathbf{0}$; this will eventually return a Nash equilibrium (z,z) of C, and the corresponding Nash equilibrium $(x,y) = z$ of (A,B).

We can also follow the "traditional" version of the Lemke-Howson algorithm; a very clear exposition of this can be found (once again) in Shapley [19].

Let P and Q be the best response polytopes of (A,B) as in 1.5. We then move alternately on P and Q, starting from the couple of vertices $(\mathbf{0},\mathbf{0})$. Since we move in $\mathbb{R}^m$ and $\mathbb{R}^n$ instead of $\mathbb{R}^{m+n}$, this version is much easier to visualize.

ex Savani - von Stengel, pag. 11; fig 8 are Schegel diagrams of BR polytopes.

The dual version of the Lemke-Howson algorithm ?? and of proposition 4 is quite straightforward.

Algorithm 2: Lemke-Howson algorithm on facets

input : A simplicial m-polytope Q with n vertices. A labeling $l_v : [n] \to [d]$ of the vertices of P. A vertex F_0 of Q, completely labeled for l.

output: A completely labeled facet $F \neq F_0$ of Q.

```
choose any label k ∈ [d]
pivot on label k from F_0 to F
while x is not completely labeled do
    pivot on the duplicate label h from F to F' ≠ x_0
    set F_0 = x, F = F'
return x
```

Proposition 5. *Algorithm ?? returns a solution to the problem* ANOTHER COMPLETELY LABELED FACET.

Furthermore, the number of completely labeled facets in a simplicial polytope with labeled vertices is even.

Example 2.2. example: octahedron! so we're ready for index!

In the case of unit vector games, theorems 5 and 6 imply that it is enough to apply algorithm ?? to the polytope $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ in (**??**), or the dual algorithm ?? to the polytope $P^\Delta = \mathrm{conv}(\{e_1, \ldots, e_m\}) \cup \{c_1, \ldots, c_n\})$ in 1.13. This allows to find one Nash equilibrium by pivoting on P^l or P^Δ, instead of considering the product of the best response polytopes P and Q. The projection of a Lemke path for a missing label $i \in [m]$ on $P \times Q$

to P defines a Lemke path in P^l; however, $P \times Q$ has $m+n$ labels, therefore how? there could be Lemke paths for a missing label $m+j$ with $j \in [n]$ on $P \times Q$ that are "lost" in the projection on P^l. The following theorem, by Savani and von Stengel [18], shows that there is no loss of generality in studying Lemke paths on P^l; an analogous result holds for the dual case.

Theorem 9. *Let (U, B) be a unit vector game, with $U = (e_{l(1)} \cdots e_{l(n)})$ for a labeling $l : [n] \to [m]$; let $P = \{x \in \mathbb{R}^m | x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ and $Q = \{y \in \mathbb{R}^n | y \geq \mathbf{0},\ Ay \leq \mathbf{1}\}$, as in 1.5; and let $P^l = \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}\}$ as in **??**. Then the Lemke path on $P \times Q$ for the missing label k projects to a path on P that is the Lemke path on P^l for missing label k if $k \in [m]$, and for missing label $l(j)$ if $k = m + j$ with $j \in [n]$.*

We finally focus on the case of unit vector games where the simplicial polytope Q is cyclic; that is, the case that we can study from the point of view of Gale strings.

Consider $s \in G(m, n)$ with d even as a "loop", and let $s(i) = 1$ for an index $i \in [n]$. Then, by Gale evenness condition, there is an odd run of $\mathbf{1}$'s in s either on the left or on the right of position i; let j be the first index after this run. A *pivot on s* is then defined as setting $s(i) = 0$ and $s(j) = 1$. Given a labeling $l_s : [n] \to [m]$, we say that we *pivot on label $l(i)$*, *dropping label $l(i)$* and *picking up label $l(j)$*. The *Lemke-Howson for Gale algorithm* is defined in **??**.

The analogous of propositions **??** and **??** is the following.

Proposition 6. *The Lemke-Howson for Gale algorithm* **??** *returns a solution to the problem* Another Gale.

Furthermore, the number of completely labeled Gale strings $s \in G(d, n)$, where d is even, is even.

Algorithm 3: Lemke-Howson for Gale algorithm

input : A labeling $l_s : [n] \to [d]$ such that there is a completely labeled Gale string $s_0 \in G(d, n)$.

output: A completely labeled Gale string $s \in G(d, n)$ such that $s \neq s_0$.

choose a label $k \in [d]$

pivot on label k from s_0 to s

while *s is not completely labeled* **do**

  pivot on the duplicate label h from s to $s' \neq s_0$

  rename $s_0 = s$, $s = s'$

return s

Example 2.3. example

something with more than two equilibria

Morris

2.2 Complexity of Lemke-Howson Algorithms

complexity of LH algorithms

oiks as in edmonds (mention).

lh is exchange on d - 2, lh dual is exchange on d - 1 (or d-1 and d?) oik.

pbl index (that gives PPA vs PPAD):

endpoints of LHG have opposite sign (Z graph) so the graph of "what eq can you reach from eq" is bipartite (not nec connected, ex in shapley 74)

but - ex octahedron - every rp reachable from each other rp

2.3 The Complexity of Gale and Another Gale

We will now give our main result: Another Gale can be solved in polynomial time; therefore, it takes polynomial time to find a Nash Equilibrium of a bimatrix game with dual cyclic best response polytope. Our proof will rely on the construction of a graph and, if possible, a perfect matching for it. A *perfect matching* of a multigraph $G = (V, E)$ is a set $M \subseteq E$ of pairwise non-adjacent edges so that every vertex $v \in V$ is incident to exactly one edge in M. A theorem by Edmonds ([6]) gives the complexity of the associated problem PERFECT MATCHING.

PERFECT MATCHING	
input :	A multigraph $G = (V, E)$.
output:	A perfect matching for G, or NO if there is no possible perfect matching for G.

Theorem 10. (Edmonds [6]) *The problem* PERFECT MATCHING *can be solved in polynomial time.*

To prove our main result on ANOTHER GALE, we will first focus on the accessory problem GALE, and we will use theorem 10 to prove that it is solvable in polynomial time. We will consider every Gale string as a "loop."

GALE	
input :	A labeling $l : [n] \to [d]$, where d is even and $d < n$.
output:	A Gale string $s \in G(d, n)$ that is completely labeled by l

Theorem 11. *The problem* GALE *is solvable in polynomial time.*

Proof. We give a reduction of GALE to PERFECT MATCHING.

Consider the multigraph $G = (V, E)$ with $V = [d]$, so that the vertices of G correspond to the labels $l(i) \in [d]$, and $E = \{(l(i), l(i + 1))$ for $i \in [n]\}$, so that there is an edge between two vertices if and only if the corresponding labels are next to each other at some index i. Let $s \in G(d, n)$ be a completely labeled Gale string. By Gale evenness condition, every run of s corresponds uniquely to $d/2$ pairs of indices $(i, i+1)$ with $s(i) = s(i+1) = 1$, and since s is completely labeled, all labels $l(i) \in [d]$ occur at exactly one of these indices. Then the edges $(l(i), l(i+1))$ form a perfect matching of G.

Conversely, let $l : [n] \to [d]$ be a labeling, and let M be a perfect matching for G. Consider a bitstring s with $s(i) = s(i+1)$ for every $(l(i), l(i+1)) \in M$ and $s(i) = 0$ otherwise. Since M is a matching, all the $(l(i), l(i+1)) \in M$ are disjoint, so, considering s as a "loop," every run of s is of even length, thus satisfying the Gale evenness condition. Since M is perfect, every vertex $v \in [d]$ is the endpoint of an edge $(l(i), l(i+1))$, so s has exactly d bits equal to $\mathbf{1}$, so it is completely labeled.

We have therefore reduced the problem GALEto PERFECT MATCHING, that by theorem 11 can be solved in polynomial time. $\square$

We give two examples of the construction used in theorem 11.

Example 2.4. Consider the labeling $l : [6] \to [4]$ with $l = 123423$. To find a Gale string $s \in G(4, 6)$ that is completely labeled by l we must look for a perfect matching M of $G = ([4], \{e_i = (l(i), l(i+1)) \mid i \in [6]\})$, as seen in figure 2.1.

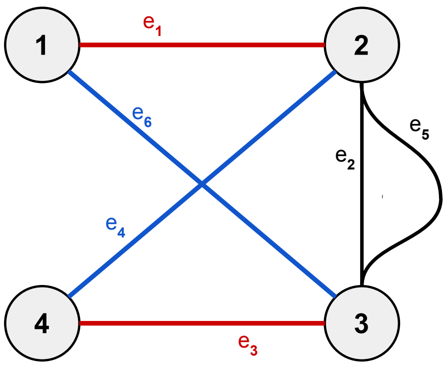

Figure 2.1 The graph G associated to the labeling $l = 123423$.

The matching M, in turn, will give the completely labeled Gale string s as $s(i) = s(i+1) = 1$ for $e_i \in M$, $s(j) = 0$ otherwise.

For instance, if we take the perfect matching $M = \{e_1, e_3\}$, we have the string $s = 111100$. If we take the perfect matching $M' = \{e_4, e_6\}$ instead, we have the string $s' = 1000111$.

A perfect matching for a graph, and therefore a Gale string for a labeling, is not always possible, as shown in the next example.

Example 2.5. Consider the labeling $l = 121314$. The graph G is shown in figure 2.2

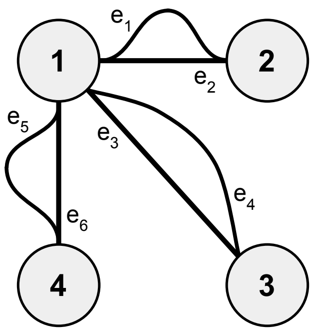

Figure 2.2 The graph for the labeling $l = 121314$

Since there aren't any disjoint edges, it's not possible to find a perfect matching for G. We have already seen in example 1.10 that there isn't any possible completely labeled Gale string for $l = 121314$.

We finally extend the proof of theorem 11 to Another Gale.

Theorem 12. *The problem* Another Gale *is solvable in polynomial time.*

Proof. Let $G = (V, E)$ be the graph corresponding to the labeling $l : [n] \to [d]$ as in the proof of theorem 11 and let M be the perfect matching of G corresponding to the completely labeled Gale string $s \in G(d, n)$.

If there are two edges $e, e' \in E$ such that $e \in M$, both e and e' have endpoints $l(i), l(i+1)$, but $e \neq e'$ (recall that G can be a multigraph), the matching $M' = (M \setminus \{e\}) \cup \{e'\}$ is perfect. The corresponding completely labeled Gale string $s' \in G(d, n)$ satisfies $s' \neq s$, since in s the **1**'s corresponding to the labels $l(i), l(i+1)$ are in the positions given by the edge e, while in s' they are in the positions given by $e' \neq e$. It takes time $d/2$ to check all edges of M, the time required is still polynomial.

We now assume that all the edges in every perfect matching M for G don't have a parallel edge. Since by theorem 6 there is an even number of completely labeled Gale strings, the existence of s guarantees the existence of another completely labeled Gale string $s' \neq s$ and the corresponding perfect matching $M' \neq M$. Since $M' \neq M$, there is at least one edge $e' \in M$ such that $e' \notin M'$. Consider the $d/2$ graphs $G_i = (V, E_i)$, where $E_i = E \setminus \{e_i\}$ for $e_i \in M$. Since $V(G) = V(G')$ and $E(G) \subset E(G')$, every perfect matching for one of these G_i is a perfect matching for G as well. With a brute force approach, we look for a perfect matching in each G_i; this will be M'. Since there are $i \in [d/2]$, the time to find it will be still polynomial. □

We give two examples of the construction of theorem 12.

Example 2.6. The labeling $l = 1234324$ gives the graph G in figure 2.3. Suppose that Edmonds' algorithm returns the matching $M = \{e_1, e_3\}$, associated to the completely labeled Gale string $s = 1111000$. The edge e_3 has a parallel edge, e_4; we immediately have a second perfect matching in $M' = \{e_1, e_4\}$, associated to the Gale string $s' = 1101100$.

A case in which every edge in every perfect matching does not have a parallel one is the one given in example 2.4.

Example 2.7. Consider the labeling $l = 123423$; the associated graph is shown on the left in figure 2.4. There are only two possible matchings, and neither has a parallel edge. Note that G is a multigraph: we look for parallel edges in the matching, not in all the edges of the graph. Suppose that Edmonds' algorithm returns the perfect matching $M = \{e_1, e_3\}$; we can then delete the edge e_1 to obtain the graph G_1, seen on figure 2.4 right. The graph G_1 has a perfect matching in $M' = \{e_4, e_6\}$, that is also a perfect matching of G,

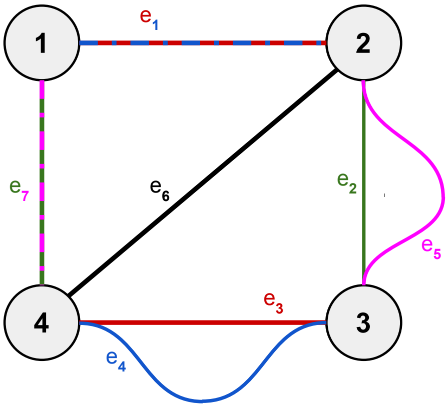

Figure 2.3 The graph for the labeling $l = 1234324$.

associated to the string $s' = 100111$.

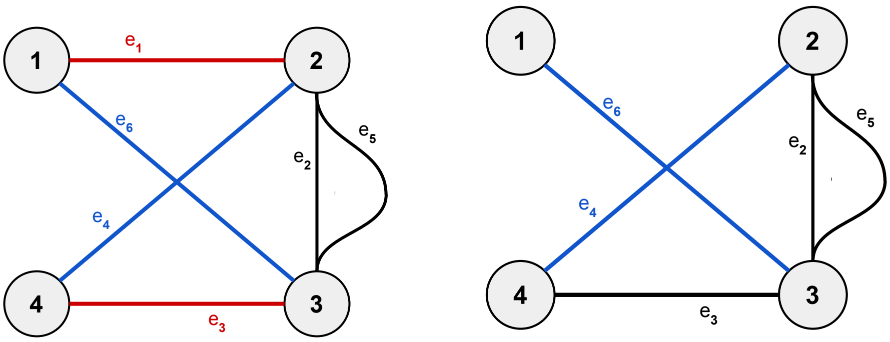

Figure 2.4 Left: the graph $G = (V, E)$ for the labeling 123423.
Right: the graph $G_1 = (V, E \setminus \{e_1\})$.

Chapter 3

Further results

A Note on the PPAD Completeness of NASH

better title

appendix/ppad-msc

Polytopes

better title

We denote the transpose of a matrix A as $A^\top$. We consider vectors $u, v \in \mathbb{R}^d$ as column vectors, so $u^\top v$ is their scalar product. A vector in $\mathbb{R}^d$ for which all components are 0's will be denoted as $\mathbf{0}$; similarly, a vectors for which all components are 1's will be denoted as $\mathbf{1}$. The *unit vector* e_i is the vector that has i-th component $e_{ii} = 1$ and $e_{ij} = 0$ for all other components. When writing an inequality of the form $u \geq v$ (and analogous), we mean that it holds for every component; that is, $u_i \geq v_i$ for all $i \in [d]$.

An *affine combination* of points in an Euclidean space $z_1, \ldots, z_n$ is

$$\sum_{i=1}^{n} \lambda_i z_i \quad \text{where } \lambda_i \in \mathbb{R} \text{ such that } \sum_{i=1}^{n} \lambda_i = 1$$

The points $z_1, \ldots, z_n$ are *affinely independent* if none of them is an affine combination of the others.

A *convex combination* of points $z_1, \ldots, z_n$ is an affine combination where $\lambda_i \geq 0$ for all $i \in [n]$. Note that such λ_i's can be seen as a probability distribution over the z_i's.

A set of point Z is *convex* if it is closed under forming convex combinations, that is, if $\bar{z} = \sum_{i=1}^{n} \lambda_i z_i$, where $z_i \in Z$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, then $\bar{z} \in Z$. A convex set has *dimension* d if it has exactly $d + 1$ affinely independent points.

convex hull (needed for def cyclic poly);

pow hyperplanes;

polyhedra, polytopes

simplex

simple and simplicial polytopes

polar: $Q = \{x \in \mathbb{R}^d \mid x^\top c_i \leq 1,\ i \in [k]\}$

with $c_i \in \mathbb{R}^d$. Then the polar (Ziegler, 1995) of Q is given by

$Q^\Delta = \mathrm{conv}\{c_i, i \in [k]\}$

from here: notes - copy-paste

A (d-dimensional) *simplicial polytope* P is the convex hull of a set of at least $d+1$ points v in $\mathbb{R}^d$ in general position, that is, no $d+1$ of them are on a common hyperplane.

If a point v cannot be omitted from these points without changing P then v is called a *vertex* of P. A *facet* of P is the convex hull $\mathrm{conv}\, F$ of a set F of d vertices of P that lie on a hyperplane $\{x \in \mathbb{R}^d \mid a^T x = a_0\}$ so that $a^T u < a_0$ for all other vertices u of P; the vector a (unique up to a scalar multiple) is called the *normal vector* of the facet. We often identify the facet with its set of vertices F.

simple polytope in dimension m: no point of P lies on more than m facets and no point of Q lies on more than n facets. A facet is obtained by turning one of the inequalities that define the polytope into an equality, provided the inequality is irredundant, that is, cannot be omitted without changing the polytope.

Acknowledgements

appendix/acknowledgments

Index of Symbols

appendix/symbols

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