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\subsection{Cyclic Polytopes and Gale Strings}\label{gs-ssect}

\begin{example}\label{no-clgs}
For $l = 121314$, there are no completely labeled Gale strings.
\end{example}
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0.1 The Complexity of Gale and Another Gale

We will now give our main result: Another Gale can be solved in polynomial time. Therefore, it takes polynomial time to find a Nash Equilibrium of a bimatrix game for which the best response polytope is cyclic.

Our proof will be based on a simple graph construction.

Definition 1. A *perfect matching* for a graph $G = (V, E)$ is a set $M \subseteq E$ of pairwise non-adjacent edges so that every vertex $v \in V$ is incident to exactly one edge in M.

We define the problem Perfect Matching as follows:

Perfect Matching	
input :	A graph $G = (V, E)$.
output:	A perfect matching for G.

The complexity of Perfect Matching has been proven to be in P by Edmonds [4].

Theorem 1 ([4])**.** *The problem* Perfect Matching *is solvable in polynomial time.*

We will first consider the accessory problem Gale, and we will show that it is solvable in polynomial time by using theorem 1.

Gale

input : A labeling $l : [n] \to [d]$, where d is even and $d < n$.

output: A completely labeled Gale string s in $G(d, n)$ associated with l.

Theorem 2. *The problem* Gale *is solvable in polynomial time.*

Proof. We give a reduction of Gale to Perfect Matching.

In the following, we will consider every Gale string as a "loop," as seen in section **??**, so $n + 1 = 1$.

Given the labeling $l : [n] \to [d]$, let $V = [d]$, let $E = \{(l(i), l(i+1))$ for $i \in [n]$ for every i such that $l(j) \neq l(i + 1)\}$, and consider the multigraph $G = (V, E)$.

Let $s \in G(d, n)$ be a completely labeled Gale string. Then every run of s splits uniquely into $d/2$ pairs $(i, i + 1)$ such that the labels $l(i)$ satisfy the condition $l(i) \neq l(i + 1)$, and all the labels $l(i) \in [d]$ occur. Then the labels will correspond to all the vertices of G, and the pairs will correspond to the edges of a perfect matching for G.

Conversely, let $l : [n] \to [d]$ be a labeling, and let M be a perfect matching for G as above. We can construct a string s such that $s(i) = s(i + 1)$ for every $(l(i), l(i+1)) \in M$ and $s(i) = 0$ otherwise. Since M is a matching, all the $(l(i), l(i + 1)) \in M$ are disjoint, so, considering s as a "loop," every run is of even length. Furthermore, since M is a perfect matching, every vertex $v \in [d]$ is the endpoint of an edge $(l(i), l(i + 1))$, so s is completely labeled.

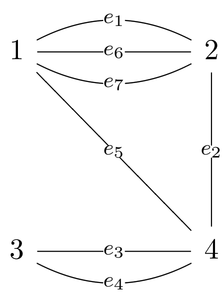
We have a reduction from Gale to the problem Perfect Matching, which is polynomial-time solvable by theorem 1. Finding a Gale string for a given labeling, or deciding that there isn't one, can therefore be done in polynomial time. □

We give two examples of the construction used in theorem 2.

Example 0.1. Let $l = 12343122$ be a string of labels. Then the egdes e_i of the graph G obtained from the construction in the proof of theorem 2 will be as follow:

$$\underbrace{1\overbrace{\quad}^{e_1}2\overbrace{\quad}^{e_2}3\overbrace{\quad}^{e_3}4\overbrace{\quad}^{e_4}3\overbrace{\quad}^{e_5}1\overbrace{\quad}^{e_6}2\overbrace{\quad}^{cycle}2}_{e_7}$$

Given the vertices $v \in [4]$, the graph G will be:

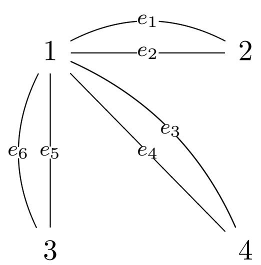


The perfect matching for G given by $M = \{e_1, e_4\}$ will then correspond to the completely labeled Gale string 11011000.

$$\underbrace{\mathbf{1}\overbrace{\quad}^{e_1}\mathbf{1}\overbrace{\quad}0\overbrace{\quad}\mathbf{1}\overbrace{\quad}^{e_4}\mathbf{1}\overbrace{\quad}0\overbrace{\quad}0\overbrace{\quad}0}$$

A perfect matching for a graph, and therefore a Gale string for a labeling, is not always possible, as shown in the next example.

Example 0.2. Let us consider the labeling $l = 121314$. The associated graph G will be



Since there aren't any disjoint edges, it's not possible to find a perfect matching for G. Analogously, we have seen in example **??** that there isn't any possible completely labeled Gale string for the labeling l.

We finally extend the proof of theorem 2 to show that Another Gale is polynomial-time solvable.

Theorem 3. *The problem* Another Gale *is solvable in polynomial time.*

Proof. proof from old draft: to edit!

The reduction for Another Gale is an extension of this. Consider the given completely labeled Gale string s and the matching M for it. If G has multiple edges between two nodes and one of them is in M, simply replace that edge by a parallel edge to obtain another completely labeled Gale string s'. Hence, we can assume that M has no edges that have a parallel edge. Another completely labeled Gale string s' exists by Theorem **??**. The corresponding matching M' does not use at least one edge in M. Hence, at least one of the $d/2$ graphs G which have one of the edges of M removed has a perfect matching M', which is a perfect matching of G, and which defines a completely labeled Gale string s' different from s. The search for M' takes again polynomial time. □

examples for second PM (one w/ double edges, one without)

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