Double Pendulum

Verifying the Lagrangian

The components of the bob positions and velocities are:

$$x_1 = \ell \sin\phi_1 \qquad \dot{x}_1 = \ell\dot{\phi}_1 \cos\phi_1 \tag{1}$$

$$y_1 = -\ell \cos\phi_1 \qquad \dot{y}_1 = \ell\dot{\phi}_1 \sin\phi_1 \tag{2}$$

$$x_2 = \ell \sin\phi_1 + \ell \sin\phi_2 \qquad \dot{x}_2 = \ell\dot{\phi}_1 \cos\phi_1 + \ell\dot{\phi}_2 \cos\phi_2 \tag{3}$$

$$y_2 = -\ell \cos\phi_1 - \ell \cos\phi_2 \qquad \dot{y}_2 = \ell\dot{\phi}_1 \sin\phi_1 + \ell\dot{\phi}_2 \sin\phi_2 \tag{4}$$

The kinetic energy can be expressed as:

$$T = \frac{1}{2}m{v_1}^2 + \frac{1}{2}m{v_2}^2 \tag{1}$$

where m is the mass of the two pendulum bobs and v_1 and v_2 are the velocities of the two pendulum bobs. Using the velocities given above, the kinetic energy can be calculated as:

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}m\left[(\ell\dot{\phi}_1\cos\phi_1)^2 + (\ell\dot{\phi}_1\sin\phi_1)^2\right] + \frac{1}{2}m\left[(\ell\dot{\phi}_1\cos\phi_1 + \ell\dot{\phi}_2\cos\phi_2)^2 + (\ell\dot{\phi}_1\sin\phi_1 + \ell\dot{\phi}_2\sin\phi_2)^2\right] \\
&= \frac{1}{2}m\left[\ell^2\dot{\phi}_1^2\cos^2\phi_1 + \ell^2\dot{\phi}_1^2\sin^2\phi_1\right] \\
&\quad + \frac{1}{2}m\left[(\ell^2\dot{\phi}_1^2\cos^2\phi_1 + \ell^2\dot{\phi}_2^2\cos^2\phi_2 + 2\ell^2\dot{\phi}_1\dot{\phi}_2\cos\phi_1\cos\phi_2)\right. \\
&\quad \left.+(\ell^2\dot{\phi}_1^2\sin^2\phi_1 + \ell^2\dot{\phi}_2^2\sin^2\phi_2 + 2\ell^2\dot{\phi}_1\dot{\phi}_2\sin\phi_1\sin\phi_2)\right]
\end{aligned} \tag{2}$$

Using the trigonometric identities:

$$\cos(\phi_1 - \phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2 \tag{3}$$

$$\cos^2\phi + \sin^2\phi = 1 \tag{4}$$

we can simplify the kinetic energy to give:

$$\begin{aligned}
T &= \frac{1}{2}m\ell^2\dot{\phi}_1^2 + \frac{1}{2}m\left(\ell^2\dot{\phi}_1^2 + \ell^2\dot{\phi}_2^2 + 2\ell^2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right) \\
&= \frac{1}{2}m\ell^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right]
\end{aligned} \tag{5}$$

The potential energy is simply the gravitational energy of the system which is given by

$$V = mgy_1 + mgy_2 = mg(-\ell\cos\phi_1) + mg(-\ell\cos\phi_1 - \ell\cos\phi_2) = -mg\ell(2\cos\phi_1 + \cos\phi_2) \tag{6}$$

Therefore, the Lagrangian of the double pendulum is defined as:

$$\mathcal{L} = T - V = \frac{1}{2}m\ell^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right] + mg\ell\left(2\cos\phi_1 + \cos\phi_2\right) \tag{7}$$

Runge-Kutta Method

Detailed description

To solve the equations of motion of the double pendulum, we can use a numerical method known as the Runge-Kutta method, or RK4. This method only accepts first order differential equations, such as $x' = f(x)$, so we must convert the system of two first-order order equations into four first-order equations. We can do this by a simple substitution of variables, letting $\phi_n' = \omega_n$, and thus $\phi_n'' = \omega_n'$ for $n \in [1, 2]$.

After some rearranging, the following expressions for $\dot{\omega_1}$ and $\dot{\omega_2}$ are obtained:

$$\dot{\omega_1} = \frac{-3g\sin\phi_1 - g\sin(\phi_1 - 2\phi_2) - 2\ell\sin(\phi_1 - \phi_2)(\omega_2^2 + \omega_1^2\cos(\phi_1 - \phi_2))}{\ell(3 - \cos(2(\phi_1 - \phi_2)))} \tag{8}$$

$$\dot{\omega_2} = \frac{2\sin(\phi_1 - \phi_2)\left[2(\ell\omega_1^2 + g\cos\phi_1) + \omega_2^2 m\ell\cos(\phi_1 - \phi_2)\right]}{\ell(3 - \cos(2(\phi_1 - \phi_2))} \tag{9}$$

Now we have arrived at a point where the equations of motion are suitable for being solved via numerical methods. To accomplish this, we will use a method from the Runge-Kutta family know as RK4 (or The Runge-Kutta Method). This family describes general techniques for approximating the solution to equations of the form $x' = f(t, x)$. Notice the rate of change of the state value (x') depends only on the current state and other constants or independent variables (like time or mass). We need equations in this form because they reveal precisely how the system is changing given its current state. This allows us to compute the next state by extrapolating a change in the system over a very small time interval $(h = \delta t)$ given only the information that is currently present. We can repeat this process ad infinitum, and as long as our h is small enough, our simulation will remain accurate. Often we need to have more than one equation of the above form to completely describe our system. In this case, we just just treat the components of the above equation as column vectors, resulting in the following equation :

$$x' = f(t, x) \tag{10}$$

$$\text{where} \quad x = [x_1, x_2, \ldots, x_n]^T, \quad x' = \frac{dx}{dt}, \quad \text{and} \quad f = [f_1, f_2, \ldots, f_n]^T \tag{11}$$

The Runge-Kutta form a family of methods because they assert different but related ways of going about approximating a solution to equations of the above form. The methods in this family of solutions range from relatively-innaccurate but computationally-trivial to very-accurate but slightly more computationally complex. The RK4 is somewhere in the middle, taking a weighted average of the rate of change at 4 different points for each interval to compute the next state. If we were working with systems that are more complex than a double-pendulum, a different member from the Runge-Kutta family that takes more samples per interval may be preferable.

Brief description

The fourth-order Runge-Kutta method is a widely used technique for solving initial value problems in ODEs. It's particularly effective for solving ODEs that cannot be solved analytically.

The basic idea is to use weighted averages of function values at different points within a small time interval to approximate the solution. The most common form is the fourth-order Runge-Kutta method, which involves four evaluations of the derivative function at different points.

For a single variable ODE $\frac{dy}{dt} = f(t, y)$, the update step at each iteration is given by

$$y_{n+1} \approx y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{12}$$

where h is the step size, t_n is the current time, y_n is the current solution, and k_1, k_2, k_3 and k_4 are the estimates of the slope at different point.

The values of k_i for $i = 1, 2, 3, 4$ are computed as follows:

$$k_1 = hf(t_n, y_n) \tag{13}$$

$$k_2 = hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \tag{14}$$

$$k_3 = hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \tag{15}$$

$$k_4 = hf(t_n + h, y_n + k_3) \tag{16}$$

This method provides a good balance between accuracy and computational efficiency and is widely used for solving ODEs in various scientific and engineering applications. The accuracy of the Runge-Kutta method makes it suitable for simulating physical systems and other dynamic processes.