$$L = \frac{1}{2}ml^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right] + mgl(2\cos\phi_1 + \cos\phi_2)$$

For small angle approximations:

$$L = \frac{1}{2}ml^2[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2] + mgl\left(2\left(1 - \frac{\phi_1^2}{2}\right) + \left(1 - \frac{\phi_2^2}{2}\right)\right)$$

$$L = \frac{1}{2}ml^2[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2] + mgl\left(3 - \phi_1^2 - \frac{\phi_2^2}{2}\right)$$

$$L = ml^2\dot{\phi}_1^2 + \frac{1}{2}ml^2\dot{\phi}_2^2 + ml^2\dot{\phi}_1\dot{\phi}_2 + 3mgl - mgl\phi_1^2 - mgl\frac{\phi_2^2}{2}$$

EOM:

$$\frac{\partial L}{\partial \phi_1} = -2mgl\phi_1, \quad \frac{\partial L}{\partial \dot{\phi}_1} = 2ml^2\dot{\phi}_1 + ml^2\dot{\phi}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) = 2ml^2\ddot{\phi}_1 + ml^2\ddot{\phi}_2$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) - \frac{\partial L}{\partial \phi_1} = 0 \therefore 2ml^2\ddot{\phi}_1 + ml^2\ddot{\phi}_2 + 2mgl\phi_1 = 0 \quad (1)$$

$$\frac{\partial L}{\partial \phi_2} = -mgl\phi_2, \quad \frac{\partial L}{\partial \dot{\phi}_2} = ml^2\dot{\phi}_2 + ml^2\dot{\phi}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) = ml^2\ddot{\phi}_2 + ml^2\ddot{\phi}_1$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) - \frac{\partial L}{\partial \phi_2} = 0 \therefore ml^2\ddot{\phi}_2 + ml^2\ddot{\phi}_1 + mgl\phi_2 = 0 \quad (2)$$

$$\left.\begin{aligned}(1) &\Rightarrow \ddot{\phi}_1 + \frac{1}{2}\ddot{\phi}_2 + \frac{g}{l}\phi_1 = 0 \\ (2) &\Rightarrow \ddot{\phi}_2 + \ddot{\phi}_1 + \frac{g}{l}\phi_2 = 0\end{aligned}\right\}$$ 2^{nd} order ODE corresponding to a doub. pend. under small angle approximations

$\therefore$ can be solved analytically

compact matrix form: $\begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix}\ddot{\phi}_1 \\ \ddot{\phi}_2\end{bmatrix} + \begin{bmatrix} g/l & 0 \\ 0 & g/l \end{bmatrix}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = 0$

$\hookrightarrow M\ddot{\phi} + K\phi = 0$ for $M = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$, $K = \frac{g}{l}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\phi = \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}$, $\ddot{\phi} = \begin{bmatrix}\ddot{\phi}_1 \\ \ddot{\phi}_2\end{bmatrix}$

$\hookrightarrow$ solutions of the form:

$$\phi = \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = Re\left(\begin{bmatrix}x_{11} \\ x_{12}\end{bmatrix}e^{i\omega_1 t} + \begin{bmatrix}x_{12} \\ x_{22}\end{bmatrix}e^{i\omega_2 t}\right), \quad \omega_1, \omega_2 = \text{normal frequencies}$$

$\left.\begin{array}{l}\omega_1, \omega_2 = \text{eigenvalues} \\ \begin{bmatrix}x_{1,1} \\ x_{1,2}\end{bmatrix}, \begin{bmatrix}x_{1,2} \\ x_{2,2}\end{bmatrix} = \text{eigenvectors}\end{array}\right\}$ of the eq: $|K - \omega^2 M| = 0$

$$|K - w^2 M| = 0 \Rightarrow \begin{vmatrix} g/L - w^2 & 0 - w^2/2 \\ 0 - w^2 & g/L - w^2 \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{g}{L} - w^2\right)\left(\frac{g}{L} - w^2\right) - \left(-\frac{w^2}{2}\right)\left(-w^2\right) = 0$$

$$\Rightarrow \left(\frac{g}{L} - w^2\right)^2 - \left(\frac{w^2}{\sqrt{2}}\right)^2 = 0$$

$$\Rightarrow \left(\frac{g}{L} - w^2 - \frac{w^2}{\sqrt{2}}\right)\left(\frac{g}{L} - w^2 + \frac{w^2}{\sqrt{2}}\right) = 0$$

$$\Rightarrow \frac{g}{L} = w^2 + \frac{w^2}{\sqrt{2}} \quad \text{and} \quad \frac{g}{L} = w^2 - \frac{w^2}{\sqrt{2}}$$

$$\Rightarrow \frac{g}{L} = \frac{(\sqrt{2}+1)w^2}{\sqrt{2}} \quad \text{and} \quad \frac{g}{L} = \frac{(\sqrt{2}-1)}{\sqrt{2}} w^2$$

$$\Rightarrow w^2 = \frac{\sqrt{2}}{\sqrt{2}+1}\frac{g}{L} \quad \text{and} \quad w^2 = \frac{\sqrt{2}}{\sqrt{2}-1}\frac{g}{L}$$

$$\Rightarrow w^2 = \frac{\sqrt{2}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\frac{g}{L} \quad \text{and} \quad w^2 = \frac{\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\frac{g}{L}$$

$$\Rightarrow w^2 = \frac{(2-\sqrt{2})}{2-1}\frac{g}{L} \quad \text{and} \quad w^2 = \frac{2+\sqrt{2}}{(2-1)}\frac{g}{L}$$

$$\Rightarrow w^2 = \frac{g}{L}(2-\sqrt{2}),\ \frac{g}{L}(2+\sqrt{2})$$

$$\Rightarrow w = \underbrace{\sqrt{\frac{g}{L}}\sqrt{2-\sqrt{2}}}_{=w_1},\ \underbrace{\sqrt{\frac{g}{L}}\sqrt{2+\sqrt{2}}}_{=w_2}$$

eigenvalues = normal frequencies corresponding to this system, now need eigenvectors

For w_1: $[K - w_1^2 M]x = 0 \Rightarrow \begin{bmatrix} g/L - w_1^2 & -w_1^2/2 \\ -w_1^2 & g/L - w_1^2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} g/L - g/L(2-\sqrt{2}) & -\frac{1}{2}\frac{g}{L}(2-\sqrt{2}) \\ -g/L(2-\sqrt{2}) & g/L - g/L(2-\sqrt{2}) \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{g}{L}\begin{bmatrix} 1 - 2 + \sqrt{2} & -1 + \frac{1}{\sqrt{2}} \\ -2 + \sqrt{2} & 1 - 2 + \sqrt{2} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 + \sqrt{2} & -1 + 1/\sqrt{2} \\ -2 + \sqrt{2} & -1 + \sqrt{2} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\sqrt{2}R_1 + R_2 \rightarrow R_2$

$$\Rightarrow \begin{bmatrix} -1 + \sqrt{2} & -1 + 1/\sqrt{2} \\ \begin{matrix} -\sqrt{2} + 2 \\ -2 + \sqrt{2} \end{matrix} & \begin{matrix} -\sqrt{2} + 1 \\ -1 + \sqrt{2} \end{matrix} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 + \sqrt{2} & -1 + 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow x_{11}(-1+\sqrt{2}) + \left(-1+\frac{1}{\sqrt{2}}\right)x_{21} = 0$$

$$\Longrightarrow x_{11}(-1+\sqrt{2}) = \left(1-\frac{1}{\sqrt{2}}\right)x_{21}$$

$$\Longrightarrow \frac{x_{21}}{x_{11}} = \frac{(-1+\sqrt{2})}{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)} = \frac{\cancel{\sqrt{2}-1}}{\cancel{\sqrt{2}-1}} \times \sqrt{2} = \sqrt{2}$$

$\therefore$ eigenvector for ω_1 : $\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$

For ω_2: $[K - \omega_2^2 M]X = 0 \Longrightarrow$

$$\Longrightarrow \begin{bmatrix} g/l - \frac{g}{l}(2+\sqrt{2}) & -1/2\, g/l\,(2+\sqrt{2}) \\ -g/l\,(2+\sqrt{2}) & g/l - g/l\,(2+\sqrt{2}) \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \frac{g}{l} \begin{bmatrix} 1-2-\sqrt{2} & -1-1/\sqrt{2} \\ -2-\sqrt{2} & 1-2-\sqrt{2} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -1-\sqrt{2} & -1-1/\sqrt{2} \\ -2-\sqrt{2} & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\sqrt{2}R_1 - R_2 \rightarrow R_2$

$$\Longrightarrow \begin{bmatrix} -1-\sqrt{2} & -1-1/\sqrt{2} \\ -\sqrt{2}-2+2-\sqrt{2} & -\sqrt{2}-1+1+\sqrt{2} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -1-\sqrt{2} & -1-1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow (1-\sqrt{2})x_{21} - \left(1+\frac{1}{\sqrt{2}}\right)x_{22} = 0$$

$$\Longrightarrow (1-\sqrt{2})x_{21} = \left(1+\frac{1}{\sqrt{2}}\right)x_{22}$$

$$\Longrightarrow \frac{x_{22}}{x_{11}} = -\frac{(\cancel{1+\sqrt{2}})}{\left(\frac{\cancel{\sqrt{2}+1}}{\sqrt{2}}\right)} = -\sqrt{2}$$

$$\Longrightarrow x_{22} = -\sqrt{2}\,x_{21}$$

$\therefore$ eigenvector for ω_2 : $\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$

$\omega_1 = \sqrt{g/l}\,\sqrt{2-\sqrt{2}}$
$\omega_2 = \sqrt{g/l}\,\sqrt{2+\sqrt{2}}$

$\therefore$ solutions of
$$\left.\begin{aligned}\ddot{\phi}_1 + \tfrac{1}{2}\ddot{\phi}_2 + \tfrac{g}{l}\phi_1 &= 0\\ \ddot{\phi}_2 + \ddot{\phi}_1 + \tfrac{g}{l}\phi_2 &= 0\end{aligned}\right\}$$

in compact matrix form $M\ddot{\phi} + K\phi = 0$

General solution: $\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \mathrm{Re}\left(\begin{bmatrix}1\\ \sqrt{2}\end{bmatrix}e^{i\omega_1 t} + \begin{bmatrix}1\\ -\sqrt{2}\end{bmatrix}e^{i\omega_2 t}\right)$

$(e^{ix} = \cos x + i\sin x)$

$$\therefore \begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix} = c_1\begin{bmatrix}1\\ \sqrt{2}\end{bmatrix}\cos\omega_1 t + c_2\begin{bmatrix}1\\ -\sqrt{2}\end{bmatrix}\cos\omega_2 t$$

$\phi_1(t) = c_1\cos\omega_1 t + c_2\cos\omega_2 t$
$\phi_2(t) = \sqrt{2}c_1\cos\omega_1 t - \sqrt{2}c_2\cos\omega_2 t$

Linear combinations of 2 sinusoidal waves

$\therefore$ There are 2 distinct harmonic oscillations that combine together in different proportions to create the general solution of the double pendulum

$$\underbrace{\sqrt{2}\phi_1 + \phi_2}_{\text{lin. combo of } \phi_1 \text{ and } \phi_2} = \sqrt{2}c_1\cos\omega_1 t + \sqrt{2}c_2\cos\omega_2 t + \sqrt{2}c_1\cos\omega_1 t - \sqrt{2}c_2\cos\omega_2 t$$
$$= 2\sqrt{2}c_1\underbrace{\cos\omega_1 t}_{\text{pure sine wave}}$$

$$\sqrt{2}\phi_1 - \phi_2 = \sqrt{2}c_1\cos\omega_1 t + \sqrt{2}c_2\cos\omega_2 t - \sqrt{2}c_1\cos\omega_1 t + \sqrt{2}c_2\cos\omega_2 t$$
$$= 2\sqrt{2}c_2\underbrace{\cos\omega_2 t}_{\text{sinusoidal variation}}$$

$\phi_1 \rightarrow$ Not SHM
$\phi_2 \rightarrow$ Not SHM
$\sqrt{2}\phi_1 + \phi_2 \rightarrow$ SHM
$\sqrt{2}\phi_1 - \phi_2 \rightarrow$ SHM

$\Rightarrow$ Double pendulum is behaving as a coupled oscillator
$\therefore$ there are certain combos. of coordinates that execute a SHM
The coords of the individual particles themselves do not execute a SHM but some lin combo of the coords will execute a SHM.

individual solutions don't execute as HM.

Motion of the particles m_1, m_2 are not executing SHM but the lin. combin of their coords exhibits some kind of sinusoidal vartion $\therefore$ these are normal coords general solution is a lin combo of simple harmonic oscillations

By giving a specific initial condition, we can make the doub. pendulum execute SHM. The initial conditions and configurations are known as normal modes.

Normal modes:

$\phi_1(t) = c_1 \cos\omega_1 t + c_2 \cos\omega_2 t$

$\phi_2(t) = \sqrt{2}c_1 \cos\omega_1 t - \sqrt{2} c_2 \cos\omega_2 t$

can choose some initial condition s.t. $c_2 = 0$

At $t=0$, $\phi_1(t=0) = \phi_0 = 15°$

$\phi_2(t=0) = \sqrt{2}c_1 \cos\omega_1 t = \sqrt{2}\phi_1$

$= \sqrt{2}\phi_0 = \sqrt{2}(15°)$

This initial condition gives some kind of sinusoidal variation of the entire double pendulum system ∴ solution is a simple harmonic oscillation

Both varying sinusoidally. So for the specific IC, the entire system executes simple harmonic oscillations. This is known as normal mode.

In general, the general solutions for different initial conditions aren not SHM, they are a lin combo of two SHM but for this specific initial condition, the entire set up is executing SHM. wrt a freq of ω_1

choosing IC s.t. $c_1 = 0$, at $t=0$, $\phi_1(t=0) = \phi_0 = 15°$

$\phi_2(t=0) = -\sqrt{2}c_2 \cos\omega_2 t$

$= -\sqrt{2}\phi_1(t=0)$

$= -\sqrt{2}\phi_0$

$= -\sqrt{2}(15°)$.