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The Geometry of Hurwitz Space

presented by **Anand Pankaj Patel**

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The Geometry of Hurwitz Space

A dissertation presented

by

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I dedicate this thesis
to my grandparents and Devin

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The Geometry of Hurwitz Space

Abstract

We explore the geometry of certain special subvarieties of spaces of branched covers which we call the *Maroni* and *Casnati-Ekedahl* loci. Our goal is to understand the divisor theory on compactifications of Hurwitz space, with the aim of providing upper bounds for slopes of sweeping families of d-gonal curves.

Contents

1. Introduction

The moduli space of genus g curves, $\mathcal{M}_g$, has been a central object of study in algebraic geometry for over a century. Classically, geometers studied a curve C by considering, roughly speaking, all of the embeddings of C to projective spaces. For example, it was well known that all projective curves C can be realized as a plane curve of some degree having only nodes as singularities, and such planar representations were used to demonstrate the unirationality of $\mathcal{M}_g$ for $g \leq 10$.

There was another way to realize an algebraic curve - not as sitting inside a projective space, but as a simply-branched covering of the projective line $\alpha \colon C \longrightarrow \mathbf{P}^1$. Fixing the degree d, genus g, and branch locus determines finitely many covers $\alpha \colon C \longrightarrow \mathbf{P}^1$ with the given invariants, so the dimension of the space of such coverings, *Hurwitz space* $\mathcal{H}_{d,g}$, is three less than the number of branch points, or $2g + 2d - 5$. By studying the monodromy of the branch point map $Br \colon \mathcal{H}_{d,g} \longrightarrow \mathcal{M}_{0,b}$, Clebsch proved the irreducibility of $\mathcal{H}_{d,g}$, and thereby gave the first proof of the irreducibility of $\mathcal{M}_g$, since every genus g curve admits a map to $\mathbf{P}^1$ of some uniform large degree.

In the late 70's Harris and Mumford published one of the most celebrated results about $\mathcal{M}_g$ - that it is of general type for $g \geq 22$ [14]. The technique of proof boiled down to computing the classes of particular effective divisors in the rational Picard group of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. These divisors were intimately related to the spaces of *admissible covers*, $\overline{\mathcal{H}}_{d,g}$, a compactification of $\mathcal{H}_{d,g}$. Specifically, for odd genera $g = 2k+1$, Harris and Mumford considered the divisor $\mathcal{D}_{k+1}$ consisting of the closure of the locus of curves possessing a degree $k + 1$ map to $\mathbf{P}^1$, i.e. the image of the natural forgetful map $F \colon \overline{\mathcal{H}}_{k+1,g} \longrightarrow \overline{\mathcal{M}}_g$. For even genera $g = 2k$, the Gieseker-Petri divisor $\mathcal{GP}^1_{k+1}$, which can be realized as the branch locus of the (generically finite) map $F \colon \overline{\mathcal{H}}_{k+1,g} \longrightarrow \overline{\mathcal{M}}_g$, played an analogous role. Our ultimate goal is to understand the effective divisor theory of compactifications of Hurwitz space. The *ramification* divisor of the forgetful map F will be one of the two "types" of effective divisors we will study. In fact, one of the results we will

see in this first chapter is that this ramification divisor is always irreducible, hence the same is true for the Gieseker-Petri divisors.

The divisor theory of $\mathcal{H}_{d,g}$ turns out to be more interesting (and less understood) when d is small compared to g, and our general investigation naturally occurs in this range. The starting point of our exploration is a classical construction, which we now describe.

By the "geometric" interpretation of Riemann-Roch, for any $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g}$ the fibers of α span $\mathbf{P}^{d-2}$'s in $\mathbf{P}^{g-1}$, the canonical space of C. The totality of these $\mathbf{P}^{d-2}$'s forms the *associated scroll* containing C.

In this way we obtain a function:

$$\Psi\colon \Big\{\text{Degree } d \text{ Covers}\Big\} \longrightarrow \Big\{\text{Isomorphism types of } \mathbf{P}^{d-2}\text{-bundles over } \mathbf{P}^1\Big\}$$

By fixing an isomorphism type of associated scrolls, (which we will eventually denote by $\mathbf{P}\mathcal{E}$), we consider the closed *Maroni locus* $\mathcal{M}(\mathcal{E}) \subset \mathcal{H}_{d,g}$, consisting of the (closure of the) locus of covers $[\alpha\colon C \longrightarrow \mathbf{P}^1]$ whose associated scroll is isomorphic to $\mathbf{P}\mathcal{E}$. The Gieseker-Petri divisors $\mathcal{GP}^1_{k+1}$ mentioned earlier will be instances of these subvarieties $\mathcal{M}(\mathcal{E}) \subset \mathcal{H}_{d,g}$. One central question which we explore in the first chapter is: What, in general, can be said about the geometry of the subvarieties $\mathcal{M}(\mathcal{E})$?

It will become quite apparent that the study of the Maroni loci will naturally lead to the study of certain other subvarieties which we call the *Casnati-Ekedahl* loci. In fact, as we will point out in the last section, the study of these two sets of subvarieties is only the first step in understanding a fascinating and completely mysterious decomposition of Hurwitz space by *higher syzygy loci*.

We will begin our explorations in chapter 1 by recalling a structure theorem, due to Casnati and Ekedahl [3], which describes the resolution of the ideal of the curve C in its associated scroll. This will immediately lead to the central objects of study: the Maroni loci $\mathcal{M}(\mathcal{E})$ and the Casnati-Ekedahl loci $\mathcal{C}(\mathcal{F})$. We then examine the necessary conditions (mainly due to Ohbuchi), Conditions $1, 2$, and 3, for a scroll $\mathbf{P}\mathcal{E}$ to occur as an associated scroll of some

cover, i.e. in the image of Ψ. (Condition 3 is actually first mentioned in section 3, where we study the degree 4 case carefully, thereby giving some motivation for the condition.)

We have devoted a few sections of chapter 1 to the study of Maroni and Casnati-Ekedahl loci in low degree cases. We provide examples showing that the Maroni loci are often not of expected codimension, and may have multiple components. We also classify all reducible Casnati-Ekedahl loci in $\mathcal{H}_{4,g}$, and describe the components of these loci in terms of the resolvent cubic morphism.

In section 1.7, we provide the most general result known about the geometry of the Maroni loci in $\mathcal{H}_{d,g}$. The key input comes from a theorem of Tyomkin [23] stating that all Severi varieties on all Hirzebruch surfaces are irreducible. Using this theorem, we prove that the only Maroni *divisor* are the "expected" ones, i.e. although some Maroni loci may have larger dimension than expected, there is never an instance of an unexpected Maroni *divisor*. As a corollary to the main theorem in section 5, we deduce the irreducibility of the Gieseker-Petri divisors $\mathcal{GP}^1_{k+1}$.

Section 1.8 explores what is known about the Maroni and Casnati-Ekedahl loci in $\mathcal{H}_{5,g}$. In particular, we will see that all Casnati-Ekedahl loci are irreducible, and that there are "no unexpected Casnati-Ekedahl divisors". This result, along with the aforementioned theorem on the nonexistence of unexpected Maroni divisors is used in proving the first results of chapter 2, which we now summarize.

In chapter 2 we introduce the well-known "Picard Rank Conjecture" for $\mathcal{H}_{d,g}$ stating that the rational Picard group is trivial. When the degree d is large compared to the genus g, Mochizuki [18] shows that the conjecture is equivalent to Harer's theorem [13]. We verify the Picard Rank Conjecture for $\mathcal{H}_{3,g}$, $\mathcal{H}_{4,g}$, and $\mathcal{H}_{5,g}$. Section 2.2 relates a conjecture about Picard groups of Severi varieties on Hirzebruch surfaces with the Picard Rank Conjecture. As a corollary, we recover some results of Edidin [12].

Chapter 3 is devoted to the study of the boundary divisors occurring in the admissible cover compactification $\overline{\mathcal{H}}_{d,g}$. We prove that the boundary divisors are independent, thereby putting the divisor calculations of chapter 4 on firm ground. We also introduce the idea of a

partial pencil family - a technique of constructing test families which have "controlled" and easily determined intersections with boundary divisors. Partial pencil families are heavily used in chapter 4.

Chapter 4 investigates enumerative questions about the geometry of Hurwitz space. In particular, we find the classes of the Maroni and Casnati-Ekedahl divisors, and use them to produce sharp upper bounds for slopes of sweeping 4-gonal and 5-gonal curves. These results generalize the result of Stankova [22].

We work over an algebraically closed field k of characteristic 0.

1.1. **Objects of study.** The Hurwitz space (or "small" Hurwitz space) $\mathcal{H}_{d,g}$ is the Deligne-Mumford stack of dimension $2d + 2g - 5$ which represents the functor of degree d, genus g *simply-branched* covers of $\mathbf{P}^1$. More precisely, for a scheme S, the objects of $\mathcal{H}_{d,g}(S)$ are diagrams:

$$\mathcal{C} \xrightarrow{\alpha} \mathcal{P}$$
$$\varphi \searrow \quad \swarrow \pi$$
$$S$$

where π is a $\mathbf{P}^1$-bundle, φ is a flat family of smooth genus g curves, and α is a finite, flat, degree d map which restricts to a simply-branched map on all geometric fibers of φ.

By associating to a cover $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g}$ its branch divisor $[B \subset \mathbf{P}^1] \in \mathcal{M}_{0,b}$, we arrive at the *branch morphism*

$$Br \colon \mathcal{H}_{d,g} \longrightarrow \mathcal{M}_{0,b}$$

which is finite and unramified. For us, $\mathcal{M}_{0,b}$ will denote the moduli space of unordered b-tuples of distinct points in $\mathbf{P}^1$, modulo PGL_2 equivalence.

In this first chapter, we will be dealing primarily with the geometry of the underlying coarse space. The distinction between the stack and coarse space will not be relevant until chapter 4, where we will compute intersection numbers. Since the coarse space of $\mathcal{M}_{0,b}$ is affine, the (coarse) Hurwitz space is also an affine variety of dimension $2d + 2g - 5$.

Now let $[\alpha\colon C \longrightarrow \mathbf{P}^1]$ be any cover and consider the sequence of sheaves on $\mathbf{P}^1$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow \alpha_*\mathcal{O}_C \longrightarrow \mathcal{E}^\vee \longrightarrow 0$$

Dualizing, and using Serre duality to identify $(\alpha_*\mathcal{O}_C)^\vee \simeq \alpha_*\omega_\alpha$, we obtain:

$$0 \longrightarrow \mathcal{E} \longrightarrow \alpha_*\omega_\alpha \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow 0$$

Pulling back to C and composing with the relative evaluation map $\mathrm{ev}_\alpha\colon \alpha^*\alpha_*\omega_\alpha \longrightarrow \omega_\alpha$, we obtain the map of sheaves on C:

$$\mathrm{ev}_\alpha|_{\mathcal{E}}\colon \alpha^*\mathcal{E} \longrightarrow \omega_\alpha$$

This relative evaluation homomorphism $\mathrm{ev}_\alpha|_{\mathcal{E}}$ is shown to be surjective in the work of Casnati and Ekedahl [3] and therefore defines a relative embedding:

(1.1)
$$C \xrightarrow{\ i\ } \mathbf{P}\mathcal{E}$$
$$\alpha \searrow \quad \downarrow \pi$$
$$\mathbf{P}^1$$

Furthermore, in [3] it is shown that the fibers of α are arithmetically Gorenstein in the $\mathbf{P}^{d-2}$ fibers of π, and that the ideal sheaf $\mathcal{I}_C$ of C has the following type of resolution:

(1.2) $\quad 0 \longrightarrow \pi^*\mathrm{det}\mathcal{E}(-d) \longrightarrow \pi^*\mathcal{N}_{d-3}(-d+2) \longrightarrow ... \longrightarrow \pi^*\mathcal{N}_1(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0$

In every fiber of π, this resolution restricts to the minimal resolution of d general points in $\mathbf{P}^{d-2}$. There is a natural duality among the syzygy bundles: $\mathcal{N}_i \simeq (\mathcal{N}_{d-2-i})^\vee \otimes \det \mathcal{E}$.

We will call $\mathcal{E}$ the *reduced direct image* of α, and

$$\mathcal{F} := \mathcal{N}_1 = \pi_* \mathcal{I}_C(2)$$

the *bundle of quadrics* of α. To emphasize the dependence on α, we will sometimes use the notation $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$. We have the following exact sequence relating the two bundles:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^2\mathcal{E} \longrightarrow \alpha_*\omega_\alpha^{\otimes 2} \longrightarrow 0$$

Throughout this thesis, the natural hyperplane class associated to a projective bundle will be denoted by ζ, and a fiber class will be denoted by f. We use post-Grothendieck conventions for projective bundles: Surjections $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$ onto line bundles correspond to geometric sections $s \colon \mathbf{P}^1 \longrightarrow \mathbf{P}\mathcal{E}$.

1.2. **The Maroni and Casnati-Ekedahl loci.** For a fixed rank $d-1$, degree $g + d - 1$ locally free sheaf $\mathcal{E}$ on $\mathbf{P}^1$, we define the subvariety $\mathcal{M}(\mathcal{E})$ as:

$$\mathcal{M}(\mathcal{E}) := \overline{\left\{ [\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g} \mid \mathcal{E}_\alpha = \mathcal{E} \right\}}$$

Maroni [17] first implicitly considered these subvarieties in the trigonal setting, so we will call $\mathcal{M}(\mathcal{E})$ the *Maroni locus* of $\mathcal{E}$.

Similarly, for a fixed $\mathcal{F}$ of the appropriate rank and degree define $\mathcal{C}(\mathcal{F})$ as:

$$\mathcal{C}(\mathcal{F}) := \overline{\left\{ [\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g} \mid \mathcal{F}_\alpha = \mathcal{F} \right\}}$$

We will call this subvariety the *Casnati-Ekedahl locus* of $\mathcal{F}$. The main objective of this first chapter is to understand general geometric properties of $\mathcal{M}(\mathcal{E})$ and $\mathcal{C}(\mathcal{F})$. In particular, we will review three conditions for the non-emptiness of $\mathcal{M}(\mathcal{E})$ essentially due to [20]. (Our method of arriving at these conditions is more geometric.) We will also see several examples which illustrate the complexity of the geometry of these varieties.

1.2.1. *Preliminary Observations.* We will now guide the reader through some basic properties of the bundles associated to a cover. Much of what follows is simply a unified treatment of well-known observations and results. We will provide references as we go - anything stated without reference is new (at least as far as the author can tell).

From the basic setup it follows that $\mathcal{E}$ is a rank $d-1$ locally free sheaf of degree $g+d-1$ and that the rank of $\mathcal{F}$ is $\frac{d(d-3)}{2}$. Furthermore, from the exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^2\mathcal{E} \longrightarrow \alpha_*\omega_\alpha^{\otimes 2} \longrightarrow 0$$

we see that the degree of $\mathcal{F}$ is $(d-3)(g+d-1)$.

As a locally free sheaf on $\mathbf{P}^1$, we may write

$$\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus ... \oplus \mathcal{O}_{\mathbf{P}^1}(a_{d-1})$$

where $a_1 \leq a_2 \leq ... \leq a_{d-1}$. The integers a_i (or slight variants thereof) are known as "scrollar invariants" in the existing literature. (The reader should note that knowing the sequence $(a_1, ... a_{d-1})$ is equivalent to knowing the sequence of numbers $h^0(C, \alpha^*\mathcal{O}(m))$.) We first notice that, if C is connected, all a_i's will be positive.

We immediately provide a first example which will be useful for us in the future.

Example 1.1 (Rational Covers)**.** Let $\alpha\colon R \longrightarrow \mathbf{P}^1$ be a degree d cover where R is a rational curve. The degree of $\mathcal{E}_\alpha$ is $d-1$. Therefore, since all summands of $\mathcal{E}_\alpha$ are positive, we

conclude that

$$\mathcal{E}_\alpha = \mathcal{O}(1)^{\oplus d-1}$$

The bundle $\mathbf{P}\mathcal{E}$ is isomorphic to the split bundle $\mathbf{P}^{d-2} \times \mathbf{P}^1$. Projection onto $\mathbf{P}^{d-2}$ embeds R as a rational normal curve of degree $d-2$. In particular, the curve $R \subset \mathbf{P}\mathcal{E}$ rests as a $(d,1)$ curve in the surface $S = \mathbf{P}^1 \times \mathbf{P}^1$ embedded in $\mathbf{P}^{d-2} \times \mathbf{P}^1$ via the relative Veronese embedding (relative to the first projection) of degree $d-2$. This allows us to compute $\mathcal{F}_\alpha$. By considering the sequence

$$0 \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_R(2) \longrightarrow \mathcal{I}_{R \subset S}(2) \longrightarrow 0$$

it easy to show that

$$\mathcal{F}_\alpha = \mathcal{O}(1)^{\oplus d-3} \oplus \mathcal{O}(2)^{\oplus \binom{d-2}{2}}$$

We simply note that the global sections of $\mathcal{F}_\alpha(-2)$ correspond to quadrics in $\mathbf{P}^{d-2}$ containing the rational normal curve R.

The line bundle $\mathcal{O}_{\mathbf{P}\mathcal{E}}(1) \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(-a_{d-1})$ is then the lowest degree *effective* hyperplane divisor class on $\mathbf{P}\mathcal{E}$. Since C is arithmetically Gorenstein over $\mathbf{P}^1$, the fibers of α are always in general linear position in $\mathbf{P}^{d-2}$. In particular, $\mathcal{O}_{\mathbf{P}\mathcal{E}}(1) \otimes \pi^* \mathcal{O}_{\mathbf{P}^1}(-a_{d-1})$ must restrict to a line bundle of nonnegative degree on C. Observing that $\mathcal{O}_{\mathbf{P}\mathcal{E}}(1)|_C = \omega_\alpha$, we obtain the first condition on the $(d-1)$-tuple $(a_1, a_2, ..., a_{d-1})$:

Condition 1. If $\mathcal{E}$ is the reduced direct image of a cover $[\alpha \colon C \longrightarrow \mathbf{P}^1]$ with C irreducible, then

$$a_{d-1} \leq \frac{2g + 2d - 2}{d}$$

Now we shift our attention to the minimal degree of a summand, a_1. In order to see the condition on a_1, we will need a construction which is interesting in its own right and which will be appear again in another chapter.

1.3. **The associated Hirzebruch surfaces.** The construction which we describe in this section can first be found in the work of Ohbuchi. We would like to give two interpretations

of the construction, one which is "algebraic", and one which is "geometric." The geometric interpretation seems to be new. Consider the inclusion of sheaves

$$\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-a_1) \hookrightarrow \alpha_* \mathcal{O}_C.$$

From this we obtain a map of $\mathcal{O}_{\mathbf{P}^1}$-algebras,

$$\mathbf{Spec}\left(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-3a_1) \oplus ...\right) \longrightarrow \mathbf{Spec}\left(\alpha_* \mathcal{O}_C\right)$$

thereby defining a morphism $j : C \longrightarrow \mathbf{F}_{a_1}$ whose image avoids the directrix σ.

1.3.1. *Geometric realization.* We view C as naturally embedded $C \subset \mathbf{P}\mathcal{E}$. Pick a surjection

$$\mathcal{E} \longrightarrow \mathcal{O}(a_1) \longrightarrow 0$$

This corresponds to picking a section $\sigma \subset \mathbf{P}\mathcal{E}$ of minimal degree (with respect to any fixed hyperplane divisor).

The section σ together with the curve C provide $d+1$ points $\sigma_t \cup \alpha_t$ in the fibers $\mathbf{P}_t^{d-2} \subset \mathbf{P}\mathcal{E}$, where t is representing a coordinate on $\mathbf{P}^1$. Since we assume that the map $\alpha : C \longrightarrow \mathbf{P}^1$ is simply branched, monodromy considerations show that the general such set of $d + 1$ points must be in general linear position. Therefore, there is a unique rational normal curve $R_t \subset \mathbf{P}_t^{d-2}$ passing through the $d + 1$ points $\sigma_t \cup \alpha_t$ for $t \in \mathbf{P}^1$ general.

The closure of the union of these rational normal curves R_t is a birationally ruled surface $S \subset \mathbf{P}\mathcal{E}$ containing C and σ. When we blow down those components of the fibers of $\pi : S \longrightarrow \mathbf{P}^1$ which *avoid* the section σ, we arrive at a Hirzebruch surface $\mathbf{F}_n$, and a birational map $j : C \longrightarrow \mathbf{F}_n$. In fact, it is not hard to show that $n = a_1$, and that this geometric procedure provides the same morphism $j : C \longrightarrow \mathbf{P}^1$ which was constructed algebraically in the previous subsection.

Either way, if we assume that j is birational onto its image then we obtain a condition on the integer a_1. The image $j(C)$ has fixed divisor class $d \cdot \tau$ where τ is the unique section

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class satisfying $\tau^2 = a_1$. Comparing the geometric genus of $j(C)$ with its arithmetic genus, we immediately arrive at:

Condition 2. If $\mathcal{E}$ is the reduced direct image of a cover $[\alpha\colon C \longrightarrow \mathbf{P}^1]$, and if α does not factor as the composite of two finite maps, then

$$a_1 \geq \frac{g + d - 1}{\binom{d}{2}}.$$

(Note that for prime d or for a simply branched cover, one can obviously drop the extra hypothesis about α.)

We will arrive at the final nontrivial condition on $\mathcal{E}$, due to Ohbuchi, after we explore degree 4 Hurwitz spaces in section 1.5. For now we observe that, since $\mathcal{H}_{d,g}$ is irreducible, the general cover $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g}$ has a well-defined generic reduced direct image $\mathcal{E}_{gen}$ and bundle of quadrics $\mathcal{F}_{gen}$. One may expect that these bundles are as balanced as possible. As we will see in Proposition 1.1, this will always be true for the generic reduced direct image $\mathcal{E}_{gen}$, but unfortunately will not always hold for bundles of quadrics, as the following example shows:

Example 1.2 $(h^1(\mathcal{E}nd\,\mathcal{F}_{gen}) \neq 0)$**.** Consider the Hurwitz space $\mathcal{H}_{d,4}$ with $d \geq 6$, and let $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,4}$ be a general cover. The reduced direct image is

$$\mathcal{E} = \mathcal{O}(1)^{\oplus d-5} \oplus \mathcal{O}(2)^{\oplus 4}$$

The bundle of quadrics $\mathcal{F}$ is a subbundle of

$$\mathcal{S}^2\mathcal{E} = \mathcal{O}(2)^{\oplus M} \oplus \mathcal{O}(3)^{\oplus N} \oplus \mathcal{O}(4)^{\oplus 10}$$

of rank $\frac{d(d-3)}{2}$ and degree $(d-3)(d+3)$.

In the factorization

$$
\begin{array}{ccc}
C & \xrightarrow{\ i\ } & \mathbf{P}\mathcal{E} \\
& \searchow{\alpha} & \downarrow{\pi} \\
& & \mathbf{P}^1
\end{array}
$$

The linear system $|\zeta - 2f|$ on $\mathbf{P}\mathcal{E}$ restricts to the complete canonical series on C. Furthermore, the system $|2\zeta - 4f|$ is simply the symmetric square of the system $|\zeta - 2f|$, and so we see that there is exactly one nonzero section of $\mathcal{O}_{\mathbf{P}\mathcal{E}}(2\zeta - 4f)$ vanishing on C. Therefore, $\mathcal{O}(4)$ occurs as a summand of $\mathcal{F}$ exactly once, and it is a maximal degree summand. The remaining summands of $\mathcal{F}$ cannot all be $\mathcal{O}(3)$'s, since the degrees would not add up. Therefore, $h^1(\mathcal{E}nd\,\mathcal{F}_{gen}) \neq 0$.

The reader may notice that the reason we took $g = 4$ is simply because the canonical model of a genus 4 curve is contained in a quadric, thereby forcing $\mathcal{F}$ to have $\mathcal{O}(4)$ as a summand. From here, we simply took d large enough so that $\mathcal{F}$ was forced to also have $\mathcal{O}(2)$ summands.

The example above indicates that there are hidden surprises in the analysis of $\mathcal{C}(\mathcal{F})$ for d large compared to g. On the other hand, for g large compared to d, which will be the central concern of most of this thesis, we are more fortunate:

Theorem 1.3 (Generic Behavior). *$\mathcal{E}_{gen}$ is rigid, i.e. $h^1(\mathcal{E}nd\,\mathcal{E}_{gen}) = 0$. $\mathcal{F}_{gen}$ is rigid for $\mathcal{H}_{4,g}$ and $\mathcal{H}_{5,g}$. For g sufficiently large compared to d, $\mathcal{F}_{gen}$ is rigid.*

Proof. The proof will be via degeneration. We will use our understanding of rational covers to deduce the first part of the proposition.

Write $g = (k-1)(d-1) + e$, where $0 \leq e < (d-1)$, and consider a chain of $k+1$ rational curves $\mathbf{P} := \cup_{i=1}^{k+1}\mathbf{P}_i^1$. Above $\mathbf{P}$ we will construct an appropriate degree d, arithmetic genus g cover, component by component. For each $i \leq k$, choose a general simply branched, degree d rational cover $\alpha_i \colon R_i \longrightarrow \mathbf{P}_i^1$, and glue the R_i appropriately (always generically) so as to construct a finite, arithmetic genus $(k-1)(d-1)$ cover $\alpha_L \colon R \longrightarrow \cup_{i=1}^{k}\mathbf{P}^1$. (The subscript L should be read as "left.")

Above the final component $\mathbf{P}_{k+1}^1$, consider a disjoint union of $d - e - 1$ "constant" rational curves $\cup_j A_j$ along with a rational curve B mapping $[e+1 \colon 1]$ onto $\mathbf{P}_{k+1}^1$. Let $X := \sqcup_j A_j \sqcup B$ be the disjoint union, and denote by $\alpha_R \colon X \longrightarrow \mathbf{P}_{k+1}^1$ the degree d map which is the union of degree 1 maps on the components A_j and the degree $e + 1$ map on B. We now choose a

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generic gluing $Y = X \cup R$, maintaining a finite map $\alpha \colon Y \longrightarrow \mathbf{P}$ which restricts to α_L and α_R.

As we have already seen, the reduced direct image $\mathcal{E}_{\alpha_L}$ restricts on every component $\mathbf{P}_i^1$, $(i \leq k)$, as

$$\mathcal{E}_{\alpha_i} = \mathcal{O}(1)^{\oplus d-1}$$

Furthermore, the reduced direct image of α_R is

$$\mathcal{E}_{\alpha_R} = \mathcal{O}(1)^{\oplus e} \oplus \mathcal{O}^{\oplus d-e-1}$$

It follows that the reduced direct image $\mathcal{E}_\alpha$ satisfies $h^1(\mathbf{P}, \mathcal{E}nd\,\mathcal{E}_\alpha) = 0$. By upper semi-continuity, we deduce the same for $\mathcal{E}_{gen}$.

Now we switch to the proof of the statements regarding $\mathcal{F}_{gen}$. Let us consider the broken curve R of arithmetic genus $(k-1)(d-1)$ which was used above. R is built from rational covers R_i, and as we have seen, these covers have bundles of quadrics $\mathcal{F}_{\alpha_i}$ of the form:

$$\mathcal{F}_{\alpha_i} = \mathcal{O}(1)^{\oplus d-3} \oplus \mathcal{O}(2)^{\oplus \binom{d-2}{2}}$$

Recall that $\mathcal{E}_{\alpha_L}$ is the reduced direct image of the broken cover $\beta \colon R \longrightarrow \cup_{i=1}^k \mathbf{P}_i^1$.

We now "transfer" all information into one fiber of the broken scroll $\mathbf{P}\mathcal{E}$. Consider the d points $Z := R_1 \cap R_2$ lying above the node $\mathbf{P}_1^1 \cap \mathbf{P}_2^1$, and let $\mathbf{P}_Z^{d-2}$ denote the fiber of $\mathbf{P}\mathcal{E}$ containing Z. The rational curves R_1 and R_2 both project into $\mathbf{P}_Z^{d-2}$ to a pair of rational normal curves passing through Z. By repeatedly projecting, we may view *all* of the R_i as rational normal curves in the projective space $\mathbf{P}_Z^{d-2}$. The resulting curve is abstractly isomorphic to R itself, with each component a general rational normal curve of degree $d-2$. For each component R_i, consider the vector space $V_i \subset H^0(\mathbf{P}_Z^{d-2}, \mathcal{O}(2))$ of quadrics containing R_i. Let $Z_{i,i+1} := R_i \cap R_{i+1}$, let $V_{Z_{i,i+1}}$ be the space of quadrics containing $Z_{i,i+1}$. Clearly, V_i and V_{i+1} are subspaces of $V_{Z_{i,i+1}}$.

It is easy to see that the calculation of $h^1(\mathcal{E}nd\,\mathcal{F})$ depends on the relative positions of the vector spaces V_i in the direct sum $\oplus_i V_{Z_{i,i+1}}$. For general choices of R_i, we can guarantee that all intersections will be proper, and therefore $h^1(\mathcal{E}nd\,\mathcal{F})$ will be zero for the resulting curve.

Now suppose $\gamma\colon E \longrightarrow \mathbf{P}^1$ is an *elliptic* degree d cover. The bundle of quadrics for γ is easily seen to be perfectly balanced:

$$\mathcal{F}_\gamma = \mathcal{O}(2)^{\oplus \frac{d(d-3)}{2}}$$

Therefore, by gluing together chains of elliptic covers to chains of rational covers, we may deduce that $h^1(\mathcal{E}nd\,\mathcal{F}_{gen}) = 0$ for all genera of the form $g = ad + b(d-1)$ with $a, b \geq 0$. Eventually, we may express every sufficiently large integer g as such a combination.

We can prove $h^1(\mathcal{E}nd\,\mathcal{F}_{gen}) = 0$ for $\mathcal{H}_{4,g}$ and $\mathcal{H}_{5,g}$ by similar arguments. As we've seen, elliptic covers always have perfectly balanced bundles of quadrics. Therefore, for $\mathcal{H}_{5,g}$, for example, we must only show that $\mathcal{F}_{gen}$ is rigid for $0 \leq g \leq 4$. We may then obtain the result for all genera by constructing chains of genus 1 covers followed by a genus $0, 1, 2, 3$, or 4 cover. The low genus cases are easily established using similar reasoning as in Example 1.2 above, so we leave them out.

$\hfill\square$

Very little is known about $\mathcal{M}(\mathcal{E})$ or $\mathcal{C}(\mathcal{F})$ in general. However, for low degrees (3, 4, and perhaps 5), a fairly complete picture can be obtained. Furthermore, by studying the low degree cases carefully, we will begin to see interesting phenomenon occurring in the geometry of these varieties. For this reason, we devote a few sections to the study of these low degree cases before returning to arbitrary d.

1.4. **The decomposition of $\mathcal{H}_{3,g}$ by Maroni loci.** We briefly review the geometry of the decomposition of $\mathcal{H}_{3,g}$ by the Maroni loci. In this case, Conditions 1 and 2 are equivalent in light of the constraint $a_1 + a_2 = g + 2$. The difference $a_2 - a_1$ is known in the literature as the *Maroni invariant* of the cover α.

Suppose $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{3,g}$ has reduced direct image $\mathcal{E}$. The associated scroll $\mathbf{P}\mathcal{E}$ is a Hirzebruch surface, so C as a member of the fixed linear series $|3\zeta - (g+2)f|$ on $\mathbf{P}\mathcal{E}$. (Recall that ζ will always denote the divisor class associated to $\mathcal{O}_{\mathbf{P}\mathcal{E}}(1)$, and f will denote the class of a fiber.) The normal bundle $N_{C/\mathbf{P}\mathcal{E}}$ is nonspecial, i.e. $H^1(C, N_{C/\mathbf{P}\mathcal{E}}) = 0$, therefore basic deformation theory tells us that *every deformation of $\mathbf{P}\mathcal{E}$ carries C along with it*. If we introduce the map on (mini)versal deformation spaces

$$\Psi\colon \underline{\mathrm{Def}}(\alpha) \longrightarrow \underline{\mathrm{Def}}(\mathbf{P}\mathcal{E})$$

we conclude that Ψ is formally smooth, and in particular *if $\mathbf{P}\mathcal{E}'$ specializes to $\mathbf{P}\mathcal{E}$, then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{E}')$*. Therefore, a *dense* open subset of $\mathcal{M}(\mathcal{E})$ can be presented as the quotient of an open subset $U \subset |3\zeta - (g+2)f|$ by $G = Aut\,\mathbf{P}\mathcal{E}$, the automorphism group of the surface $\mathbf{P}\mathcal{E}$. This greatly simplifies the study of the Maroni loci, and allows for a rather complete understanding of their geometry. We have

Proposition 1.4. *If $\mathcal{E} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$ is such that $a_1 < \frac{g+2}{3}$, then $\mathcal{M}(\mathcal{E})$ is empty. Otherwise, the Maroni loci $\mathcal{M}(\mathcal{E}) \subset \mathcal{H}_{3,g}$ are irreducible and of expected codimension $h^1(\mathcal{E}nd\,\mathcal{E})$. If $\mathcal{E}$ specializes to $\mathcal{E}'$, then $\mathcal{M}(\mathcal{E}') \subset \mathcal{M}(\mathcal{E})$, and, away from the locus of covers having automorphisms, the union of such $\mathcal{M}(\mathcal{E}')$ forms the singular locus of $\mathcal{M}(\mathcal{E})$.*

Proof. The linear series $|3\zeta - (g+2)f|$ contains smooth divisors if and only if $a_1 \geq \frac{g+2}{3}$, hence the first statement. The rest follows from the formal smoothness of Ψ, and the well-known description of the decomposition of the deformation space $\underline{\mathrm{Def}}(\mathbf{P}\mathcal{E})$ by isomorphism type of scroll. □

Therefore, we see that the decomposition (in fact, stratification) of $\mathcal{H}_{3,g}$ by Maroni loci behaves exactly as we would expect and hope. As we will see in the next section, almost every aspect of Proposition 1.4 fails in higher degree.

1.4.1. *The Maroni Divisor.* When the genus $g = 2k$ is even, the general reduced direct image $\mathcal{E}_{gen}$ is *perfectly* balanced with $\mathcal{E}_{gen} = \mathcal{O}_{\mathbf{P}^1}(k+1) \oplus \mathcal{O}_{\mathbf{P}^1}(k+1)$. Therefore, for the special bundle $\mathcal{E}_{div} := \mathcal{O}_{\mathbf{P}^1}(k) \oplus \mathcal{O}_{\mathbf{P}^1}(k+2)$ the Maroni locus $\mathbf{M} := \mathcal{M}(\mathcal{E}_{div})$ is irreducible of codimension 1 and is appropriately called the *Maroni divisor* in the existing literature.

When $k = 2$, the Maroni divisor $\mathbf{M} \subset \mathcal{H}_{3,4}$ is the locus of *Petri curves*, i.e. those genus 4 curves which lie on a quadric cone $Q \subset \mathbf{P}^3$ in canonical space. The resolution of the quadric cone is the scroll $\mathbf{P}\mathcal{E}_{div}$.

The Maroni divisor plays a central role in understanding the birational geometry of the locus of trigonal curves $\mathcal{T}_3 \subset \mathcal{M}_g$ [10] [22]. In particular, by computing the class of its closure in $\overline{\mathcal{T}}_3 \subset \overline{\mathcal{M}}_g$, Stankova [22] produced the sharp upper bound of $7 + \frac{6}{g}$ for the slope δ/λ of a sweeping curve in $\overline{\mathcal{T}}_3$. We will generalize Stankova's result in chapter 4 after studying the divisor theory on the *Hurwitz space of admissible covers*, $\overline{\mathcal{H}}_{d,g}$ for larger d.

1.5. The decomposition of $\mathcal{H}_{4,g}$ by Maroni loci.

The decomposition of $\mathcal{H}_{4,g}$ by Maroni loci is more complicated. We begin by reviewing the essential (for our purposes) geometric property of a degree four cover $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{4,g}$: The domain curve C is a complete intersection in the $\mathbf{P}^2$-bundle $\mathbf{P}\mathcal{E}$.

1.5.1. *The geometry of a degree 4 cover.* For $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{4,g}$, the natural factorization (1.2) from section 1

$$
\begin{array}{ccc}
C & \xrightarrow{\ i\ } & \mathbf{P}\mathcal{E} \\
& {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\pi} \\
& & \mathbf{P}^1
\end{array}
$$

expresses $C \subset \mathbf{P}(\mathcal{E})$ as a family of 4 points varying in the fibers of the $\mathbf{P}^2$-bundle $\mathbf{P}\mathcal{E}$. Furthermore, the Casnati-Ekedahl resolution (1.2) from Section 1.1 has the following form:

$$(1.3) \qquad\qquad 0 \longrightarrow \pi^*\det\mathcal{E}(-4) \longrightarrow \pi^*\mathcal{F}(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0$$

where $\mathcal{F}$ is the rank 2 bundle of quadrics (conics) associated to α. Since $\mathcal{F}$ splits, C *is always a complete intersection of two relative conic divisors in the bundle* $\mathbf{P}\mathcal{E}$. Specifically, if $\mathcal{F} = \mathcal{O}_{\mathbf{P}^1}(u) \oplus \mathcal{O}_{\mathbf{P}^1}(v)$, where $u + v = g + 3$, then C is a complete intersection of a pair of divisors $Q_u \in |2\zeta - uf|$ and $Q_v \in |2\zeta - vf|$. We will assume that $u \leq v$.

In order to understand the geometry of $\mathcal{M}(\mathcal{E})$, we first consider the subvarieties

$$\mathcal{M}(\mathcal{E}, \mathcal{F}) := \overline{\left\{ [\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{4,g} \mid \mathcal{E}_\alpha = \mathcal{E}, \mathcal{F}_\alpha = \mathcal{F} \right\}}$$

In other words, we consider the closure of the locus of covers with a prescribed reduced direct image $\mathcal{E}$ and quadric bundle $\mathcal{F}$. The loci $\mathcal{M}(\mathcal{E}, \mathcal{F})$ are easily described, given the fact that degree four covers occur as complete intersections of relative quadrics:

Proposition 1.5. $\mathcal{M}(\mathcal{E}, \mathcal{F})$ *is an irreducible subvariety of* $\mathcal{H}_{4,g}$ *of codimension* $h^1(\mathcal{E}nd\,\mathcal{E}) + h^1(\mathcal{E}nd\,\mathcal{F}) - h^1(\mathcal{F}^\vee \otimes \mathcal{S}^2\mathcal{E})$.

Proof. A general section, modulo scaling, of the bundle $\mathcal{F}^\vee \otimes \mathcal{S}^2\mathcal{E}$ provides a complete intersection $Q_u \cup Q_v$ in the scroll $\mathbf{P}\mathcal{E}$. Applying automorphisms in $Aut\,\mathbf{P}\mathcal{F}/\mathbf{P}^1$ gives rise to the same complete intersection. Secondly, we must mod out by automorphisms of the scroll $\mathbf{P}\mathcal{E}$. Straightforward calculation yields the result.

$\square$

Since $\mathcal{M}(\mathcal{E}) = \bigcup_i \mathcal{M}(\mathcal{E}, \mathcal{F}_i)$ for finitely many $\mathcal{F}_i$, we can establish the dimension/irreducibility of $\mathcal{M}(\mathcal{E})$ if we know the inclusion relationships among the subvarieties $\mathcal{M}(\mathcal{E}, \mathcal{F}_i)$.

With this in mind, let us fix $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus \mathcal{O}_{\mathbf{P}^1}(a_3)$ with $a_1 \leq a_2 \leq a_3$, and $a_1 + a_2 + a_3 = g + 3$. Condition 1 says that $a_3 \leq \frac{g+3}{2}$, which is equivalent to saying that the *second gap* $m_2 := a_3 - a_2$ must be less than or equal to the lowest degree, a_1. What is slightly less obvious is that the *first gap* $m_1 := a_2 - a_1$ also cannot exceed a_1, which we now explain.

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Suppose, on the contrary, that $m_1 > a_1$, or equivalently $2a_1 < a_2$. Then, since $a_3 \leq \frac{g+3}{2}$, we must have $a_1 + a_3 < 2a_2$. Now consider $\mathcal{S}^2\mathcal{E}$:

$$\mathcal{S}^2\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(2a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_1 + a_2) \oplus \mathcal{O}_{\mathbf{P}^1}(a_1 + a_3)$$

$$\oplus \mathcal{O}_{\mathbf{P}^1}(2a_2) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2 + a_3) \oplus \mathcal{O}_{\mathbf{P}^1}(2a_3)$$

The degrees of the summands as presented are now in increasing order. Let X, Y, Z denote the relative homogeneous "coordinates" of $\mathbf{P}\mathcal{E}$ corresponding to the projections of $\mathcal{E}$ to the summands $\mathcal{O}_{\mathbf{P}^1}(a_1), \mathcal{O}_{\mathbf{P}^1}(a_2)$, and $\mathcal{O}_{\mathbf{P}^1}(a_3)$, respectively. If $\mathcal{F} = \mathcal{O}_{\mathbf{P}^1}(u) \oplus \mathcal{O}_{\mathbf{P}^1}(v), (u \leq v)$ is a potential quadric bundle for α, then $u \leq 2a_1$ must hold, otherwise the section $[X : Y : Z] = [1 : 0 : 0]$ would be contained in the intersection of Q_u and Q_v. Therefore, $v \geq g + 3 - 2a_1$. However, this forces the equation of Q_v to be an expression of the form

$$Q_v = pY^2 + qYZ + rZ^2$$

where the degrees of (p, q, r) are $(2a_2 - v, a_2 + a_3 - v, 2a_3 - v)$. In other words, Q_v, viewed as fibered over $\mathbf{P}^1$ under the projection π, will be a family of *reducible* conics.

Let $\tilde{\gamma}\colon Q_v^\nu \longrightarrow \mathbf{P}^1$ be the natural map from the normalization Q_v^ν of Q_v to $\mathbf{P}^1$, and let $\gamma\colon Q_v^\nu \longrightarrow E \longrightarrow \mathbf{P}^1$ be its Stein factorization. The curve $C = Q_u \cap Q_v$ is then a double cover of E. This means that C will necessarily be *bi-hyperelliptic*, i.e. the cover $\alpha\colon C \longrightarrow \mathbf{P}^1$ will factor though E, which forces the branching to be non-simple. This provides a simply geometric reason explaining some of the results in [8].

Remark 1.6. In fact, we can easily describe the genus of the curve E: The number of "double lines" occurring in the family of reducible conics $[\pi\colon Q_v \longrightarrow \mathbf{P}^1]$ is the number of zeros of the polynomial $q^2 - 4pr$, which is $2(a_2 + a_3) - 2v$. Since the cover C is assumed to be smooth, we must have $u = 2a_1$, so this implies that the number of double lines is $2a_1$. Therefore, the genus of E is $a_1 - 1$.

The conlusion is: for simply branched degree 4 covers, $m_1 \leq a_1$. In fact, Ohbuchi shows more generally for all degrees d and genera g:

Condition 3 (Ohbuchi [20]). For every $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g}$, the ith difference $m_i := a_{i+1} - a_i$ never exceeds a_1.

(We emphasize: The only way Condition 3 can fail to be satisfied is if α is composite, but we are only considering simply branched covers.)

Now we will present some examples to illustrate the complexity of the decomposition by Maroni loci.

Example 1.7 (Failure of Expected Codimension). : Suppose $a_1 < \frac{g+2}{5}$, and consider the bundle $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(2a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(g+3-3a_1)$. An argument parallel to that used to arrive at Condition 3 shows that the only possible bundle of quadrics $\mathcal{F}$ has degrees $u = 2a_1$ and $v = g + 3 - 2a_1$. However, Proposition 1.5 and direct calculation gives

$$\operatorname{codim} \mathcal{E} = \operatorname{codim} \mathcal{M}(\mathcal{E}, \mathcal{F})$$

$$= h^1(\mathcal{E}nd\,\mathcal{E}) - (g+2-5a_1)$$

$$< h^1(\mathcal{E}nd\,\mathcal{E})$$

It then remains to show that there are actually smooth curves arising as $Q_u \cap Q_v$, but this is an application of Bertini's theorem on Q_v.

Example 1.8 (Failure of Irreducibility). Consider $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(5) \oplus \mathcal{O}_{\mathbf{P}^1}(7)$. Then both $\mathcal{F}_0 = \mathcal{O}_{\mathbf{P}^1}(6) \oplus \mathcal{O}_{\mathbf{P}^1}(9)$ and $\mathcal{F}_1 = \mathcal{O}_{\mathbf{P}^1}(5) \oplus \mathcal{O}_{\mathbf{P}^1}(10)$ are bundles of quadrics for covers in $\mathcal{M}(\mathcal{E}) \subset \mathcal{H}_{4,12}$. Calculating the codimensions of $\mathcal{M}(\mathcal{E}, \mathcal{F}_i)$ using Proposition 1.5 gives

$$\operatorname{codim} \mathcal{M}(\mathcal{E}, \mathcal{F}_i) = h^1(\mathcal{E}nd\,\mathcal{E})$$

Therefore, $\mathcal{M}(\mathcal{E})$ has expected dimension, yet is reducible with components $\mathcal{M}(\mathcal{E}, \mathcal{F}_i)$. In fact, $\mathcal{M}(\mathcal{E}, \mathcal{F}_1) \cap \mathcal{M}(\mathcal{E}, \mathcal{F}_2) = \emptyset$, because a cover $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{M}(\mathcal{E}, \mathcal{F}_0) \cap \mathcal{M}(\mathcal{E}, \mathcal{F}_1)$ must have a reduced direct image $\mathcal{E}' \neq \mathcal{E}$ which is a specialization of $\mathcal{E}$. Therefore, $a_1 = 3$ must drop to 2, but this violates Condition 2. This is the first example of a reducible Maroni locus. It is also the first example of a disconnected Maroni locus.

Example 1.9 (Failure of Expected Codimension and Irreducibility). Consider $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(7) \oplus \mathcal{O}_{\mathbf{P}^1}(10)$. The two possible bundles of quadrics are $\mathcal{F}_0 = \mathcal{O}_{\mathbf{P}^1}(8) \oplus \mathcal{O}_{\mathbf{P}^1}(13)$ and $\mathcal{F}_1 = \mathcal{O}_{\mathbf{P}^1}(7) \oplus \mathcal{O}_{\mathbf{P}^1}(14)$. $\mathcal{M}(\mathcal{E}, \mathcal{F}_i)$ both have codimension 1 less than expected, and form the two irreducible components of $\mathcal{M}(\mathcal{E})$. The two components do not intersect in $\mathcal{H}_{4,18}$ just as in the previous example.

We can trivially generalize these examples to show that $\mathcal{M}(\mathcal{E})$ can have more than two components: Fix integers $j < m$ and let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(m + j) \oplus \mathcal{O}_{\mathbf{P}^1}(2m + j) \oplus \mathcal{O}_{\mathbf{P}^1}(3m + j)$. The possible bundles of quadrics are $\mathcal{F}_i = \mathcal{O}_{\mathbf{P}^1}(2m + 2j - i) \oplus \mathcal{O}_{\mathbf{P}^1}(4m + j + i), 0 \leq i \leq j$. Each component $\mathcal{M}(\mathcal{E}, \mathcal{F}_i)$ has codimension $h^1(\mathcal{E}nd\,\mathcal{E}) + m - j - 1$.

These examples indicate that both reducibility and unexpected dimension are to be expected when considering the subvarieties $\mathcal{M}(\mathcal{E})$. However, all is not lost - we can at least establish irreducibility for a particular collection of $\mathcal{M}(\mathcal{E})$. We will do this in the next section. For now, we provide a classical example:

Example 1.10 (The $g = 6$ Gieseker-Petri locus). A general genus 6 curve C rests in its canonical embedding as a quadric hypersurface section of a unique quintic Del Pezzo surface $S \subset \mathbf{P}^5$. The (closure of the) locus of curves C lying on a singular Del Pezzo surface forms the *Gieseker-Petri divisor* $\mathcal{GP}_4^1 \subset \mathcal{M}_6$.

If we realize S as $Bl_{\{p_1, p_2, p_3, p_4\}}\mathbf{P}^2$, the blow up of the plane at 4 general points, then the divisor class of C is $[C] = 6H - \sum_i 2E_i$. Therefore, C is realized as a sextic plane curve

having 4 nodes. The five different g_4^1's on C are given by the linear series $|H - E_i|$ and $|2H - \sum_i E_i|$. If p_1 becomes incident with p_2, or if three (say p_2, p_3, p_4) of the four points become collinear, then the Del Pezzo surface S becomes singular, acquiring an ordinary double point. Furthermore, the five distinct g_4^1's no longer remain distinct: In the first case, $|H - E_1|$ coincides with $|H - E_2|$, while in the second case $|2H - \sum_i E_i| = |H - E_1|$.

On the other hand, in $\mathcal{H}_{4,6}$ the general reduced direct image is $\mathcal{E}_{gen} = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3)$. The *Maroni divisor* is $\mathcal{M}(\mathcal{E}_{div})$, where $\mathcal{E}_{div} := \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(4)$.

Fix a cover $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{4,6}$ and consider the sequence of sheaves on C:

$$0 \longrightarrow T_C \longrightarrow \alpha^* T_{\mathbf{P}^1} \longrightarrow N_\alpha \longrightarrow 0$$

The boundary map

$$\delta \colon H^0(C, N_\alpha) \longrightarrow H^1(C, T_C)$$

is the differential of the map on versal deformation spaces

$$F \colon \underline{\mathrm{Def}}(\alpha) \longrightarrow \underline{\mathrm{Def}}(C)$$

If $\mathcal{E}_\alpha = \mathcal{E}_{gen}$, the kernel $\ker \delta$ is 3 dimensional (coming from infinitesimal automorphisms of $\mathbf{P}^1$. However, when $\mathcal{E}_\alpha = \mathcal{E}_{div}$, $\ker \delta$ becomes 4 dimensional, so F is ramified. Therefore, *the Maroni divisor is the ramification divisor of the forgetful map $F \colon \mathcal{H}_{4,6} \longrightarrow \mathcal{M}_6$.* Notice that this analysis does not depend on the pair g, d satisfying $g = 2(d-1)$, so *the Maroni divisor is always the ramification divisor of the dominant map $F \colon \mathcal{H}_{d,2(d-1)} \longrightarrow \mathcal{M}_{2(d-1)}$.*

Returning to our specific setting, consider $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{4,6}$ such that $\mathcal{E}_\alpha = \mathcal{E}_{gen}$. The only acceptable bundle of quadrics is $\mathcal{F} = \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(5)$. So, in $\mathbf{P}\mathcal{E}_\alpha$, C is the complete intersection of $Q_4 \in |2\zeta - 4f|$ and $Q_5 \in |2\zeta - 5f|$. Furthermore, the linear series $|\zeta - 2f|$ restricts to ω_C on C, and maps $\mathbf{P}\mathcal{E}_\alpha$ to the geometric scroll mentioned in the Introduction. Under $|\zeta - 2f|$, Q_5 embeds as the quintic Del Pezzo containing C.

If $\mathcal{E}_\alpha = \mathcal{E}_{div}$, then the linear series $|\zeta - 2f|$ contracts the section $[X : Y : Z] = [1 : 0 : 0]$ which is contained in Q_5 as a -2 curve, and therefore we recover the singular Del Pezzo surface which contains C. For our purposes, the important take away from this classical example is that the Maroni divisor dominates the Gieseker-Petri locus. We will revisit this in general degree and genera later.

1.6. **The Casnati-Ekedahl loci** $\mathcal{C}(\mathcal{F}) \subset \mathcal{H}_{4,g}$. The Casnati-Ekedahl loci first appear in $\mathcal{H}_{4,g}$. In the natural factorization

$$C \xrightarrow{\ i\ } \mathbf{P}\mathcal{E}$$
$$\alpha \searrow \quad \downarrow \pi$$
$$\mathbf{P}^1$$

we may interpret the bundle map $\beta \colon \mathbf{P}\mathcal{F} \longrightarrow \mathbf{P}^1$ as the family of conics containing the fibers of α (four points) in each fiber of π. There is a distinguished curve $D \subset \mathbf{P}\mathcal{F}$ parametrizing the *singular* conics containing the fibers of α. The restriction $\beta \colon D \longrightarrow \mathbf{P}^1$ clearly has degree 3, and is simply branched precisely above the points where α is branched. By Riemann-Hurwitz, the genus of D is $g + 1$. In fact the function field, $K(D)$, is isomorphic to the "resolvent cubic" field(s) associated to the extension of function fields $K(\mathbf{P}^1) \subset K(C)$. Furthermore, since $\deg \mathcal{F} = g + 3 = (g + 1) + 2 = g(D) + 2$, $\mathcal{F}$ *must be the reduced direct image $\mathcal{E}_\beta$ of* $[\beta \colon D \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{3,g+1}$. In this way, we obtain the classical morphism

$$\Phi \colon \mathcal{H}_{4,g} \longrightarrow \mathcal{H}_{3,g+1}$$

$$\Phi([\alpha \colon C \longrightarrow \mathbf{P}^1]) := [\beta \colon D \longrightarrow \mathbf{P}^1]$$

which we will call the *resolvent cubic map*. The *Recillas construction* [21] naturally identifies the fiber $\Phi^{-1}([\beta \colon D \longrightarrow \mathbf{P}^1])$ with the nontrivial 2-torsion points in the Jacobian $Jac(D)$. Therefore Φ is etale, of degree $2^{2(g+1)} - 1$.

So we immediately see that Casnati-Ekedahl loci have a beautifully simple geometric description in terms of the resolvent cubic morphism:

$$\Phi \colon \mathcal{H}_{4,g} \longrightarrow \mathcal{H}_{3,g+1}$$

$$\mathcal{C}(\mathcal{F}) = \Phi^{-1}(\mathcal{M}(\mathcal{F}))$$

Therefore, it follows from Proposition 1.4 that *the Casnati-Ekedahl loci are always of the expected codimension $h^1(\mathcal{E}nd\,\mathcal{F})$. Furthermore if $\mathcal{F}$ specializes to $\mathcal{F}'$, then $\mathcal{C}(\mathcal{F}') \subset \mathcal{C}(\mathcal{F})$.*

The question of irreducibility is only slightly more subtle. Recall the irreducible varieties

$$\mathcal{M}(\mathcal{E},\mathcal{F}) := \overline{\left\{ [\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{4,g} \mid \mathcal{E}_\alpha = \mathcal{E}, \mathcal{F}_\alpha = \mathcal{F} \right\}}$$

The varieties $\mathcal{M}(\mathcal{E},\mathcal{F})$ serve as building blocks for Maroni and Casnati-Ekedahl loci, in the sense that *if a Maroni or Casnti-Ekedahl locus is irreducible, the components must obviously be of the form $\mathcal{M}(\mathcal{E},\mathcal{F})$*. Since all Casnati-Ekedahl loci are pure dimensional, we can count dimensions, finding instances when the codimension of $\mathcal{M}(\mathcal{E},\mathcal{F})$ is equal to $h^1(\mathcal{E}nd\,\mathcal{F})$. From Proposition 1.5 we see that

$$h^1(\mathcal{E}nd\,\mathcal{E}) = h^1(\mathcal{F}^\vee \otimes \mathcal{S}^2\mathcal{E})$$

must hold. By checking all cases, we may find all instances of *reducible* Casnati-Ekedahl loci.

Proposition 1.11. $\mathcal{C}(\mathcal{F})$ *is irreducible if and only if $g \equiv 0 \mod 3$ and*

$$\mathcal{F} = \mathcal{O}\left(\frac{g+3}{3}\right) \oplus \mathcal{O}\left(\frac{2(g+3)}{3}\right)$$

(In other words, $\mathcal{C}(\mathcal{F})$ is the smallest dimensional nonempty Casnati-Ekedahl locus.)

In these situations, $\mathcal{C}(\mathcal{F})$ has precisely two components:

$$\mathcal{C}(\mathcal{F}) = \mathcal{M}(\mathcal{E}_{gen}, \mathcal{F}) \cup \mathcal{M}(\mathcal{E}_{div}, \mathcal{F})$$

(In other words, the Maroni divisor $\mathcal{M}(\mathcal{E}_{div})$ does not intersect $\mathcal{C}(\mathcal{F})$ properly.

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Proof. We simply check all instances of pairs of bundles $(\mathcal{E}, \mathcal{F})$ which satisfy the equality

$$h^1(\mathcal{E}nd\,\mathcal{E}) = h^1(\mathcal{F}^\vee \otimes \mathcal{S}^2\mathcal{E})$$

We leave this to the reader, as it is not very enlightening. $\qquad\square$

We recover the observation made by Vakil in [24]: The preimage of the Maroni divisor under the resolvent cubic map $\Phi \colon \mathcal{H}_{4,3} \longrightarrow \mathcal{H}_{3,4}$ has exactly two components.

Proposition 1.11 sheds light on the monodromy of the resolvent cubic map Φ. By the Recillas construction, we may view the resolvent cubic morphism $\Phi \colon \mathcal{H}_{4,g} \longrightarrow \mathcal{H}_{3,g+1}$ as the total space of the local system of $(\mathbb{Z}/2\mathbb{Z})$-valued relative first homology of the universal curve over $\mathcal{H}_{3,g+1}$. Using this description, we can think about the reducibility of $\mathcal{C}(\mathcal{F})$ as a statement about the monodromy of Lefschetz pencils.

We look at the following situation: On the Hirzebruch surface $\mathbf{F}_k$ consider the linear system $|3\tau + af|$, $a \geq 0$. Here τ is the unique section class which avoids the directrix σ. Let $U \subset |3\tau + af|$ be the open subset parametrizing smooth divisors, i.e. the complement of the discriminant divisor. Let $u : \mathscr{C} \to U$ be the universal curve over U, and fix a point $x \in U$. Proposition 4.1 then says that the action of $\pi_1(U)$ on $R^1u_*(\mathbb{Z}/2\mathbb{Z})_x \setminus \{0\}$ is transitive, *unless* $a = 0$, in which case there are exactly two orbits.

Let $P \subset |3\tau+af|$ be a general pencil containing x, and denote by P_U the intersection $P \cap U$. It is a well-known fact, due to Lefschetz [16], that the natural map $g \colon \pi_1(P_U) \longrightarrow \pi_1(U)$ is surjective. Therefore, we may replace $u : \mathscr{C} \to U$ with the total space of the pencil $p \colon X \to P$. $X = \mathrm{Bl}_B\mathbf{F}_k$ is the blow up of $\mathbf{F}_k$ along the base locus B of the pencil P. Let C_x be the fiber of p at x.

Assume, for the moment, that *all singular fibers of the pencil p are simply nodal.* Under this assumption, by copying Lefschetz's argument for the monodromy of plane curves, one can similarly show that $\pi_1(P_U)$ acts on $H_1(C_x, \mathbb{Z})$ as the full symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. The general pencil in the linear series $|3\tau + af|$ will have simply nodal singular elements if and

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only if $a \neq 0$. When $a = 0$, the linear series $|3\tau|$ is trivial when restricted to the directrix σ, so any pencil of 3τ curves will violently split off the directrix σ at a certain point. This is where the obstruction for full monodromy comes from.

Another explanation for why there are exactly two orbits in $R^1 u_*(\mathbb{Z}/2\mathbb{Z})_x \setminus \{0\}$ comes from applying adjunction to a 3τ curve C. We see that

$$\omega_C = \mathcal{O}_{\mathbf{F}_k}(C + K_{\mathbf{F}_k})|_C = 2(k-1)f|_C$$

so the line bundle $(k-1)f_C$ is a theta characteristic for C which is even or odd depending on the parity of k. Therefore, we may translate all theta characteristics to the two-torsion of the Jacobian $J(C)$, and furthermore, we can do this globally over the parameter space U. The monodromy action must preserve even and odd theta characteristics, and Proposition 1.5 tells us that the action on these two subsets of $R^1 u_*(\mathbb{Z}/2\mathbb{Z})_x \setminus \{0\}$ is transitive.

1.7. **Irreducibility of some Maroni loci.** The associated Hirzebruch surface construction $j \colon C \longrightarrow \mathbf{F}_{a_1}$ from Section 1.3 allows us to establish the irreducibility of a particular class of Maroni loci. Before going on, let us make a definition: We will call a nondecreasing sequence $(a_1, a_2, ..., a_{d-1})$ of $d-1$ positive integers *acceptable* if it satisfies Conditions $1, 2, 3$ and the degree constraint $a_1 + a_2 + ... + a_{d-1} = g + d - 1$.

What we have seen from our explorations and from the work of Ohbuchi is that *all reduced direct image bundles for simply branched covers must have an acceptable sequence of summands*. However, it seems to be an open question whether the converse holds: Are all *acceptable* sequences realized by a reduced direct image for a simply branched cover? The only result known to us seems to be a theorem of Coppens [6] which we now explain.

Fix $k := a_1$, an integer satisfying Condition 2. Let $(k, a_2', a_3', ..., a_{d-1}')$ be the unique acceptable sequence maximizing the weighted sum $(d-1)k + (d-2)a_2 + (d-3)a_3 + (d-4)a_4 + ... + a_{d-1}$. The reader should think of this acceptable sequence as the most "general" among acceptable sequences having k as the first coordinate. This can be made precise by

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an upper semi-continuity for the corresponding bundles. Define

$$\mathcal{E}_{gen}(k) := \mathcal{O}(k) \oplus \mathcal{O}(a_2') \oplus \mathcal{O}(a_3') \oplus ... \oplus \mathcal{O}(a_{d-1}')$$

The main theorem of Coppens [6], reformulated into our language, is:

Theorem 1.12 (Coppens [6]). *The Maroni locus $\mathcal{M}(\mathcal{E}_{gen}(k)) \subset \mathcal{H}_{d,g}$ is nonempty.*

The bundles $\mathcal{E}_{gen}(k)$ are particularly nice, because it turns out that *any* reduced direct image $\mathcal{E}'$ having $\mathcal{O}(k)$ as a minimal degree summand will be a specialization of $\mathcal{E}_{gen}(k)$, i.e. $\mathcal{M}(\mathcal{E}') \subset \mathcal{M}(\mathcal{E}_{gen}(k))$. This will be a consequence of the stronger statement that $\mathcal{M}(\mathcal{E}_{gen}(k))$ is irreducible.

Theorem 1.13 (Irreducibility of $\mathcal{M}(\mathcal{E}_{gen}(k))$). *For $k < \left\lfloor \frac{g+d-1}{d-1} \right\rfloor$, $\mathcal{M}(\mathcal{E}_{gen}(k)) \subset \mathcal{H}_{d,g}$ is an irreducible subvariety of codimension $g - (d-1)k + 1$. Furthermore, any reduced direct image $\mathcal{E}'$ having $\mathcal{O}_{\mathbf{P}^1}(k)$ as a least degree summand will be a specialization of $\mathcal{E}_{gen}(k)$, and furthermore, $\mathcal{M}(\mathcal{E}') \subset \mathcal{M}(\mathcal{E}_{gen}(k))$.*

Proof. The proof of the theorem relies on Coppens theorem mentioned above, and the following result of Tyomkin [23]:

Theorem 1.14 (Tyomkin [23]). *All Severi varieties parametrizing irreducible curves on Hirzebruch surfaces are irreducible and of expected dimension.*

Let $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g}$ be any cover such that k is a minimal degree of a summand of $\mathcal{E}_\alpha$. (The summand may not be unique.) A choice of inclusion $\mathcal{O}(-k) \subset \mathcal{E}_\alpha^\vee$ gives rise to a map $j \colon C \longrightarrow \mathbf{F}_k$ which is birational onto its image, by the "associated Hirzebruch surface construction" in section 1.3. Letting V denote the Severi variety parametrizing irreducible, geometric genus g curves in the linear system $|d\tau|$ on $\mathbf{F}_k$, we arrive at a rational map

$$q \colon V \longrightarrow \mathcal{H}_{d,g}$$

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whose image contains the cover α. Tyomkin's theorem says that V is irreducible, and hence so is the image $q(V) \subset \mathcal{H}_{d,g}$.

Coppens' theorem cited above then shows that the generic point of $q(V)$ parametrizes covers α such that $\mathcal{E}_\alpha = \mathcal{E}_{gen}(k)$. Therefore, $\overline{q(V)} = \mathcal{M}(\mathcal{E}_{gen}(k))$.

Finally, assuming $k < \lfloor \frac{g+d-1}{d-1} \rfloor$, the choice of inclusion $\mathcal{O}(-k) \subset \mathcal{E}_\alpha^\vee$ will be unique, so the map $q \colon V \longrightarrow \mathcal{H}_{d,g}$ is birationally a G-torsor over its image, where $G = Aut\,\mathbf{F}_k/\mathbf{P}^1$. By Tyomkin's theorem, we know that V is of expected dimension. A direct calculation then gives the statement about the codimension of $\mathcal{M}(\mathcal{E}_{gen}(k))$.

$\square$

Two important observations follow from Theorem 1.13. First, we have:

Theorem 1.15 (No unexpected Maroni divisors)**.** *The only divisorial Maroni locus is* $\mathcal{M}(\mathcal{E}_{div}(k))$, *where*

$$\mathcal{E}_{div}(k) := \mathcal{O}(k) \oplus \mathcal{O}(k+1)^{\oplus d-3} \oplus \mathcal{O}(k+2)$$

Proof. By Theorem 1.13 above, the codimension of the Maroni locus $\mathcal{M}(\mathcal{E}_{gen}(k))$ is $g - (d-1)k + 1$ when $k < \lfloor \frac{g+d-1}{d-1} \rfloor$. By setting this equal to 1, we see that

$$g = (d-1)k$$

which means that $\mathcal{E}_{gen}(k)$ is precisely $\mathcal{E}_{div}(k)$ from the statement of the theorem.

It remains to check that there is no reduced direct image $\mathcal{E}'$ which: (1) is *not* of the form $\mathcal{E}_{gen}(k)$, (2) has minimal summand $k = \lfloor \frac{g+d-1}{d-1} \rfloor$, and (3) is such that $\mathcal{M}(\mathcal{E}')$ is divisorial. The first two conditions on $\mathcal{E}'$ imply that

$$\mathcal{E}' = \mathcal{O}(k)^r \oplus \mathcal{N}$$

for some $r \geq 2$.

Recall the map

$$q \colon V \longrightarrow \mathcal{H}_{d,g}$$

from the proof of Proposition 5.1. Above the locus $\mathcal{M}(\mathcal{E}')$, the map q (rationally) factors through a $\mathbf{P}^{r-1}$-bundle $\mathbf{P}\mathcal{G}^r \longrightarrow \mathcal{M}(\mathcal{E}')$ parametrizing "choices of inclusions" $\mathcal{O}(-k) \subset (\mathcal{E}')^\vee$.

Since we assume $r \geq 2$, the dimensions of the fibers of q are jumping, so Tyomkin's irreducibility theorem for the Severi variety V implies that $\mathcal{M}(\mathcal{E}')$ must not be a divisor. $\square$

We will call $\mathcal{M}(\mathcal{E}_{div}(k))$ the *Maroni divisor*. In the special case $k = 2$, we observe:

Corollary 1.16 (Irreducibility of the Gieseker-Petri divisors $\mathcal{GP}_d^1$)**.** *Assume $g = 2(d - 1)$. The Maroni divisor $\mathcal{M}(\mathcal{E}_{div}(2))$ is irreducible and dominates the Gieseker-Petri divisor $\mathcal{GP}_d^1 \subset \mathcal{M}_g$ under the forgetful map $F \colon \mathcal{H}_{d,g} \longrightarrow \mathcal{M}_g$. Therefore, $\mathcal{GP}_d^1$ is irreducible.*

Proof. The corollary follows from the observations made in Example 1.10, and Theorem 1.13. $\square$

1.7.1. *An interesting open question.* As we see from the irreducibility theorem above, the subvarieties $\mathcal{M}(\mathcal{E}_{gen}(k))$ are, in general, not of expected codimension, yet we may still ask the following question:

Question: Is it always true that $\mathcal{M}(\mathcal{E}_{gen}(k - 1)) \subset \mathcal{M}(\mathcal{E}_{gen}(k))$?

1.8. **Degree 5 covers.** We now explore the Maroni and Casnati-Ekedahl loci in $\mathcal{H}_{5,g}$. The main tool is a well-known linear algebraic description of degree 5 covers, which we review.

Let $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{d,g}$ be any cover. The Casnati-Ekedahl resolution from section 1.1 has the form:

$$0 \longrightarrow \det \mathcal{E}(-5) \longrightarrow \pi^*\mathcal{F}^\vee \otimes \det \mathcal{E}(-3) \longrightarrow \pi^*\mathcal{F}(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0$$

The central map between rank 5 bundles is well-known to be skew symmetric, i.e., after pushing down to $\mathbf{P}^1$ under π defines a nonzero section s of the bundle $\wedge^2\mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee$. The curve C is the "rank 2" locus of this skew symmetric matrix.

The matrix associated to the section s may be written as

$$M = \begin{pmatrix} 0 & L_{1,2} & L_{1,3} & L_{1,4} & L_{1,5} \\ -L_{1,2} & 0 & L_{2,3} & L_{2,4} & L_{2,5} \\ -L_{1,3} & -L_{2,3} & 0 & L_{3,4} & L_{3,5} \\ -L_{1,4} & -L_{2,4} & -L_{3,4} & 0 & L_{4,5} \\ -L_{1,5} & -L_{2,5} & -L_{3,5} & -L_{4,5} & 0 \end{pmatrix}$$

where the sections $L_{i,j}$ restrict to linear forms on the $\mathbf{P}^3$-fibers of the scroll $\pi \colon \mathbf{P}\mathcal{E} \longrightarrow \mathbf{P}^1$. The equations cutting out C are the five 4×4 sub-Pfaffians of the matrix M.

Geometrically, we may view the construction of degree 5 covers as follows. The section s may be viewed as a surjection

$$s \colon \wedge^2 \mathcal{F}^\vee \longrightarrow \mathcal{E} \otimes \det \mathcal{E}^\vee \longrightarrow 0$$

which provides an inclusion of bundles:

$$\mathbf{P}(\mathcal{E} \otimes \det \mathcal{E}^\vee) \simeq \mathbf{P}\mathcal{E} \hookrightarrow \mathbf{P}(\wedge^2 \mathcal{F}^\vee)$$

In the $\mathbf{P}^9$-bundle $\mathbf{P}(\wedge^2 \mathcal{F}^\vee)$, we may consider the locus of rank 2 tensors, also known as the relative Grassmanian:

$$\mathbf{G} := Grass(2, \mathcal{F}^\vee) \hookrightarrow \mathbf{P}(\wedge^2 \mathcal{F}^\vee)$$

The original curve C is simply the intersection $C = \mathbf{P}\mathcal{E} \cap \mathbf{G} \subset \mathbf{P}(\wedge^2 \mathcal{F}^\vee)$.

As in section 1.6, we define the subvarieties

$$\mathcal{M}(\mathcal{E}, \mathcal{F}) := \overline{\left\{ [\alpha \colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{5,g} \mid \mathcal{E}_\alpha = \mathcal{E}, \mathcal{F}_\alpha = \mathcal{F} \right\}}$$

Casnati [4] proves a general Bertini-type existence theorem for degree 5 covers. We state the version we need as:

Theorem 1.17 (Casnati [4]). *If the bundle $\wedge^2\mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee$ is globally generated, then there exists a cover $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \mathcal{H}_{5,g}$ such that $\mathcal{E}_\alpha = \mathcal{E}$ and $\mathcal{F}_\alpha = \mathcal{F}$.*

The coarse geometric properties of the subvarieties $\mathcal{M}(\mathcal{E}, \mathcal{F})$ are now easily established.

Proposition 1.18. *The subvarieties $\mathcal{M}(\mathcal{E}, \mathcal{F})$ are irreducible and of codimension $h^1(\mathcal{E}nd\,\mathcal{E}) + h^1(\mathcal{E}nd\,\mathcal{F}) - h^1(\wedge^2\mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$.*

Proof. The proof is completely analogous to the proof of Proposition 1.5. We begin with the projective space of "skew symmetric matrices" $\mathbf{P}(\wedge^2\mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$, and then quotient out successively by $Aut\,\mathbf{P}\mathcal{F}/\mathbf{P}^1$ and $Aut\,\mathbf{P}\mathcal{E}$ to obtain a birational description of $\mathcal{M}(\mathcal{E}, \mathcal{F})$. The proposition follows easily from here.

$\square$

In section Proposition 1.11, we encountered some reducible Casnati-Ekedahl loci in $\mathcal{H}_{4,g}$. However, it is a remarkable fact that Casnati-Ekedahl loci are *always irreducible* in $\mathcal{H}_{5,g}$:

Proposition 1.19. *All Casnati-Ekedahl loci $\mathcal{C}(\mathcal{F}) \subset \mathcal{H}_{5,g}$ are irreducible.*

Proof. In the $\mathbf{P}^9$-bundle $\mathbf{P}(\wedge^2\mathcal{F}^\vee)$, we consider all $\mathbf{P}^3$-subbundles $\Lambda \simeq \mathbf{P}\mathcal{E}$ having the appropriate class in the Chow ring $A_*\mathbf{P}(\wedge^2\mathcal{F}^\vee)$. The totality of these subbundles forms an irreducible open subset of the appropriate Hilbert scheme. The conditions that force $\Lambda \cap \mathbf{G}$ to be a smooth, nondegenerate genus g curve are all open conditions, so the proposition follows. $\square$

As a consequence of the two propositions above, we have:

Theorem 1.20 (No unexpected Casnati-Ekedahl divisors). *The only divisorial Casnati-Ekedahl loci are of the form $\mathcal{C}(\mathcal{F}_{div})$, where $\mathcal{F}_{div} = \mathcal{O}(m) \oplus \mathcal{O}(m+1)^{\oplus 3} \oplus \mathcal{O}(m+2)$.*

Proof. A divisorial Casnati-Ekedahl locus $\mathcal{C}(\mathcal{F})$ must be of the form $\mathcal{M}(\mathcal{E}, \mathcal{F})$ for some $\mathcal{E}$, by Proposition 1.19. By the Theorem 1.15 on "No unexpected Maroni divisors", we know that $\mathcal{E}$ must either be $\mathcal{E}_{gen}$ or $\mathcal{E}_{div}$. We may eliminate the latter case by Casnati's existence

theorem cited above: There will always exist covers $[\alpha\colon C \longrightarrow \mathbf{P}^1]$ such that $\mathcal{E}_\alpha = \mathcal{E}_{div}$, and $\mathcal{F}_\alpha = \mathcal{F}_{gen}$. However, $\mathcal{M}(\mathcal{E}_{div}, \mathcal{F}_{gen})$ is clearly not $\mathcal{C}(\mathcal{F})$ for any $\mathcal{F}$.

So we may assume $\mathcal{E} = \mathcal{E}_{gen}$. We will indicate how the task of "checking all relevant cases" can be reduced to a much simpler check.

First, we observe that $\mathcal{F}$ must be imbalanced, i.e. $h^1(\mathcal{E}nd\,\mathcal{F}) \neq 0$. Write

$$\mathcal{F} = \mathcal{O}(x_1) \oplus ... \oplus \mathcal{O}(x_5)$$

with $x_1 \leq x_2 \leq ... \leq x_5$ and suppose

$$\mathcal{E}_{gen} = \mathcal{O}(k)^{\oplus r} \oplus \mathcal{O}(k+1)^{4-r}$$

Let us interpret a section of the bundle $\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee$ as a skew symmetric matrix

$$M = \begin{pmatrix} 0 & L_{1,2} & L_{1,3} & L_{1,4} & L_{1,5} \\ -L_{1,2} & 0 & L_{2,3} & L_{2,4} & L_{2,5} \\ -L_{1,3} & -L_{2,3} & 0 & L_{3,4} & L_{3,5} \\ -L_{1,4} & -L_{2,4} & -L_{3,4} & 0 & L_{4,5} \\ -L_{1,5} & -L_{2,5} & -L_{3,5} & -L_{4,5} & 0 \end{pmatrix}$$

where $L_{i,j} \in H^0(\mathbf{P}^1, \mathcal{E} \otimes \det \mathcal{E}^\vee \otimes \mathcal{O}(x_i + x_j))$. In other words, the $L_{i,j}$ restrict to linear forms on the fibers of $\pi\colon \mathbf{P}\mathcal{E} \longrightarrow \mathbf{P}^1$ with "coefficients" read off by the splitting type of the rank 4 bundle $\mathcal{E} \otimes \det \mathcal{E}^\vee \otimes \mathcal{O}(x_i + x_j)$.

In $\mathbf{P}\mathcal{E}$, the curve C is cut out by the 4×4 sub-Pfaffians of the matrix M. Suppose both $L_{1,2}$ and $L_{1,3}$ are zero. Then the Pfaffian of the sub matrix obtained by eliminating the fifth row and column is

$$Q_5 = L_{1,2}L_{3,4} - L_{1,3}L_{2,4} + L_{2,3}L_{1,4} = L_{2,3}L_{1,4}$$

which is a *reducible* quadric. This would force the cover $\alpha\colon C \longrightarrow \mathbf{P}^1$ to have smaller monodromy group.

Therefore, the maximum of the (two) degrees of the summands of $\mathcal{E} \otimes (\det \mathcal{E}^\vee) \otimes \mathcal{O}(x_1 + x_3)$ must be nonzero, meaning

$$x_1 + x_3 \geq (g + 4) - (k + 1)$$

The $x_i + x_j$ are increasing, so we deduce the same inequality for almost all pairs:

(1.4)
$$\forall \{i, j\} \neq \{1, 2\}, \quad x_i + x_j \geq (g + 4) - (k + 1)$$

Since $\mathcal{E}$ is balanced, this means that $h^1(\mathcal{E} \otimes (\det \mathcal{E}^\vee) \otimes \mathcal{O}(x_1 + x_3)) = 0$. Therefore, *all of the contribution of $h^1(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$* comes from the smallest 4 summands, i.e.

$$h^1(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) = h^1(\mathcal{E} \otimes \det \mathcal{E}^\vee \otimes \mathcal{O}(x_1 + x_2))$$

Now, the divisorial assumption, along with Proposition 6.2 say that

$$1 = h^1(\mathcal{E}nd\, \mathcal{E}) + h^1(\mathcal{E}nd\, \mathcal{F}) - h^1(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$$

Under our assumptions, $h^1(\mathcal{E}nd\, \mathcal{E}) = 0$, so we simply must show

$$h^1(\mathcal{E}nd\, \mathcal{F}) - h^1(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) = 1$$

However, from the considerations above, we know that this is equivalent to

$$h^1(\mathcal{E}nd\, \mathcal{F}) - h^1(\mathcal{E} \otimes \det \mathcal{E}^\vee \otimes \mathcal{O}(x_1 + x_2)) = 1$$

From here, it is straightforward to check using (1.4) that in the remaining few possibilities if $h^1(\mathcal{E}nd\, \mathcal{F}) \geq 2$ then

$$h^1(\mathcal{E}nd\, \mathcal{F}) - h^1(\mathcal{E} \otimes \det \mathcal{E}^\vee \otimes \mathcal{O}(x_1 + x_2)) \geq 2$$

$\square$

Remark 1.21. In the next chapter we will prove the Picard Rank Conjecture for degrees $3, 4$, and 5, i.e. that $\mathrm{Pic}_{\mathbb{Q}} \mathcal{H}_{3,g} = \mathrm{Pic}_{\mathbb{Q}} \mathcal{H}_{4,g} = \mathrm{Pic}_{\mathbb{Q}} \mathcal{H}_{5,g} = 0$. The theorem above will be needed for the degree 5 case.

1.9. **Further Directions.** The reader may notice that the Casnati-Ekedahl resolution (1.2) from section 1.1 actually provides a much more comprehensive association

$$\Phi \colon \left\{ \text{degree } d \text{ covers} \right\} \longrightarrow \left\{ \left\lfloor \frac{d}{2} \right\rfloor - \text{tuples of vector bundles on } \mathbf{P}^1 \right\}$$

whose formula is:

$$\Phi([\alpha \colon C \longrightarrow \mathbf{P}^1]) := (\mathcal{E}_\alpha, \mathcal{F}_\alpha, \mathcal{N}_2, \mathcal{N}_3, ...)$$

We may define *higher syzygy loci* to be the fibers of this association and its projections to the various factors. The resulting decomposition of Hurwitz space seems, in full generality, to be intractable, but we still think it a worthy subject of research.

As we will see in the final chapter, as far as "divisorial" enumerative questions are concerned, the bundles $\mathcal{E}$ and $\mathcal{F}$ play a more fundamental role than the rest of the syzygy bundles. Before we tackle enumerative questions however, we will need to work in a compactification of Hurwitz space. Understanding the admissible cover compactification will be the subject of the third chapter.

2. Expressing Hurwitz Space as a Quotient

The goals of this chapter are to (1) establish the Picard Rank Conjecture of the Hurwitz space parametrizing simply branched covers of $\mathbf{P}^1$ of degree 3, 4, and 5, and (2) understand the relationship between the Picard Rank Conjecture and the Diaz-Harris conjectures about Picard groups of Severi varieties.

As far as the first goal is concerned, the strategy is the same in all three cases: We will begin with a vector space V which parametrizes "presentations of ideals of covers in their scrolls". Then we will quotient out successively by "equivalence of presentations", and then by automorphisms of the associated scroll $\mathbf{P}\mathcal{E}$. Many of the ideas of the proof come from conversations with Anand Deopurkar.

The second goal is achieved by analyzing the "associated Hirzebruch surface" construction from chapter 1, section 1.3. Via the associated Hirzebruch surface construction, we may think of Hurwitz space as a "quotient" of an appropriate Severi variety, and this explains the title of the chapter.

The proofs of the statements in this chapter will require a few results from chapters 3 and 4. The results referred to are not dependent on the results of this chapter, as the reader can check for themselves.

2.1. The Picard Rank Conjecture.
The main outstanding conjecture about Picard groups of Hurwitz spaces is:

Conjecture 2.1 (Picard Rank Conjecture). *The rational Picard group* $\mathrm{Pic}_{\mathbb{Q}} \mathcal{H}_{d,g}$ *is zero for all pairs* (d, g).

The proofs of the statements in this chapter will require a few results from chapter 4. The results referred to in chapter 4 are not dependent on the results of this chapter, as the reader can check for themselves.

For pairs (d, g) satisfying $d \geq g+1$, Mochizuki [18] showed that the Picard Rank Conjecture is equivalent by Harer's theorem stating that $\mathrm{Pic}_{\mathbb{Q}} \mathcal{M}_g = \mathbb{Q}\langle\lambda\rangle$ [13]. However, for d small

compared to g, the conjecture is mostly wide open. In this section, we prove the conjecture for $d = 3, 4$, and 5.

2.1.1. *The Picard Rank Conjecture for $\mathcal{H}_{3,g}$.* For completeness, and to illustrate the ideas which will be used in the other two cases, we will review the proof of the Picard Rank Conjecture in degree 3. We let $\mathcal{H}_{d,g}^\circ$ denote the space of *all* degree d covers $[\alpha \colon C \longrightarrow \mathbf{P}^1]$ having *smooth* domain C. It is a quasi projective variety containing the simple Hurwitz space $\mathcal{H}_{d,g}$ as a dense open subset. The complement $\mathcal{H}_{d,g}^\circ \backslash \mathcal{H}_{d,g}$ is the union of two divisors T, D which generically parametrize covers having a triple ramification point and covers having "double", or (2,2) ramification above a single point in $\mathbf{P}^1$, respectively. Of course, when $d = 3$ the divisor D does not exist.

As we have seen, the generic reduced direct image sheaf $\mathcal{E} = \mathcal{E}_{gen}$ is rigid, i.e. $h^1(\mathcal{E}nd\,\mathcal{E}) = 0$. The vector space

$$V = H^0(\mathcal{O}_{\mathbf{P}\mathcal{E}}(3\zeta - (g+2)f)) = H^0(\mathbf{P}^1, \mathcal{S}^3\mathcal{E}(-g-2))$$

parametrizes "equations for trigonal curves" C embedded in the Hirzebruch surface $\mathbf{P}\mathcal{E}$ via the natural Casnati-Ekedahl factorization.

The complement of the locus of smooth curves in the linear system $\mathbf{P}V$ is an irreducible divisor (the discriminant divisor), and therefore $\mathrm{Pic}_\mathbb{Q} U = 0$. We have the natural map

$$q \colon U \longrightarrow \mathcal{H}_{3,g}^\circ$$

which is (away from a codimension 2 subset) a G-torsor, where $G = Aut\,\mathbf{P}\mathcal{E}$. The image $q(U) \subset \mathcal{H}_{3,g}^\circ$ is the locus of covers $[\alpha \colon C \longrightarrow \mathbf{P}^1]$ satisfying $\mathcal{E}_\alpha = \mathcal{E}_{gen}$.

We then appeal to a theorem which is due to Knopf, Kraft, and Vust [15]:

Theorem 2.2 (Knop, Kraft, Vust [15])**.** *Let $q \colon X \longrightarrow Y$ be a Zariski-locally trivial G-torsor, G a connected algebraic group. Let $\mathrm{Pic}_G X$ denote the the space of G-linearized line bundles on X. Then the pullback map $q^* \colon Y \longrightarrow \mathrm{Pic}_G X$ is injective. Furthermore, there exists an*

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exact sequence:

(2.1)
$$\chi(G) \longrightarrow \operatorname{Pic}_G X \longrightarrow \operatorname{Pic} X$$

where $\chi(G)$ is the character group of G.

As a consequence of the theorem, we notice that the following inequality holds:

(2.2)
$$\operatorname{rk} \operatorname{Pic}_{\mathbb{Q}} Y \leq \operatorname{rk} \operatorname{Pic}_{\mathbb{Q}} X + \operatorname{rk} \chi(G)$$

Applying Theorem 1 in our context, we conclude

$$\operatorname{rk} \operatorname{Pic}_{\mathbb{Q}} q(U) \leq \operatorname{rk} \chi(G)$$

We divide our analysis into two cases: g even, and g odd.

When g is even, the bundle $\mathbf{P}\mathcal{E}_{gen}$ is perfectly balanced. Therefore, $G = PGL_2 \times PGL_2$, which means $\chi(G) = 0$. The complement of the image $\mathbf{M} = \mathcal{H}^\circ_{3,g} \backslash q(U)$ is an old friend, the Maroni divisor $\mathbf{M} := \mathcal{M}(\mathcal{E}_{div})$. Proposition 1 and Proposition 2 from chapter 4 tell us that the divisor classes of $\mathbf{M}$ and T are multiples of each other in $\operatorname{Pic}_{\mathbb{Q}} \mathcal{H}^\circ_{d,g}$. This allows us to conclude that $\operatorname{Pic}_{\mathbb{Q}} \mathcal{H}_{3,g} = 0$ for g even, because the complement $\mathcal{H}^\circ_{3,g} \setminus \mathcal{H}_{3,g}$ is precisely the divisor T.

When g is odd, we have $\chi(G) = \mathbb{Z}$, while the complement $\mathcal{H}^\circ_{3,g} \backslash q(U)$ is codimension 2 by Proposition 2.1 in chapter 1. Therefore,

$$\operatorname{rk} \operatorname{Pic}_{\mathbb{Q}} \mathcal{H}^\circ_{3,g} = \operatorname{rk} \operatorname{Pic}_{\mathbb{Q}} q(U) \leq 1$$

However, Proposition 1 from chapter 4 tells us that the divisor class of T is nontrivial, so $\operatorname{rk} \operatorname{Pic}_{\mathbb{Q}} \mathcal{H}^\circ_{3,g} = 1$, and the Picard Rank Conjecture follows.

2.1.2. *The Picard Rank Conjecture for $\mathcal{H}_{4,g}$.* We follow exactly the same line of reasoning as above. We let $\mathcal{E} = \mathcal{E}_{gen}$, and $\mathcal{F} = \mathcal{F}_{gen}$ and consider

$$V = H^0(\mathbf{P}^1, \mathcal{F}^\vee \otimes \mathcal{S}^2\mathcal{E})$$

The vector space V, modulo scaling parametrizes "ideals of complete intersections" in the scroll $\mathbf{P}\mathcal{E}$ of the type we saw in section 3 of chapter 1. As before, there is a dense open subset $U \subset \mathbf{P}V$ parametrizing smooth complete intersections whose complement is a divisor. Therefore, $\mathrm{Pic}_{\mathbb{Q}}\, U = 0$. We may change parametrizations of the same ideal, which means we must quotient out by $G_1 = Aut\, \mathbf{P}\mathcal{F}/\mathbf{P}^1$. Then we must further quotient out by $G_2 = Aut\, \mathbf{P}\mathcal{E}$ to obtain the total composite quotient map:

$$q\colon U \longrightarrow \mathcal{H}_{4,g}^{\circ}$$

There are two possibilities for $\chi(G_1)$ depending on the parity of g (splitting type of $\mathcal{F}$) and two possibilities for $\chi(G_2)$ depending on whether g is a multiple of three or not. We will explain how we deal with one of the four possible scenarios, leaving the other three to the reader.

If g is odd, then $\mathcal{F}_{gen}$ is perfectly balanced, and $G_1 \simeq PGL_2$, which has no nontrivial characters. If, furthermore, g is divisible by 3, then $\mathcal{E}_{gen}$ will be perfectly balanced, and so G_2 will be $PGL_2 \times PGL_3$, which also has no nontrivial characters. The Maroni divisor $\mathbf{M}$ and the Casnati-Ekedahl divisor $\mathbf{CE}$ will comprise two components of the complement of the image $q(U)$. Theorem 1 tells us

$$\mathrm{rk}\,\mathrm{Pic}_{\mathbb{Q}}\, q(U) \le \mathrm{rk}\,\mathrm{Pic}_{\mathbb{Q}}\, U + \mathrm{rk}\,\chi(G_1) + \mathrm{rk}\,\chi(G_2) = 0$$

However, Proposition 2 in chapter 4 and the theorem on independence of boundary divisors in chapter 3 together tell us that: 1) $\mathbf{M}$ and $\mathbf{CE}$ are linear combinations of T and D in $\mathrm{Pic}_{\mathbb{Q}}\, \mathcal{H}_{4,g}^{\circ}$ and 2) the divisors T and D are linearly independent in $\mathrm{Pic}_{\mathbb{Q}}\, \mathcal{H}_{4,g}^{\circ}$.

To summarize: Theorem 1 provides an upper bound for the Picard rank. This upper bound is then realized by exhibiting enough independent divisor classes in the image. The divisorial components of the complement of the image $q(U)$ are either the Maroni or Casnati-Ekedahl divisor, which are shown to be linear combinations of T and D in chapter 4.

2.1.3. *The Picard Rank Conjecture for* $\mathcal{H}_{5,g}$. We follow the same reasoning: Begin with

$$V = H^0(\mathbf{P}^1, \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$$

the space of "skew symmetric matrices." Consider the open set $U \subset \mathbf{P}V$ of matrices whose rank 2 locus is a smooth curve. Again, $\text{Pic}_\mathbb{Q} U = 0$. We quotient out by $G_1 = Aut\, \mathbf{P}\mathcal{F}/\mathbf{P}^1$, and then by $G_2 = Aut\, \mathbf{P}\mathcal{E}$ resulting in the map

$$q \colon U \longrightarrow \mathcal{H}_{5,g}^\circ$$

There are again 2 possibilities for $\chi(G_i)$ depending on whether or not the bundles $\mathcal{E}_{gen}$ and $\mathcal{F}_{gen}$ are perfectly balanced.

   We conclude by the exact analogues of 1) and 2) from the previous section. One small point to bring to the reader's attention: In order to finish the argument, we must know whether there exist unexpected divisorial Maroni or Casnati-Ekedahl loci. Thankfully, Theorem 1.15 and Theorem 1.20 guarantee that this does not happen.

2.2. **From Severi varieties to Hurwitz spaces.** In this section, we would like to understand a morphism relating Severi varieties on Hirzebruch surfaces with Hurwitz spaces. Recall the construction of the "associated Hirzebruch surface" from chapter 1, section 1.3. Let us write $\mathcal{E}_{gen} = \mathcal{O}(k)^{\oplus r} \oplus \mathcal{O}(k+1)^{\oplus d-r-1}$, and assume that $r > 0$. Then the map of $\mathcal{O}_{\mathbf{P}^1}$-algebras given by

$$\text{Spec}\, Sym\, \mathcal{O}(-k) \longrightarrow \text{Spec}\, \alpha_* \mathcal{O}_C$$

provides a map

$$j \colon C \longrightarrow \mathbf{F}_k$$

which factors the original covering α. Furthermore, the image of j avoids the directrix, forcing the the divisor class $[j(C)]$ to be $d\tau$ where τ is the section class satisfying $\tau^2 = k$. The space of all inclusions $\mathcal{O}(-k) \subset \mathcal{E}^\vee$, up to scalar multiplication, forms a projective space

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$\mathbf{P}^{r-1}$. Therefore, *once such an inclusion has been selected*, the resulting map $j\colon C \longrightarrow \mathbf{F}_k$ is determined up to the action of $G = Aut\,\mathbf{F}_k/\mathbf{P}^1$.

Let $V = V_{g,d\tau}^{irr}$ be the Severi variety parametrizing irreducible, nodal curves of geometric genus g in the linear series $|d\tau|$ and consider the natural map

$$q\colon V \longrightarrow \widetilde{\mathcal{H}}_{d,g}$$

which, to every nodal curve $[X] \in V$ assigns the obvious $d:1$ map from the normalization X^ν induced by the projection from $\mathbf{F}_k$ to $\mathbf{P}^1$. Notice that the image $q(V)$ is precisely the locus of covers $\alpha\colon C \longrightarrow \mathbf{P}^1$ such that the least degree of a summand of $\mathcal{E}_\alpha$ is k. From the theorem of "No unexpected Maroni divisors" Theorem 1.15, we observe that the complement of the image $q(V)$ is a divisor if and only if $r = d - 1$.

2.2.1. *A rephrasing of the Picard Rank Conjecture.* Using the "quotient map" q, we will relate the Picard groups of V and $\widetilde{\mathcal{H}}_{d,g}$. Although we have already introduced the Picard Rank Conjecture, we will reformulate it in a more "intrinsic" way, following Diaz-Harris [11]. Consider the *framed* Hurwitz space $\widetilde{\mathcal{H}}_{d,g}^{fr}$ which parametrizes branched covers of $\mathbf{P}^1$, *without* modding out by the automorphisms of $\mathbf{P}^1$. The dense open subset $\mathcal{H}_{d,g}^{fr} \subset \widetilde{\mathcal{H}}_{d,g}^{fr}$ parametrizes framed *simply-branched* covers with smooth domain curve. $\widetilde{\mathcal{H}}_{d,g}^{fr}$ is a PGL_2-torsor over $\widetilde{\mathcal{H}}_{d,g}$, so the rational Picard groups remain unchanged. Over $\widetilde{\mathcal{H}}_{d,g}^{fr}$, there is the universal branched cover:

(2.3)
$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \alpha\ } & \mathbf{P}^1 \times \widetilde{\mathcal{H}}_{d,g}^{fr} \\
& {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle g} & \\
& \widetilde{\mathcal{H}}_{d,g}^{fr} &
\end{array}
$$

(Throughout this chapter, we will implicitly omit higher codimension loci. Therefore, we are allowed to invoke the universal branched covering.) The "intrinsic" divisor classes on $\widetilde{\mathcal{H}}_{d,g}^{fr}$

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are those which are obtained "solely" from the diagram (2.3):

$$\lambda_H = c_1(f_* c_1(\omega_f))$$

$$\kappa_H = f_*(c_1^2(\omega_f))$$

$$\xi_H := f_*(\alpha^* c_1(\mathcal{O}_{\mathbf{P}^1}(1)) \cdot c_1(\omega_f))$$

(We include the subscript H to emphasize the context of working with Hurwitz space.) Proposition 4.1 tells us that the Picard Rank Conjecture is equivalent to:

Conjecture H. The rational Picard group $\mathrm{Pic}_{\mathbb{Q}} \widetilde{\mathcal{H}}_{d,g}^{fr}$ is generated by the intrinsic classes λ_H, κ_H, and ξ_H.

Next, we formulate the analogous conjecture for the Severi varieties V.

2.2.2. *A conjecture for Severi varieties.* The "small" Severi variety V parametrizing irreducible, geometric genus g, nodal curves in the linear system $|d\tau|$ is not compact in codimension one. Exactly as in Diaz-Harris [11], we must allow three types of codimension one singular phenomenon to occur: cusps CU, tacnodes TAC, and triple points TP. We must also allow curves to acquire an additional node, yielding the divisor which we will call Δ. Unlike for Severi varieties on $\mathbf{P}^2$, *we must also allow our curves to split off the directrix σ of* $\mathbf{F}_k$, yielding the divisor of reducible curves, RED. Let

$$\widetilde{V} := V \cup CU \cup TAC \cup TP \cup \Delta \cup RED$$

be the partial compactification of V. $\widetilde{V}$ is now compact in codimension one.

In order to formulate our conjecture, we would like to invoke the universal genus g curve parametrized by $\widetilde{V}$. However, we may not do so immediately, since there is no way, for example, of distinguishing *which* subset of nodes of a curve parametrized by Δ ought to be normalized. As shown in Diaz-Harris [11], the solution to this problem is to work over the normalization $\widetilde{W}$ of $\widetilde{V}$. Then, we have a family of arithmetic genus g curves:

(2.4)

$$\mathcal{C} \xrightarrow{\ j\ } \mathbf{F}_k \times \widetilde{W}$$

with maps φ to $\widetilde{W}$ and ψ to $\widetilde{W}$.

Let $\pi\colon \mathbf{F}_k \longrightarrow \mathbf{P}^1$ denote the projection. The Picard group of $\mathbf{F}_k$ is generated by the class of a fiber f, and the directrix σ. Following Diaz-Harris [11], we define the intrinsic divisor classes to be:

$$\lambda_S = c_1(f_* c_1(\omega_\varphi))$$

$$\kappa_S = f_*(c_1^2(\omega_\varphi))$$

$$\xi_S := f_*(j^*(f) \cdot c_1(\omega_\varphi))$$

$$\epsilon_S := f_*(j^*(\sigma) \cdot c_1(\omega_\varphi))$$

$$\rho_S := f_*(j^*(\sigma \cdot f))$$

Finally, we may formulate

Conjecture S. The rational Picard group $\mathrm{Pic}_{\mathbb{Q}}\, \widetilde{W}$ is generated by the intrinsic classes $\lambda_S, \kappa_S, \xi_S, \epsilon_S,$ and ρ_S.

Letting $W \subset \widetilde{W}$ be the locus parametrizing *irreducible* curves, we immediately notice that ϵ_S and ρ_S are both trivial in $\mathrm{Pic}_{\mathbb{Q}}\, W$. The quotient morphism

$$q\colon W \longrightarrow \widetilde{\mathcal{H}}_{d,g}^{fr}$$

will be the central tool used to prove the main result of the next section which will compare Conjecture H and Conjecture S.

2.3. **A comparison of two conjectures.** Now that we have this setup, we would like to compare Conjecture H with Conjecture S using the quotient map q. We first show that Conjecture H implies Conjecture S.

Lemma 2.3. *Conjecture H implies Conjecture S for the appropriate Severi variety.*

Proof. Conjecture H implies $\mathrm{Pic}_{\mathbb{Q}}\, \mathcal{H}^{fr}_{d,g} = 0$, by Proposition 4.1. Restrict attention to the open subset $U \subset \mathcal{H}^{fr}_{d,g}$ parametrizing covers $[\alpha\colon C \longrightarrow \mathbf{P}^1]$ satisfying $\mathcal{E}_\alpha = \mathcal{E}_{gen}$. If

$$\mathcal{E}_{gen} = \mathcal{O}(k)^{\oplus r} \oplus \mathcal{O}(k+1)^{\oplus d-r-1}$$

then the complement of U has codimension two or more unless $r = d - 1$, in which case the complement is the Maroni divisor $\mathbf{M}$. In either case, $\mathrm{Pic}_{\mathbb{Q}}\, U = 0$, assuming Conjecture H.

Consider the quotient map

$$q\colon W \longrightarrow \widetilde{\mathcal{H}}^{fr}_{d,g}$$

This map is *not* a G-torsor - it factors through the space which parametrizes "choices of inclusions" $\mathcal{O}(-k) \subset \mathcal{E}^\vee$, or equivalently "choices of surjections" $\mathcal{E} \longrightarrow \mathcal{O}(k) \longrightarrow 0$. Restricting our attention over the subset $U \subset \widetilde{\mathcal{H}}^{fr}_{d,g}$, we can make this "choice of surjections" space precise as follows.

Over the open set $U \times \mathbf{P}^1$, the universal reduced direct image bundle $\mathcal{E}_\alpha$ has a distinguished surjection $\mathcal{E}_\alpha \longrightarrow \mathcal{G}^r \longrightarrow 0$ onto the well-defined rank r bundle $\mathcal{G}^r$ which restricts to $\mathcal{O}_{\mathbf{P}^1}(k)^{\oplus r}$ on the fibers of g. The projective bundle $\mathbf{P}\mathcal{G}^r$ is the desired "choice of sections" space, and by applying the "associated Hirzebruch surface" construction globally over $\mathbf{P}\mathcal{G}^r$ we arrive at a family:

(2.5)
$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;j'\;} & \mathbf{P}\mathcal{G}^r \times \mathbf{F}_k \\
& {\scriptstyle f'}\searrow \quad \swarrow {\scriptstyle g'} & \\
& \mathbf{P}\mathcal{G}^r &
\end{array}
$$

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The quotient map q, when restricted to the preimage Y of U, factors as

$$q \colon Y \longrightarrow \mathbf{P}\mathcal{G}^r \longrightarrow U$$

and the first map is a trivial G-torsor because the family (2.5) above provides a section. Since $\mathrm{Pic}_{\mathbb{Q}}\, G = 0$, we conclude that $\mathrm{Pic}_{\mathbb{Q}}\, Y = \mathbb{Q}\langle \zeta \rangle$ if $r > 1$ and $\mathrm{Pic}_{\mathbb{Q}}\, Y = 0$ if $r = 1$. Here ζ is the hyperplane class on $\mathbf{P}\mathcal{G}^r$. In the space $\mathbf{P}\mathcal{G}^r$, we may eliminate the choices of surjections $\mathcal{E} \longrightarrow \mathcal{O}(k) \longrightarrow 0$ which lead to a triple point under the associated Hirzebruch surface construction. This is a "horizontal" divisor in $\mathbf{P}\mathcal{G}^r$, so eliminating it will reduce the rational Picard group of Y to 0. Therefore, let Y' be $Y \setminus TP$. Then, what we have concluded is that $\mathrm{Pic}_{\mathbb{Q}}\, Y' = 0$, assuming Conjecture H.

To recap: The open subset $Y' \subset W$ is obtained by removing q-preimages of some divisors in $\widetilde{\mathcal{H}}_{d,g}^{fr}$ along with the divisor TP. It is not hard to show (following Diaz-Harris [11]) that the divisor TP is a combination of intrinsic divisor classes. Finally, observe that $q^*(\lambda_H) = \lambda_S, q^*(\kappa_H) = \kappa_S$, and $q^*(\xi_H) = \xi_S$. This observation together with the conclusion that $\mathrm{Pic}_{\mathbb{Q}}\, Y' = 0$, implies Conjecture S.

$\square$

Now we prove the opposite direction.

Lemma 2.4. *Conjecture S for the appropriate Severi variety implies Conjecture H.*

Proof. Let us first assume $r < d - 1$. We consider the open subsets $Y' \subset Y \subset W$, and $U_k \subset \widetilde{\mathcal{H}}_{d,g}^{fr}$ defined as follows: U_k is the locus of covers $\alpha \colon C \longrightarrow \mathbf{P}^1$ such that the least degree of a summand of $\mathcal{E}_\alpha$ is k. $Y := q^{-1}(U_k)$, where

$$q \colon W \longrightarrow \widetilde{\mathcal{H}}_{d,g}^{fr}$$

is the quotient map, and $Y' = Y \setminus TP$.

Let $U \subset U_k$ be the locus of covers satisfying $\mathcal{E}_\alpha = \mathcal{E}_{gen} = \mathcal{O}(k)^{\oplus r} \oplus \mathcal{O}(k+1)^{\oplus d-r-1}$. Put $Y'' := q^{-1}(U)$. As before, we see that the quotient map q, when restricted to Y'', factors as

$$q \colon Y'' \longrightarrow \mathbf{P}\mathcal{G}^r \longrightarrow U$$

The $\mathbf{P}^{r-1}$-bundle $\mathbf{P}\mathcal{G}^r$ is the space of "choices of surjections" $\mathcal{E} \longrightarrow \mathcal{O}(k) \longrightarrow 0$, so the first map is a G-torsor. Applying Theorem 2.2 to the first map gives

(2.6) $$\operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} \mathbf{P}\mathcal{G}^r \leq \operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} Y'' + \operatorname{rk}\chi(G) = \operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} Y'' + 1$$

Now, *if* $r > 1$, then we observe that $\operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} \mathbf{P}\mathcal{G}^r = \operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} U + 1$, so we obtain the inequality

$$\operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} U \leq \operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} Y''$$

However, Conjecture S implies that $\operatorname{Pic}_{\mathbb{Q}} Y''$ is generated by $\lambda_S, \kappa_S,$ and ξ_S, which are all pulled back from U via q. Therefore, we see that the inequality above must be an equality, and Conjecture H follows.

The case $r = d - 1$ essentially requires the same argument. The only thing we need to know in the end is that the Maroni divisor $\mathbf{M}$ is an intrinsic divisor in $\widetilde{\mathcal{H}}_{d,g}^{fr}$. This follows from Proposition 4.1. We let the reader spell it out. The interesting remaining case is $r = 1$, which we now deal with.

Suppose $r = 1$, and consider the locally closed locus $Z \subset \widetilde{\mathcal{H}}_{d,g}^{fr}$ parametrizing covers $\alpha \colon C \longrightarrow \mathbf{P}^1$ satisfying

$$\mathcal{E}_\alpha = \mathcal{O}(k)^{\oplus 2} \oplus \mathcal{O}(k+1)^{\oplus d-4} \oplus \mathcal{O}(k+2)$$

An upper bound for the codimension of Z is $h^1(\mathcal{E}nd\,\mathcal{E}_\alpha) = 2$. Notice that the "space of surjections" has jumped dimension to become a $\mathbf{P}^1$. Therefore, the preimage $X := q^{-1}(Z) \subset W$ is a divisor which is contracted under the quotient map. The divisor X must be nontrivial in $\operatorname{Pic}_{\mathbb{Q}} W$. X is visibly contained in the complement of Y'', and so, assuming Conjecture S,

we see that $\operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} Y' \le 2$. From the inequality (6), we obtain

$$\operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} U \le \operatorname{rk}\operatorname{Pic}_{\mathbb{Q}} Y'' + 1 \le 3$$

($\mathbf{P}\mathcal{G}^r$ is isomorphic to U.) Conjecture H follows by noting that λ_H, κ_H and ξ_H are independent classes, which follows from results of chapter 3.

$\square$

We have therefore proved the main theorem of the chapter:

Theorem 2.5 (A Comparison of Conjectures). *The Picard Rank Conjecture for $\mathcal{H}_{d,g}$ is equivalent to Conjecture H for $\widetilde{\mathcal{H}}_{d,g}^{fr}$, which in turn is equivalent to Conjecture S for the appropriate Severi variety.*

Remark 2.6. By the work of Mochizuki [18], we know that Conjecture H is known to hold for $d \ge g + 1$. In particular, it holds when $d \ge g + 2$ in which case the generic reduced direct image $\mathcal{E}_{gen}$ has $\mathcal{O}(1)$ as a least degree summand. Therefore, the associated Hirzebruch surface will be $\mathbf{F}_1$, i.e. the plane $\mathbf{P}^2$ blown up at a point. The linear series $|d\tau|$ is essentially the series of plane curves of degree d, and the theorem above allows us to deduce the Harris-Diaz conjecture for certain Severi varieties of plane curves this way, recovering some results of Edidin [12].

3. Compactifying Hurwitz Space

We now introduce the compactifications of the Hurwitz space $\mathcal{H}_{d,g}$ which will appear in chapter 4, where we will attempt an enumerative study of Hurwitz space.

The most natural codimension one phenomenon we expect in complete families of covers is two branch points crashing. Therefore, we must add covers $\alpha\colon C \longrightarrow \mathbf{P}^1$ which have a triple ramification, two stacked ramification points, or which have a singular, simply nodal domain curve C. Such a moduli space exists, and we denote it by $\widetilde{\mathcal{H}}_{d,g}$. The space $\widetilde{\mathcal{H}}_{d,g}$ may be constructed as follows. Consider the Kontsevich space $\mathcal{K}_{g,d} := \mathcal{M}_g(\mathbf{P}^1, d)$ parametrizing degree d covers of $\mathbf{P}^1$ by smooth genus g curves. We will assume that $g \geq 1, d \geq 3$, to avoid stacky complications. Let $\overline{\mathcal{K}}_{d,g}$ be the compactification by stable maps, and let $\overline{\mathcal{K}}_{d,g}^+$ be the open substack parametrizing *finite* stable maps. We notice that PGL_2 acts on the stack $\overline{\mathcal{K}}_{d,g}^+$ by automorphisms of the target $\mathbf{P}^1$, so we define $\widetilde{\mathcal{H}}_{d,g}$ to be the quotient

$$\widetilde{\mathcal{H}}_{d,g} := [\overline{\mathcal{K}}_{d,g}^+/PGL_2]$$

To be sure that the reader understands exactly what the moduli problem is, we indicate precisely what the functor $\widetilde{\mathcal{H}}_{d,g}$ is: An object in $\widetilde{\mathcal{H}}_{d,g}(S)$, S a scheme, is a family

$$\mathcal{C} \xrightarrow{\ \alpha\ } \mathcal{P}$$
$$\varphi \searrow \quad \swarrow \pi$$
$$S$$

where φ is a flat family of connected, arithmetic genus g nodal curves, π is a $\mathbf{P}^1$-bundle, and α is a finite, flat, degree d morphism. The morphisms are "commuting squares." We let $\mathcal{H}_{d,g}^\circ \subset \widetilde{\mathcal{H}}_{d,g}$ be the stack quotient $[\mathcal{K}_{d,g}/PGL_2]$ parametrizing covers with smooth domain curve.

The space $\widetilde{\mathcal{H}}_{d,g}$ is compact in codimension one, and contains $\mathcal{H}_{d,g}^\circ$ as an open dense subset. The components of $\widetilde{\mathcal{H}}_{d,g} \setminus \mathcal{H}_{d,g}^\circ$ will collectively be called the *boundary divisors* of $\widetilde{\mathcal{H}}_{d,g}$. Note,

however, that $\widetilde{\mathcal{H}}_{d,g}$ is not complete - if enough branch points come together, we will either be forced out of the realm of nodal curves, or out of the realm of finite maps.

There are two boundary divisors which we will encounter more often than others in what follows. First, there is the divisor δ_{irr} parametrizing *irreducible*, nodal covers of $\mathbf{P}^1$. THen there is the divisor B parametrizing degree $d-1$ maps "with a basepoint". These are covers $\alpha\colon C \longrightarrow \mathbf{P}^1$ with $C = L \cup C'$ where $L \simeq \mathbf{P}^1$ maps with degree 1, C' is a genus g curve mapping with degree $d-1$, and $L \cap C'$ is one point, the "basepoint" of the degree $d-1$ map on C'.

In the interior of $\widetilde{\mathcal{H}}_{d,g}$ there are the divisors T and D which parametrize covers with a triple ramification point, and covers with stacked simple ramification, respectively. (T stands for "triple". D stands for "double".)

The reader is warned that there are many more boundary divisors than the two mentioned above. The boundary divisors which have not been mentioned will sometimes be referred to as *higher boundary divisors* of $\widetilde{\mathcal{H}}_{d,g}$.

For the reader's convenience, we include a figure to illustrate some boundary divisors in $\widetilde{\mathcal{H}}_{3,g}$ (thanks to Anand Deopurkar):

(a) T (b) δ_{irr} (c) B
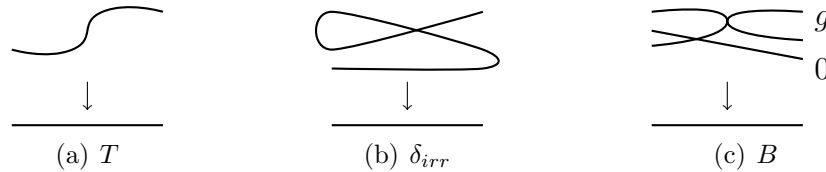
3.1. **The admissible cover compactification $\overline{\mathcal{H}}_{d,g}$.** We recall the *admissible cover* compactification of $\mathcal{H}_{d,g}$ which was first introduced in Harris-Mumford [14]. We are not interested in ordering the branch points in this thesis, so technically we are working modulo the symmetric group S_b, where $b = 2d + 2g - 2$. Let $\widetilde{\mathcal{M}}_{0,b}$ be the moduli space of Deligne-Mumford stable rational curves with b unordered marked points. The functor $\overline{\mathcal{H}}_{d,g}$ is described as follows [14]:

An object of $\overline{\mathcal{H}}_{d,g}(S)$ is a family

$$\mathcal{C} \xrightarrow{\ \alpha\ } (\mathcal{P}, br)$$

$$\varphi \searrow \quad \swarrow \pi$$

$$S$$

where φ is a flat family of connected, arithmetic genus g nodal curves, $[br \subset \mathcal{P}] \in \widetilde{\mathcal{M}}_{0,b}$, and α is a finite, degree d morphism satisfying the following "admissibility conditions":

(1) Above the divisor br, the map α is simply-branched, and all other branching occurs over the singular locus of $[\pi : \mathcal{P} \longrightarrow S]$.

(2) The set-theoretic singular locus $Sing(\mathcal{C}/S)$ is the preimage of $Sing(\mathcal{P}/S)$, and analytically around a point $s \in S$ and a point $x \in \mathcal{C}_s$ where $\mathcal{C}/S$ is singular, the equations of $x \in \mathcal{C}$, $y := \alpha(x) \in \mathcal{P}$, and the map α are:

$$\mathcal{C} : xy = a, \ a \in \widehat{\mathcal{O}}_s, \ x, y \text{ generate } \widehat{m}_{x,\mathcal{C}}$$

$$\mathcal{P} : uv = a^e, \ a \in \widehat{\mathcal{O}}_s \text{ same as above}, \ u, v \text{ generate } \widehat{m}_{x,\mathcal{P}}$$

$$\alpha : u = x^e, \ v = y^e$$

The space $\overline{\mathcal{H}}_{d,g}$ carries a forgetful morphism $F : \overline{\mathcal{H}}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$ while also retaining a finite map $Br : \overline{\mathcal{H}}_{d,g} \longrightarrow \widetilde{\mathcal{M}}_{0,b}$. The space of admissible covers is unfortunately not normal: the smooth Deligne-Mumford stack of *twisted stable maps* [1] provides a "normalization" on which our divisor theory will make sense. The Weil divisor $\overline{\mathcal{H}}_{d,g} \setminus \mathcal{H}_{d,g}$ will be called the *boundary*. Components of the boundary will be identified with their reduced preimage Cartier divisor in the normalization.

3.2. **Boundary Divisors in $\overline{\mathcal{H}}_{d,g}$.** In this section, we will give a description of the boundary divisors occurring in the admissible cover compactification $\overline{\mathcal{H}}_{d,g}$. We also provide a proof of the independence of the boundary divisors in the rational Picard group. Although Theorem

1 is not very interesting in its own right, the strategy of exploiting the inductive structure of the boundary may be of some interest to some readers. Roughly speaking, the difficulty in working with the boundary divisors stems from 3 sources: (1) There are simply too many boundary divisors, (2) for degrees $d > 5$, the construction of test families becomes nontrivial due to a lack of "linear algebraic" parametrizations of Hurwitz space and (3) If one is to use test families, one must somehow maintain control over the intersection of test families with boundary divisors. We overcome these difficulties by "concentrating" issues (2) and (3) into "genus 0" families, and "hyperelliptic" families. The reader will see precisely what this means in Lemma 3.3.

Before carrying on, we will introduce a graph-theoretic representation of admissible covers (and boundary divisors).

3.3. **The decorated dual graph.** Recall that the branch morphism $Br : \overline{\mathcal{H}}_{d,g} \longrightarrow \widetilde{M}_{0,b}$ is finite. This means that every boundary divisor E will lie over a unique boundary divisor, $Br(E) \subset \widetilde{M}_{0,b}$. Therefore, the generic admissible cover $[\alpha : C \longrightarrow \mathbf{P}]$ parametrized by E will have a fixed *branching profile* which is the union of two marked $\mathbf{P}^1$'s: $b_L \subset \mathbf{P}^1_L$ and $b_R \subset \mathbf{P}^1_R$. The curve $\mathbf{P}$ is simply $\mathbf{P}^1_L$ glued to $\mathbf{P}^1_R$ at a point x which is not any of the branch points. Furthermore, C breaks up into two halves C_L and C_R which are the preimages of $\mathbf{P}^1_L$ and $\mathbf{P}^1_R$. Since C is a nodal curve, it has a dual graph Γ_C whose vertices v are marked by the geometric genera g_v of the corresponding components. We furthermore label every vertex as either an L-vertex or an R-vertex depending on whether it parametrizes a component of C_L or C_R respectively. Furthermore, we label every vertex v with its *degree d_v*. Since the nodes of C must lie over the node x, we see that every edge in Γ_C must join an L-vertex with an R-vertex. Finally, we label every edge e with the index of ramification r_e occurring at the corresponding node. (The index of ramification of $z \to z^m$ is $m - 1$)

We may arrange the vertices of the graph Γ_E in two columns, the *left*, and the *right* side. In this way, we may easily represent the boundary divisors by their decorated dual graphs. One small detail is worth mentioning: The locus of covers with fixed dual graph

is irreducible. This follows from the fact that the moduli space of branched covers of $\mathbf{P}^1$, simply branched everywhere except perhaps at one point, is irreducible.

Labeling every vertex and edge of Γ_E will often be notationally cumbersome and unnecessary. Therefore, we will adopt the following rule: Genus 0 vertices and unramified edges will usually be left undecorated. Furthermore, we note that since the degree of a vertex v is determined by all ramification indices of edges containing v, we need not specify *both* the degrees and ramification indices.

For example, in $\overline{\mathcal{H}}_{3,g}$, we may consider the boundary divisor E generically parametrizing covers of the form:
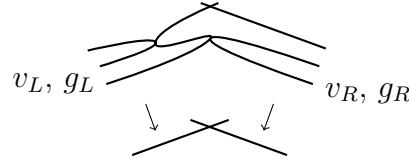


The corresponding dual graph Γ_E would be:
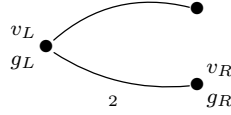
3.4. **Some more examples.** We provide some examples for the reader's convenience. These examples are not selected randomly: they all lie above $\Delta_2 \subset \widetilde{M}_{0,b}$, i.e. two branch points are colliding. In enumerative settings, the divisors lying over $\Delta_2 \subset \widetilde{M}_{0,b}$ show up most frequently, so we refer to them as the "enumeratively relevant" boundary divisors. For the reader's convenience, we indicate the dual graphs of 4 basic enumeratively relevant divisors.
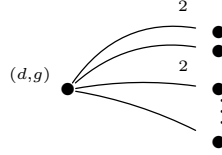
(1) Δ_{irr}, admissible covers having dual graph $\Gamma_{\Delta_{irr}}$:
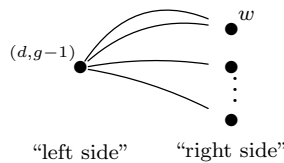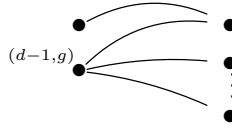
(2) T_{ad}, "triple ramification". These are covers with dual graph $\Gamma_{T_{ad}}$:
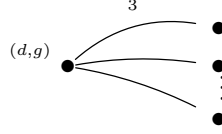
(3) D_{ad}, "stacked $(2,2)$ ramification". These covers have dual graph $\Gamma_{D_{ad}}$:

(4) B_{ad}, "basepoint". These covers have dual graph Γ_B:

We will briefly explain the interpretation of $\Gamma_{\Delta_{irr}}$, just for the reader's convenience. The vertex $w \in \Gamma_{\Delta_{irr}}$ is unlabeled, so has genus 0. Furthermore, both edges emanating from w are unramified, therefore the degree d_w of w is 2. So the rational curve curve associated to w is attached to the curve C_v at two points. The stable model is therefore an irreducible nodal curve, hence the label "Δ_{irr}".

Pick any boundary divisor D, and choose a general admissible cover $[\alpha \colon C \longrightarrow P] \in D$. Let $p \in P$ be the unique node, and let $(d_1, d_2, ..., d_k)$ be the local degrees of α occurring at the nodes $(q_1, ..., q_k)$ above n. The description of the versal deformation space $\mathbf{Def}\,\alpha$ is given in Harris-Mumford [14] as

$$\Delta^{b-4} \times \mathbf{Spec}\,k[[t_1, t_2, ..., t_k, s]]/(t_1^{d_1} = t_2^{d_2} = ... = t_k^{d_k} = s)$$

where Δ^{b-4} is a smooth germ. Above $\mathbf{Def}\,\alpha$, we have a versal family

$$
\begin{array}{ccc}
C \subset \mathcal{C} & \xrightarrow{\ \tilde{\alpha}\ } & P \subset \mathcal{P} \\
& \varphi \searrow \quad \swarrow \pi & \\
& \mathbf{Def}\,\alpha &
\end{array}
$$

Locally around the node $n \in P \subset \mathcal{P}$, $\mathcal{P}$ has local equation $uv = s$. At the nodes $q_i \in C \subset \mathcal{C}$, $i = 1, 2, ..., k$, the total space $\mathcal{C}$ has local equation $x_i y_i = t_i$.

This local description of the boundary allows us to understand intersection multiplicities. In particular, we consider the following situation. Let F_1

$$
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{\ \alpha_1\ } & \mathcal{P}_1 \\
\sigma_i \nwarrow\ \searrow & \swarrow\ \nearrow \tau_1 & \\
& T &
\end{array}
$$

be a family of d sheeted covers having k sections σ_i, $i = 1, ..., k$ mapping to the marked section τ_1, where, around the section σ_i the map α_1 has local degree d_i over τ_1. Let F_2 be another family of the same type over the same base T:

$$
\begin{array}{ccc}
\mathcal{C}_2 & \xrightarrow{\ \alpha_2\ } & \mathcal{P}_2 \\
\sigma_i \nwarrow\ \searrow & \swarrow\ \nearrow \tau_2 & \\
& T &
\end{array}
$$

Suppose we glue the two families of covers along the sections σ_i and $\tau_{1,2}$ to obtain a family of admissible covers

$$
\begin{array}{ccc}
\mathcal{C}_1 \cup \mathcal{C}_2 & \xrightarrow{\ \alpha_1 \cup \alpha_2\ } & \mathcal{P}_1 \cup \mathcal{P}_2 \\
& \searrow \quad \swarrow & \\
& T &
\end{array}
$$

The admissibility condition tells us that

$$
(N_{\sigma_i/\mathcal{C}_1} \otimes N_{\sigma_i/\mathcal{C}_2})^{\otimes d_i} = (N_{\sigma_j/\mathcal{C}_1} \otimes N_{\sigma_j/\mathcal{C}_2})^{\otimes d_j} = N_{\tau/\mathcal{P}_1} \otimes N_{\tau/\mathcal{P}_2}
$$

When we pass to the normalization of $\overline{\mathcal{H}}_{d,g}$, the family T will have intersection number

$$\deg \left(N_{\tau/\mathcal{P}_1} \otimes N_{\tau/\mathcal{P}_2} \right)/lcm(d_1, ..., d_k)$$

with the boundary divisor E in which it is contained. (We assume that T does not have additional isolated intersections with E, in which case we must add these contributions appropriately.) This follows from normalizing the local equations defining $\mathbf{Def}\,\alpha$.

From the local equations, we notice that the map $Br\colon \overline{\mathcal{H}}_{d,g} \longrightarrow \overline{\mathcal{M}}_{0,b}$ is unramified along a boundary divisor E exactly when the dual graph Γ_E has no edges e with $r_e > 0$. In other words, the general admissible cover parametrized by E has no ramification occurring at the nodes. Therefore, we will call a boundary divisor E *unramified* if the generic admissible cover parametrized by E has no ramification occurring at the nodes. Otherwise, we will call E a *ramified* divisor. Finally, if Γ_E has exactly 2 vertices, we will call E a *2-vertex* divisor.

3.5. **Admissible Reduction.** In this section, we explain the process of "admissible reduction". The general problem is stated as follows: Suppose we have a family of degree d covers

(3.1)
$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \alpha\ } & \mathbf{P}^1_\Delta \\
 & {}_{\varphi}\searrow \quad \swarrow_{\pi} & \\
 & \Delta = \mathbf{Spec}\,k[[t]] &
\end{array}$$

where $\mathcal{C}$ is a family of generically smooth nodal curves (having smooth total space) mapping $d:1$ onto $\mathbf{P}^1_\Delta$. In other words, we have a map $t\colon \Delta \longrightarrow \widetilde{\mathcal{H}}_{d,g}$, with the image of the punctured disc Δ^* lying in the interior $\mathcal{H}_{d,g}$. How do we arrive at the corresponding family of admissible covers?

3.5.1. *Replacing a node.* Suppose the central fiber $C_0 \subset \mathcal{C}$ has a node $x \in C_0$, and suppose the local equation of $\mathcal{C}$ around x is $uv - t$. Let's assume that x is the unique node of C_0,

just for simplicity - the general procedure is basically the same. In other words, the arc Δ intersects the total boundary δ transversely at $t = 0$ along exactly one component.

We first make an order 2 base change $\mathbf{Spec}\, k[[s]] \longrightarrow \mathbf{Spec}\, k[[t]]$ given by $t \to s^2$, and consider the new family

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\ \alpha'\ } & \mathbf{P}^1_{\Delta'} \\
& \searrow^{\varphi'} \quad \swarrow^{\pi'} & \\
& \Delta' = \mathbf{Spec}\, k[[s]] &
\end{array}
$$

We may consider the point x as living in $\mathcal{C}'$, where it is an A_1 singularity of $\mathcal{C}'$. Let $y := \alpha(x) \in \mathbf{P}^1_{\Delta'}$ be the image. To arrive at a family of admissible covers, we blow up the point $y \in \mathbf{P}^1_{\Delta'}$, the reduced point $x \in \mathcal{C}'$, and the $d - 2$ remaining points of $\alpha^{-1}(y)$. The reader can check that in this way we obtain an admissible cover $[\alpha^{ad} \colon \mathrm{Bl}\,\mathcal{C}' \longrightarrow \mathrm{Bl}\,\mathbf{P}^1_{\Delta'}] \in \overline{\mathcal{H}}_{d,g}(\Delta')$.

3.5.2. *Replacing higher ramification.* Now that we know how to replace nodes, we describe how to replace higher ramification. Suppose

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \alpha\ } & \mathbf{P}^1_{\Delta} \\
& \searrow^{\varphi} \quad \swarrow^{\pi} & \\
& \Delta = \mathbf{Spec}\, k[[t]] &
\end{array}
$$

is a general family such that there exists a (unique, say) point $x \in C_0 \subset \mathcal{C}$ such that the finite cover $\alpha_0 \colon C_0 \longrightarrow \mathbf{P}^1$ is locally given by $z \to z^e$ where $e \geq 3$. Let $y := \alpha(x) \in \mathbf{P}^1_{\Delta}$ We first perform a base change $\mathbf{Spec}\, k[[s]] \longrightarrow \mathbf{Spec}\, k[[t]]$ given by $t \to s^{e-1}$ to arrive at the pullback family as before:

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\ \alpha'\ } & \mathbf{P}^1_{\Delta'} \\
& \searrow^{\varphi'} \quad \swarrow^{\pi'} & \\
& \Delta' = \mathbf{Spec}\, k[[s]] &
\end{array}
$$

We let $Z \subset \mathbf{P}^1_{\Delta'}$ be the scheme theoretic preimage of y. Therefore $Z \simeq \mathbf{Spec}\, k[\epsilon]/(\epsilon^{e-1})$. We then blow up $\mathbf{P}^1_{\Delta'}$ along Z and blow up $\mathcal{C}'$ at the reduced preimage of Z. The reader

can check that there is then a map $\alpha^{ad}\colon \mathrm{Bl}\,\mathcal{C}' \longrightarrow \mathrm{Bl}_Z \mathbf{P}^1_{\Delta'}$, which provides the admissible replacement.

Notice in both situations that a base change is always required. The reason why this is true is that the branch morphism $Br\colon \overline{\mathcal{H}}_{d,g} \longrightarrow \widetilde{M}_{0,b}$ is generically representable, and so we need to, at the very least, extend the induced maps $\Delta^* \longrightarrow \widetilde{M}_{0,b}$, which generically requires an order $e-1$ base change to separate the sheets of the $e-1$ branch points coming together.

The admissible reduction process allows us to understand the relationship between $\widetilde{\mathcal{H}}_{d,g}$ and $\overline{\mathcal{H}}_{d,g}$. There is a birational map $R\colon \overline{\mathcal{H}}_{d,g} \dashrightarrow \widetilde{\mathcal{H}}_{d,g}$ which extends in codimension one over the enumeratively relevant boundary divisors. The admissible reduction process shows that $R^*(T) = 2T_{ad}$, $R^*(D) = 2D_{ad}$, and $R^*(\delta_{irr}) = 2\Delta_{irr}$. In fact, the pullback of any component of the total boundary δ is twice the "corresponding" divisor in the admissible cover compactification.

3.6. **Independence of the boundary divisors.** It appears as though the proof of independence of boundary divisors is not easy to find in the existing literature. Therefore, we provide a proof in this section. The reader is of course invited to skip this section. The main point of interest, in the author's opinion, is the choice of test families which will be used. We take a moment to describe a class of test families which we call *partial pencils*.

3.7. **Partial pencils.** We now describe a method for constructing test families which have a "controlled" intersection with boundary divisors. Consider the linear system of (m,d) curves on $\mathbf{P}^1 \times \mathbf{P}^1$. Suppose we pick a distinct d-tuple of points Z *lying entirely in a* $(1,0)$ *ruling curve*, which we call F. Consider a general linear pencil $\mathbf{P}^1_t \subset |(m,d)|$ of curves containing Z in its base locus B. The total space of the pencil

$$p\colon X \longrightarrow \mathbf{P}^1_t$$

is the blow up of $\mathbf{P}^1 \times \mathbf{P}^1$ along B. Since we have forced d points of F to be in the base locus, the pencil will be constant along F, so there will be some value , say $t = 0$, where the fiber X_0 of p splits off F as a component. In other words, there is a unique element of the

pencil, X_0 which is a union of F and a $(m, d - 1)$ residual curve R. We may suppose that $X_0 = F \cup R$ is general among such reducible curves.

The utility of the family p comes from the fact that the fixed locus Z provides d sections σ_i along which we may attach "the rest" of an admissible cover. When we try to construct families of admissible covers, we generally want to avoid the combinatorial analysis that comes with "uncontrolled" incidences of branch points. Therefore, we prefer to use the pencils as described above. Notice that whenever a ramification point approaches any of the sections σ_i (i.e. points of Z), the *entire* fiber F is forced to split off, which will provide an intersection with only one other boundary divisor which is easy to describe. The rest of the pencil, i.e. away from $t = 0$, only provides intersections with enumeratively relevant divisors: triple ramification, stacked $(2, 2)$ ramification, and simple nodes may occur.

In the sequel, we will see other instances of similar families arising. The ambient surface $\mathbf{P}^1 \times \mathbf{P}^1$ may be different (e.g. other Hirzebruch surfaces), but the general construction will almost always be the same: We force the "maximal" number of base points in a "fiber" forcing the fiber to split off. This gives us control over the divisors with which the resulting family will interact. A family of admissible covers which is created from such a construction (by, say, attaching a constant family at the sections σ_i) will be called a *partial pencil* family. (The name comes from thinking about the family as a pencil, but only varying "one half" of an admissible cover.)

Theorem 3.1. *The boundary divisors are independent in* $\mathrm{Pic}_{\mathbb{Q}} \overline{\mathcal{H}}_{d,g}^{\nu}$ *for* $g \geq 3$.

Proof. We will use the method of test families. Begin with a dependence relation

$$0 = \sum_E c(E) \cdot E$$

Let us rewrite the dependence relation as:

(3.2) $$a\lambda + b\delta + cD + \sum_{E'} c(E') \cdot E' + \sum_{E} c(E) \cdot E = 0$$

where the first sum is over all enumeratively relevant divisors *other than* T_{ad}, D_{ad}, and Δ_{irr}. We will call the coefficients $c(E')$, along with a, b, and c the *enumeratively relevant coefficients*. The second sum is over all other boundary divisors, and the divisors classes λ and δ are pulled back via the forgetful morphism $F \colon \overline{\mathcal{H}}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$. We may perform this change of basis thanks to Proposition 1.4. The first claim is that the coefficients $c(E)$ are linear combinations of a, b, c and the $c(E')$. Before proving this reduction, we will need a definition. Define the *excess*, $ex(E)$, of a boundary divisor E to be

(3.3) $$ex(E) := min\Big\{ \sum_{L-vertices} g_v + r_v, \ \sum_{R-vertices} g_v + r_v \Big\}.$$

Therefore, the divisors with $ex(E) = 0$ are precisely those which are unramified and have only rational components on one side.

Lemma 3.2. *If $ex(E) = 0$, then the coefficient $c(E)$ is a linear combination of the enumeratively relevant coefficients.*

Proof. Pick a rational vertex $v \in \Gamma_E$ which has degree $d_v \geq 2$. Vary v in a pencil of $(1, d_v)$ curves on $\mathbf{P}^1 \times \mathbf{P}^1$, with d_v distinct basepoints forced to lie in a $(1, 0)$ fiber. The resulting test family γ has negative intersection with E and has 0 intersection with λ. Otherwise, the test family γ intersects D, δ, and perhaps some other enumeratively relevant divisors, but no others. Therefore, we obtain a relation among $c(E), c(E'), a, b$, and c, which is what we wanted. $\square$

Before continuing on with the proof of the theorem, we will introduce a useful construction of test families which will help us "reduce" the ramification occurring above a node. On $\mathbf{P}^1 \times \mathbf{P}^1$, consider a pencil $\mathbf{P}^1_t$ of $(1, d)$ curves such that the base locus contains a $\mathbf{Spec}\, k[\epsilon]/(\epsilon^d)$

subscheme of a $(1, 0)$ ruling line. Suppose $\mathbf{P}_t^1$ is general among such pencils. Let

$$p\colon X \longrightarrow \mathbf{P}_t^1$$

be the total space of the pencil. X will have exactly one A_{d-1} singular point $s \in X$ which will be a singular point of the corresponding fiber X_0 of p. The curve X_0 has two components S_0 and T_0, where S_0 is a $(0, 1)$ curve and T_0 is a $(1, d-1)$ curve. The two components intersect transversally at the reduction of the $k[\epsilon]/(\epsilon^d)$ point in the base locus, and T_0 contains a $\mathbf{Spec}\, k[\epsilon]/(\epsilon^{d-1})$ subscheme of $\mathbf{Spec}\, k[\epsilon]/(\epsilon^d)$, i.e. is totally ramified under the projection to the first factor.

The family $p\colon X \longrightarrow \mathbf{P}_t^1$ has a section σ corresponding to the reduced basepoint underlying $\mathbf{Spec}\, k[\epsilon]/(\epsilon^d)$. The singular point $s \in X$ is contained in this section σ. If we partially blow up the point s, we arrive at a family $p\colon \widetilde{X} \longrightarrow \mathbf{P}_t^1$ with an A_{d-2} singular point $\widetilde{s} \in \widetilde{X}$, and a new central fiber $\widetilde{X}_0 = S_0 \cup T_0 \cup P$, where $P \simeq \mathbf{P}^1$ is the exceptional divisor of the blow up. The singular point $\widetilde{s}$ is precisely $S_0 \cap P$. Furthermore, the section σ now passes through a general point of P. (We are considering strict transforms everywhere.)

The rational component P has a well defined pencil of degree d on it. Indeed, set $x := \sigma \cap P$, $y := P \cap S_0$, and $z := P \cap T_0$. Then the divisors $d \cdot x$ and $y + (d-1) \cdot z$ span a well-defined pencil of degree d on P.

Notice that the degree d map on P is ramified maximally at the point x (which will eventually serve as a point of attachment with the rest of an admissible cover) and ramified to order $d - 1$ at the point z. Furthermore, all other elements of the pencil p are totally ramified along the section σ. Therefore, the family $p\colon \widetilde{X} \longrightarrow \mathbf{P}_t^1$ will serve (perhaps after some finite base changes) as a "ramification reducing" test family, as we see in the next lemma.

Lemma 3.3. *Let E have excess $ex(E) = n > 0$. Then $c(E)$ is a linear combination of the enumeratively relevant coefficients and one more coefficient, $c(E_i)$, where $ex(E_i) < n$.*

Proof. If the divisor E is unramified, pick a vertex $v \in \Gamma_E$ (on the side with smaller sum of genera) of genus $g_v > 0$. Suppose the dual graph, locally around v, had the following form:
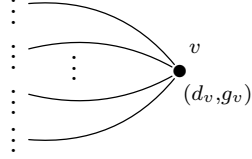

We will now "concentrate" all of the genus of the vertex v into a hyperelliptic vertex, which allows us to vary the vertex in a family. So, we will replace this part of the dual graph with the following picture.
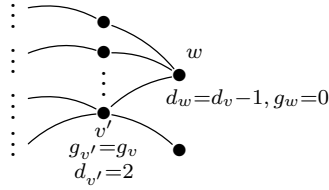

Now we vary the genus g_v hyperelliptic curve v' in a pencil of $|2\tau + f|$ curves on $\mathbf{F}_{g_v}$, with two pairs of prescribed basepoints lying in two chosen fibers of the natural projection $\pi \colon \mathbf{F}_{g_v} \longrightarrow \mathbf{P}^1$. The resulting test family γ provides the reduction we seek because it has nonzero intersection with E and with two higher boundary divisors with strictly smaller excess.

If the divisor E is ramified, we concentrate our attention at a vertex v, such that locally around v the dual graph has the form:
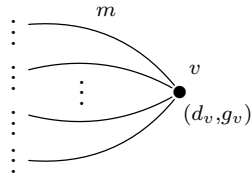
(The top edge in the diagram has ramification index m.) We replace this local picture with:
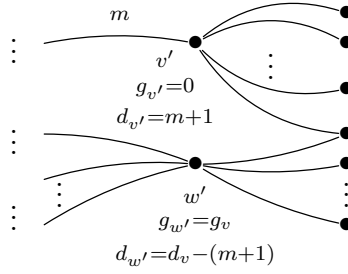
We vary the totally ramified, degree $m + 1$ rational vertex v' in $\mathbf{P}^1 \times \mathbf{P}^1$ with an $(m+1)$-fold basepoint fixed in a fiber. The resulting test family will intersect E negatively, and will intersect other higher boundary divisors which have strictly smaller total ramification index. $\qquad\square$

Finally, we show that the enumeratively relevant coefficients are zero.

Lemma 3.4. *The enumeratively relevant coefficients are zero.*

Proof. We will argue by reducing the degree d incrementally. First we note that using partial pencil families like those in section 4.4 we may easily deduce Lemma 3 when $d = 3$ and $d = 4$. (In fact, we can also show it for $d = 5$, but we don't need this.) Therefore, we assume $d \geq 5$. Furthermore, we must assume $g \geq 3$ because there is a relation between λ and δ when $g \leq 2$ which persists in Hurwitz space.

We switch to the partial compactification $\widetilde{\mathcal{H}}_{d,g}$ (which is compact in codimension 1) where we deduce from relation (1) the relation $a\lambda + b\delta + cD + \sum c(\delta') \cdot \delta' = 0$ in $\mathrm{Pic}_{\mathbb{Q}} \widetilde{\mathcal{H}}_{d,g}$. We may do this because, as we have seen, one only needs to make an order two base change to pass from a family $T \subset \widetilde{\mathcal{H}}_{d,g}$ to a family of admissible covers. Therefore, the relation remains the same, up to multiplication by 2. Here we are writing δ' to refer to the remaining boundary divisors. Among the divisors δ' is the divisor B parametrizing "covers with a basepoint". A general cover $[\alpha \colon C \longrightarrow \mathbf{P}^1] \in B$ has a singular domain curve $C = L \cup C'$ where $L \simeq \mathbf{P}^1$ maps isomorphically to $\mathbf{P}^1$ under α, C' is a genus g, degree $d - 1$ cover of $\mathbf{P}^1$, and $L \cap C'$ is one point, which we call the "basepoint". The coefficient $c(B)$ is easily seen to be 0: Consider the family S obtained by "sliding" the basepoint along C'. This family intersects

B negatively, and has trivial intersection number with λ, δ, and D, and all other boundary divisors.

Now consider a general complete curve $T \subset \widetilde{\mathcal{H}}_{d-1,g}$ obtained from a family

(3.4)
$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \alpha\ } & \mathcal{P} \\
& {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle g} & \\
& T &
\end{array}
$$

After a suitable base change (which may be performed globally over $\widetilde{\mathcal{H}}_{d-1,g}$) we may assume that the family f has a section s satisfying $s^2 \neq 0$. We attach a constant "basepoint" $\mathbf{P}^1$ to the family f along s to obtain a family $T' \subset B \subset \widetilde{\mathcal{H}}'_{d,g}$. The relation (2) then implies the same for the family $T \subset \widetilde{\mathcal{H}}_{d-1,g}$. Since we would obtain relation (2) for any complete curve $T \subset \widetilde{\mathcal{H}}_{d-1,g}$ avoiding a codimension 2 locus (where the section intersects triple ramification points for example), we would deduce that the relation (3.2) must also hold in $\mathrm{Pic}_{\mathbb{Q}}\widetilde{\mathcal{H}}_{d-1,g}$. $\qquad\square$

The independence of the boundary now follows from Lemmas 1, 2, and 3. $\qquad\square$

Remark 3.5. Another way to phrase the argument in Lemma 3 is: The boundary divisor B parametrizing "basepoints" is birational to the universal genus g curve over $\widetilde{\mathcal{H}}_{d-1,g}$. Thus, by restricting relation (3.2) to B, we obtain a relation on $\widetilde{\mathcal{H}}_{d-1,g}$, which has the same coefficients as the original relation.

4. ENUMERATIVE GEOMETRY

After describing the relationships between various "natural" divisor classes on Hurwitz spaces, we will go on to describe the divisor classes of the Maroni and Casnati-Ekedahl divisors. These descriptions will then allow us to determine sharp upper bounds for slopes of sweeping families of trigonal, tetragonal, and pentagonal curves, extending the Cornalba-Harris bound for hyperellitpic curves, and Stankova's bound for trigonal curves. All intersections will be considered as occurring on the appropriate Deligne-Mumford stack.

4.1. **The natural divisor classes on $\widetilde{\mathcal{H}}_{d,g}$.** Let's consider the universal smooth d sheeted cover of genus g:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ u\ } & \mathcal{P} \\
& {\scriptstyle \varphi}\searrow \quad \swarrow {\scriptstyle \pi} & \\
& \widetilde{\mathcal{H}}_{d,g} &
\end{array}
$$

On the $\mathbf{P}^1$-bundle $\mathcal{P}$, we have the universal reduced direct image $\mathcal{E}_u$ and bundle of quadrics $\mathcal{F}_u$. In $\widetilde{\mathcal{H}}_{d,g}$, we have the following divisor classes:

(1) λ, pulled back from $\overline{\mathcal{M}}_g$.

(2) δ, the total boundary pulled back from $\overline{\mathcal{M}}_g$.

(3) T, the locus of branched covers which have a triple ramification point.

(4) D, the locus of branched covers which have stacked $(2,2)$ ramification somewhere.

(5) $\pi_*(c_1^2 \mathcal{E}_u)$

(6) $\pi_*(c_2 \mathcal{E}_u)$

(7) $\pi_*(c_1^2 \mathcal{F}_u)$

(8) $\pi_*(c_2 \mathcal{F}_u)$

(9) $\varphi_*(c_1^2(\omega_u))$

(10) $\varphi_*(c_1(\omega_u) \cdot u^* c_1(\omega_\pi))$

(The divisor class $\varphi_*(u^* c_1^2(\omega_\pi))$ is 0 because π is a $\mathbf{P}^1$-bundle.) We will call these 10 classes the *natural divisor classes* on $\widetilde{\mathcal{H}}_{d,g}$.

Proposition 4.1. *All natural divisor classes are expressible in terms of $\{\lambda, \delta, D\}$. In particular, the following relations hold:*

(1) $\varphi_*(c_1^2(\omega_u)) = \frac{12b+24}{b-10}\lambda - \frac{b+6}{b-10}\delta + \frac{4}{b-10}D$

(2) $\frac{b-10}{3}T = 8(b-1)\lambda - (b-2)\delta + 2D$

(3) $\pi_*(c_1^2\mathcal{E}_u) = \frac{18b}{b-10}\lambda - \frac{2b}{b-10}\delta + \frac{b}{2(b-10)}D$

(4) $\pi_*(c_2\mathcal{E}_u) = \frac{8(b-1)}{b-10}\lambda - \frac{b-2}{b-10}\delta + \frac{(b-2)}{4(b-10)}D$

Proof. For ease of notation, in what follows we often choose to suppress pullback and pushforward symbols. Suppose

(4.1)
$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \alpha\ } & \mathcal{P} \\
& {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle g} & \\
& T &
\end{array}
$$

is a family of generically smooth covers over a smooth one dimensional base curve T. We will apply the Grothendieck-Riemann-Roch formula to the map $\alpha\colon \mathcal{C} \longrightarrow \mathcal{P}$. So let us begin by applying G-R-R to ω_α^N:

$$ch[\alpha_*(\omega_\alpha^N)] \cdot Td(T_\mathcal{P}) = \alpha_*[ch(\omega_\alpha^N) \cdot Td(T_\mathcal{C})]$$

$$\Rightarrow [d + c_1\alpha_*(\omega_\alpha^N)t + ch_2(\omega_\alpha^N)t^2 + ...] \cdot \left[1 - \frac{K_\mathcal{P}}{2}t + \frac{K_\mathcal{P}^2 + c_2 T_\mathcal{P}}{12}t^2 + ...\right]$$

$$= \alpha_*\left[\left(1 + N\omega_\alpha t + \frac{N^2}{2}\omega_\alpha^2 t^2\right)\left(1 - \frac{K_\mathcal{C}}{2}t + \frac{K_\mathcal{C}^2 + c_2 T_\mathcal{C}}{12}t^2\right)\right]$$

$$\Rightarrow c_1\alpha_*(\omega_\alpha^N) - \frac{d}{2}K_\mathcal{P} = \alpha_*(Nc_1\omega_\alpha - \frac{K_\mathcal{C}}{2})$$

$$\Rightarrow \boxed{c_1[\alpha_*(\omega_\alpha^N)] = (2N-1)c_1\mathcal{E}}$$

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Note: From the exact sequence

$$0 \to \mathcal{F} \to \mathcal{S}^2\mathcal{E} \to \alpha_*(\omega_\alpha^2) \to 0$$

we immediately deduce the conjecture stated in Casnati's hexagonal covers paper:

$$\mathcal{F} = (d-3)c_1\mathcal{E}$$

If we consider the degree 2 part in the G-R-R calculation, we arrive at the relation:

$$\boxed{[ch(\alpha_*(\omega_\alpha^N))]_2 = [ch\mathcal{E}]_2 + \binom{N}{2}c_1^2(\omega_\alpha)}$$

In particular, we notice that $[ch\mathcal{F}]_2$ is expressible in terms of $[ch\mathcal{E}]_2$ and $c_1^2(\omega_\alpha)$, and therefore $c_2\mathcal{F}$ is expressible in terms of $[ch\mathcal{E}]_2, c_1^2(\omega_\alpha)$, and $c_1^2\mathcal{E}$.

By applying G-R-R to f, we see

$$\lambda = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_T)\chi(\mathcal{O}_{C_\eta})$$

where $\mathcal{C}_\eta$ where denotes the generic fiber of f. Applying G-R-R to α as before, we can show:

$$\lambda = \left(\frac{1}{2} - \frac{1}{b}\right)c_1^2(\mathcal{E}) - c_2(\mathcal{E})$$

Where $b = 2d + 2g - 2$, the number of branch points.

We now compute κ. By definition, $\kappa = \omega_f^2 = (\omega_\alpha + \alpha^*(\omega_g))^2 = \omega_\alpha^2 + 2(\omega_\alpha \cdot \alpha^*\omega_g)$. Now, we know that $\frac{c_1^2(\mathcal{E})}{b} = c_1\mathcal{E} \cdot \left[\frac{-\omega_g}{2}\right]$, and that $\alpha_*c_1\omega_\alpha = 2c_1\mathcal{E} = B$, the class of the branch divisor. Combining this information gives:

$$\kappa = \omega_\alpha^2 - \frac{8c_1^2(\mathcal{E})}{b}$$

Mumford's relation [19] $\delta = 12\lambda - \kappa$ gives:

$$\delta = 12\left(\frac{c_1^2(\mathcal{E})}{2} - c_2(\mathcal{E}) - \frac{c_1^2(\mathcal{E})}{b}\right) - \left(\omega_\alpha^2 - \frac{8c_1^2(\mathcal{E})}{b}\right)$$

$$= \left(6 - \frac{4}{b}\right)c_1^2\mathcal{E} - 12c_2\mathcal{E} - \omega_\alpha^2$$

In order to understand the divisors D and T, we use the double point formula for the ramification divisor $R := c_1(\omega_\alpha)$ of α mapping onto the branch divisor $B \subset \mathcal{P}$. (Equivalently, we interpret simple cusps and nodes occurring in B as intersections with T and D, respectively.)

Adjunction, both on $\mathcal{C}$ and $\mathcal{P}$, gives the following:

$$2p_a(R) - 2 = \omega_\alpha \cdot (\omega_\alpha + K_\mathcal{C}) = \omega_\alpha \cdot (2\omega_\alpha + \alpha^* K_\mathcal{P})$$

$$2p_a(B) - 2 = B \cdot (B + K_\mathcal{P})$$

$$\implies p_a(B) - p_a(R) = \frac{B^2}{2} - \omega_\alpha^2 = 2c_1^2(\mathcal{E}) - \omega_\alpha^2 = T + D$$

Next, note that the ramification of $f: R \to T$ gives the number $T + \delta$:

$$T + \delta = \omega_\alpha \cdot (\omega_\alpha + \omega_f) = \omega_\alpha \cdot (2\omega_\alpha + \alpha^* \omega_g) = 2\omega_\alpha^2 + \omega_\alpha \cdot \alpha^* \omega_g = 2\omega_\alpha^2 - \frac{4c_1^2(\mathcal{E})}{b}$$

Solving for T, and using the expressions above gives

(4.2)
$$T = 3\omega_\alpha^2 - 12\left(\frac{c_1^2(\mathcal{E})}{2} - c_2(\mathcal{E})\right)$$

(4.3)
$$D = 8c_1^2(\mathcal{E}) - 12c_2(\mathcal{E}) - 4\omega_\alpha^2$$

After manipulating the various relations above, we arrive at the following:

$$(4.4) \qquad \omega_\alpha^2 = \frac{12b + 24}{b - 10}\lambda - \frac{b + 6}{b - 10}\delta + \frac{4}{b - 10}D$$

$$(4.5) \qquad \frac{b - 10}{3}T = 8(b - 1)\lambda - (b - 2)\delta + 2D$$

$$(4.6) \qquad c_1^2(\mathcal{E}) = \frac{18b}{b - 10}\lambda - \frac{2b}{b - 10}\delta + \frac{b}{2(b - 10)}D$$

$$(4.7) \qquad c_2(\mathcal{E}) = \frac{8(b - 1)}{b - 10}\lambda - \frac{b - 2}{b - 10}\delta + \frac{(b - 2)}{4(b - 10)}D$$

$$(4.8) \qquad (ch\mathcal{E})_2 = \frac{b + 8}{b - 10}\lambda - \frac{2}{b - 10}\delta + \frac{1}{2(b - 10)}D$$

$$\square$$

Remark 4.2. Note that we have shown that $c_1\mathcal{F} = (d - 3)c_1\mathcal{E}$. The proof was not specific to our situation, so the conjecture appearing in [5] is settled. In fact, it follows that all chern classes of the higher syzygy bundles $\mathcal{N}_i$ are expressible in terms of chern classes of $\mathcal{E}$ and $\mathcal{F}$.

4.1.1. *The divisor classes* **M** *and* **CE**. We may now express the divisors **M** and **CE** in terms of λ, δ, and D. Since $\pi \colon \mathcal{P} \longrightarrow \widetilde{\mathcal{H}}_{d,g}$ is a $\mathbf{P}^1$-bundle, we may use the Bogomolov expressions for $\mathcal{E}_u$ and $\mathcal{F}_u$, respectively, to find our expressions. For a rank r locally free sheaf $\mathcal{G}$ on $\mathcal{P}$, the Bogomolov expression for $\mathcal{G}$ is

$$Bog(\mathcal{G}) := c_2\mathcal{G} - \frac{(r - 1)}{2r}c_1^2\mathcal{G}$$

(It is, up to scaling, the unique linear combination of c_2 and c_1^2 which is invariant under tensoring by pullbacks of line bundles from the base.) The Bogomolov expression detects the change of splitting type of $\mathcal{G}$ on the fibers of the $\mathbf{P}^1$-bundle $\mathcal{P}$ [2]. Using Proposition 4.1, we conclude:

Proposition 4.3. *In* $\mathrm{Pic}_{\mathbb{Q}} \, \widetilde{\mathcal{H}}_{d,g}$*, the divisors* $\mathbf{M}$ *and* $\mathbf{CE}$ *are expressible in terms of* $\lambda, \delta,$ *and* D*. These expressions are:*

$$\mathbf{M} = \left(\frac{\frac{10-d}{d-1}b - 8}{b-10} \right) \lambda - \left(\frac{\frac{1}{d-1}b - 2}{b-10} \right) \delta + \left(\frac{\frac{1}{d-1}b - 2}{4(b-10)} \right) D$$

$$\mathbf{CE} = \left(\frac{(21 - d - \frac{54}{d})b - 8d + 24}{b-10} \right) \lambda - \left(\frac{(2 - \frac{6}{d})b - 2d + 6}{b-10} \right) \delta + \left(\frac{(1 - \frac{6}{d})b - 2d + 16}{4(b-10)} \right) D$$

Proof. $\square$

Remark 4.4. By an earlier remark, the Bogomolov expressions for *every* syzygy bundle $\mathcal{N}_i$ are expressible in terms of $\lambda, \delta,$ and D. This is to be expected if we believe the Picard Rank Conjecture. We will explore the possible implications of this observation in the last section.

4.2. **Slope bounds.** Suppose we are given a family of stable genus $g \geq 2$ curves $f \colon \mathcal{C} \longrightarrow T$, where T is a smooth, complete curve. One way to measure the topological complexity of the fibration is to count the number of singular fibers occurring in the family. This can be done by considering the quantity

$$\chi_{top}(\mathcal{C}) - \chi_{top}(\mathcal{C}_\eta)\chi_{top}(T)$$

which will give the intersection number $\delta \cdot T$. ($\mathcal{C}_\eta$ denotes the general fiber.) This quantity is zero precisely when the family T parametrizes smooth curves.

On the other hand, one way to measure the change in "moduli" is to consider the difference

$$\chi(\mathcal{O}_\mathcal{C}) - \chi(\mathcal{O}_{\mathcal{C}_\eta})\chi(\mathcal{O}_T)$$

which, by a Grothendieck-Riemann-Roch calculation turns out to be $\lambda \cdot T$.

One measure of the complexity by which the curve T rests inside $\overline{\mathcal{M}}_g$ is simply the ratio of the two quantities, or the *slope*:

$$s(T) := \frac{\delta \cdot T}{\lambda \cdot T}$$

For example, the (projective) Satake compactification of $\mathcal{M}_g$ adds a codimension 2 boundary and therefore $\mathcal{M}_g$ is swept out by complete curves T which, obviously, have slope 0. On the

other hand, the Mumford relation [19]

$$12\lambda = \delta + \kappa$$

and the fact that κ is positive [**?**] together imply that the slope is bounded above by 12.

Whereas it is easy, theoretically, to prove the existence of many families having slope 0, one quickly realizes that families with *higher* slopes are more "special." This is made more precise by the following theorem of Cornalba and Harris [7]:

Theorem 4.5 (Cornalba, Harris [7]). *Let $f\colon \mathcal{C} \longrightarrow T$ be a generically smooth family of stable curves. Then the slope is bounded above by*

$$s(T) \leq 8 + \frac{4}{g}$$

Furthermore, equality is achieved if and only if T is a family of hyperelliptic curves.

In other words, the maximal slope is achieved for families which are entirely contained in some proper subvariety of $\overline{\mathcal{M}}_g$, namely the hyperelliptic locus. Furthermore, it is easy to see that the hyperelliptic locus is actually *swept out* by families of slope $8+4/g$, i.e. there exists a curve T in the hyperelliptic locus whose algebraic deformations pass through the general point of the hyperelliptic locus.

One way to understand the relationship slopes of families and special subvarieties of $X \subset \overline{\mathcal{M}}_g$ is to ask the "reverse" question. Given a subvariety $X \subset \overline{\mathcal{M}}_g$, what is the maximum slope of a sweeping curve $T \subset X$?

Taking the result of Cornalba and Harris as a starting point, one may ask the general question for the locus of d-gonal curves $\mathcal{T}_d \subset \overline{\mathcal{M}}_g$. In Stankova's thesis [22], the following theorem was proved:

Theorem 4.6 (Stankova [22]). *Let $g \geq 4$ be even. For a sweeping curve $T \subset \mathcal{T}_3$,*

$$s(T) \leq 7 + \frac{6}{g}$$

In this section, we will extend Stankova's results by establishing sharp upper bounds for slopes of sweeping curves T in $\mathcal{T}_4$ and $\mathcal{T}_5$. The divisors $\mathbf{M}$ and $\mathbf{CE}$ will play an essential role. In particular, our slope bounds will only hold under the appropriate congruence conditions which must be met in order for $\mathbf{M}$ and $\mathbf{CE}$ to both exist as effective divisors.

After examining the expressions in Proposition 4.3, we observe that *when $d \leq 5$, the coefficient of the boundary divisor D is positive in $\mathbf{M}$, and negative in $\mathbf{CE}$.* Consider the (unique, up to scaling) effective combination $\mathbf{X}$ of $\mathbf{M}$ and $\mathbf{CE}$ which eliminates the coefficient of D,and write it as

$$\mathbf{X} := a\lambda - b\delta$$

If we think of $\mathbf{X}$ as an effective divisor in $\overline{\mathcal{H}}_{d,g}$ (by taking the closure), we may include the higher boundary divisors in the divisor class expression for $\mathbf{X}$ by writing

$$\mathbf{X} = a\lambda - b\delta - \mathbf{Y}$$

Where $\mathbf{Y}$ is some combination of higher boundary divisors. Finally, observe that if $\mathbf{Y}$ is an *effective* combination of higher boundary divisors, then the ratio $\frac{a}{b}$ will be an upper bound for the slope of a sweeping curve $T \subset \overline{\mathcal{H}}_{d,g}$. Fortunately, we have:

Theorem 4.7. $\mathbf{Y}$ *is an effective combination of higher boundary divisors.*

As corollaries of Theorem 4.7, we conclude:

Corollary 4.8 (Slope bounds for trigonal curves)**.** *If $T \subset \overline{\mathcal{H}}_{3,g}$ is a sweeping curve, and g is even, then the slope $s(T)$ is bounded above by $\frac{21g+18}{3g}$. Furthermore, this bound is sharp.*

Corollary 4.8 is Stankova's result.

Corollary 4.9 (Slope bounds for tetragonal curves)**.** *If $T \subset \overline{\mathcal{H}}_{4,g}$ is a sweeping curve, and if $g \equiv 3 \mod 6$, then the slope $s(T)$ is bounded above by $\frac{26g+30}{4g}$. Furthermore, this bound is sharp.*

Corollary 4.10 (Slope bounds for pentagonal curves). *If $T \subset \overline{\mathcal{H}}_{5,g}$ is a sweeping curve, and if $g \equiv 16 \mod 20$, then the slope $s(T)$ is bounded above by $\frac{31g+44}{5g}$. Furthermore, this bound is sharp.*

We have written the slope bounds in nonreduced form because by doing so, a definite pattern emerges. If we include the original Cornalba-Harris bound $\frac{16g+8}{2g}$ for slopes of hyperelliptic families, the pattern is further reinforced. Therefore, we make the conjecture:

Conjecture 4.11. *When both $\mathbf{M}$ and $\mathbf{CE}$ exist as effective divisors in $\overline{\mathcal{H}}_{d,g}$, the slope of a sweeping curve $T \subset \overline{\mathcal{H}}_{d,g}$ is bounded above by*

$$s_{conj} = \frac{(5d+6)g + d^2 + 5d - 6}{dg}$$

Further evidence for Conjecture 4.11 will be provided in the last section.

One interesting observation should be made regarding these slopes. [14] The canonical class $K_{\overline{\mathcal{M}}_g}$ is [14]:

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta$$

The results above tell us that both the trigonal locus $\mathcal{T}_3$ and the tetragonal locus $\mathcal{T}_4$ are swept out by $K_{\overline{\mathcal{M}}_g}$-negative curves. However, the locus $\mathcal{T}_5$ is not swept out by $K_{\overline{\mathcal{M}}_g}$-negative curves. This may be of some interest in the program of understanding the birational geometry of $\overline{\mathcal{M}}_g$.

We will prove Theorem 4.7 for each degree $d = 3, 4, 5$ separately, beginning with $d = 3$. The reader should note that all three arguments are inductive in nature: We carefully choose test families which allow us to deduce the nonnegativity of certain coefficients in $\mathbf{Y}$ by knowing the nonnegativity of other coefficients.

Since we are only trying to show effectivity of $\mathbf{Y}$, we will not try to calculate the exact class of $\mathbf{Y}$. To see an exact description of the class of $\mathbf{M}$ in $\overline{\mathcal{H}}_{3,g}$, we refer the reader to [9].

Before we begin with the proofs, we want to mention that it is not enough to show that the enumeratively relevant boundary divisors have nonnegative coefficients. In other words, we may not restrict ourselves to sweeping families of complete curves in $\widetilde{\mathcal{H}}_{d,g}$, because there

may be sweeping families in the d-gonal locus $\mathcal{T}_d$ which pass through a point lying outside of the image of the nonproper forgetful map $F\colon \widetilde{\mathcal{H}}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$. This is why we must deal with the monstrous boundary of the admissible cover compactification.

4.3. **Slope bounds:** $d = 3$. When the degree is 3 and the genus is even, the effective divisor class

$$\mathbf{X} := a\lambda - b\delta - \mathbf{Y}$$

is simply a multiple of the Maroni divisor $\mathbf{M}$. Furthermore, the reader can check that $a = 7g + 6$ and $b = g$.

For a boundary divisor $E \subset \overline{\mathcal{H}}_{d,g}$, we will denote by $c(E, \mathbf{Y})$ the coefficient of E in the divisor class expression of $\mathbf{Y}$. We wish to show that $c(E, \mathbf{Y}) \geq 0$ for all higher boundary divisors E.

Rather than create names for every boundary divisor, we will adopt the following simplifying naming scheme: The letter E will always be understood to mean the divisor which is "currently" of primary concern. All decorations on the letter E, (e.g. E', E_1, etc.) will indicate auxiliary boundary divisors occurring in the analysis of $c(E, \mathbf{Y})$. Every section will be devoted to showing $c(E, \mathbf{Y}) \geq 0$ for a specified type of boundary divisor.

We begin by considering unramified, 2-vertex divisors.

4.3.1. *Unramified, 2-vertex Divisors.* Let E have dual graph Γ_E:

We let g_R, g_L denote the genus of the components v_R, and v_L. Furthermore, by symmetry we may assume that $g_R \leq g_L$. We will vary the right side in a partial pencil family. Consider a pencil $\mathbf{P}_t^1$ of genus g_R trigonal curves on $\mathbf{F}_0$ or $\mathbf{F}_1$ (depending on the parity of g_R) with three points p_1, p_2, p_3 of the base locus lying on a fiber F_0. After performing the admissible reduction procedure to the pencil, and then attaching a constant family on the left, we arrive

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at a family of admissible covers γ which has the following intersection numbers:

$$\lambda \cdot \gamma = 2g_R$$

$$\delta \cdot \gamma = 2(7g_R + 3)$$

$$E \cdot \gamma = -2$$

$$E' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

Here, E' is the boundary divisor which has dual graph $\Gamma_{E'}$:

Notice that we have simply decreased the genus of the right side by two. This was the reason we impose the "collinear" basepoints. The fact that $\mathbf{X} \cdot \gamma \geq 0$ follows from the observation that the families of type γ sweep out the divisor E. (In fact, it can be shown that $\mathbf{X} \cdot \gamma = 0$, but we do not need this.) The test family γ provides the relation:

(4.9) $$c(E, \mathbf{M}) \geq c(E', \mathbf{M}) + 3g - 6g_R$$

Since we are assuming that $g_R \leq g_L$, and since $g_L + g_R + 2 = g$, we immediately conclude that $c(E', \mathbf{Y}) \geq 0$ implies $c(E, \mathbf{Y}) \geq 0$. Suppose g_R was even. Then by repeatedly applying relation (4.9), we may conclude by showing that $c(E_0, \mathbf{Y}) \geq 0$ for the boundary divisor E_0 which has the dual graph Γ_{E_0}:

We use the same type of pencil as was used to construct γ above. Fortunately, the split fiber does not contribute any intersection with further boundary components, because the

residual curve R_0 is a disjoint union of $(1, 0)$ curves which will be blown down. Therefore, the resulting partial pencil family of admissible covers, γ_0 has the following intersection numbers:

$$\lambda \cdot \gamma_0 = 0$$

$$\delta \cdot \gamma_0 = 6$$

$$E_0 \cdot \gamma_0 = -2$$

$$\mathbf{X} \cdot \gamma_0 = 0$$

From these calculations we easily see that $c(E_0, \mathbf{Y}) \geq 0$, and this implies $c(E, \mathbf{Y}) \geq 0$. The analysis for g_R is completely analogous. We simply need to show that $c(E_1, \mathbf{Y}) \geq 0$ where E_1 has an elliptic curve as the right hand component. Again, we just use a partial pencil family.

4.3.2. *Simply Ramified* 2-*vertex divisors.* Let E, a simply ramified 2-vertex divisor, have dual graph Γ_E:

As before, we let g_R, g_L denote the genera of v_R, v_L and assume that $g_R \leq g_L$. Vary v_R in a pencil $\mathbf{P}_t^1$ of genus g_R trigonal curves in $\mathbf{F}_0$ or $\mathbf{F}_1$. This time, however, we impose the condition that the base locus of our pencil contains a *nonreduced* "triple" $\{2p_1, p_2\}$ of points in a fiber F_0.

The total space $p \colon \mathcal{C}_t \longrightarrow \mathbf{P}_t^1$ of the pencil has a unique A_1 singular point s occurring above a point $x \in \mathbf{P}_t^1$, and a split element $F_0 \cup R_0$ occurring at, say, $t = 0$. The fiber of $C_x := p^{-1}(x)$ is a nodal curve which is singular exactly at the point s. Furthermore, there are two sections s_1, s_2 corresponding to the points p_1 and p_2. After resolving the singularity s and blowing up $s_2 \cap C_x$, we arrive at a new family $\tilde{p} \colon \widetilde{\mathcal{C}}_t \longrightarrow \mathbf{P}_t^1$, which has two sections $\tilde{s}_1$, $\tilde{s}_2$, the strict transforms of s_1 and s_2. We note that $\tilde{s}_1^2 = -1$.

After an order two base change (as usual in the admissible reduction process) we may attach a "constant" genus g_L trigonal curve $[\alpha_L \colon C_L \longrightarrow \mathbf{P}_L^1]$ at the sections $\tilde{s}_1$ and $\tilde{s}_2$ to

obtain our test family γ of admissible covers. Obviously, we must attach a simple ramification point of α_L to the section $\tilde{s}_1$. Notice that γ intersects two higher boundary divisors other than E. The split element provides an intersection with the unramified boundary divisor E' which has dual graph $\Gamma_{E'}$:

$$
\overset{v'_L}{\underset{g_L+1}{\bullet}} \overbrace{} \overset{v'_R}{\underset{g_R-2}{\bullet}}
$$

The fiber corresponding to the point x provides an intersection with the unramified boundary divisor E'' which has dual graph $\Gamma_{E''}$:

$$
\overset{v''_L}{\underset{g_L}{\bullet}} \overbrace{} \overset{v''_R}{\underset{g_R-1}{\bullet}}
$$

The intersection numbers of γ are as follows:

$$\lambda \cdot \gamma = 2g_R$$

$$\delta \cdot \gamma = 2(7g_R + 4)$$

$$E \cdot \gamma = -2$$

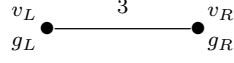$$E' \cdot \gamma = 2$$

$$E'' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

This provides us with the relation

(4.10) $$c(E, \mathbf{Y}) \geq c(E', \mathbf{Y}) + c(E'', \mathbf{Y}) + 4g - 6g_R$$

We already know from the previous section that $c(E', \mathbf{Y}) \geq 0$ and $c(E'', \mathbf{Y}) \geq 0$. Furthermore. we are assuming $g_R \leq g_L$. Therefore $c(E, \mathbf{Y}) \geq 0$.

4.3.3. *Triply ramified 2-vertex divisors.* Now let E be a triply ramified, 2-vertex divisor with dual graph Γ_E:

$$\underset{g_L}{\overset{v_L}{\bullet}} \overset{3}{\rule{3cm}{0.4pt}} \underset{g_R}{\overset{v_R}{\bullet}}$$

Once again, we begin by varying v_R in a pencil on the appropriate Hirzebruch surface ($\mathbf{F}_0$ or $\mathbf{F}_1$ depending on the parity of g_R.) We impose a triple point "$3p_1$" (isomorphic to $\mathbf{Spec}\, k[\epsilon]/(\epsilon^3)$) in a fiber F_0 to be contained in the base locus of the pencil. We arrive at a total space $p \colon \mathcal{C}_t \longrightarrow \mathbf{P}^1_t$ which has a unique A_2 singularity s occurring in the fiber $C_x := p^{-1}(x)$, for some $x \in \mathbf{P}^1_t$. Furthermore, the nodal curve C_x is not only singular at the point s, but is also *simply tangent* to the fiber F_0 along one of its branches. The base point p_1 provides a section s_1, and as usual we have a split element $F_0 \cup R_0$ at $t = 0$.

Consider the resolution of s, $\tilde{p} \colon \widetilde{\mathcal{C}}_t \longrightarrow \mathbf{P}^1_t$. The fiber $\tilde{p}^{-1}(x)$ is now $\widetilde{C}_x \cup E_1 \cup E_2$, where $E_1 \cup E_2$ is the chain of exceptional divisors coming from resolving an A_2 singularity, and $\widetilde{C}_x$ is the normalization of C_x. $\widetilde{C}_x$ intersects E_1 at a point a and E_2 at the point b. Let's assume that a corresponds to the ramified branch of C_x. Then the strict transform of the section, which we will continue to call s_1, has self intersection -1, and passes through the exceptional divisor E_1.

Now blow down E_2 to acquire a family $p' \colon \mathcal{C}'_t \longrightarrow \mathbf{P}^1_t$. The fiber $p'^{-1}(x)$ is now the union $\widetilde{C}_x \cup E_1$, where $\widetilde{C}_x \cap E_1 = \{a, b\}$. The total space $\mathcal{C}'_t$ has an A_1 singularity at the point b. After an order two base change (admissible reduction), we may glue a constant, triply ramified trigonal curve $\alpha_L \colon C_L \longrightarrow \mathbf{P}^1_L$ to p' along the section s_1, to obtain our family γ of admissible covers. The split fiber of γ provides an intersection with the unramified boundary divisor E' with dual graph $\Gamma_{E'}$:

$$\underset{g_L}{\overset{v'_L}{\bullet}}\ \underset{g_R-2}{\overset{v'_R}{\bullet}}$$
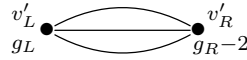
The fiber corresponding to the point x provides an intersection with the *simply ramified* boundary divisor E'' which has dual graph $\Gamma_{E''}$:
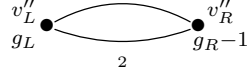
The intersection numbers for γ are now:

$$\lambda \cdot \gamma = 2g_R$$

$$\delta \cdot \gamma = 2(7g_R + 5)$$

$$E \cdot \gamma = -2$$

$$E' \cdot \gamma = 2$$
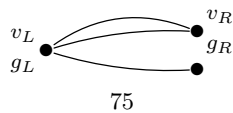
$$E'' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

As a result, we obtain the relation

(4.11) $$c(E, \mathbf{Y}) \geq c(E', \mathbf{Y}) + c(E'', \mathbf{Y}) + 6g - 5g_R$$

From this relation we conclude $c(E, \mathbf{Y}) \geq 0$.

4.3.4. *More than 2 vertices.* Now let's suppose E has a dual graph with more than 2 vertices. Then one of the vertices, v_R, must have degree 2, i.e. it must correspond to a hyperelliptic component. Let us assume Γ_E is:

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A priori, there are two types of 3-vertex divisors: Either v_R is ramified, or not. (We will only consider the analysis for the unramified case - the reader will by now be able to make the necessary minor adjustments for the ramified case.)

We vary the genus g_R hyperelliptic curve v_R in a $\mathbf{P}_t^1$ pencil on $\mathbf{F}_{g_R+1}$ with two specified points p_1, p_2 in a fiber F_0 lying in the base locus. The resulting family of admissible covers, γ, will intersect a new boundary divisor E' which has dual graph $\Gamma_{E'}$:

(Notice that we are simply reducing the genus of v_R by one.) The intersection numbers for γ are:

$$\lambda \cdot \gamma = 2g_R$$

$$\delta \cdot \gamma = 2(8g_R + 2)$$

$$E \cdot \gamma = -2$$

$$E' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

We end up with the relation

(4.12) $$c(E, \mathbf{Y}) \geq g(g_R + 2) - 6g_R + c(E', \mathbf{Y})$$

The expression $g(g_R+2) - 6g_R$ is always non-negative for $0 \leq g_R \leq g-1$, $g \geq 2$. Therefore, if $c(E', \mathbf{Y}) \geq 0$, then $c(E, \mathbf{Y}) \geq 0$. By repeatedly use relation (4.12), we must only show

that $c(E_0, \mathbf{Y}) \geq 0$ for E_0 having dual graph Γ_{E_0}:

But notice E_0 is just Δ_{irr}, and clearly $c(\Delta_{irr}, \mathbf{Y}) = 0$, so we are done.

4.3.5. 4 *vertices.* Now suppose the dual graph for E looks like Γ_E:

As usual, assume that $g_R \leq g_L$. Again, vary v_R in a partial pencil of hyperelliptic curves. We obtain a family of admissible covers γ which intersects one other higher boundary divisor E' whose dual graph is $\Gamma_{E'}$:

In other words, E' is of the type discussed in the previous subsection - the left sides of the admissible covers represented by E' have only one vertex. The intersection numbers for γ are:

$$\lambda \cdot \gamma = 2g_R$$

$$\delta \cdot \gamma = 2(8g_R + 3)$$

$$E \cdot \gamma = -2$$

$$E' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

Therefore, we arrive at the relation:

(4.13)
$$c(E, \mathbf{Y}) \geq g(g_R + 3) - 6g_R + c(E', \mathbf{Y})$$

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For all acceptable pairs (g_R, g), the expression $g(g_R + 3) - 6g_R$ is easily checked to be nonnegative, so we conclude that $c(E, \mathbf{Y}) \geq 0$.

This concludes the proof of Stankova's theorem for $\overline{\mathcal{H}}_{3,g}$, g even.

4.4. **Slope bounds:** $d = 4$. The proof of Theorem 4.7 for $\overline{\mathcal{H}}_{4,g}$ runs almost exactly as the proof of the $d = 3$ case. After seeing the general inductive method in the previous section, we pause to notice a few things. First, the underlying pencil used to construct the appropriate test families obviously depended on the degree of the vertex v_R which was being varied: pencils of trigonal curves in $\mathbf{F}_0$ or $\mathbf{F}_1$ for degree 3 vertices, and pencils of genus g_R hyperelliptic curves in $\mathbf{F}_{g_R+1}$ for degree 2 vertices. When we switched to considering *hyperelliptic* pencils, the quantity $b\delta - a\lambda$ becomes "much more positive" than it is when we consider pencils of trigonal curves. This essentially happens because the slope $\frac{8g_R+4}{g_R}$ of a hyperelliptic family is larger than $\frac{a}{b} = \frac{7g+6}{g}$.

Now that we are considering the degree 4 situation, we notice that a similar phenomenon occurs. For a divisor E whose dual graph has more than 2 vertices, we may use the same types of pencils used in the previous case: either pencils of trigonal curves in $\mathbf{F}_{0,1}$, or pencils of hyperelliptic curves of genus g_R in $\mathbf{F}_{g_R+1}$. Because $\frac{a}{b} = \frac{13g+15}{2g}$ is fundamentally smaller than $\frac{7g_R+6}{g_R}$ and $\frac{8g_R+4}{g_R}$, arguments exactly like those found in the previous section show that $c(E, \mathbf{Y}) \geq 0$ if we know positivity for "low genus" examples of such E. The low genus cases are easy to establish, so we will leave them to the reader.

Therefore, it is the 2-vertex boundary divisors which are of primary interest. We will need to exhibit fundamentally new partial pencil families, heavily relying on the description of degree 4 covers as complete intersections in their associated scrolls.

4.4.1. *Constructing sweeping families of tetragonal curves.* Recall that a tetragonal, genus g_R cover $[\alpha\colon C \longrightarrow \mathbf{P}^1]$ rests in its associated scroll $\pi\colon \mathbf{P}\mathcal{E} \longrightarrow \mathbf{P}^1$ as a complete intersection of two relative conic divisors Q_u and Q_v in the linear systems $|2\zeta - uf|$ and $|2\zeta - vf|$, respectively. The integers u and v are such that $\mathcal{F}_\alpha = \mathcal{O}(u) \oplus \mathcal{O}(v)$. Let us assume, as usual,

that $u \leq v$, so that $v = \lceil \frac{g_R+3}{2} \rceil$. The line bundle $\mathcal{O}(2\zeta - vf)$ is always globally generated on $\mathbf{P}\mathcal{E}$, while $\mathcal{O}(2\zeta - uf)$ is always very ample.

We fix a smooth element $S \in |2\zeta - vf|$, and consider a pencil $\mathbf{P}_t^1 \subset |2\zeta - uf|$ on S. Let $p: \widetilde{S} \longrightarrow \mathbf{P}_t^1$ denote the total space of the pencil. So p is a family of tetragonal curves, and $\widetilde{S}$ is just the blow up of S at the base locus of the pencil. We compute the invariants δ_p and λ_p using standard methods.

The Euler characteristic, $\chi(\mathcal{O}_{\widetilde{S}}) = 1$, so

$$\lambda_p = \chi(\mathcal{O}_{\widetilde{S}}) - \chi(\mathcal{O}_{\mathbf{P}_t^1})\chi(\mathcal{O}_{C_{gen}}) = g_R$$

where C_{gen} denotes the curve in the pencil.

To compute δ_p, we do the following. Letting $L := \mathcal{O}_S(2\zeta - uf)$, we recall that:

$$
\begin{aligned}
\delta_p &= c_2(L + L \otimes \Omega_S) = c_1(L) \cdot c_1(L \otimes \Omega_S) + c_2(L \otimes \Omega_S) \\
&= c_1(L)[K_S + 2c_1(L)] + c_2(L \otimes \Omega_S) \\
&= c_1(L)[K_S + 2c_1(L)] + c_2(\Omega_S) + c_1(L)K_S + c_1^2(L) \\
&= 3c_1^2(L) + 2c_1(L) \cdot K_S + c_2(\Omega_S) \\
&= 2(2g_R - 2) + c_1^2(L) + c_2(\Omega_S)
\end{aligned}
$$

To compute $c_1^2(L)$, we just intersect

$$
\begin{aligned}
c_1^2(L) &= (2\zeta - uf)^2(2\zeta - vf) = [4\zeta^2 - 4u\zeta f][2\zeta - vf] \\
&= 8\zeta^3 - 8u - 4v = 8(u + v) - 8u - 4v = 4v \\
\Rightarrow \delta_p \cdot \gamma &= 2(2g_R - 2) + 4v + c_2(\Omega_S)
\end{aligned}
$$

In order to compute $c_2(\Omega_S)$, we use the exact sequence of sheaves on $\mathbf{P}\mathcal{E}$:

$$0 \to \mathcal{O}_S(-2\zeta + vf) \to \Omega_{\mathbf{P}\mathcal{E}}|_S \to \Omega_S \to 0$$

Put $M := \mathcal{O}_S(-2\zeta + vf)$. Then:

$$c(\Omega_S) \cdot (1 + c_1(M)t) = c(\Omega_{\mathbf{P}\mathcal{E}}|_S)$$

$$\Rightarrow \quad c(\Omega_S) = (1 - c_1(M)t + c_1^2(M)t^2) \cdot (1 + K_{\mathbf{P}\mathcal{E}}|_S \cdot t + c_2(\Omega_{\mathbf{P}\mathcal{E}})|_S \cdot t^2)$$

$$\Rightarrow \quad c_2(\Omega_S) = c_1^2(M) - K_{\mathbf{P}\mathcal{E}}|_S \cdot c_1(M) + c_2(\Omega_{\mathbf{P}\mathcal{E}})|_S$$

The canonical class of $\mathbf{P}\mathcal{E}$, $K_{\mathbf{P}\mathcal{E}}$, is $-3\zeta + (u + v - 2)f$, so the first part of this expression is:

$$c_1^2(M) - K_{\mathbf{P}\mathcal{E}}|_S \cdot c_1(M) = [2\zeta - vf]^2 \cdot [2\zeta - vf - 3\zeta + (u + v - 2)f]$$

$$= [2\zeta - vf]^2 \cdot [-\zeta + (u - 2)f] = [4\zeta^2 - 4v\zeta f] \cdot [-\zeta + (u - 2)f]$$

$$= -4\zeta^3 + 4u + 4v - 8 = -8$$

In order to compute $c_2(\Omega_{\mathbf{P}\mathcal{E}})|_S$, consider

$$0 \to \Omega_\pi \to \pi^*\mathcal{E}(-1) \to \mathcal{O} \to 0$$

and

$$0 \to \mathcal{O}(-2f) \to \Omega_{\mathbf{P}\mathcal{E}} \to \Omega_\pi \to 0$$

Then,

$$
\begin{aligned}
c(\Omega_{\mathbf{P}\mathcal{E}}) &= (1 - 2f \cdot t) \cdot c(\Omega_\pi) = (1 - 2f \cdot t) \cdot c(\pi^*\mathcal{E}(-1)) \\
&= (1 - 2f \cdot t) \cdot (1 + [-3\zeta + (g_R + 3)f] \cdot t + [3\zeta^2 - 2(g_R + 3)\zeta f] \cdot t^2 + ...) \\
\Rightarrow \quad c_2(\Omega_{\mathbf{P}\mathcal{E}}) &= [3\zeta^2 - 2(g_R + 3)\zeta f] - [-3\zeta + (g_R + 3)f] \cdot (2f) = 3\zeta^2 - 2g_R\zeta f \\
\Rightarrow \quad c_2(\Omega_{\mathbf{P}\mathcal{E}})|_S &= [3\zeta^2 - 2g_R\zeta f] \cdot [2\zeta - vf] = 3v + 6u - 4g_R
\end{aligned}
$$

We conclude

$$c_2(\Omega_S) = 3v + 6u - 4g_R - 8$$

and so:

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Calculation (Invariants of a pencil of degree 4 covers). For a general pencil $p \colon \widetilde{S} \longrightarrow \mathbf{P}^1_t$ of the type described above

$$\lambda_p = g_R$$

$$\delta_p = v + 6g_R + 6$$

We will want to construct partial pencil families of admissible covers with pencils of type p varying on the right side. Therefore, we will impose a 4-tuple of points in a fiber of $\pi \colon S \longrightarrow \mathbf{P}^1$ to lie in the base locus of our pencil p, just as we did in the degree 3 test family calculations.

4.4.2. *Unramified, 2-vertex divisors.* Let E be an unramified, 2-vertex divisor with dual graph Γ_E:

Assume that $g_L \geq g_R \geq 3$. Vary v_R in a pencil as above while keeping v_L fixed. We obtain a family of admissible covers γ which intersects one other higher boundary divisor E' with dual graph $\Gamma_{E'}$:
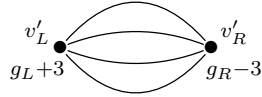
The intersection numbers for γ are:

$$\lambda \cdot \gamma = 2g_R$$

$$\delta \cdot \gamma = 2(v + 6g_R + 2)$$

$$E \cdot \gamma = -2$$

$$E' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

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Recall that the divisor $\mathbf{X}$ has the form

$$\mathbf{X} = a\lambda - b\delta - \mathbf{Y}$$

where $\frac{a}{b} = \frac{13g+15}{2g}$. The test curve γ gives us the relation

(4.14) $$c(E, \mathbf{Y}) \geq c(E', \mathbf{Y}) + b(v + 6g_R + 2) - a(g_R)$$

Unfortunately, we may not run the induction directly from relation (4.14): the quantity $b(v + 6g_R + 2) - a(g_R)$ is not positive for all g_R in the assumed range. Luckily, the fraction of g_R's making this quantity negative is quite small, so we may use relation (4.14) multiple times, reducing genera of the right hand sides of boundary divisors by three each time, and add all the resulting inequalities. So write $g_R = 3i + k$, with $k = 0, 1, 2$ depending on g_R mod 3. Let E_l denote the unramified, 2-vertex boundary divisor having dual graph Γ_{E_l}:

(So our original boundary divisor E is E_i.) Relation (4.14) gives

$$c(E_l, \mathbf{Y}) \geq c(E_{l-1}, \mathbf{Y}) + b(v_l + 6(3l + k) + 2) - a(3l + k)$$

for any l. We may add these relations together for $0 \leq l \leq i$ to obtain the following expression

(4.15) $$c(E_i, \mathbf{Y}) \geq c(E_0, \mathbf{Y}) + \sum_{l=1}^{i}(18b - 3a)l + bv_l + (6bk + 2b - ka)i$$

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In the above expression, $v_l := \lceil \frac{3l+k+3}{2} \rceil$. Therefore, if we replace v_l by $\frac{3l+k+3}{2}$ and if the right hand side of (4.15) remains positive, we will be able to conclude what we want by showing $c(E_0, \mathbf{Y}) \geq 0$. Indeed, substituting v_l with $\frac{3l+k+3}{2}$ gives

$$(4.16) \qquad c(E_i, \mathbf{Y}) \geq c(E_0, \mathbf{Y}) + 3\left(\frac{13}{2}b - a\right) \cdot \binom{i+1}{2} + \left(\left(\frac{13}{2}k + \frac{7}{2}\right)b - ka\right) \cdot i$$

From here, it is straightforward to check that

$$3\left(\frac{13}{2}b - a\right)\binom{i+1}{2} + \left(\left(\frac{13}{2}k + \frac{7}{2}\right)b - ka\right)i \geq 0$$

for i in the required range, i.e. $3i+k \leq \frac{g-3}{2}$. Therefore, we must only show that $c(E_0, \mathbf{Y}) \geq 0$. These three remaining cases are easily dealt with: we may interpret the divisor E_{-1} as having more than 2 vertices. In other words, the split fibers appearing in the pencils have disconnected residual curves. The relation

$$c(E_0, \mathbf{Y}) \geq c(E_{-1}, \mathbf{Y}) + b(v_0 + 6k + 2) - ak$$

then allows us to conclude $c(E_0, \mathbf{Y}) \geq 0$.

4.4.3. *Ramified, 2-vertex divisors.* The positivity of $c(E, \mathbf{Y})$ for ramified 2-vertex divisors E will now follow easily. We create a partial pencil family γ by imposing a 4-tuple of basepoints of the form $\{2p_1, p_2, p_3\}$ in a fiber of $\pi \colon S \longrightarrow \mathbf{P}^1$. Suppose Γ_E is:
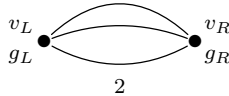
Then the intersection numbers for the resulting family γ are:

$$\lambda \cdot \gamma = 2g_R$$
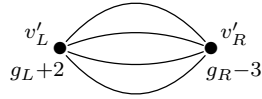
$$\delta \cdot \gamma = 2(v + 6g_R + 3)$$

$$E \cdot \gamma = -2$$

$$E' \cdot \gamma = 2$$

$$E'' \cdot \gamma = 2$$

$$\mathbf{X} \cdot \gamma \geq 0$$

Here E', originating from the split element, has dual graph $\Gamma_{E'}$:

and E'' has dual graph $\Gamma_{E''}$:

The relation we obtain from γ is

(4.17) $$c(E, \mathbf{Y}) \geq c(E', \mathbf{Y}) + c(E'', \mathbf{Y}) + 9g - 15g_R$$

Since we assume $g_R \leq \frac{g-3}{2}$, we easily conclude from the previous section that $c(E, \mathbf{Y}) \geq 0$.

For higher ramification, we simply impose nonreduced points in the base loci of our partial pencils, just as we did in the degree 3 case. The resulting partial pencil family will relate the ramified divisor with a divisor having less ramification. The analysis is equivalent to the one in the proof of the independence of boundary divisors in chapter 3.

4.5. Slope bounds: $d = 5$. Now we move on to the proof of Theorem 4.7 for $\overline{\mathcal{H}}_{5,g}$. We must assume that $g \equiv 0 \mod 4$ and $g \equiv 1 \mod 5$ for both **M** and **CE** to exist as effective divisors. In the divisor class expression

$$\mathbf{X} = a\lambda - b\delta - \mathbf{Y}$$

we see that $\frac{a}{b} = \frac{31g+44}{5g}$. Therefore, for exactly the same reason that we gave in the proof of the $d = 4$ case of Theorem 4.7, we need only consider the coefficients $c(E, \mathbf{Y})$ for 2-vertex divisors E.

We will heavily use the determinantal presentation of degree 5 covers, which we now recall. For an arbitrary degree 5 cover $[\alpha\colon C \longrightarrow \mathbf{P}^1] \in \widetilde{\mathcal{H}}_{5,g}$ (arbitrary g), the resolution of the ideal sheaf $\mathcal{I}_C$ of $C \subset \mathbf{P}\mathcal{E}$ has the form:

(4.18) $\quad 0 \longrightarrow \pi^*(\det\mathcal{E})(-5) \longrightarrow \pi^*(\mathcal{F}^\vee \otimes \det\mathcal{E})(-3) \longrightarrow \pi^*(\mathcal{F})(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0$

The central map is shown to be skew symmetric in [4]. Conversely, starting from a general skew symmetric map

$$M\colon \pi^*(\mathcal{F}^\vee \otimes \det\mathcal{E})(-3) \longrightarrow \pi^*(\mathcal{F})(-2)$$

the rank 2 locus of M will define a genus g pentagonal curve $C \subset \mathbf{P}\mathcal{E}$ if the bundle $\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det\mathcal{E}^\vee$ is globally generated on $\mathbf{P}^1$ [4]. Therefore, as we saw in chapter 2, if $\mathcal{E} = \mathcal{E}_{gen}$ and $\mathcal{F} = \mathcal{F}_{gen}$ we may think of the projective space

$$\mathbf{P}^N := \mathbf{P}H^0(\mathbf{P}^1, \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det\mathcal{E}^\vee)$$

as a parameter space for general genus g pentagonal curves.

Now consider a linear pencil $\mathbf{P}^1_t \subset \mathbf{P}^N$. Letting $\pi\colon \mathbf{P}\mathcal{E} \longrightarrow \mathbf{P}^1_s$ denote the projection, we see that the total space of the resulting family of pentagonal curves $p\colon \mathcal{C}_t \longrightarrow \mathbf{P}^1_t$ is a 5

sheeted cover of $\alpha\colon \mathcal{C}_t \longrightarrow \mathbf{P}^1_t \times \mathbf{P}^1_s$. So we have the diagram

$$\mathcal{C}_t$$

$$\downarrow{\scriptstyle\alpha}$$

$$\mathbf{P}^1_t \times \mathbf{P}^1_s \xrightarrow{\ \pi\ } \mathbf{P}^1_s$$

$$\downarrow{\scriptstyle p}$$

$$\mathbf{P}^1_t$$

The reduced direct image of α, as a rank 4 locally free sheaf on $\mathbf{P}^1_t \times \mathbf{P}^1_s$ is

$$\mathcal{E}_\alpha = \pi^*(\mathcal{E}) \otimes p^*(\mathcal{O}(5))$$

and the bundle of quadrics is

$$\mathcal{F}_\alpha = \pi^*(\mathcal{F}) \otimes p^*(\mathcal{O}(8))$$

Indeed

$$\wedge^2 \mathcal{F}_\alpha \otimes \mathcal{E}_\alpha \otimes (\det \mathcal{E}_\alpha)^\vee = \pi^*(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det\mathcal{E}^\vee) \otimes p^*(\mathcal{O}(1))$$

so a section of this bundle exactly corresponds to a linear pencil $\mathbf{P}^1_t \subset \mathbf{P}^N$. We use the expressions for λ and δ in terms of chern classes of $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ in the proof of Proposition 4.1 to obtain:

Calculation. For a pencil $p\colon \mathcal{C}_t \longrightarrow \mathbf{P}^1_t$ obtained from considering a linear $\mathbf{P}^1_t \subset \mathbf{P}^N$

$$\lambda_p = 5g$$

$$\delta_p = 31g + 44$$

This calculation holds for arbitrary genera g. These pencils will be the starting point for the construction of partial pencil test families which will allow us to run the inductive argument.

4.5.1. *Unramified and simply-ramified, 2-vertex divisors.* Let E be a boundary divisor with dual graph Γ_E:
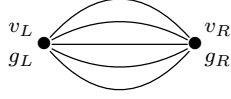
Assume, as usual, that $g_R \leq g_L$. We vary v_R in a general pencil $p\colon \mathcal{C}_t \longrightarrow \mathbf{P}^1_t$ of genus g_R pentagonal curves as described in the previous section. Therefore, we have the following diagram

$$
\begin{array}{ccc}
\mathcal{C}_t & & \\
& \searrow{\scriptstyle \alpha} & \\
& \mathbf{P}^1_t \times \mathbf{P}^1_s & \xrightarrow{\ \pi\ } \mathbf{P}^1_s \\
& {\scriptstyle p}\big\downarrow & \\
& \mathbf{P}^1_t &
\end{array}
$$

Pick a general fiber f_s of the projection $\pi\colon \mathbf{P}^1_t \times \mathbf{P}^1_s \longrightarrow \mathbf{P}^1_s$, and consider its preimage

$$
D := \alpha^{-1}(f_s) \subset \mathcal{C}_t
$$

Under $p\colon \mathcal{C}_t \longrightarrow \mathbf{P}^1_t$, D is expressed as a five sheeted cover of $\mathbf{P}^1_t$, simply branched along a subset $B \subset \mathbf{P}^1_t$. We will need to perform a base change in order to make the points of D into sections of the family. First, we make an order two base change $X \longrightarrow \mathbf{P}^1_t$ branched along the subset B. For each branch point b, we blow up the node occurring in the preimage D_X of D over b. We also blow up the three tangent vectors of D_X lying above $b \in X$, and let $\widetilde{D}_X$ denote the strict transform of D_X. (We are performing admissible reduction.) Finally, we make a $5!\colon 1$ base change $Y \longrightarrow X$ to kill the monodromy of the cover $\widetilde{D}_X \longrightarrow \mathbf{P}^1_t$. The base changed family, which we will denote by $p_Y\colon \mathcal{C}_Y \longrightarrow Y$, now has five sections, s_i, along which we attach a "constant" admissible cover on the left. In this way, we arrive at a family

of admissible covers γ with intersections:

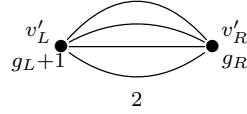$$\lambda \cdot \gamma = 240 \cdot 5 g_R$$

$$\delta \cdot \gamma = 240(31 g_R + 44) - 3 \cdot 120 B$$

$$E \cdot \gamma = -2(120)B$$

$$E' \cdot \gamma = 120 B$$

$$\mathbf{X} \cdot \gamma = 0$$

Here, E' is a *ramified* boundary divisor having dual graph $\Gamma_{E'}$:

Notice that we are claiming that $\mathbf{X} \cdot \gamma = 0$. This follows from the following lemma.

Lemma 4.12. *The admissible covers corresponding to $\gamma \cap E'$ in the family γ are not Maroni or Casnati-Ekedahl special.*

Proof. Let $[\alpha \colon Z \longrightarrow \mathbf{P}]$ be the general admissible cover occurring in intersection of the two boundary divisors $E \cap E'$. Then the dual graph Γ_Z looks like:
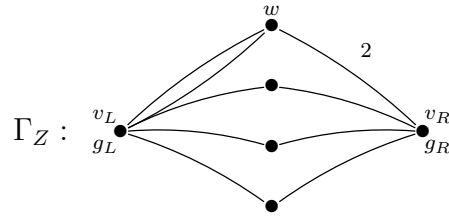
The vertex w is a rational curve mapping P_w mapping $2\colon 1$ to the middle component of the target tree of $\mathbf{P}^1$'s, (which we are calling $\mathbf{P}$). Let C_L denote the curve corresponding to v_L and C_R the curve corresponding to v_R. Then the first observation is that Z does *not* live in a projective bundle over $\mathbf{P}$, because any admissible cover α which has ramification above the nodes is *not* a flat morphism.

However, there is a natural way of flattening an admissible cover having simple ramification occurring at the nodes [22]: One simply attaches an intermediate *ribbon*, R, between P_w and C_R along the (nonreduced) ramified points. The ribbon R is the same ribbon as the double exceptional divisor in a blow up of a smooth surface.

Let us introduce three more $\mathbf{P}^1$'s at the three unramified points of C_R in the same fiber as the ramified node, to ultimately construct a flat cover of a chain of now 4 $\mathbf{P}^1$'s. We denote this cover by $\tilde{Z}$, and we draw a dual graph representation for it as follows:
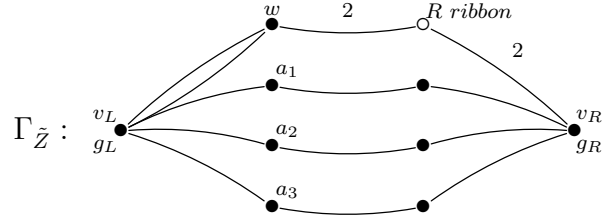
The cover $\tilde{\alpha} \colon \tilde{Z} \longrightarrow \tilde{\mathbf{P}}$ is now flat, and has Gorenstein fibers, so the Casnati-Ekedahl factorization applies to it. Therefore, there is a factorization:

(4.19)
$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\ i\ } & \mathbf{P}\mathcal{E} \\
& {}_{\tilde{\alpha}}\searrow & \downarrow{}^{\pi} \\
& & \tilde{\mathbf{P}}
\end{array}
$$

The broken projective bundle $\mathbf{P}\mathcal{E}$ has four components, which we will call $\mathbf{P}\mathcal{E}_L$, $\mathbf{P}\mathcal{E}_w$, $\mathbf{P}\mathcal{E}_{Ribbon}$, and $\mathbf{P}\mathcal{E}_R$. Since we are assuming that Z is general, we know from results of the first chapter we may assume that $\mathcal{E}_L$, and $\mathcal{E}_R$ are rigid. Furthermore the quadric bundles $\mathcal{F}_L$ and $\mathcal{F}_R$ are also rigid. It is easy to see that $\mathcal{E}_w = \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1)$, and $\mathcal{E}_{Ribbon} = \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-1)$.

For a rigid bundle $\mathcal{G}$, define the *directrix* $\Sigma_{\mathcal{G}} \subset \mathbf{P}\mathcal{G}$ to be the intersection of all minimal degree effective hyperplane divisors. Then, the first observation is that the directrices $\Sigma_{\mathcal{E}_w} \subset \mathbf{P}\mathcal{E}_w$ and $\Sigma_{\mathcal{E}_{Ribbon}} \subset \mathbf{P}\mathcal{E}_{Ribbon}$ are disjoint, i.e. their restrictions to the intersection $\mathbf{P}^3_{w,Ribbon} := \mathbf{P}\mathcal{E}_w \cap \mathbf{P}\mathcal{E}_{Ribbon}$ are disjoint. Secondly, we may choose C_L and C_R general so that the directrices $\Sigma_{\mathcal{E}_L}$ and $\Sigma_{\mathcal{E}_R}$ meet the directrices $\Sigma_{\mathcal{E}_w}$ and $\Sigma_{\mathcal{E}_{Ribbon}}$ properly. The resulting bundle $\mathbf{P}\mathcal{E}$ will then be rigid, i.e. the cover $\tilde{Z}$ will be Maroni general.

The same type of analysis applies for $\mathcal{F}$. First, it is easy to show that

$$\mathcal{F}_w = \mathcal{O}(1)^{\oplus d - 3} \oplus \mathcal{O}^{\oplus N}$$

and that

$$\mathcal{F}_{Ribbon} = \mathcal{O}(-1)^{\oplus d - 3} \oplus \mathcal{O}^{\oplus N}$$

The distinguished subvector bundle $\mathcal{O}(1)^{\oplus d - 3} \hookrightarrow \mathcal{F}_w$ parametrizes "quadrics splitting off the plane spanned by the points a_1, a_2, and a_3." The sub vector bundle $\mathcal{O}^{\oplus N} \hookrightarrow \mathcal{F}_{Ribbon}$ parametrizes "quadrics singular at the reduced point of the ribbon".

When restricted to the intersection $\mathbf{P}^3_{w,Ribbon} := \mathbf{P}\mathcal{E}_w \cap \mathbf{P}\mathcal{E}_{Ribbon}$, these two distinguished vector subspaces of $\mathcal{F}$ are complementary: The quadrics which split off the plane spanned by a_1, a_2, and a_3 and which contain the $\mathbf{Spec}\, k[\epsilon]/(\epsilon^2)$ point of $P_w \cap R$ will never be singular at the reduced point of $P_w \cap R$, because the fibers of $\tilde{\alpha}$ are in general linear position.

The vector subbundles described in the previous paragraph give a description of the directrices of $\mathbf{P}\mathcal{F}_w$ and $\mathbf{P}\mathcal{F}_{Ribbon}$. Using this description, it is straightforward to show that one can select C_L and C_R generically so that the directrices of $\mathbf{P}\mathcal{F}_L$, $\mathbf{P}\mathcal{F}_w$, $\mathbf{P}\mathcal{F}_{Ribbon}$, and $\mathbf{P}\mathcal{F}_R$ meet properly.

The directrices of $\mathbf{P}\mathcal{F}_w$ and $\mathbf{P}\mathcal{F}_{Ribbon}$ are disjoint, as we've seen. So we must show two things. (1) There exists a genus g_L degree 5 cover $\alpha_L\colon C_L \longrightarrow \mathbf{P}^1$ and a triple of points (a_1, a_2, a_3) in a fiber $\alpha^{-1}(x)$ such that the vector space of quadrics splitting off the plane spanned by (a_1, a_2, a_3) and containing $\alpha^{-1}(x)$, properly intersects the distinguished vector subspace of quadrics corresponding to the directrix subbundle of $\mathbf{P}\mathcal{F}_{\alpha_L}$. (2) There exists a genus g_R degree 5 cover $\alpha_R\colon C_R \longrightarrow \mathbf{P}^1$ and a *ramified* fiber $\alpha^{-1}(y) = (2u_1, u_2, u_3, u_4)$ (by this we mean $u_1 \in C_R$ is a ramification point), such that the vector space of quadrics *singular* at u_1 and containing $\alpha^{-1}(y)$ intersects the distinguished vector space of quadrics corresponding to the directrix subbundle of $\mathbf{P}\mathcal{F}_{\alpha_R}$.

For both of these, we simply refer the reader to the broken covers constructed in the proof of Theorem 1.1 in chapter 1. By making these general choices, the cover $\tilde{Z}$ will be Casnati-Ekedahl general. □

The relation resulting from our test family γ is

$$(4.20) \qquad c(E', \mathbf{Y}) = 2c(E, \mathbf{Y}) + \frac{11}{10}g_R + \frac{2}{5}g$$

(Note that we are relating a ramified divisor on the left hand side of the equation with an unramified divisor on the right hand side of the equation.)

Now we use the same type of test family for the unramified, 2-vertex divisor E^- which has dual graph Γ_{E^-}:
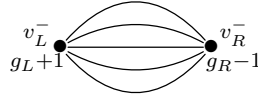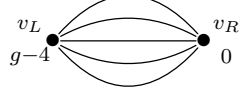

where *we vary the left vertex v_L^-*. As before, we obtain the relation

$$(4.21) \qquad c(E', \mathbf{Y}) = 2c(E^-, \mathbf{Y}) + \frac{11}{10}(g_L + 1) + \frac{2}{5}g$$

Subtracting relations, we arrive at

$$(4.22) \qquad c(E, \mathbf{Y}) - c(E^-, \mathbf{Y}) = \frac{11}{20}(g_L + 1 - g_R)$$

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Since we assume $g_R \leq g_L$, we can conclude that $c(E, \mathbf{Y}) \geq 0$ once we know $c(E_0, \mathbf{Y}) \geq 0$, where Γ_{E_0} is:

However, one just uses a pencil of $(1, 5)$ curves on $\mathbf{P}^1 \times \mathbf{P}^1$ with 5 basepoints in a fiber, as in chapter 3, section 3.1, to conclude $c(E_0, \mathbf{Y}) \geq 0$.

Finally, notice that relation (4.20) also allows us to conclude that $c(E', \mathbf{Y}) \geq 0$ for any simply-ramified 2-vertex divisor E'.

4.5.2. *Arbitrary 2-vertex divisors.* For an arbitrary 2-vertex divisor, we use the same sort of test family as we did in the previous section. We will only provide the starting point of the construction, and will leave the details of the admissible reduction process to the reader. Begin with a pencil $p \colon \mathcal{C}_t \longrightarrow \mathbf{P}_t^1$ such that, for $t = 0$, the pentagonal curve C_0 has a prescribed ramification type. For example, if the divisor under consideration had dual graph Γ_E:
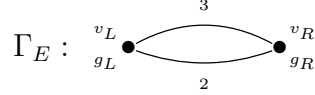
then we would require our pencil $\mathbf{P}_t^1 \subset \mathbf{P}H^0(\mathbf{P}^1, \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee)$ to have $(2, 3)$ ramification occurring somewhere, say $s = 0$, in the fiber C_0. We pick $\mathbf{P}_t^1$ general among such pencils. Again, we obtain a diagram

We pick the "section" curve $D := \alpha^{-1}\{s = 0\} \subset \mathcal{C}_t$ so that, under the map p, D also has ramification profile $(2, 3)$ above $t = 0$. After a suitable base change and blowups, we arrive at a family $\tilde{p}$ with sections along which we may glue a constant family of curves on the

left. (The procedure is essentially the same as the one described earlier.) The resulting test family of admissible covers will entirely live inside an unramified 2-vertex divisor E^{urn} , and will intersect a simply-ramified 2-vertex divisor E^{ram} finitely many times. The relationship between E^{ram} and E^{unr} is exactly the same relationship as the one between E' and E^- above. The relation (4.20) shows that:

$$(4.23) \qquad\qquad c(E^{ram}, \mathbf{Y}) = 2c(E^{unr}, \mathbf{Y}) + \frac{11}{10}g_R + \frac{2}{5}g$$

From this, and the relation coming from the test family γ constructed above, the positivity of $c(E, \mathbf{Y})$ will follow.

4.6. **Some numerical observations.** The main obstructions for determining slope bounds for arbitrary d are: (1) The effective divisor class $\mathbf{X}$ is useless: the coefficient of D is not negative, and (2) even if we had a useful candidate effect divisor class $\mathbf{X}$, we would need to exhibit sufficiently general test families to ensure that all coefficients of the divisor class expression for $\mathbf{X}$ were negative.

Although the second obstruction seems completely beyond reach, we may still have a chance of finding useful effective divisor classes $\mathbf{X}$. Our first observation is this: For a complete, one dimensional family $T \subset \widetilde{\mathcal{H}}_{d,g}$ of covers, let

$$\mathcal{C} \xrightarrow{\ \alpha\ } \mathcal{P}$$
$$\varphi \searrow \qquad \swarrow \pi$$
$$T$$

be the total diagram of the family. Let $\mathcal{E}$ and $\mathcal{F}$ denote the reduced direct image and bundle of quadrics for the cover $\alpha\colon \mathcal{C} \longrightarrow \mathcal{P}$. By Proposition 4.1, we see that the slope may be written as:

$$s(T) = \frac{\delta \cdot T}{\lambda \cdot T} = \frac{A(c_1^2(\mathcal{E}), c_2(\mathcal{E}), c_2(\mathcal{F}))}{B(c_1^2(\mathcal{E}), c_2(\mathcal{E}), c_2(\mathcal{F}))}$$

where A and B are linear expressions. The reader may check that after we set the Bogomolov expressions $Bog(\mathcal{E})$ and $Bog(\mathcal{F})$ equal to zero, the the slope *automatically* becomes

$$s_{conj} = \frac{(5d+6)g + d^2 + 5d - 6}{dg}$$

Therefore, a complete family of curves $T \subset \widetilde{\mathcal{H}}_{d,g}$ (passing through the interior) which *completely avoids* the Maroni and Casnati-Ekedahl divisors (when they exist) will automatically attain the above conjectural slope bound.

On the other hand, for degrees $d > 5$, there is no *effective* combination of **M** and **CE** which makes the coefficient of D zero. Therefore, we may have to venture beyond the world of Maroni and Casnati-Ekedahl divisors. It is unclear how to proceed, however. For example, in $\widetilde{\mathcal{H}}_{6,g}$, we may want consider the higher syzygy divisor (when it exists) corresponding to the Bogomolov expression for the bundle $\mathcal{N}_3$ occurring in the Casnati-Ekedahl resolution of $\mathcal{I}_C$. After calculating, however, it unfortunately turns out that $Bog(\mathcal{N}_2) = 2(Bog(\mathcal{F}))$.

However, all is not lost. One really should consider the Bogomolov expressions for *all* vector bundles which arise from exact sequences among all tensor products of $\mathcal{E}$ and $\mathcal{F}$. For example, one may consider the Bogomolov expressions for the kernels of the "evaluation" maps

$$\mathcal{S}^N \mathcal{E} \longrightarrow \alpha_*(\omega_\alpha^{\otimes N}) \longrightarrow 0$$

Hopefully, one might be able to find a Bogomolov expression which, when written in terms of λ, δ, D and higher boundary divisors, has negative contribution coming from δ, D, and the rest of the boundary.

4.7. **The congruence conditions on** (g, d)**.** Finally, we would like to comment on the congruence conditions between g and d occurring in corollaries 4.8, 4.9, and 4.10. In [9], a sharp slope bound for $\overline{\mathcal{H}}_{3,g}$ is obtained for *odd* g. This bound is $7 + 20/(3g + 1)$, and the divisor **X** (which doesn't exist) is replaced by the divisor of trigonal curves which are *tangent* to the directrix of $\mathbf{F}_1$.

4.8. **A final fun example.** We end by returning to the example in 1.4.1. The Maroni divisor $\mathbf{M} \subset \mathcal{H}_{3,4}$ parametrizes genus 4 curves which lie on a singular quadric in their canonical embeddings. The branch divisor of a Maroni special cover $\alpha \colon C \longrightarrow \mathbf{P}^1$ is a set of 12 points on $\mathbf{P}^1$ whose defining equation has the form $4a^3 + 27b^2$. In [9], we calculate the exact class of $\mathbf{M}$ in $\mathrm{Pic}_{\mathbb{Q}} \, \widetilde{\mathcal{H}}_{3,4}$. (This can also be done using partial pencil families.) By pushing this class to $\mathcal{M}_{0,12}$, we recover a result originally due to Zariski [25]: The locus of 12 points on $\mathbf{P}^1$ which are zeros of a polynomial of the form $4a^3 + 27b^2$ is 3762. This beautiful example initiated our study of Maroni and Casnati-Ekedahl loci, so we find it a fitting way to end.

References

[1] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers. *Comm. Algebra*, 31(8):3547–3618, 2003.

[2] F. A. Bogomolov. Stable vector bundles on projective surfaces. *Mat. Sb.*, 185(4):3–26, 1994.

[3] G. Casnati and T. Ekedahl. Covers of algebraic varieties I. A general structure theorem, covers of degree 3, 4 and Enriques' surfaces. *J. Algebraic Geom.*, 5:439–460, 1996.

[4] Gianfranco Casnati. Covers of algebraic varieties. II. Covers of degree 5 and construction of surfaces. *J. Algebraic Geom.*, 5(3):461–477, 1996.

[5] Gianfranco Casnati. Covers of algebraic varieties. IV. A Bertini theorem for Scandinavian covers. *Forum Math.*, 13(1):21–36, 2001.

[6] Marc Coppens. Existence of pencils with prescribed scrollar invariants of some general type. *Osaka J. Math.*, 36(4):1049–1057, 1999.

[7] Maurizio Cornalba and Joe Harris. Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. *Ann. Sci. École Norm. Sup. (4)*, 21(3):455–475, 1988.

[8] Andrea Del Centina and Alessandro Gimigliano. Scrollar invariants and resolutions of certain d-gonal curves. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 39:187–201 (1995), 1993.

[9] A. Deopurkar and A. Patel. Sharp slope bounds for sweeping families of trigonal curves. *ArXiv e-prints*, November 2012.

[10] Anand Deopurkar. Modular compactifications of the space of marked trigonal curves. *Preprint*, 2012.

[11] Steven Diaz and Joe Harris. Geometry of Severi varieties. II. Independence of divisor classes and examples. In *Algebraic geometry (Sundance, UT, 1986)*, volume 1311 of *Lecture Notes in Math.*, pages 23–50. Springer, Berlin, 1988.

[12] Dan Edidin. Picard groups of Severi varieties. *Comm. Algebra*, 22(6):2073–2081, 1994.

[13] John L. Harer. The cohomology of the moduli space of curves. In *Theory of moduli (Montecatini Terme, 1985)*, volume 1337 of *Lecture Notes in Math.*, pages 138–221. Springer, Berlin, 1988.

[14] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982.

[15] Friedrich Knop, Hanspeter Kraft, and Thierry Vust. The Picard group of a G-variety. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 77–87. Birkhäuser, Basel, 1989.

[16] S. Lefschetz. *L'analysis situs et la géométrie algébrique*. Gauthier-Villars, Paris, 1950.

[17] Arturo Maroni. Le serie lineari speciali sulle curve trigonali. *Ann. Mat. Pura Appl. (4)*, 25:343–354, 1946.

[18] Shinichi Mochizuki. The geometry of the compactification of the Hurwitz scheme. *Publ. Res. Inst. Math. Sci.*, 31(3):355–441, 1995.

[19] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[20] Akira Ohbuchi. On some numerical relations of d-gonal linear systems. *J. Math. Tokushima Univ.*, 31:7–10, 1997.

[21] Sevin Recillas. Jacobians of curves with g_4^1's are the Prym's of trigonal curves. *Bol. Soc. Mat. Mexicana (2)*, 19(1):9–13, 1974.

[22] Zvezdelina E. Stankova-Frenkel. Moduli of trigonal curves. *J. Algebraic Geom.*, 9(4):607–662, 2000.

[23] Ilya Tyomkin. On Severi varieties on Hirzebruch surfaces. *Int. Math. Res. Not. IMRN*, (23), 2007.

[24] Ravi Vakil. Twelve points on the projective line, branched covers, and rational elliptic fibrations. *Math. Ann.*, 320(1):33–54, 2001.

[25] Oscar Zariski. On Hyperelliptic θ-Functions with Rational Characteristics. *Amer. J. Math.*, 50(3):315–344, 1928.