The Acceleration Principle and the Nature of Business Cycles

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THE ACCELERATION PRINCIPLE AND THE NATURE
OF BUSINESS CYCLES *

GREGORY C. CHOW

I. INTRODUCTION

Half a century ago, J. M. Clark stressed the importance of the
acceleration principle in business cycle theory.[1] Almost thirty years
ago P. A. Samuelson pointed out that the acceleration relation can
interact with the multiplier to generate oscillations in national
income.[2] This paper presents a formulation of the acceleration
principle and some strong supporting evidence. It attempts to
ascertain whether the acceleration principle is *necessary* for the
generation of oscillations in national output. It also extends the
discussion of the nature of economic fluctuations from the non-
stochastic to the stochastic case.

Recently I constructed a simple macroeconometric model of the
United States economy in the form of a system of linear stochastic
difference equations in which the acceleration principle plays an
important role.[3] The present paper is partly the outcome of an
analytical study of the dynamic properties of such a linear stochastic
system. The tools employed are the autocovariances and the spectral
densities of the time series so generated. The presentation of results
on these matters will be elementary and self-contained.

1. J. M. Clark, "Business Acceleration and the Law of Demand: A
Technical Factor in Economic Cycles," *Journal of Political Economy*, XXV
(Mar. 1917), 217–35.
2. P. A. Samuelson, "Interactions between the Multiplier Analysis and
the Principle of Acceleration," *Review of Economic Statistics*, XXI (May
1939), 75–78.
3. G. C. Chow, "Multiplier, Accelerator, and Liquidity Preference in the
Determination of National Income in the United States," *Review of Economics
and Statistics*, XLIX (Feb. 1967), 1–15.

This paper begins with a formulation of the acceleration principle and its supporting evidence in Section II. Section III shows that, without an acceleration equation, a system of nonstochastic linear demand equations for the components of national output cannot produce oscillations in the variables. This amounts to showing that the characteristic roots of the system are all real and positive. The discussion is generalized to allow for random disturbances in the equations. In Sections IV and V, respectively, the autocovariance matrix and the spectral density matrix will be derived in an elementary fashion. It is shown that random shocks, when introduced into an otherwise nonoscillatory system, can generate cycles. Thus the acceleration principle is no longer necessary. While Sections IV and V are confined to the stationary case (with all roots less than one in absolute value), Section VI contains brief comments on the nonstationary case as well as some concluding remarks on the nature of business cycles.

II. Demand Equations and Evidence for Acceleration

To ascertain whether a system of nonstochastic economic equations can produce oscillations without acceleration, one obviously has to specify the nature of the equations. It seems relevant to restrict the equations in a macroeconomic model to demand equations for different components y_i of national product $Y = \Sigma y_i$ which obey simple distributed lags: [4]

(1) $\qquad y_{it} = a_i \, Y_t + b_i y_{i, \, t-1}$.

Of course, demand equations for investment goods and for some consumer durables are not of this type because of acceleration, but our purpose is precisely to find out whether there could be oscillations if all demand equations were of the form (1). In other words, while Samuelson [5] demonstrated that the interaction of the multiplier and the accelerator can produce oscillations, we ask whether the latter is necessary, in the context of a system of equations.

For investment goods and possibly consumer durables, the demand functions for *total stock* s_i are assumed to be of the same form:

(2) $\qquad s_{it} = a_i \, Y_t + b_i \, s_{i, \, t-1}$.

The flow of net investment $y_{it}{}^n$ is $\triangle s_{it} = s_{it} - s_{i, \, t-1}$:

4. The intercept will be omitted, as exogenous variables are omitted in this paper which treats business cycles as the result of interactions of endogenous forces, holding the exogenous variables constant.

5. Samuelson, *op. cit.*

(3) $\qquad y_{it}{}^n = \triangle s_{it} = a_i \triangle Y_t + b_i y_i{}^n, {}_{t-1}$

which is our formulation of the acceleration principle. If one chooses to work with gross investment $y_{it} = \triangle s_{it} + \delta_i s_{i, t-1} = s_{it} - (1 - \delta_i) s_{i, t-1}$, where δ_i is the percentage rate of depreciation, the demand equation will be

(4) $\qquad y_{it} = a_i \triangle Y_t + a_i \delta_i Y_{t-1} + b_i y_{i,t-1}$
$\qquad\qquad = a_i [Y_t - (1 - \delta_i) Y_{t-1}] + b_i y_{i,t-1}.$

In the previous macroeconometric study, I tested equation (4) using annual observations (1931–40 and 1948–63) on new construction, on gross private domestic investment in producers' durable equipment combined with change in business inventories, and on the last two variables separately, the Y variable being gross national product minus taxes net of transfers.[6] In each case, the coefficient of Y_{t-1} turned out to be of opposite sign, and slightly smaller in absolute value, as compared with the coefficient of Y_t. In other words, when $\triangle Y_t$ and Y_{t-1} were used, the coefficient of the latter turned out to be a small fraction of the coefficient of the former.

Using annual data (1922–41 and 1948–57) on net investment in automobiles obtained in previous work,[7] I have found, by the method of least squares,

(5) $\qquad y_t{}^n = .0155 Y_t - .0144 Y_{t-1} - .0239 p_t$
$\qquad\qquad\quad (.0022) \quad\;\; (.0020) \qquad\;\; (.0056)$
$\qquad\qquad + .0199 p_{t-1} + .351 y_{t-1}{}^n + \text{const.}$
$\qquad\qquad\;\; (.0040) \qquad\;\; (.101)$

where Y_t is real disposable personal income per capita, p_t is a relative price index for automobiles, and $y_t{}^n$ is per capita net investment in passenger automobiles. Here again, the coefficients of Y_{t-1} and p_{t-1} are practically the negatives of the coefficients of Y_t and p_t.

The strong empirical support for the acceleration principle as formulated in equations (3) and (4) should not be surprising to those willing to accept the stock adjustment equation (2), as (3) and (4) are the results of differencing (2). In earlier work,[8] I presented demand functions for automobile stock in the form of equation (2), and for the purchase of new automobiles by subtracting $(1 - \delta_i)$ $s_{i, t-1}$ from both sides of (2) while keeping $s_{i, t-1}$ as an explanatory variable; the price variable p_t was used in addition to Y_t in the stock adjustment equation. The calculation presented in equation (5) is

6. Chow, *op. cit.*
7. G. C. Chow, *Demand for Automobiles in the United States* (Amsterdam: North-Holland Publishing Co., 1957), and "Statistical Demand Functions for Automobiles and Their Use for Forecasting," in A. C. Harberger, ed., *Demand for Durable Goods* (Chicago: University of Chicago Press, 1960).
8. *Ibid.*

essentially the first difference of (2), rather than the result of sub-tracting $(1 - \delta_i)s_{i, t-1}$ from both sides.

III. The Nonstochastic Case: no Oscillations without Acceleration

To set the stage for later analyses of the cyclical properties of a system of linear difference equations, with or without acceleration, write the nonstochastic system as

(6) $y_t = Ay_{t-1}$

where y_t is a p-component column vector of dependent variables at time t, and A is a p by p matrix of coefficients. Any higher order system to start with is understood to have been reduced to first order by redefining variables. This paper is confined to the interactions of endogenous variables, leaving out the possible effects of exogenous variables.

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the characteristic roots of A, assumed to be distinct, D_λ the diagonal matrix with these roots on the diagonal, and $B = (b_{ij})$ the matrix whose columns are the (right) characteristic vectors corresponding to these roots:

(7) $AB = BD_\lambda$ or $A = BD_\lambda B^{-1}$.

If the initial values y_0 at time 0 are given, the solution of (6) is

(8) $y_t = A^t y_0 = BD_\lambda{}^t B^{-1} y_0$.

The solution (8) can be interpreted in terms of a set of canonical variables $z_t = B^{-1} y_t$ that are linear combinations of the original variables:

$$(9) \qquad z_t = B^{-1} y_t = B^{-1} A y_{t-1}$$
$$= D_\lambda B^{-1} y_{t-1} = D_\lambda z_{t-1} = D_\lambda{}^t z_0.$$

The solution for the i-th canonical variable is $z_{it} = z_{i0}\lambda_i{}^t$. The solutions for the original variables $y_t = Bz_t$ are thus the same linear combination B of the solutions $z_{it} = z_{i0}\lambda_i{}^t$,

(10) $y_t = Bz_t = BD_\lambda{}^t z_0$

as we have in (8). The components of (10) are

(11) $y_{it} = b_{i1}z_{10}\lambda_1{}^t + b_{i2}z_{20}\lambda_2{}^t$
            $+ \ldots + b_{ip}z_{p0}\lambda_p{}^t$ $\qquad (i = 1, \ldots, p)$

which are linear combinations of $\lambda_j{}^t$ $(j = 1, \ldots, p)$. If the roots λ_j are all real and positive, there will be no prolonged oscillations in the solution (11). With only real and positive roots, some negative coefficients $b_{ij}z_{j0}$ could produce a few swings in the time path (11) initially, but the solution will soon behave like an exponential function according to the largest root.

Returning to our problem of a system of demand equations (1) without acceleration,

(12)
$$\begin{bmatrix} 1-a_1 & -a_1 & -a_1 \ldots & -a_1 \\ -a_2 & 1-a_2 & -a_2 \ldots & -a_2 \\ & & \cdots & \\ -a_p & -a_p & -a_p \ldots 1-a_p \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \\ \cdot \\ \cdot \\ \cdot \\ y_{pt} \end{bmatrix} = \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ & & \cdot \\ & & \cdot \\ 0 & & b_p \end{bmatrix} \begin{bmatrix} y_{1,\,t-1} \\ y_{2,\,t-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{p,\,t-1} \end{bmatrix}$$

or

$$G y_t = D_b y_{t-1},$$

we inquire whether, and under what conditions, the roots of $A = G^{-1} D_b$ will all be real and positive, assuming that the coefficients a_i and b_i are real and positive. First to be established is the non-singularity of G. This is done by evaluating the roots of G.[9] Write $G = I - F$, where the i-th row of F is $(a_i\, a_i \ldots a_i)$. F has rank one, and hence only one nonzero root. This root is $\overset{p}{\underset{i=1}{\Sigma}}\, a_i$, as it can be easily checked. The roots of G satisfy $|G - \lambda I| = 0$ or $|F - (1 - \lambda)I| = 0$, and are equal to one minus the roots of F, i.e., $1 - \overset{p}{\underset{i=1}{\Sigma}}\, a_i$, $1, \ldots, 1$. Thus, G is nonsingular if and only if $\Sigma\, a_i \neq 1$.

The main results are, first, that *the roots of system (12) are all real*; and, second, they are *all positive if and only if the sum* $\overset{p}{\underset{i=1}{\Sigma}}\, a_i$ *of the marginal propensities is less than unity*.

To show that the roots of A are real, consider $A^{-1} = D_b^{-1} G$. Let D_a be the diagonal matrix with $a_1, \ldots, a_p$ on its diagonal. $D_b^{-1} G = (D_b^{-1} D_a)\,(D_a^{-1} G)$ is the product of two real symmetric matrices, the former $D_b^{-1} D_a$ being positive definite. Therefore, the roots of $D_b^{-1} G$, and of $G^{-1} D_b$, are all real.[1]

To obtain a necessary and sufficient condition for all positive roots, observe that the roots of $D_b^{-1} G$ can be obtained from the following equivalent characteristic equations:

$$|\, D_b^{-1} G - \mu I\,| = 0$$
$$|D_b^{-\frac{1}{2}} G\, D_b^{-\frac{1}{2}} - \mu I\,| = 0$$
$$|\, D_b^{-\frac{1}{2}} D_a^{\frac{1}{2}}\, (D_a^{-1} G)\, D_a^{\frac{1}{2}} D_b^{-\frac{1}{2}} - \mu I\,| = 0 .$$

Therefore, they are the same as the roots of the last symmetric matrix. The latter roots are all positive if and only if the roots of the symmetric matrix $D_a^{\frac{1}{2}}(D_a^{-1} G)D_a^{\frac{1}{2}}$ are; but the roots of

9. This argument is due to A. J. Hoffman.
1. See, for example, R. Bellman, *Introduction to Matrix Analysis* (New York: McGraw-Hill, 1960), p. 36, Exercise 4.

$D_a^{\frac{1}{2}}(D_a^{-1}G)D_a^{\frac{1}{2}}$ are indeed those of G. As we have evaluated, the roots of G are all positive if and only if $1 - \Sigma a_i$ is positive.

It has been shown that the solution of system (12) cannot oscillate, under the assumption $\overset{n}{\underset{i=1}{\Sigma}} a_i < 1$. By a numerical example, of two variables say, one can easily demonstrate that, if a demand equation (1) is replaced by an acceleration equation (3) or (4) in the system (12), complex roots can be generated.

IV. AUTOCOVARIANCES OF A LINEAR STOCHASTIC SYSTEM

To generalize our discussion to the stochastic case is important not only for the study of dynamic properties of linear econometric models, but for business cycle theory. In the words of Frisch:

> The examples we have discussed . . . show that when an [deterministic] economic system gives rise to oscillations, these will most frequently be damped. But in reality the cycles . . . are generally not damped. How can the maintenance of the swings be explained? . . . One way which I believe is particularly fruitful and promising is to study what would become of the solution of a determinate dynamic system if it were exposed to a stream of erratic shocks . . .
>
> Thus, by connecting the two ideas: (1) the continuous solution of a determinate dynamic system and (2) the discontinuous shocks intervening and supplying the energy that may maintain the swings — we get a theoretical setup which seems to furnish a rational interpretation of those movements which we have been accustomed to see in our statistical time data.[2]

It turns out that oscillations in the deterministic system are neither necessary nor sufficient in producing "cycles" in the stochastic system, but the importance of the stochastic element should be stressed.

In the stochastic case, we add a random vector u_t to the right side of system (6):

(13) $\qquad y_t = Ay_{t-1} + u_t$

with $Eu_t = 0$, $Eu_t u_t' = V$, and $Eu_t u'_{t-k} = 0$ for $k \neq 0$. It is further assumed that the roots of A are all smaller than one in absolute value, so that [3] the covariances $\gamma_{ij,\ k} = Ey_{it}\, y_{j,\ t-k}$ exist and are independent of t, leaving a brief discussion of the nonstationary case (with some roots of A greater than one) to Section VI. The covariances $\gamma_{ij,\ k}$ do provide information about the dynamic properties of the system. For example, if $\gamma_{ii,\ k}$ is an "approximately" periodic function of the lag k, with forty months as the length of the

2. Ragnar Frisch, "Propagation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honour of Gustav Cassel* (London: Allen & Unwin, 1933), pp. 197 and 202–3.
3. A proof will be provided following equation (18).

period, say, then observations on the time series y_{it} made forty months apart are more highly correlated than those made at other intervals — the sense in which a forty-month cycle can be claimed for y_{it} will be further specified by the spectral density function of y_{it}, to be derived in Section V.

It is quite straightforward to derive the autocovariance matrix [4]

(14) $\Gamma_k = E y_t\, y'_{t-k} = (\gamma_{ij,\,k}) = \Gamma'_{-k}$

in terms of the parameters A and V of the system (13). Post-multiply each term of (13) by y'_{t-k}, and take expectation:

(15) $E\, y_t\, y'_{t-k} = AE\, y_{t-1}\, y'_{t-k} + E\, u_t\, y'_{t-k}$.

For $k = 0$, the last term is V, and for $k = 1, 2, \ldots$, the last term is 0. Note also that $\Gamma_{-1} = \Gamma_1'$. Hence, (15) can be written as

(16) $\Gamma_0 = A\Gamma_{-1} + V = A\Gamma'_1 + V$, and

(17) $\Gamma_k = A\Gamma_{k-1}$ \qquad $(k = 1, 2, 3, \ldots)$.

Once Γ_0 (the covariance matrix of y_t) is known, (17) can be used to find Γ_k $(k = 1, 2, \ldots)$, i.e., $\Gamma_k = A^k \Gamma_0$.

To find Γ_0, we use (16) to iterate, noting $\Gamma_1 = A\Gamma_0$ and $\Gamma_1' = \Gamma_0' A' = \Gamma_0 A'$,

(18) $\Gamma_0 = V + A\Gamma_1'$
        $= V + A\Gamma_0\, A' = V + A(V + A\Gamma_0\, A')A'$
        $= V + AVA' + A^2 V A'^2 + \cdots$

From (18), we will observe that Γ_0 exists if the roots of A are all less than one in absolute value. Using (7) and letting $W = (w_{ij})$ be the symmetric positive semidefinite matrix $B^{-1}\,VB^{-1'}$, we write (18) as

(19) $\Gamma_0 = \displaystyle\sum_{s=0}^{\infty} A^s V A'^s$
        $= \displaystyle\sum_{s} BD_{\lambda^s} B^{-1}\, VB^{-1'}\, D_{\lambda^s}\, B'$
        $= B(\displaystyle\sum D_{\lambda^s}\, WD_{\lambda^s})B'$
        $= B(\displaystyle\sum_{s=0}^{\infty} w_{ij}\lambda_i^s\lambda_j^s)B'$.

Hence, the matrix in parentheses exists if all the roots are less than one in absolute value. Assuming this condition to hold, we can write out the autocovariance matrices explicitly:

(20) $\Gamma_0 = B\,\Big(\dfrac{w_{ij}}{1 - \lambda_i\lambda_j}\Big)\, B'$

_____

4. Many results on the autocovariance matrix can be found, for example, in M. H. Quenouille, *The Analysis of Multiple Time-Series* (London: Charles Griffin, 1957).

(21) $\Gamma_k = A^k \Gamma_0 = BD_\lambda^k B^{-1} \Gamma_0$

$\qquad = BD_\lambda^k \left(\dfrac{w_{ij}}{1 - \lambda_i \lambda_j} \right) B'$.

It is useful to interpret (21) in terms of the canonical variables $z_t = B^{-1} y_t$. Premultiplying (13) by B^{-1} shows that the canonical variables satisfy

(22) $z_t = D_\lambda z_{t-1} + B^{-1} u_t = D_\lambda z_{t-1} + \epsilon_t$.

The covariance matrix of the residuals ϵ_t is

(23) $E \epsilon_t \epsilon_t' = EB^{-1} u_t u_t' B^{-1'} = B^{-1} V B^{-1'} = W$

where the matrix W has been previously defined. Analogous to (17) and (19), the autocovariance matrix of z_t, denoted by Γ_k^*, satisfies the relations

(24) $\Gamma_k^* = D_\lambda \Gamma_{k-1}^* = D_\lambda^k \Gamma_0^*$ $(k = 1, 2, 3, \ldots)$

and

(25) $\Gamma_0^* = \sum\limits_{s=0}^{\infty} D_\lambda^s W D_\lambda^s = \left(\sum\limits_{s=0}^{\infty} w_{ij} \lambda_i^s \lambda_j^s \right) = \left(\dfrac{w_{ij}}{1 - \lambda_i \lambda_j} \right)$.

The relation between Γ_k and Γ_k^* is

(26) $\Gamma_k = E y_t y'_{t-k} = EB z_t z'_{t-k} B' = B\Gamma_k^* B'$

or $\Gamma_k = BD_\lambda^k \Gamma_0^* B'$ $(k = 0, 1, 2, \ldots)$

which is the same as (21).

By (24), each covariance $\gamma_{mn}^*,_k$ of the canonical variables is $\gamma_{mn}^*,_0 \lambda_m^k$. By (26), the convariance $\gamma_{ij},_k$ of the original variables is a linear combination of λ_m^k:

(27) $\gamma_{ij,\,k} = \sum\limits_{m} \sum\limits_{n} b_{im} b_{jn} \gamma_{mn}^* ,_k = \sum\limits_{m} b_{im} \left(\sum\limits_{n} b_{jn} \gamma_{mn}^*,_0 \right) \lambda_m^k$

$\qquad = \sum\limits_{m} d_{ij,\,m} \lambda_m^k$ $(k = 0, 1, 2, \ldots)$

and

$\qquad \gamma_{ij,\,-k} = \gamma_{ji,k} = \sum d_{ji,m} \lambda_m^k$ $(k = 1, 2, 3, \ldots)$.

Comparing the covariance function (27) and the time path (11) of the nonstochastic model, we see a formal resemblance: both are linear combinations of λ_m^k. In the nonstochastic case, the coefficients of (11) depend on the initial values y_{i0}, or on the initial values z_{i0} and b_{ij}. In the stochastic case, the coefficients of (27) depend on the covariances $\gamma_{ij,\,0}$, or, in terms of canonical variables, on $\gamma^*_{ij,\,0}$ and b_{ij}.

If complex roots exist, they come in conjugate pairs. The contribution of a pair of complex roots to the linear combination (27) can be simply stated. Let the pair be

(28) $\lambda_1 = re^{i\theta}, \lambda_2 = re^{-i\theta}$ $(i = \sqrt{-1})$.

Their contribution to $\gamma_{ij,\,k}$ is

(29) $d_{ij,\,1} r^k e^{i\theta k} + d_{ij,\,2} r^k e^{-i\theta k}$.

If this contribution is real, we let the coefficients $d_{ij,\,1}$ and $d_{ij,\,2}$ be conjugate complex, say,

(30) $\qquad d_{ij, 1} = se^{i\rho}, \ d_{ij, 2} = se^{-i\rho}.$

Substituting (30) into (29) gives

(31) $\qquad sr^k(e^{i(\theta k + \rho)} + e^{-i(\theta k + \rho)}) = 2sr^k \cos (\theta k + \rho).$

Thus, with the absolute value r of the roots less than one, the contribution (31) to $\gamma_{ij, k}$ is a damped cosine function of k, with frequency θ. In the nonstochastic model, the complex roots (28) would generate a component of the time path which is a damped cosine function of t, with the same frequency θ, but the phase would in general be different from ρ.

Damped oscillations in (a component of) the time path in the nonstochastic model, due to a pair of complex roots, will necessarily be associated with damped oscillations in (a component of) the autocovariance function $\gamma_{ii, k}$. It is in this sense that the former oscillations could be maintained in the stochastic model. The interesting questions are whether damped oscillations in the deterministic model are necessary, and sufficient, to generate distinct "cycles" in the stochastic time series in the sense of their spectral densities. We will turn to this subject in the next section.

V. SPECTRAL DENSITIES OF A LINEAR STOCHASTIC SYSTEM [5]

There are certainly many interesting aspects of business cycles, but this paper is confined mainly to their periodicities. For this purpose, the spectral density is a useful tool. Before going into the periodicities in the relations among time series, it is well to begin with the periodicities of an individual time seres y_{it}. The spectral density of a discrete time series y_{it} is a weighted sum of its autocovariance function (omitting a normalization constant),

$$
(32) \qquad f_{ii} (\omega) = \sum_{k = -\infty}^{\infty} \gamma_{ii, k} \cos \omega k
$$

where the weights $\cos \omega k$ are periodic, with frequency ω (or with period length $2\pi/\omega$). Intuitively speaking, if the autocovariance function $\gamma_{ii, k}$ happens to be approximately periodic with frequency θ (or period length $2\pi/\theta$), then weighing it by a periodic function $\cos \theta k$ with the same frequency will give a large sum, i.e., $f_{ii}(\omega)$ will be large for $\omega = \theta$. For a discrete time series, the shortest cycle that can be detected is of length 2, i.e., with frequency π. Thus, the range of ω for the spectral density $f_{ii} (\omega)$ is from 0 to π, indicating the strength of the very long cycles and two-period cycles respectively.

5. For a discussion of spectral densities, see, for example, D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes* (New York: Wiley, 1965), Chap. 8, and P. Whittle, *Prediction and Regulation by Linear Least-Square Methods*, (New York: D. Van Nostrand, 1963), Chap. 2.

The periodic weighing function $\cos \omega k$ is appropriate only for the autocovariance function $\gamma_{ii,\, k}$ since both have a maximum at $k = 0$, and there is no question of being out of phase. For a cross-covariance function $\gamma_{ij,\, k}$, which may not have a maximum at $k = 0$, the phase problem in the weighing function arises, and the function

(33) $\cos(\omega k + \psi) = (\cos \psi) \cos \omega k - (\sin \psi) \sin \omega k$

for some phase ψ may be appropriate. This suggests using the weighing function

(34) $e^{-i\omega k} = \cos \omega k - i \sin \omega k$

which contains the two components $\cos \omega k$ and $\sin \omega k$. For the auto-covariance function $\gamma_{ii,\, k} = \gamma_{ii,\, -k}$,

(35) $\displaystyle\sum_{k=-\infty}^{\infty} \gamma_{ii,\, k}\, e^{-i\omega k} = \sum_{k=-\infty}^{\infty} \gamma_{ii,\, k} \cos \omega k = f_{ii}(\omega)$

since $\sin(-\omega k) = -\sin \omega k$. Using $e^{-i\omega k}$ rather than $\cos \omega k$ as the weighing function will also simplify algebra.

The spectral density matrix is defined as

(36) $F(\omega) = \displaystyle\sum_{k=-\infty}^{\infty} \Gamma_k\, e^{-i\omega k}$

where the diagonal elements are the spectral densities of the individual time series, and the off-diagonal elements are the cross-spectral densities. Although the cross-spectral densities will also be derived, our discussion will be mainly confined to the spectral densities. To evaluate (36), we will first evaluate the spectral density matrix $F^*(\omega)$ of the canonical variables which is related to $F(\omega)$ by

(37) $F(\omega) = \displaystyle\sum_{k=-\infty}^{\infty} B\Gamma_k^*B'e^{-i\omega k} = B \sum_{k=-\infty}^{\infty} \Gamma_k^* e^{-i\omega k}B'$

                $= BF^*(\omega)B'$

in view of the relation (26).

By (24) and (25), the spectral density matrix of the canonical variables is

(38) $F^*(\omega) = \displaystyle\sum_{k=0}^{\infty} \Gamma_k^* e^{-i\omega k} + \sum_{k=0}^{\infty} \Gamma_{-k}^* e^{i\omega k} - \Gamma_0^*$

                $= \displaystyle\sum_{k=0}^{\infty} D_\lambda^k \Gamma_0^* e^{-i\omega k} + \sum_{k=0}^{\infty} \Gamma_0^* D_\lambda^k e^{i\omega k} - \Gamma_0^*$

                $= \left[\left(\dfrac{1}{1 - \lambda_i e^{-i\omega}} + \dfrac{1}{1 - \lambda_j e^{i\omega}} - 1 \right) \gamma^*_{ij,\, 0} \right]$

                $= \left[\dfrac{1 - \lambda_i \lambda_j}{(1 - \lambda_i e^{-i\omega})\ (1 - \lambda_j e^{i\omega})} \cdot \dfrac{w_{ij}}{1 - \lambda_i \lambda_j} \right]$

                $= \left[\dfrac{w_{ij}}{(1 - \lambda_i e^{-i\omega})\ (1 - \lambda_j e^{i\omega})} \right].$

With $F^*(\omega)$ in (37) so substituted, the spectral density matrix of the original variables is

$$(39) \qquad F(\omega) = B \left[\frac{w_{ij}}{(1 - \lambda_i e^{-i\omega})(1 - \lambda_j e^{i\omega})} \right] B' .$$

While the result (39) gives the entire spectral density matrix, the diagonal elements are of special interest and can be expressed in an alternative way. Let b_m be the m-th row of B.

$$(40) \qquad \begin{aligned} f_{mm}(\omega) &= b_m F^*(\omega) b'_m \\ &= \tfrac{1}{2} b_m [f^*_{ij}(\omega) + f^*_{ji}(\omega)] b'_m . \end{aligned}$$

Using the expression after the third equality sign in (38) for $f^*_{ij}(\omega)$,

$$(41) \qquad [f^*_{ij}(\omega) + f^*_{ji}(\omega)]$$

$$= \left[\left(\frac{1}{1 - \lambda_i e^{-i\omega}} + \frac{1}{1 - \lambda_j e^{i\omega}} \right. \right.$$

$$\left. + \frac{1}{1 - \lambda_j e^{-i\omega}} + \frac{1}{1 - \lambda_i e^{i\omega}} - 2 \right) \gamma^*_{ij,\,0} \right]$$

$$= \left[\frac{1 - \lambda_i^2}{(1 - \lambda_i e^{-i\omega})(1 - \lambda_i e^{i\omega})} \gamma^*_{ij,\,0} \right.$$

$$\left. + \frac{1 - \lambda_j^2}{(1 - \lambda_j e^{-i\omega})(1 - \lambda_j e^{i\omega})} \gamma^*_{ij,\,0} \right].$$

Define

$$(42) \qquad g_i = \frac{1 - \lambda_i^2}{(1 - \lambda_i e^{-i\omega})(1 - \lambda_i e^{i\omega})} = \frac{1 - \lambda_i^2}{1 + \lambda_i^2 - 2\lambda_i \cos \omega}$$

and let D_g be the diagonal matrix with g_i as its i-th diagonal element. Then (41) becomes

$$(43) \qquad [f^*_{ij}(\omega) + f^*_{ji}(\omega)] = [g_i \gamma^*_{ij,\,0} + g_j \gamma^*_{ij,\,0}]$$

and the spectral density (40) can be written as

$$(44) \qquad \begin{aligned} f_{mm}(\omega) &= \tfrac{1}{2} b_m [g_i \gamma^*_{ij,\,0} + g_j \gamma^*_{ij,\,0}] b'_m \\ &= \tfrac{1}{2} b_m D_g \Gamma^*_0 b'_m + \tfrac{1}{2} b_m \Gamma^*_0 D_g b'_m \\ &= b_m D_g \Gamma^*_0 b'_m = \sum_i d_{mm,\,i} g_i. \end{aligned}$$

Observe that (44) is a linear combination of g_i, and that, by (43), g_i is proportional to the spectral density

$$(45) \qquad f^*_{ii}(\omega) = \gamma^*_{ii,\,0} \, g_i$$

of the i-th canonical variable. Therefore, $f_{mm}(\omega)$ is a linear combination of $f_{ii}^*(\omega)$, $i = 1, \ldots, p$.

There is a striking resemblance between (44) and the mth diagonal element of (26), i.e.,

$$(46) \qquad y_{mm,\,k} = b_m D_\lambda^k \Gamma^*_0 b'_m = \sum_i d_{mm,\,i} \lambda_i^k .$$

g_i in the former replaces λ_i^k in the latter. Thus, whereas the auto-covariance function is a linear combination of λ_i^k, the spectral density function is the same linear combination of g_i. This result is expected because we could have obtained $f_{mm}(\omega)$ directly as a

weighted sum (over k) of (46); the weighted summation could be performed on each λ_i^k,

$$(47) \quad \sum_{k=0}^{\infty} \lambda_i^k e^{-i\omega k} + \sum_{k=0}^{\infty} \lambda_i^k e^{i\omega k} - 1$$

$$= \frac{1}{1 - \lambda_i e^{-i\omega}} + \frac{1}{1 - \lambda_i e^{i\omega}} - 1$$

yielding g_i.

Once the spectral density has been expressed in terms of the roots, it is convenient to discuss the relationships between prolonged oscillations in the deterministic model (generated by complex roots) and distinct cycles in the stochastic model (defined by a peak of the spectral density function). Are complex roots necessary or sufficient for generating a peak in the spectral density?

To answer this question, the well-known Hansen-Samuelson multiplier-accelerator model will first be used as an example.[6] The second-order stochastic difference equation is written as a first-order system:

$$(48) \quad \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{1,\,t-1} \\ y_{2,\,t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ 0 \end{bmatrix}.$$

Since, for this system, y_{2t} is simply $y_{1,\,t-1}$, we have

$$(49) \quad \gamma_{11,\,k} = \gamma_{22,\,k} = \gamma_{12,\,k-1} = \gamma_{21,\,k+1}$$

which, when substituted into the homogenous system (17), gives

$$(50)$$
$$\begin{bmatrix} \gamma_{12,\,k-1} & \gamma_{12,\,k} \\ \gamma_{11,\,k-1} & \gamma_{11,\,k} \end{bmatrix} = A \begin{bmatrix} \gamma_{12,\,k-2} & \gamma_{12,\,k-1} \\ \gamma_{11,\,k-2} & \gamma_{11,\,k-1} \end{bmatrix} \quad (k = 1, 2, 3, \ldots)$$

(50) implies that, for $k = 1$,

$$(51) \quad \begin{bmatrix} \gamma_{12,\,0} \\ \gamma_{11,\,0} \end{bmatrix} = A \begin{bmatrix} \gamma_{12,\,-1} \\ \gamma_{11,\,-1} \end{bmatrix}.$$

The solution for $\gamma_{11,\,k}$ is, by (27),

$$(52) \quad \gamma_{11,\,k} = d_{11,\,1}\lambda_1^k + d_{11,\,2}\lambda_2^k \qquad (k = 0, 1, 2, \ldots)$$

which, in view of (51), is also valid for $k = -1$. Therefore,

$$(53) \quad \gamma_{11,\,-1} = d_{11,\,1}\lambda_1^{-1} + d_{11,\,2}\lambda_2^{-1} = \gamma_{11,\,1} = d_{11,\,1}\lambda_1 + d_{11,\,2}\lambda_2.$$

The condition (53) will be used to simplify the spectral density function.

As given by equation (44), the spectral density function is

$$(54) \quad f_{11}(\omega) = d_{11,\,1}g_1 + d_{11,\,2}g_2$$

$$= \frac{d_{11,\,1}(1 - \lambda_1^2)}{1 + \lambda_1^2 - 2\lambda_1 \cos \omega} + \frac{d_{11,\,2}(1 - \lambda_2^2)}{1 + \lambda_2^2 - 2\lambda_2 \cos \omega}$$

6. Samuelson, *op. cit.*

$$= \frac{d_{11,\,1}(1 - \lambda_1{}^2)\,(1 + \lambda_2{}^2) + d_{11,\,2}(1 - \lambda_2{}^2)\,(1 + \lambda_1{}^2)}{(1 + \lambda_1{}^2 - 2\lambda_1 \cos \omega)\,(1 + \lambda_2{}^2 - 2\lambda_2 \cos \omega)}.$$

Note that the numerator of the last expression is not a function of $\cos \omega$, as a consequence of (53). To find a maximum of (54), we need only to find a minimum of the denominator.[7] The first derivative of the denominator is

(55) $\qquad 2[\,(1 + \lambda_1{}^2)\lambda_2 + (1 + \lambda_2{}^2)\lambda_1]\,\sin \omega - 8\lambda_1\lambda_2 \cos \omega \sin \omega.$

For $0 < \omega < \pi$, $\sin \omega$ is positive. Hence, the derivative (55) will be zero if and only if

(56) $\qquad (1 + \lambda_1{}^2)\lambda_2 + (1 + \lambda_2{}^2)\lambda_1 - 4\lambda_1\lambda_2 \cos \omega = 0.$

When (56) holds, the second derivative of the denominator is

(57) $\qquad 2 \sin \omega\,[\,(1 + \lambda_1{}^2)\lambda_2 + (1 + \lambda_2{}^2)\lambda_1 + 4\lambda_1\lambda_2 \sin \omega\,]\,.$

Let λ_1 and λ_2 be conjugate complex, as given by (28). Substitution into (56) gives

(58) $\qquad 2r(1 + r^2) \cos \theta - 4r^2 \cos \omega = 0$

or

$$\cos \omega = \frac{1 + r^2}{2r} \cos \theta\,.$$

When (58) holds, the second derivative (57) will be positive if and only if $(\cos \omega + \sin \omega)$ is positive, i.e., $\omega < \tfrac{3}{4}\pi$, guaranteeing a maximum in the spectral density. Since $-1 < \cos \omega < 1$, the necessary condition (58) for a maximum in the spectral density is

(59) $\qquad -1 < \dfrac{1 + r^2}{2r} \cos \theta < 1.$

If the absolute value r of the roots is close to one, so that the factor $(1 + r^2)/2r$ is also close to one, the condition (59) will be satisfied; in fact, by (58) the frequency ω that corresponds to a peak in the spectral density will be close to the frequency θ of the roots. On the other hand, if r is small, so that $(1 + r^2)/2r$ is large, the condition (59) can be met only if $\cos \theta$ is close to zero, or the frequency θ is close to $\pi/2$, which corresponds to cycles of approximately 4 periods. For example, if $\theta = 54°$, corresponding to cycles

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7. The numerator is always nonnegative since the spectral density in general is nonnegative, as the quadratic form $b_m D_g \Gamma^*{}_0 b'_m$ in (44) can be shown to be positive semidefinite, by writing
$$\tfrac{1}{2}(D_g\Gamma^*{}_0 + \Gamma^*{}_0 D_g) = D_p W D_p + D_q W D_q$$
where the diagonal matrices D_p and D_q consist respectively of
$$p_i = \frac{1 - \lambda_i \cos \omega}{(1 + \lambda_i{}^2 - 2\lambda_i \cos \omega)}$$
and
$$q_i = \frac{\lambda_i \sin \omega}{(1 + \lambda_i{}^2 - 2\lambda_i \cos \omega)}$$
and where W is positive semidefinite.

of approximately 6.67 periods, cos θ is .5878. An absolute value r of .3 will give a factor of 1.817, violating the necessary condition (59). An absolute value of .4 will give cos $\omega = 1.450(.5878) =$ .8523, or $\omega = 31.5°$, corresponding to cycles of 11.4 periods, much longer than the 6.67-period cycle of the deterministic model.

If the roots are real and positive in the second-order equation, the necessary condition (56) cannot be satisfied, since

$$(60) \qquad \cos \omega = \frac{(1 + \lambda_1{}^2)\lambda_2 + (1 + \lambda_2{}^2)\lambda_1}{4\lambda_1\lambda_2} > 1$$

or

$$(1 + \lambda_1{}^2)\lambda_2 + (1 + \lambda_2{}^2)\lambda_1 - 4\lambda_1\lambda_2 > 0.$$

The last inequality is equivalent to

$$(61) \qquad \lambda_1(1 - \lambda_2)^2 + \lambda_2(1 - \lambda_1)^2 > 0$$

which certainly holds under our assumption.

While real and positive roots alone in the second-order equation (48) cannot generate distinct cycles in the spectral sense, it is important to emphasize that they can indeed do so in the general linear model (13), even when the matrix A is of dimension two. To demonstrate this point, let us examine the derivative of (44), which is a linear combination of the derivatives of g_i

$$(62) \qquad \frac{dg_i}{d\omega} = \frac{-(2 \sin \omega)\lambda_i(1 - \lambda_i{}^2)}{(1 + \lambda_i{}^2 - 2\lambda_i \cos \omega)^2} = -(2 \sin \omega) h_i$$

with h_i defined as

$$(63) \qquad h_i = \frac{\lambda_i(1 - \lambda_i{}^2)}{(1 + \lambda_i{}^2 - 2\lambda_i \cos \omega)^2} \qquad (i = 1, \ldots, p).$$

From (44) and (62), denoting by D_h the diagonal matrix consisting of h_i,

$$(64) \qquad \frac{df_{mm}(\omega)}{d\omega} = (-2 \sin \omega) b_m D_h \Gamma_0{}^* b'_m.$$

As sin ω is positive for the relevant range $0 < \omega < \pi$, the derivative (64) is opposite in sign to the quadratic form

$$(65) \qquad b_m D_h \Gamma_0{}^* b'_m.$$

If this quadratic form is definite, the spectral density will have no maximum for $0 < \omega < \pi$. Otherwise, we can choose a vector b_m to make the form (65) vanish. The choice of B is independent of the choice of $W = B^{-1}VB^{-1'}$ since V can always be specified as BWB'.

The fact that all roots are real and positive does not guarantee the form (65) to be positive definite. As an example, let

$$(66) \qquad \begin{array}{ll} \lambda_1 = .1 & \lambda_2 = .9 \\ w_{11} = w_{22} = 1 = b_{m1} \\ w_{12} = .8 & b_{m2} = -.01 \,. \end{array}$$

The form (65) will not be positive definite, and the spectral density will be, by (44),

$$(67) \qquad b_m D_g \left(\frac{w_{ij}}{1 - \lambda_i \lambda_j} \right) b'_m = \frac{.9913}{1.01 - .2 \cos \omega} - \frac{.001570}{1.81 - 1.8 \cos \omega} .$$

Selected values of the density (67) are:

ω	0	$\dfrac{\pi}{16}$	$\dfrac{\pi}{8}$	$\dfrac{2\pi}{8}$	$\dfrac{3\pi}{8}$	$\dfrac{4\pi}{8}$	$\dfrac{5\pi}{8}$	$\dfrac{6\pi}{8}$	$\dfrac{7\pi}{8}$	π
$f_{mm}(\omega)$	1.067	1.183	1.191	1.138	1.061	.981	.912	.860	.829	.819

The peak (at ω slightly below $.1\pi$) is quite pronounced.

This example reminds us of the well-known result of Slutsky [8] that taking certain moving averages of a random series can generate cycles. Our model

$$(68) \qquad y_t = u_t + A u_{t-1} + A^2 u_{t-2} + \cdots$$

amounts to taking an infinite moving average of the random vectors u_t with "geometrically declining" weights, A^0, A^1, A^2, ..., in matrix terms. Although there can be no distinct cycles if A is a scalar between zero and one, the situation is different for a matrix A with roots between zero and one. To put it differently, when neither of two (canonical) variables

$$z_{1t} = \lambda_1 z_{1,\,t-1} + \epsilon_{1t}, \; z_{2,\,t} = \lambda_2 z_{2,\,t-1} + \epsilon_{2,\,t}$$

has distinct cycles (with $\lambda_1 = .1$ and $\lambda_2 = .9$, say), a linear combination, say $z_{1t} - .01 z_{2t}$, can have a peak in its spectral density.

VI. Conclusions

Before concluding, I would like to comment briefly on the case when a root, say λ_1, is greater than one. For this case, Quenouille [9] has claimed that (A) any univariate time series

$$y_t = \lambda_1 y_{t-1} + \epsilon_t \qquad\qquad (\lambda_1 > 1)$$

can be decomposed into an exponential trend plus a stationary first-order series, that (B) the trend and the stationary series are respectively $x_t = \lambda_1 x_{t-1}$ and

$$y^*_t = \lambda_1^{-1} y^*_{t-1} + \eta_t$$

and that (C), in general, any nonstationary autoregressive process with real roots may be partitioned in a similar manner, where the remaining stationary scheme is determined by his discussion on identification. If assertions (A) and (B), or even (A) alone, are

8. See, for example, M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics* (London: Charles Griffin, 1966), Vol. 3, pp. 378 and 414–15.
9. Quenouille, *op. cit*, pp. 57–58, and Sec. 3. 2.

accepted, the trend in a nonstationary canonical variable can be eliminated, and the resulting system can be treated as stationary even if some roots are complex. This subject is discussed in another paper.[1]

This paper has begun with a formulation of the acceleration principle. In a deterministic model consisting only of demand equations which obey simple distributed lags, the acceleration equation is necessary for prolonged oscillations. However, when random shocks are introduced, prolonged oscillations in the deterministic system (complex roots) will be neither necessary nor sufficient for generating distinct cycles — although, in the Hansen-Samuelson multiplier-accelerator model, complex roots are necessary, and, if their absolute value is large, sufficient. An obvious moral is that the nature of business cycles can be understood only by an integrated view of the deterministic as well as the random elements.

As far as the spectral densities of economic time series are concerned, Granger has reported on a "typical spectral shape," i.e., $f(\omega)$ as a monotone decreasing function of ω, and has suggested a set of first-order autoregressive equations, e.g.,

$$y_t = a y_{t-1} + \epsilon_t$$

to explain this shape.[2] In view of the possibilities that economic models may contain complex roots, and that, even with only real and positive roots, they may generate distinct cycles, is one willing to accept this shape as a good approximation? Only more empirical work will tell. But the approach to ascertaining the spectral shape is perhaps not to do more spectral analysis (which applies little or no restriction on the density), but to analyze the spectral densities implicit in econometric models.[3]

1. G. C. Chow and R. E. Levitan, "Spectral Properties of Non-stationary Systems of Linear Stochastic Difference Equations," IBM Research Report, RC–2059, April 1968.

2. C. W. J. Granger, "The Typical Spectral Shape of an Economic Variable," *Econometrica*, Vol. 34 (Jan. 1966), pp. 150–61.

3. See G. C. Chow and R. E. Levitan, "Nature of Business Cycles Implicit in a Linear Economic Model," IBM Research Report, RC–2085, May 1968, which also deals with cross-spectral densities.

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