MONEY AS A MEDIUM OF EXCHANGE IN AN ECONOMY WITH ARTIFICIALLY INTELLIGENT AGENTS*

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We study the exchange economies of Kiyotaki and Wright (1989) in which agents must use a commodity or fiat money as a medium of exchange if trade is to occur. Our agents are artificially intelligent and are modeled as using classifier systems to make decisions. In the assignment of credit within the classifier systems, we introduce some innovations designed to study sequential decision problems in a multi-agent environment. For most economies that we have simulated, trading and consumption patterns converge to a stationary Nash equilibrium even if agents start with random rules. In economies with multiple equilibria, the only equilibrium that emerges in our simulations is the one in which goods with low storage costs play the role of medium of exchange (i.e., the 'fundamental equilibrium' of Kiyotaki and Wright).

1. Introduction

Kiyotaki and Wright (1989) studied how three classes of rational agents would interact within a Nash–Markov equilibrium in a world of repeated 'Wicksell triangles'. This paper studies how a collection of artificially intelligent agents would learn to coordinate their activities if they were thrust into the economic environment described by Kiyotaki and Wright. Our agents'

behavior is determined by Holland (1975) classifier systems.[1] Thus, while Kiyotaki and Wright studied stationary equilibria in which beliefs about 'media of exchange' are *consistent* with trading patterns, we study economies in which particular commodities *emerge* as media of exchange. We also study an economy (Economy C) in which a good from which no agent derives utility emerges as fiat money. We want to learn whether our artificially intelligent agents can learn to play a Markovian Nash equilibrium of the Kiyotaki–Wright model. When there are multiple Nash equilibria (e.g., the 'fundamental' and 'speculative' equilibria of Kiyotaki and Wright), we want to know whether the system might converge to some but not others of these equilibria. In addition, if classifier systems do converge to Nash–Markov equilibria, they might be used to compute equilibria in other economies for which it is difficult to obtain analytic solutions. To this end, we also study enlarged versions of the Kiyotaki–Wright model.

A *classifier system* is a collection of potential decision rules, together with an accounting system for selecting which rules to use. The accounting system credits rules generating good outcomes and debits rules generating bad outcomes. The system is designed to select a 'co-adaptive' set of rules that work well over the range of situations that an agent typically encounters. In a multiple-agent environment like Kiyotaki and Wright's, the range of situations encountered by one agent depends on the actions taken by other agents. This means that the collections of rules used by the collection agents must co-adapt jointly.

We study two sorts of classifier systems, with the distinction between them being induced by the fact that for many problems enumerating all possible rules would require a very long list. The first kind of classifier system is one in which a complete enumeration of all possible rules *is* carried along. For many problems, it is not feasible to use a complete enumeration classifier system because the state and action spaces are so large that the set of all possible rules is much too big. For us, a complete enumeration is tractable because the Kiyotaki–Wright model has low-dimensional state and action spaces.

The second kind of classifier system is designed for situations in which it is not efficient to carry along a complete enumeration of decision rules. In this case, a modified version of the *genetic algorithm* of Holland is used as a device

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[1]The way in which we model agents as searching for rewarding decision rules attributes much less knowledge and rationality to them than is typical in the literature on least-squares learning in the context of linear rational-expectations models [see Bray (1982), Bray and Savin (1986), Fourgeaud, Gourieroux, and Pradel (1986), and Marcet and Sargent (1989)]. In the least-squares learning literature, agents are posited to be almost fully rational and knowledgeable about the system they operate within, lacking knowledge only about particular parameters in laws of motion of a set of variables exogenous to their own decisions, which they learn about through the recursive application of least squares. But given their estimates, agents are supposed to compute optimal decision rules by using dynamic programming or the calculus of variations.

for periodically eliminating some rules and injecting new rules into the population of rules to be operated on by the accounting system. The genetic-algorithm version of the classifier models learning in the face of unforeseen contingencies. When an unprecedented state arises which is not covered by the existing set of rules, the system contains a procedure for manufacturing and experimenting with new rules that apply in the new situation. Even though it is feasible for us to work with a complete enumeration of rules, we also study the genetic-algorithm version of the classifier.[2]

In this paper, we set up classifier systems for the Kiyotaki–Wright environment and simulate them on a computer. We also formulate definitions of equilibrium and stability for classifier systems, and pose some convergence questions. Although we suggest possible convergence theorems, we prove no theorems in this paper. Section 2 describes the Kiyotaki–Wright environment. Section 3 describes the classifier systems without genetics. Section 4 describes behavior in the stationary equilibria of Kiyotaki and Wright and also the types of classifier systems that could support that behavior. Section 5 defines stability criteria for sets of classifier systems, and discusses how these definitions can be used to formulate questions about whether systems of classifier systems converge to a stationary equilibrium. Section 6 describes classifier systems operating with genetics. Section 7 describes our simulations of systems of classifier systems. In addition to the economies studied by Kiyotaki and Wright (including one with fiat money), we study an economy with five types of agents and five goods. Section 8 concludes the paper.

2. The Kiyotaki–Wright environment

There are three types of agents, with types being indexed by $i = 1, 2, 3$. Type i agents get utility only from consuming type i good. Type i agent has access to a technology for producting type i^* good, where $i^* \neq i$. We initially specify

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[2] When the state and action spaces are small, as in the Kiyotaki–Wright model, the list of all possible rules mapping states into actions is not too long. In such a situation, one can follow Axelrod (1986) and model an agent's strategy as a single binary bit string. A pure genetic algorithm can then be applied, as in Axelrod (1986). This approach has been applied to the Kiyotaki–Wright model by Marimon and Miller (1989) and by Knez and Litterman (1989). While much easier to implement than classifier systems, this pure genetic approach also has some drawbacks. First, it is difficult to extend to situations involving 'unforeseen contingencies' and in which it is not practical to enumerate all of the histories that might be encountered in the course of play. Second, relative to the classifier, the pure genetic algorithm encodes information wastefully in the sense that an entire strategy is encoded, while the experience of play will typically throw up observations only on a small subset of the possible states. While the classifier system concentrates 'effort' on learning rules to use in frequently encountered states, the pure genetic algorithm purports to be learning about entire strategies. Third, classifiers can learn general rules that apply to several states and that offer potential informational economies.

(i, i^*) according to Kiyotaki–Wright's 'model A', namely, as follows:

i	1	2	3
i^*	2	3	1

This specification assumes no 'double coincidence of wants' and seems to call for a multilateral trading arrangement. All goods are indivisible. Each agent can store one and only one unit of only one good from one period to the next. When an agent of type i consumes good i at time t, he immediately produces good i^*, which he carries over to the next period. The net utility to an agent of type i of consuming good y and producing good i^* is given by $u_i(y)$. We assume that an agent of type i does not know his utility *function*, but does recognize utility when he experiences it. The goods are costly to store. Storing good k $(k = 1, 2, 3)$ from t to $t + 1$ imposes costs at t of s_k. Following Kiyotaki and Wright, we assume that $s_3 > s_2 > s_1 > 0$. We summarize this cost function by saying that $s(y)$ is the one-period cost of storing one unit of good y. We assume that individuals do not know this cost function, but that they do recognize costs when they bear them.

The economy and each agent within it live forever. There are large and equal numbers of agents of each type. We modify Kiyotaki and Wright's model by assuming that each agent cares about his long-run average level of utility. (Kiyotaki and Wright assumed that agents have preferences ordered by expected discounted future utilities.) Each period, there is a random matching process that assigns each and every agent in the economy to a pair with one and only one other agent in the economy. The random matching technology matches agents without regard to type. Only pairs of agents matched together can trade at a point in time.

The economy begins at $t = 0$ with agents being endowed with an arbitrary and randomly generated initial distribution of holdings of goods. At each date $t \geq 0$, each agent i has to make two decisions sequentially. First, given the good he is holding and given the good held by the partner with whom he is matched at t, he must decide whether to propose to trade. Trade occurs only if both parties propose to do so. Second, the agent must decide whether or not to consume the good with which he exits the trading process. If he doesn't consume, he simply carries the good into period $t + 1$, experiencing cost s_k. If an agent of type i does consume good y_t at t, he experiences net utility $u_i(y_t)$, produces good i^*, and experiences carrying costs s_{i^*}. We assume that $u_i(k) = 0$ if $k \neq i$ and $u_i(i) = u_i > 0$. We follow Kiyotaki and Wright both in adopting their specification of the physical environment and in considering only Markov strategies, but we drop the assumption of rational agents. Instead, our agents are assumed to be 'artificially intelligent', in the sense that they use versions of the *classifier system* introduced by Holland (1986).

Before describing the classifier systems used by our agents, we introduce some notation for naming agents and for describing the state of each individual agent in the economy. There is a collection of agents $\mathscr{A} = \{1, 2, \ldots, A\}$. A typical element of $\mathscr{A}$ is denoted a. Kiyotaki and Wright assumed that there was a continuum of each type of agent, while we assume that there is a finite number of each type. The first A_1 agents are of type I, the next A_1 are of type II, and the following A_1 are of type III. Here $A = 3A_1$.

At the beginning of time t, agent a is carrying good x_{at}. The variable x_{at} characterizes the pre-match state of agent a at time t. There is a random matching process which each period matches each agent $a \in \mathscr{A}$ with a distinct agent $\rho_t(a) \in \mathscr{A}$. For each agent in each period, the matching process induces a function $\rho_t(a): \mathscr{A} \to \mathscr{A}$. *After* the matching process, the pre-trade state of agent a is $(x_{at}, x_{\rho_t(a)t})$. The pair $(x_{at}, x_{\rho_t(a)t}) \equiv z_{at}$ records what agent a is carrying and what the agent $\rho_t(a)$ with whom a is matched at t is carrying.

At t, after being matched with agent $\rho_t(a)$, agent a decides whether or not to propose to trade. We let λ_{at} denote the trading decision of agent a at time t, where

$$\lambda_{at} = \begin{cases} 1 & \text{if} \quad a \text{ proposes to trade } x_{at} \text{ for } x_{\rho_t(a)t}, \\ 0 & \text{if} \quad a \text{ refuses to trade.} \end{cases} \tag{1}$$

Similarly, $\lambda_{\rho_t(a)t}$ summarizes the trading decision of the agent $\rho_t(a)$ with whom a is paired at t. Trade takes place if and only if $\lambda_{at} \cdot \lambda_{\rho_t(a)t} = 1$.

Let x_{at}^+ denote the post-trade (but pre-consuming decision) holdings of agent a at t. We then have that

$$x_{at}^+ = \left(1 - \lambda_{at} \cdot \lambda_{\rho_t(a)t}\right) x_{at} + \lambda_{at} \lambda_{\rho_t(a)t} x_{\rho_t(a)t}. \tag{2}$$

After leaving the trading process with holdings x_{at}^+, agent a must decide whether to consume x_{at}^+ or to carry it into the next period. We let γ_{at} denote the consumption decision of agent a at t where

$$\gamma_{at} = \begin{cases} 1 & \text{if} \quad a \text{ decides to consume } x_{at}^+, \\ 0 & \text{if} \quad a \text{ decides not to consume.} \end{cases} \tag{3}$$

If agent a decides to consume, he automatically produces good $f(a)$, which he carries into $(t + 1)$. From the specification of i^* as a function of i described above for Kiyotaki and Wright's model A, we have that $f(a)$ is a good of type 2 if a is a type I agent, $f(a)$ is a good of type 3 if a is a type II agent, and $f(a)$ is a good of type 1 if a is a type III agent. It follows that beginning-of-

period holdings of a at time $t + 1$ are described by[3]

$$x_{at+1} = \gamma_{at} f(a) + (1 - \gamma_{at})\left((1 - \lambda_{at} \cdot \lambda_{\rho_t(a)t})x_{at} + \lambda_{at} \cdot \lambda_{\rho_t(a)t} x_{\rho_t(a)t}\right).$$

$$(4)$$

3. Classifier systems for the Kiyotaki–Wright environment

We now describe the *classifier systems* that agents use to make their trading and consumption decisions.[4] Agents sequentially use two interconnected classifier systems each period. A first trading classifier system receives input in the form of the pre-trade state $z_{at} = (x_{at}, x_{\rho_t(a)t})$. This classifier system determines the trading decision λ_{at}, which interacts with the trading decision $\lambda_{\rho_t(a)t}$ of the agent $\rho_t(a)$ with whom a is paired at t to determine x_{at}^+ [see eq. (2)]. A second consumption classifier system takes x_{at}^+ as input and determines the consumption decision γ_{at}. The two classifier systems are used sequentially in this way each period, and their accounting systems are interconnected in ways to be described below.

A *classifier system* consists of the following objects:

(i)   A collection of trinary strings, called 'classifiers'.
(ii)   A system for interpreting or decoding the strings or classifiers as instructions mapping states into decisions. The first part of a string encodes a particular state or *condition*, while the second part encodes a particular action. Thus, an individual classifier or string is just an encoding of a single (state, action) pair. For a given state, there can be many classifiers present in a classifier system.
(iii) A list of 'strengths' attached to each classifier at each point in time $t = 0, 1, \ldots$.

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[3]A pure genetic algorithm could be applied to the trading decision as follows. For each agent, there are $3 \times 3 = 9$ possible values that his current state prior to trade can attain. There are two possible actions, trade or don't trade. Letting 1 mean trade and 0 mean don't trade, a binary string of length 9 can be used to encode a Markovian strategy. This is done by numbering the set of possible states $j = 1, \ldots, 9$ and then letting the jth entry in the bit string denote the decision to be made when the jth state is encountered. The genetic algorithm is then applied to select the best binary string (i.e., the best strategy). This approach, which has been applied to the Kiyotaki–Wright model [see Marimon and Miller (1989) and Knez and Litterman (1989)], has shortcomings as a starting point for studying more complex systems. Because the optimization is done over entire strategies, which include descriptions of decisions to be made in *all* possible states, even rarely visited ones, the encoding of information is wasteful and does not facilitate more intensive learning about frequently visited states.

[4]See Goldberg (1989) for a description of classifier systems and a survey of some of their applications.

Table 1

Encoding of goods in classifiers.

Code			Meaning
1	0	0	Good 1
0	1	0	Good 2
0	0	1	Good 3
0	#	#	Not good 1
#	0	#	Not good 2
#	#	0	Not good 3

(iv) A system for reading in external messages or 'states' and for determining the set of classifiers that are pertinent, i.e., the set of classifiers whose 'conditions' are satisfied or 'matched' at that state. For a given state, the system can sometimes contain several classifiers with distinct actions.

(v) An 'auction system' for determining which of the pertinent classifiers is allowed to make a decision at t. Among all classifiers whose condition part matches the actual state at t, the 'highest bidder' in the auction actually 'makes the decision' in real time.[5]

(vi) An accounting system for updating the strengths of the collection of classifiers in response to the net rewards that flow into the system as a result of the decisions that are made.

and possibly:

(vii) A genetic algorithm for introducing new strings and extinguishing old strings. This algorithm will be called whenever states are encountered for which no existing classifiers have their conditions met. The algorithm may also be applied from time to time as a device to promote experimentation.

We now describe each of these elements as applied to the Kiyotaki–Wright environment. We first consider an exchange classifier system, which maps the pre-trade state z_{at} into a trading decision. This classifier system consists of a list of trinary strings of length 7. The first three digits represent own storage, the next three represent trading partner's storage, and the seventh represents the trading decision. Table 1 displays the encoding of the goods stored. The coding is written in the trinary alphabet $(1, 0, \#)$, where $\#$ means 'don't care'. For the trading decision, the code is in the binary alphabet $(1, 0)$, where 1 means trade, while 0 means don't trade. To illustrate how the codes in table 1

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[5]This rule could easily be modified to permit a 'stochastic auction' in which the probability of winning the auction varies directly with strength.

are applied to encode particular trading classifiers, we consider the following two classifiers:

Own storage			Partner storage			Trading decision
1	0	0	0	0	1	1
1	0	0	#	#	0	0

The first classifier instructs an agent who is carrying good 1 and who is matched with someone carrying good 3 to offer to trade. The second classifier instructs an agent who is carrying good 1 not to trade if he is matched with someone who is not carrying good 3.

The exchange classifier system of agent a consists of a list of such classifiers. Let $e = 1, 2, \ldots, E_a$ index this collection of classifiers. A given classifier system has a fixed number of classifiers. For the exchange classifier system, how many distinct classifiers are possible? Evidently, $3^3 \times 3^3 \times 2 = 1458$. However, most of these classifiers are redundant in the sense that only a subset of them is needed to represent all possible trading decision rules defined on the state space $z_{at} = (x_{at}, x_{\rho_t(a)t})$ when there are three goods. All rules can be written in terms of pairs of conditions drawn from table 1. Thus, only $6 \times 6 \times 2 = 72$ strings are required to represent the complete set of possible rules for this system.

Assigned to each exchange classifier $e \in \{1, 2, \ldots, E_a\}$ is a *strength*, denoted $S_e^a(t)$. The strength $S_e^a(t)$ evolves over time in a way determined by the accounting system. The strengths attached to classifiers are used to determine the decisions made by the classifier system at t. For a given state or 'condition' $z_{at} = (x_{at}, x_{\rho_t(a)t})$, there is a collection of classifiers within the classifier system whose conditions are satisfied. We denote the set of such classifiers by $M_e(z_{at})$, defined as

$$M_e(z_{at}) = \{ e : z_{at} \text{ matches the condition part of classifier } e \}. \quad (5)$$

The members of $M_e(z_{at})$ form a class of potential 'bidders' in an 'auction' whose purpose is to determine which classifier makes the decision of agent a at time t. When state z_{at} is encountered at time t by agent a, the classifier belonging to $M_e(z_{at})$ that has the highest strength makes the decision. Let $e_t(z_{at})$ denote the index of the classifier to be used in deciding whether to trade at t. Then,

$$e_t(z_{at}) = \arg\max \{ S_e^a(t) : e \in M_e(z_{at}) \}. \quad (6)$$

We denote the action (trade or no trade) taken by classifier $e_t(z_{at})$ as λ_{at}. Eqs. (5) and (6) describe the 'auction system' by which the highest strength rule that applies in a given state is given the right to decide for agent a at t.

Because the trade and consumption classifier systems will be making payments to one another, we have to describe the consumption classifier system before describing the accounting system that updates strengths of the trade classifier system. The consumption classifier system is a collection of trinary strings of length 4. The first three positions encode x_{at}^+ using the same code that was described in table 1. The condition part of the strings is written in terms of the trinary alphabet $(0, 1, \#)$. The fourth position of a consumption string is the 'action' part, taking values of 1 (meaning consume) and 0 (meaning don't consume).

We let consumption classifier strings be indexed by $c \in \{1, 2, \ldots, C_a\}$. The strength assigned to classifier c of agent a is $S_c^a(t)$. By virtue of eq. (2), the state x_{at}^+ can be expressed in terms of $x_{at} \equiv (x_{at}, x_{p_t(a)t})$. Therefore it suffices to denote the set of 'matched' classifiers by $M_c(z_{at})$, where

$$M_c(z_{at}) = \{ c: x_{at}^+ \text{ matches the condition part of classifier } c \}. \qquad (7)$$

Let $c_t(z_{at})$ denote the classifier that makes the consumption decision at t. The highest-strength classifier gets to make the decision:

$$c_t(z_{at}) = \arg\max \{ S_c^a(t): c \in M_c(z_{at}) \}. \qquad (8)$$

Given the decisions determined by (6) and (8), x_{at} evolves according to the law of motion (4).

We attach to each exchange classifier e a 'counter' $\tau_e(t)$ which records the cumulative number of times that classifier e has won the auction as of date t. We shall change the strength of classifier e only when it actually wins the auction and thereby gets to make the exchange decision, which is why we need the counter $\tau_e(t)$. The counter $\tau_e(t)$ for classifier e is defined recursively in terms of the indicator $I_e^a(t)$, which records whether classifier e wins the auction:

$$I_e^a(t) = \begin{cases} 1 & \text{if} \quad e \text{ wins the auction } (\textit{unless} \text{ classifier} \\ & \quad e \text{ sets } \lambda_{at} = 1 \text{ while } \lambda_{p_t(a)t} = 0, \text{ so that} \\ & \quad \text{the offer to trade is not reciprocated}), \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_e^a(t) = \sum_{s=0}^{t} I_e^a(s) + 1. \qquad (9)$$

Notice that we initialize the counter of each classifier at unity. The strength of

classifier e at t will be represented as

$$S_e^a(t) = S_{e\tau_e^a(t)}^a.$$

Similarly, we attach to each consumption classifier c a 'counter' $\tau_c(t)$ which records the cumulative number of times that classifier c has won the auction as of date t. The counter $\tau_c(t)$ is defined by

$$I_e^a(t) = \begin{cases} 1 & \text{if } c \text{ wins the auction,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_c^a(t) = \sum_{s=0}^{t} I_c^a(s) + 1. \tag{10}$$

The strength of classifier c at t will be represented as

$$S_c^a(t) = S_{c\tau_c^a(t)}^a.$$

The counters $\tau_c(t)$ and $\tau_e(t)$ induce a transformation of time in terms of which the strengths of classifiers c and e are updated. At date t, a strength $S_{c\tau_c^a(t)}^a$ is attached to classifier c, while a strength $S_{e\tau_e^a(t)}^a$ is attached to classifier e of agent a. At date t, if classifier e's condition is matched [i.e., if $e \in M_e(z_{at})$], then classifier e makes a *bid* of $b_1(e)S_c^a(t)$, where $b_1(e)$ is a positive fraction that can depend on e. If classifier e wins the auction, its bid will be deducted from its strength. The winning bid will be allocated to augment the strength of other classifiers whose actions drove the system to the state that satisfied e's condition. We choose the particular bid function

$$b_1(e) = b_{11} + b_{12}\sigma_e, \tag{11a}$$

where b_{11} and b_{12} are positive constants adding up to less than one, and σ_e is a fraction which is proportional to the specificity of a particular classifier. In particular, we choose $\sigma_e = 1/(1 + \text{number of } \#\text{'s in the string})$. Similarly, we define a function $b_2(c)$ as

$$b_2(c) = b_{21} + b_{22}\sigma_c, \tag{11b}$$

where $\sigma_c = 1/(1 + \text{number of } \#\text{'s in the string})$. By the above choices of $b_1(e)$ and $b_2(c)$, we favor specific rules over more general rules that can be activated by a particular state. When $c \in M_c(z_{at})$, classifier c makes a bid of $b_2(c)S_c^a(t)$.

Only winning classifiers pay their bids by having them deducted from their strengths. The bid of the winning exchange classifier at t is paid to the

winning consumption classifier at $t - 1$, which is the classifier that is to be credited with setting the time t state to z_{at}. The bid of the winning consumption classifier at t is paid to the winning exchange classifier at time t, which is to be credited with setting the post-exchange state at t at x_{at}^+, thereby giving the winning consumption classifier a chance to bid.

We represent these payments in terms of the following difference equations:

$$S_{c\tau_c^a(t)}^a = S_{c\tau_c^a(t)-1}^a - \frac{1}{\left(\tau_c^a(t) - 1\right)}\left[(1 + b_2(c))S_{c\tau_c^a(t)-1}^a\right.$$

$$\left. - \sum_e I_e^a(t) b_1(e) S_{e\tau_e^a(t)}^a - U_a(\gamma_{ct}^a)\right], \qquad (12)$$

$$S_{e\tau_e^a(t)+1}^a = S_{e\tau_e^a(t)}^a - \frac{1}{\tau_e^a(t)}\left[(1 + b_1(e))S_{e\tau_e^a(t)}^a\right.$$

$$\left. - \sum_c I_c^a(t) b_2(c) S_{c\tau_c^a(t)}^a\right]. \qquad (13)$$

In (12), $U_a(\gamma_{ct}^a)$ is the external payoff when the winning consumption classifier c makes final consumption decision γ_{ct}^a. If the post-exchange state at t is x_{at}^+, then we have

$$U_a(\gamma_{ct}^a) = \gamma_{ct}^a[u_i(x_{at}^+) - s(f(a))] + (1 - \gamma_{ct}^a)s(x_{at}^+). \qquad (14)$$

There are several features of (12) and (13) that bear emphasizing. First, eqs. (12) and (13) are recursive formulas that make $S_{c\tau_c^a(t)}^a$ and $S_{e\tau_e^a(t)}^a$ averages of past payoffs (external rewards plus bids received from other classifiers) minus payments (bids made to other classifiers). Use of cumulative average net payoffs in this way (as opposed to *total* payoffs) is a departure from the existing literature on classifiers. Second, notice how the term $\sum_e I_e^a(t)b_1(e)S_{e\tau_e^a(t)}^a$ expresses the condition that only the winning exchange classifier at t pays the winning consumption classifier at $t - 1$. Use of the term $\sum_c I_c^a(t)b_2(e)S_{c\tau_c^a(t)}^a$ achieves an analogous purpose in (13). Third, notice how the use of the counters $\tau_c^a(t)$ and $\tau_e^a(t)$ causes changes to be made only to the strengths attached to the winning classifiers.

Fig. 1 illustrates how the exchange and consumption classifier systems interact for an individual of type I. Suppose that the state at time t is encoded 010 100. This means that the type I individual is storing good 2 and has been matched with someone who is storing good 1. The individual has two classifiers whose conditions are matched, one with strength 50, the other with strength 10. The strength 50 classifier makes the decision, which is '1', namely, offer to trade. If the individual's trading partner also offers to trade, the

Exchange classifier system

Exchange classifier for matched agent

state at t

010 100

100	#01	1	20
010	0#0	0	4
0##	1#0	0	10
010	001	0	0
01#	100	1	50

0
100
50
40
50

trade?

to consumption classifier system

strength

yes   **no**

100   010

Consumption classifier system

state

100

0##	0	40
10#	1	80
010	1	0
100	0	-1

External Payoff

$(U, s_k, ...)$

state at t + 1
010 ... ...

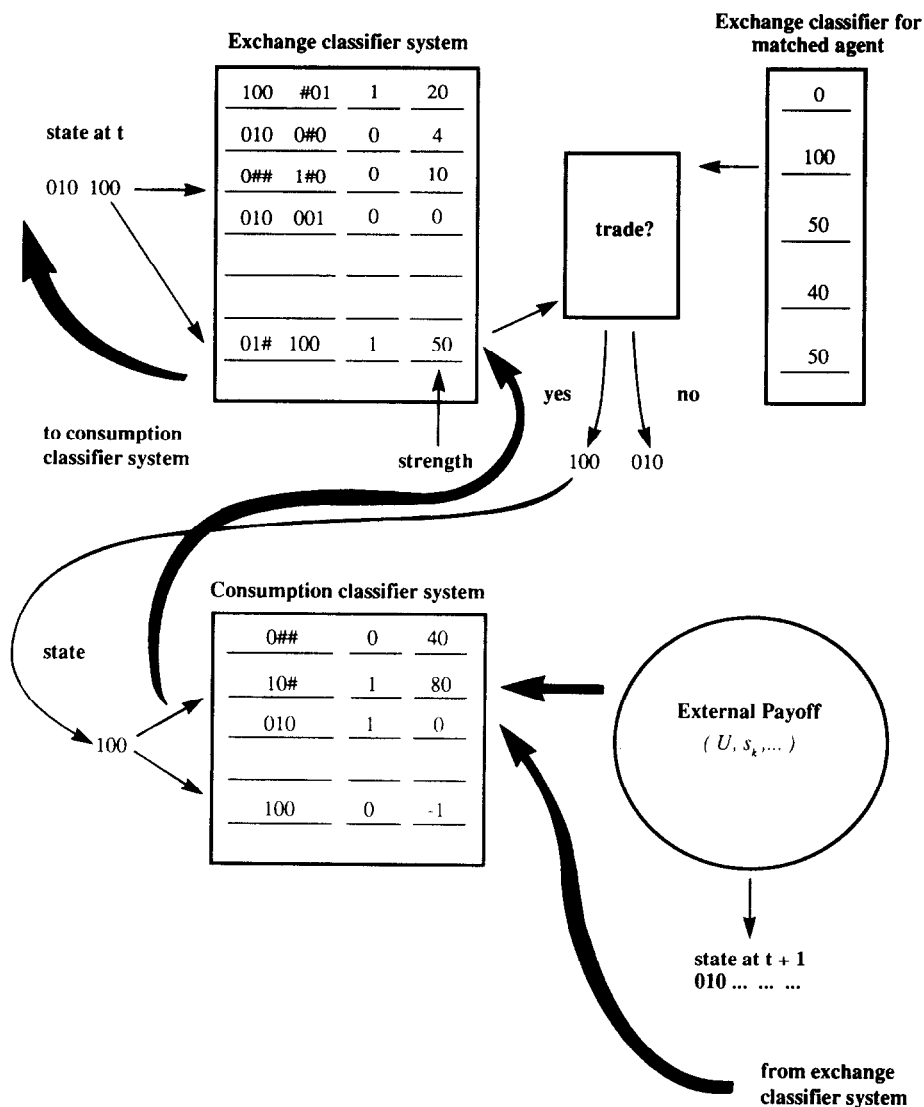**from exchange classifier system**

Fig. 1. Example of flow of payments in classifier systems for type I agent: transfer payments denoted in bold lines, decision flows denoted in thin lines.

individual exits the trading process with state 100, which is the input into type I's consumption classifier. With this input, there are two consumption classifiers whose conditions are matched, one with strength 80 that says 'consume', the other with strength -1 that says 'don't consume'. The individual consumes, initiating an external payoff. The bold lines indicate the flows of payoffs. In addition to the external payoff to the winning consumption classifier, there is a payment from the winning consumption classifier at t to the winning exchange classifier at t, and a payment from the winning exchange classifier at t to the winning consumption classifier at $t-1$.

With (12), (13), and (14), we have completed our description of the classifier system for agent a when that classifier system contains a complete enumeration of possible rules. Our version of the Kiyotaki–Wright model is formed by assuming that each agent a of type i uses the same classifier system to make decisions.[6] Thus, we can replace a superscripts and subscripts in the descriptions of strengths by i's. We start with a set of initial strengths $S^i(0)$ for $i = 1, 2, 3$. We temporarily introduce the index T to index calendar time, while t now temporarily indexes the cumulative number of matches which have occurred rather than calendar time. Our artificially intelligent agents then interact as follows for each $T = 1, 2, \ldots, L$, where L is the length of our simulation. All A agents are randomly matched into $A/2$ pairs. For each pair (a, a'), the exchange and classifier systems for agents whose types are a and a', respectively, are used mutually to determine the trading and consumption decisions of agents a and a'. After the consumption decisions have been made for the pair, the counters and the strengths of the classifier systems for both types of agents are updated according to (10), (11), (12), and (13). Also, after each pair has played, the synthetic time counter t is augmented by one. After all $A/2$ pairs have played, the counter T is augmented by one, and the entire process is repeated. The counter T records the passage of physical time, while the counter t records the cumulative number of matches that have occurred. The outcomes of all matches $t \in [(T-1)(A/2) + 1, T(A/2)]$ are interpreted as having occurred during period T.

In section 7, we shall be interested in recording various frequency distributions across agents as a function of calendar time. That is, there will be distinct frequency distributions of, e.g., holdings of goods across agents for each date in calendar time. At the risk of confusion, we choose to index calendar time by t (rather than T) in subsequent sections.

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[6] We make this specification in order to economize on computer time and space. It is possible that this specification speeds convergence relative to a setting in which each agent $a \in \mathscr{A}$ uses his own classifier system. However, relative to a setup in which each agent uses his own classifier system and so can experiment individually, the common classifier setup causes all type i agents to experiment simultaneously. This feature might delay convergence.
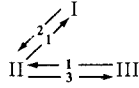
Fig. 2. Exchange patterns in fundamental equilibrium of model A.

4. Classifier systems for supporting Kiyotaki and Wright's stationary equilibria

4.1. The fundamental equilibrium

For their model A, Kiyotaki and Wright define as the *fundamental equilibrium* the trading pattern defined by the triangle depicted in fig 2. In this trading pattern, good 1, which has the lowest storage cost, serves as the general medium of exchange. In terms of storage, agents of type I only store good 2, which they exchange only for good 1; agents of type III only store good 1, which they exchange only for good 3; and agents of type II half of the time store good 1, which they exchange for good 2, and half of the time good 3, which they exchange for good 1.

We can characterize equilibria in terms of a set of probabilities that describe holding and exchanging patterns. We define

$$
\begin{aligned}
\pi_{it}^h(k) \quad &= \text{probability that a type } i \text{ agent is holding } k \text{ at } t, \\
\pi_t^h(k) \quad &= \text{probability that a randomly selected agent is holding } k \text{ at } t, \\
\pi_{it}^e(kj) \quad &= \text{probability that a type } i \text{ agent exchanges } k \text{ for } j \text{ at } t, \\
\pi_{it}^e(kj|k) &= \text{probability that a type } i \text{ agent exchanges } k \text{ for } j \text{ given that } i \text{ is holding } k \text{ at the beginning of } t, \\
\pi_{it}^c(k) \quad &= \text{probability that a type } i \text{ agent consumes } k \text{ at } t, \\
\pi_{it}^c(k|k) &= \text{probability that a type } i \text{ agent consumes } k \text{ at } t \text{ given that he holds } k \text{ after trading at } t, \\
\pi_t(i|k) \quad &= \text{probability that an agent is of type } i \text{ given that he holds } k \text{ at } t.
\end{aligned}
\tag{15}
$$

We use the notation $\hat{\pi}_{it}^e(kj|k)$ to denote the probability that a type i agent *is willing* to exchange good k for j whenever i is holding k. Note also that $\pi_t(i|k) = \pi_{it}^h(k)/\frac{1}{3}\sum_{i'}\pi_{i't}^h(k)$, $\pi_{it}^e(kj) = \pi_{it}^e(kj|k)\pi_{it}^h(k)$, and $\pi_t^h(k) = \frac{1}{3}\sum_i \pi_{it}^h(k)$.

We denote by Π_t the entire set of probabilities defined in (15) for all i, k, j. Associated with Kiyotaki and Wright's stationary equilibrium is a time-

Table 2

Behavior of type I agent in a fundamental equilibrium.

state at $t, z_{at} \equiv (x_{at}, x_{\rho(a)t})$:	$[2,1]$	$[2,-1]$
exchange classifier, $e_t(z_{at})$:	$e^1_{2,1,1}$	$e^1_{2,-1,0}$
does trade occur, i.e. is $\lambda_{\rho(a)t} \cdot \lambda_{at} = 1$?	yes	no
consumption state, x^*_{at} :	$[1]$	$[2]$
consumption classifier, $c_t(z_{at})$:	$c_{1,1}$	$c_{-1,0}$
payoff at the end of period t :	$(u_1 - s_2)$	$-s_2$
matching process	$\pi^h_{t+1}(1)$	$(1 - \pi^h_{t+1}(1))$
state at $t+1, z_{at+1}$:	$[2,1]$	$[2,-1]$

invariant $\Pi_t = \Pi$. In table 5b we display the Π's for the fundamental equilibria of Kiyotaki and Wright's model A.

We now describe how the strategies that support Kiyotaki and Wright's 'fundamental equilibrium' can be represented as a classifier system. For the classifiers we introduce the following additional notation. Let $e^i_{k,j,d}$ denote the exchange classifier for a type i agent that reads 'If I am storing good k and I meet an agent with good j, then my exchange decision is d'; $d \in \{0,1\}$. Similarly, specific consumption classifiers for type i agents are of the form $c^i_{k,d}$ for 'If, at the end of the exchange subperiod, I am storing good k, then my consumption action is d'. For more general rules, we use the subindex $-j$ to denote 'not good j' and $\#$ to denote 'any good'. That is, $e^1_{2,-1,0}$ is the classifier for type I agents that reads 'If I am storing good 2 and I meet an agent not carrying good 1, then do not trade', and $c^1_{\#,0}$ is a classifier for type I that reads 'No matter what I am storing at the end of the exchange subperiod, do not consume'.

We can simplify the study of classifiers systems by considering only a small complete set of rules. For type I agents these rules are: $e^1_{2,1,1}$; $e^1_{2,2,0}$; $e^1_{2,3,0}$; $e^1_{2,3,1}$; $e^1_{2,-1,0}$; $e^1_{3,1,1}$; $e^1_{3,-1,0}$; $c^1_{1,1}$; and $c^1_{-1,0}$. If $\{e^1_{2,1,1}; e^1_{2,-1,0}; c^1_{1,1}; c^1_{-1,0}\}$ or alternatively $\{e^1_{2,1,1}; e^1_{2,2,0}; e^1_{2,3,0}; c^1_{1,1}; c^1_{-1,0}\}$ are the only classifiers being used by agents of type I, we can say that type I agents support the fundamental equilibrium. We can also describe classifiers for type II and type III agents

that support the fundamental equilibrium. These are, for type II: $e^2_{1,2,1}$; $e^2_{1,-2,0}$; $e^2_{3,1,1}$; $e^2_{3,2,1}$; $e^2_{3,3,0}$; $c^2_{2,1}$; and $c^2_{2,0}$; and for type III: $e^3_{1,3,1}$; $e^3_{1,-3,0}$; $c^3_{3},1$; and $c^3_{-3,0}$.

It is useful to define formally the sense in which these fixed lists of classifiers support Kiyotaki and Wright's fundamental equilibrium. Let D_a be a fixed set of classifiers (exchange and consumption classifiers) for agent a. Let the set D_a have no redundancies in the sense that for every state z_{at}, D_a uniquely determines actions. Then the fixed set of classifiers D_a represents a strategy for agent a defined on z_{at}. We use the following definition of optimality for a fixed set of classifiers:

Definition.   Given the probability distributions Π and fixed sets of classifiers of the other agents $D_{a'}$ for all $a' \neq a$, a fixed set of classifiers D_a is said to be *optimal for agent a* if there exists no other set $\tilde{D}_a$ which yields higher long-run average utility for agent a.

A stationary Nash equilibrium can then be defined as follows:

Definition.   A stationary Nash equilibrium is a set of probability distributions Π and fixed sets of classifiers D_a, $a = 1, \ldots, A$, such that:

(1) Given Π and $D_{a'}$ for $a' \neq a$, D_a is optimal for agent a.

(2) $\{ D_a, a = 1, \ldots, A \}$ and the random matching technology imply that $\Pi_t = \Pi$ for all t.

This definition is just an alternative representation of Kiyotaki and Wright's definition in terms of fixed lists of classifiers.

Table 2 illustrates the sequence of events under the circumstance that the only active (i.e., auction-winning) classifiers are those that support the fundamental equilibrium for agents of type I. A natural question is whether the classifier systems of the three types of agents will actually converge to a situation in which the fundamental equilibrium of Kiyotaki and Wright is supported. Before introducing some concepts that will help us think about this question, we shall briefly describe another kind of equilibrium of Kiyotaki and Wright's model and the kind of classifier system that could support it.

4.2. Kiyotaki and Wright's 'speculative equilibrium'

Kiyotaki and Wright show that for their model A the trading pattern associated with a fundamental equilibrium, which was described in fig. 2
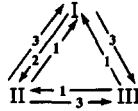
Fig. 3. Exchange patterns in speculative equilibrium of Economy A.

Table 3

Behavior of type I agent in a speculative equilibrium.

above, is not the only possible one in equilibrium. They show that there can occur a *speculative equilibrium* with a trading pattern depicted in fig. 3.[7]

In the speculative equilibrium, table 3 depicts the flow of events in the stationary rational-expectations equilibrium for a classifier system of a type I agent using a subset of rules. Notice that the flows depicted in table 3 reduce to a set of fundamental classifiers if the dotted link ($*$) replaces its alternative link ($**$), i.e., if type I agents decide not to trade good 2 for good 3.

---

[7]Kiyotaki and Wright establish a condition under which the unique stationary rational-expectations equilibrium is a fundamental equilibria. The condition, for the limiting case as the discount rate converges to zero, is that the following inequality is satisfied:

$$s_3 - s_2 > \left(\pi_1^h(3) - \pi_1^h(2) \right) \tfrac{1}{3} u_1. \qquad (16)$$

[Notice that this condition involves both parameter values (s_3, s_2, u_1) and the quantities ($\pi_1^h(3), \pi_1^h(2)$) which are endogenous variables that are determined in equilibrium.] Kiyotaki and Wright show that when the above inequality is reversed, then the unique stationary rational-expectations equilibrium is a speculative equilibrium. Vis-à-vis the fundamental equilibrium, the only change in trading patterns is that agents of type I 'speculate' by holding good 3, which is not too costly because in equilibrium inequality (16) is reversed (see fig. 3.)

In section 7 we shall provide an example of a simulated classifier economy where (16) is not satisfied (u_1 is too high), but where nevertheless when agents start with a complete set of classifiers (with homogeneous strengths), the classifier systems converge to the fundamental equilibrium and not to the speculative equilibrium. Before turning to these simulations, we define some notions of stability that are designed to distinguish alternative senses in which a set of classifier systems may be said to converge.

5. Concepts of convergence for classifier systems playing games

Given the decision rules being employed by other agents, (12)–(13) for agent a forms a system of stochastic difference equations in the strengths $S^a(t)$. Since the classifier systems of the three types of agents $i = 1, 2, 3$ are operating simultaneously, the behavior of the entire system is determined by the system of difference equations (12)–(13) for agent types $i = 1, 2, 3$. Let $S^i(t)$ denote the strengths for an agent of type i. It is natural to inquire whether the system formed by (12)–(13) for $a = i = 1, 2, 3$ converges to a stationary point in the space of strengths, and if it does so, whether that stationary point supports the fundamental equilibrium of Kiyotaki and Wright.

Although we have not yet carried out an analysis of the convergence of this system, it seems useful at this point to report our ideas about how such an analysis could be structured. In this section we indicate how ideas from the stochastic-approximation literature might be used to study convergence in the space of strengths.[8]

From the stochastic-approximation literature,[9] we know that any limit points (S_c^a, S_e^a) of (12)–(13) must satisfy

$$
\begin{aligned}
&\mathrm{E}\left[(1 + b_2(c))S_c^a - \sum_e I_e^a(t) b_1(e) S_e^a - U_a(\gamma_c^a)\right] = 0, \\
&\mathrm{E}\left[(1 + b_1(e))S_e^a - \sum_c I_c^a(t) b_2(c) S_c^e\right] = 0,
\end{aligned}
\tag{17}
$$

for all c and e. For a given a we define the solutions (S_e^a, S_c^a) of (17) as a *stationary set of strengths* for agent a. Given a stationary set of strengths for agent a, one can determine the set of classifiers $\{e, c\} = W_a(z_{at})$ that would

---

[8] It was considerations from the stochastic-approximation literature that let us to alter Holland's specification of the laws of motion of strengths so that they measure cumulative *average* past net rewards, not total rewards. The use of average rewards makes it possible for strengths to converge. Arthur (1989) uses average rewards for strengths for the same reason.

[9] Ljung and Soderstrom (1983) is a useful reference on stochastic approximation and on the 'ordinary differential-equations approach' to proving almost sure convergence. See Marcet and Sargent (1989a, b) for some applications in economics.

win the auction for each z_{at}. If a has the fixed set of classifiers $W_a(z_{at}) = D_a$, where D_a is the fixed set of classifiers that supports a's behavior in a Nash equilibrium, then the stationary set of strengths associated with $W_a(z_{at})$ can be said to support the stationary Nash equilibrium behavior of agent a.

These definitions inspire several questions. First, given stationary sets of strengths for agents of types $1, 2, 3$, is it true that $W_a(z_{at}) = D_a$ for a type i agent, for each $i = 1, 2, 3$? That is, does a stationary set of strengths for the classifier systems for agents of type $i = 1, 2, 3$ support a stationary Nash equilibrium?

Second, given fixed sets of classifiers D_a for a of type $i = 1, 2, 3$ that support a stationary Nash equilibrium, do these sets of classifiers imply the existence of a set of strengths S_a for a of type $i = 1, 2, 3$ that solve (17)? That is, is every Nash equilibrium supported as a stationary point in strengths?

Third, if we let the classifier systems of agents of type $i = 1, 2, 3$ run in real time, does the joint system formed by (12)–(13) for agents of each type converge almost surely, and if so, to which solution of (17) does it converge? We know from the theory of stochastic approximation that *if* it converges, it must converge to solutions of (17).

Associated with the third of these questions is the notion of global asymptotic stability. In future work, we plan to apply results from the stochastic approximation literature to study the global asymptotic stability of versions of our interacting classifier systems.

We also intend to study a form of stability that is distinct from and weaker than global asymptotic stability. This form of stability is summarized in the following:

Nash-like stability test.   Take a stationary equilibrium fixed by Π. Place a single-agent of a given type i in the environment formed by the fixed probabilities Π. Let a classifier system for that single agent operate until it converges. Check to see whether it converges to imply the collection of classifiers D_a for an agent a of type i which are required to support the equilibrium probabilities. In this experiment, in running the classifier system for the single agent, the decision rules of all other agents of his type are being held fixed at their equilibrium values. This experiment is to be repeated for a single agent of each type.

In the remainder of this paper, we shall not pursue the formal analysis of the stability and convergence questions formulated above.[10] Instead, we shall report computer simulations. In these computer simulations, convergence can

---

[10]Brian Arthur and Carl Simon have used stochastic approximation arguments to establish convergence of strengths for a fixed system of two classifiers playing a two-armed bandit. Their arguments do not apply to our system because their system is one in which there are competing classifiers with distinct actions, whose condition parts are identical and are always met.

be based either on the limiting behavior of the strengths, as indicated above, or on the convergence of key aspects of the empirical frequency distribution of holdings Π_t, which we defined in section 4.

6. Incomplete enumeration classifiers and the 'genetic algorithm'

We now describe how the classifier system is modified when we deal with systems in which there is not a complete enumeration of rules. In order to complete the system, two aspects must be added to the above description of the classifier system under complete enumeration. The first is a way of generating an initial population of rules. The second is a way of deleting old rules from the system and adding new rules to the system. To generate the initial population of rules, we simply use a random number generator to generate bit strings by choosing successive bits independently.

The process of adding new rules and deleting old ones is accomplished by adding four major operations to the previous description of the workings of the classifier system. We call these steps 'creation', 'diversification', 'specialization', and 'generalization', respectively. The *creation* operation is activated when there is no classifier matching the current state z_{at}, i.e., $M_e(z_{at})$ is empty. In this case, a new classifier is created with its condition part defined by the current state and its action part randomly selected. A 'weak' classifier from the set of all classifiers is deleted to keep the number of classifiers constant. The *diversification* operation is designed to inject sufficient diversity into the range of actions called for by different classifiers in a given situation. For the exchange classifier, this step occurs at the stage at which z_{at} has been observed and the set $M_e(z_{at})$ has been constructed. Recall that $M_e(z_{at})$ is the set of exchange classifiers whose condition parts match z_{at}. If all $e \in M_e(z_{at})$ have the same action (0 or 1), the diversification operation creates a classifier encoding the specific state z_{at} and an action opposite to that taken by the other classifiers in $M_e(z_{at})$. The strength of the winning classifier is assigned to this new classifier. This new classifier is added to the system, while simultaneously a 'weak' classifier from the set $M_e(z_{at})$ is deleted from the system. The weakness of a classifier is measured in terms of a combination of strength and cumulated number of times the classifier has won the bidding in the past. The diversification operation is used each time the classifier system is called upon. Notice that if this step were to be added to a complete enumeration system, it would have no effect on the population of rules because sufficient diversity is present from the beginning. The *specialization* operation is called randomly with a probability that we choose to be diminishing over time. The random variable governing specialization calls this step into action just after the winning bid has been determined. The winning classifier is checked to determine whether there are any #'s in its condition part. A new classifier is synthesized by exposing each # in the condition part to a probability of being

parent 1:	1	0	#	1	0	0	0
parent 2:	0	#	0	1	1	1	1
offspring 1:	1	0	#	1	#	0	0
offspring 2:	0	#	0	1	#	1	1

Fig. 4. The mating process for exchange classifiers who have drawn '3,6' and 'in'.

switched to a 0 or a 1, whichever one specifically encodes the particular z_{at} that was just observed or matched. This new and (probably) more specific rule is then used to replace a weak rule from the set $e \in M_e(z_{at})$ [or $c \in M_c(z_{at})$ in the case of the consumption classifier], where weakness is measured by a combination of strengths and cumulative number of times the auction has been won.

The fourth step, which we call *generalization* is a version of a 'genetic algorithm'. This step is called randomly, according to a probability that we specify to be declining through time. At a point when the step is called, the process is initiated by generating two distinct populations of classifiers: 'potential parents' and 'potential exterminants'. The potential parents are chosen to be a population of classifiers of specified size. They are chosen according to a 'fitness criterion' that weights strength and cumulative number of times that the classifier won the auction. The potential exterminants are chosen to be a population of specified size, whose members are of low fitness as measured by a fitness criterion weighting strengths and cumulative victories in past bidding.

From the population of potential parents, a specified number of pairs of parents are randomly drawn to engage in 'mating' for the purpose of creating children. How a pair of parents mates can be illustrated for two exchange classifiers, which are bit strings of length 7, is depicted in fig. 4.

The pair mates in two steps. First the pair draws two random integers uniformly distributed on [1, 7]. The two integers signify the positions between bits depicted in fig. 4. In fig. 4, we have drawn lines in between bit positions 2 and 3 and between positions 5 and 6 to signify that this couple has drawn a 3 and 6. Next, a Bernoulli random variable with probability 0.5 is drawn, assuming values of 'in' or 'out'. If 'in' is drawn, we focus on the bits *between* the two lines, i.e., the bits inside the randomly drawn interval. (In fig. 4, we have drawn an 'in'.) If 'out' is drawn, we focus on the bits *outside* the randomly drawn interval. Within the interval of focus, we complete the mating process via the following version of genetic crossover. For each of the parent strings in the area of focus, we change to a # any bits for which the parent strings disagree. (Here, # agrees with either 0 or 1, while 0 disagrees with 1). This process produces two children. The children are each assigned strengths that are the average of their parents' strengths. The children are

Table 4

Description of the economies.

	Production					Storage cost					Utility u_i[a]	Initial CS[b,c]	Equil. type[c]
	I	II	III	IV	V	1	2	3	4	5			
A1.1	2	3	1	—	—	0.1	1	20	—	—	100	F	F
A1.2	2	3	1	—	—	0.1	1	20	—	—	100	R	F
A2.1	2	3	1	—	—	0.1	1	20	—	—	500	F	S
A2.2	2	3	1	—	—	0.1	1	20	—	—	500	R	S
B.1	3	1	2	—	—	1	4	9	—	—	100	F	F/S
B.2	3	1	2	—	—	1	4	9	—	—	100	R	F/S
C	2	3	1	—	—	0.1	20	70	0	—	100	R	F
D	3	4	5	1	2	1	4	9	20	30	200	R	—

[a] Utility levels u_i are set equal for $i = 1, 2, 3$.
[b] CS denotes 'classifier system'.
[c] 'F' implies fixed enumeration and 'R' implies randomly generated rules.

introduced into the population of classifiers. For each child added to the population, a randomly selected individual from the population of 'potential exterminants' is deleted from the population.

This completes our description of the algorithm to be used for updating the classifier system when we deal with an incomplete enumeration classifier system. In the simulations to be reported in section 7, we shall employ both complete and incomplete enumeration classifier systems.

7. Simulation results

The sets of parameters defining the economies under study are summarized in table 4. The last column reports whether the Nash–Markov stationary equilibrium of Kiyotaki and Wright is speculative or fundamental for our parameter settings. Our economies differ from one another in parameter settings, and whether initial classifier systems contain a complete enumeration of states and actions or whether they are randomly generated, in which case the genetic algorithm described in section 6 is active. For all the simulations, we keep the number of exchange classifiers fixed at 72 for each type of agent, and the number of consumption classifiers fixed at 12 for each type of agent. We now briefly discuss the results.

Economy A1

Table 5 reports various probabilities that characterize the stationary equilibrium of Economy A1. These probabilities are to be compared with the empirical frequencies from our simulations, to appear in tables 6 and 7. In table 5d, we report the conditional exchange probabilities, $\pi_j^e(jk|j) =$ the

Table 5

$\pi_i^h(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	1	0
$i = 2$	0.5	0	0.5
$i = 3$	1	0	0

(b) The probability that i holds j, in the stationary equilibrium of Economy A1.

$\pi_i^e(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.167,[0,0.333],0)$	$(0,0,0)$
$i = 2$	$([0,0.25],0.167,0)$	$(0,0,0)$	$(0.167,0,[0,0.083])$
$i = 3$	$([0,0.5],0,0.167)$	$(0,0,0)$	$(0,0,0)$

(c) The probability that i holds j, meets k, and trades in the stationary equilibrium of Economy A1. The (i, j) entry in the table is the triple $(\pi_i^e(j1), \pi_i^e(j2), \pi_i^e(j3))$.

$\hat{\pi}_i^e(jk|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$([0,1],0,0)?$	$(1,[0,1],0)$	$(1,1,[0,1])?$
$i = 2$	$([0,1],1,0)$	$(0,[0,1],0)?$	$(1,1,[0,1])$
$i = 3$	$([0,1],0,1)$	$(1,[0,1],1)?$	$(0,0,[0,1])?$

(d) Equilibrium exchange strategies for Economy A1. (A question mark denotes sequential equilibrium strategies for events of zero probability.) The (i, j) entry in the table is the triple $(\pi_i^e(j1|j), \pi_i^e(j2|j), \pi_i^e(j3|j))$.

$\pi_i^c(j|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	1	0	0
$i = 2$	0	1	0
$i = 3$	0	0	1

(e) Equilibrium consumption strategies for Economies A1, A2, B, and C.

probability that a type i agent trades good j for good k, given that i is holding j at the beginning of the period. When an interval appears in this list of probabilities, it means that any probability in the interval is in equilibrium value. When a question mark appears behind a list of probabilities, it denotes that the event being conditioned upon has zero probability in equilibrium. Given the way that classifier systems learn from experience, it is too much to expect a classifier's system to learn the correct sequential equilibrium behavior for zero probability events.

Table 6 and fig. 5 report the results of a long simulation of Economy A1.1 (Economy A1 with complete enumeration of classifiers). In table 6, and in all subsequent tables of empirical frequencies, the frequencies reported are ten-period moving averages ending at the indicated time periods.[11] The tables

[11] When '–' appears in the table, it means that the event in question occurred so rarely in our simulation that no reliable frequency can be computed.

Table 6

No. of agents	$A_1 = 50$
No. of classifiers	$E_a = 72,\ C_a = 12$
Storage costs	$s_1 = 0.1,\ s_2 = 1,\ s_3 = 20$
Utility	$u_i = 100,\ \forall i$
Initial strengths	$S^a_{c_7 \gamma_e(0)} = 0,\ S^a_{c_7 c(0)} = 0$
Bids	$b_{11} = 0.025,\ b_{12} = 0.025,\ b_{21} = 0.25,\ b_{22} = 0.25$

(a) Parameter values used for Economy A1.1.

$\pi^h_{i500}(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	1	0
$i = 2$	0.502	0	0.498
$i = 3$	1	0	0

$\pi^h_{i1000}(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	1	0
$i = 2$	0.506	0	0.494
$i = 3$	1	0	0

(b) The frequency with which i holds j at $t = 500$ and $t = 1000$, $\pi^h_{it}(j)$, for Economy A1.1.

$\pi^e_{i500}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.16,0,0)$	$(0,0,0)$
$i = 2$	$(0.26,0.16,0)$	$(0,0,0)$	$(0.17,0,0)$
$i = 3$	$(0.52,0,0.17)$	$(0,0,0)$	$(0,0,0)$

$\pi^e_{i1000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.17,0,0)$	$(0,0,0)$
$i = 2$	$(0.27,0.17,0)$	$(0,0,0)$	$(0.19,0,0)$
$i = 3$	$(0.51,0,0.19)$	$(0,0,0)$	$(0,0,0)$

(c) The frequency with which i holds j, meets someone with k, and trades, $\pi^e_{it}(jk)$, for Economy A1.1, at $t = 500$ and $t = 1000$.

$\hat{\pi}^e_{i1000}(jk\,|\,j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$		$(1,0,0)$	
$i = 2$	$(1,1,0)$		$(1,1,0)$
$i = 3$	$(1,0,1)$		

(d) Actions of winning classifiers for i with condition (j, k), $\hat{\pi}^e_{it}(jk\,|\,j)$, for Economy A1.1, at $t = 1000$.

Iteration	Classifier				Strength
1000	1	0	0	1	45.8
	#	#	0	0	− 0.47
	0	#	#	1	− 9.16

(f) Highest-strength consumption classifiers that are activated for an agent of type I at iteration $t = 1000$ in Economy A1.1.

Table 6 (*continued*)

Iteration	Classifier							Strength
1000	0	#	#	1	0	0	1	11.22
	0	#	#	0	1	0	0	− 0.08
	#	#	0	0	0	1	0	− 0.08

(g) Highest-strength exchange classifiers that are activated for an agent of type I at iteration $t = 1000$ in Economy A1.1.

Iteration	Classifier				Strength
1000	1	0	0	0	0.0089
	0	1	0	1	49.04
	#	0	#	0	− 10.90

(h) Highest-strength consumption classifiers that are activated for an agent of type II at iteration $t = 1000$ in Economy A1.1.

Iteration	Classifier							Strength
1000	#	#	0	#	#	0	1	4.86
	#	#	0	#	#	0	1	4.86
	#	#	0	#	0	#	0	0.0022
	0	0	1	1	0	0	1	0.002
	0	#	#	0	1	0	1	12.72
	0	#	#	0	#	#	0	− 1.79

(i) Highest-strength exchange classifiers that are activated for an agent of type II at iteration $t = 1000$ in Economy A1.1.

Iteration	Classifier				Strength
1000	#	0	#	0	− 0.024
	#	#	0	0	− 0.412
	0	0	1	1	31.62

(j) Highest-strength consumption classifiers that are activated for an agent of type III at iteration $t = 1000$ in Economy A1.1.

Iteration	Classifier							Strength
1000	#	0	#	#	0	#	1	− 0.01
	#	0	#	0	1	0	0	− 0.01
	#	0	#	0	0	1	1	7.78

(k) Highest-strength exchange classifiers that are activated for an agent of type III at iteration $t = 1000$ in Economy A1.1.

Table 7

No. of agents	$A_1 = 50$
No. of classifiers	$E_a = 72$, $C_a = 12$
Storage costs	$s_1 = 0.1$, $s_2 = 1$, $s_3 = 20$
Utility	$u_i = 100$, $\forall i$
Initial strengths	$S^a_{e\eta_e}(0) = 0$, $S^a_{c\eta_c}(0) = 0$
Bids	$b_{11} = 0.025$, $b_{12} = 0.025$, $b_{21} = 0.25$, $b_{22} = 0.25$
Specialization[1]	$f_s(t) = 1/2\sqrt{t}$, $p_s = 0.01$
Generalization[2]	$f_g(t) = 1/2\sqrt{t}$, $p_1 = 0.2$, $p_2 = 0.7$, $p_3 = 0.2$, $p_4 = 0.5$
	$S = 0$, $N_e = 8$, $N_c = 4$

[1]$f_s(t)$ is the probability that specialization occurs in period t. Each '#' in the winning classifier has probability p_s of being switched to a zero or one depending on z_{at}.

[2]$f_g(t)$ is the probability that generalization occurs in period t. The proportion of a classifier system that can be altered is p_1. A proportion p_2 of classifiers forms the population of potential parents; they are chosen randomly in accordance with strength and frequency. Potential exterminants are those classifiers i such that $S^a_i(t) < S$ and $\tau^a_i(t) < p_3 \cdot \max_j \tau^a_j(t)$. To select a classifier from the set of potential exterminants to be replaced by an offspring, we find that classifier which is most similar in condition and most dissimilar in action. For each offspring, we determine a classifier to extinguish via the following process. For a number of N_i (where $i = c$ or e) times, we draw a proportion p_4 of the extinguishable classifiers. For each such sample, we find the classifier with lowest strength, count the total number of bits in its condition matching corresponding bits in the offspring, and add one to that total if the action of the offspring and the classifier in hand do not match. Of the N_i classifiers, the one with the highest count is eliminated from the system, to be replaced by an offspring.

(a) Parameter values used for Economy A1.2.

$\pi^h_{i1000}(j)$	$j = 1$	$j = 2$	$j = 3$	$\pi^h_{i2000}(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	0.992	0.008	$i = 1$	0	0.98	0.02
$i = 2$	0.226	0	0.774	$i = 2$	0.318	0	0.682
$i = 3$	1	0	0	$i = 3$	1	0	0

(b) The frequency with which i holds j at $t = 1000$ and $t = 2000$, $\pi^h_{it}(j)$, for Economy A1.2.

$\pi^e_{i1000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.08, 0.14, 0.01)$	$(0,0,0)$
$i = 2$	$(0,0.08,0)$	$(0,0,0)$	$(0.28, 0.01, 0.09)$
$i = 3$	$(0,0,0.28)$	$(0,0,0)$	$(0,0,0)$

$\pi^e_{i2000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.09, 0.19, 0.03)$	$(0,0,0)$
$i = 2$	$(0,0.08,0)$	$(0,0,0)$	$(0.2, 0.03, 0.06)$
$i = 3$	$(0,0.01,0.2)$	$(0,0,0)$	$(0,0,0)$

(c) The frequency with which i holds j, meets k, and trades at $t = 1000$ and $t = 2000$, $\pi^e_{it}(jk)$, for Economy A1.2.

$\hat{\pi}^e_{i1000}(jk|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0, -, -)$	$(1,1,0)$	$(1,1,0)$
$i = 2$	$(0,1,0)$	$(-, -, -)$	$(1,0,1)$
$i = 3$	$(0,0,1)$	$(-, -, 1)$	$(-, -, 1)$

Table 7 (continued)

$\hat{\pi}^e_{i2000}(jk\|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(-,-,-)$	$(1,0,1)$	$(1,1,1)$
$i = 2$	$(0,1,0)$	$(-,-,-)$	$(1,1,1)$
$i = 3$	$(0,1,1)$	$(0,1,1)$	$(-,-,1)$

(d) Actions of winning classifiers for i with condition (j, k), $\hat{\pi}^e_{it}(jk\|j)$, for Economy A1.2.

$\pi^c_{i1000}(j\|j)$	$j = 1$	$j = 2$	$j = 3$	$\pi^c_{i2000}(j\|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	1	0.011	0.049	$i = 1$	1	0	0.69
$i = 2$	0.45	1	0.366	$i = 2$	0.23	1	0.641
$i = 3$	0.033	—	1	$i = 3$	0.84	1	1

(e) The frequency with which i consumes j at t given j is held after the trading period, $t = 1000$ and $t = 2000$, $\pi^c_{it}(j\|j)$, for Economy A1.2.

Iteration	Classifier				Strength
1000	1	0	0	1	38.11
	0	1	#	0	-0.44
	0	0	1	1	-4.26
2000	1	0	0	1	41.65
	0	1	0	0	-0.43
	0	0	1	0	-8.86

(f) Highest-strength consumption classifiers that are activated for an agent of type I at iteration t in Economy A1.2.

Iteration	Classifier							Strength
1000	0	1	0	1	0	0	1	8.27
	0	1	0	0	1	0	1	-0.09
	0	1	0	0	0	1	0	-0.10
2000	0	1	0	1	0	0	1	9.05
	0	1	0	0	1	0	0	-0.10
	0	1	0	0	0	1	1	-0.10

(g) Highest-strength exchange classifiers that are activated for an agent of type I at iteration t in Economy A1.2.

Iteration	Classifier				Strength
1000	#	#	0	0	1.88
	0	#	0	1	11.48
	0	0	1	1	-9.29
2000	#	#	0	0	8.05
	0	#	0	1	38.40
	0	0	1	1	-9.01

(h) Highest-strength consumption classifiers that are activated for an agent of type II at iteration t in Economy A1.2.

Table 7 (continued)

Iteration	Classifier							Strength
1000	1	0	0	1	0	0	0	0.29
	1	0	0	0	1	0	1	1.67
	1	0	0	0	0	1	0	0.28
	0	0	1	#	#	0	1	2.23
	0	#	1	0	1	0	0	4.08
	0	0	1	0	0	1	1	−2.08
2000	1	0	0	1	0	0	0	1.32
	1	0	0	0	1	0	1	6.15
	1	0	0	0	0	1	0	1.25
	0	0	1	1	0	0	1	1.34
	0	#	1	0	1	0	1	2.98
	0	0	1	0	0	1	1	−1.99

(i) Highest-strength exchange classifiers that are activated for an agent of type II at iteration t in Economy A1.2.

Iteration	Classifier				Strength
1000	1	0	0	0	0.00
	0	0	1	1	30.94
2000	1	0	0	1	8.97
	0	0	1	1	30.55

(j) Highest-strength consumption classifiers that are activated for an agent of type III at iteration t in Economy A1.2.

Iteration	Classifier							Strength
1000	1	0	0	#	#	#	0	0.28
	1	0	0	0	#	#	0	1.77
	1	#	#	0	#	1	1	7.39
2000	1	0	0	1	#	0	0	0.14
	1	0	0	0	1	0	1	0.42
	1	#	#	0	#	1	1	7.25

(k) Highest-strength exchange classifiers that are activated for an agent of type III at iteration t in Economy A1.2.

report frequencies for the middle and the ending time periods. Table 6d is constructed from the winning classifiers at period 1000, and is to be compared with the theoretical predictions from table 5d. Finally, tables f to k show the highest-strength classifiers of each type of agent that are activated at the indicated iteration t. The listed classifiers form a system that is sufficient to determine decisions in all states that are visited with nonnegligible frequency at iteration t. Comparison of the results in table 6 and fig. 5 with the theoretical predictions of table 5 shows that the distribution of holdings rapidly converges to the stationary equilibrium distributions. Similarly, the exchange and consumption strategies implemented by the system of winning classifiers virtually coincide with the equilibrium strategies. For example, table
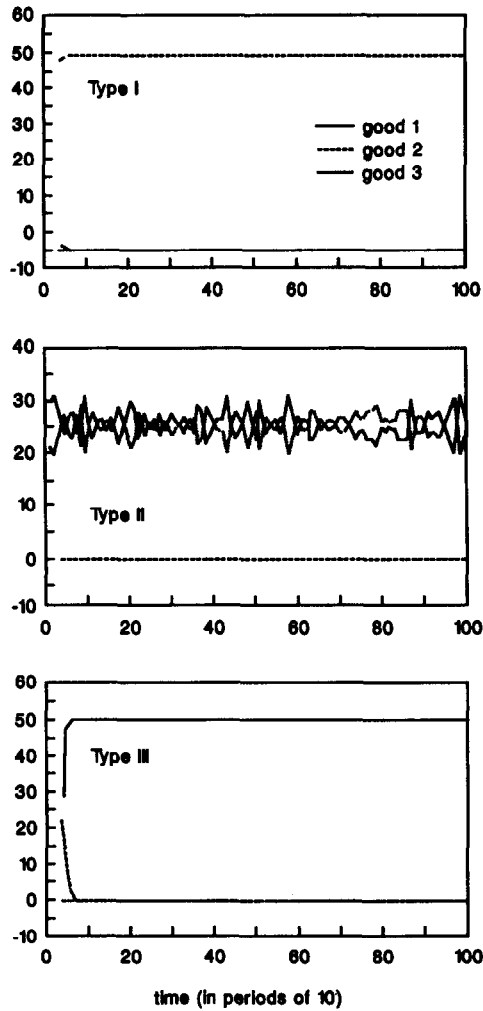
Fig. 5. Distribution of holdings for Economy A1.1.

6d is to be read as follows. When a type I agent has 1 in storage after exchange, he eats it; for both the first and second classifier in the list have their conditions matched ('good 1' and 'not good 3') but the first has higher strength and therefore determines the decisions. When the agent is storing 2, he does not eat (the second and third classifiers have their conditions matched, but the second has higher strength). When the agent is storing 3, he eats (the third classifier is the only pertinent one). This last decision is wrong, but the
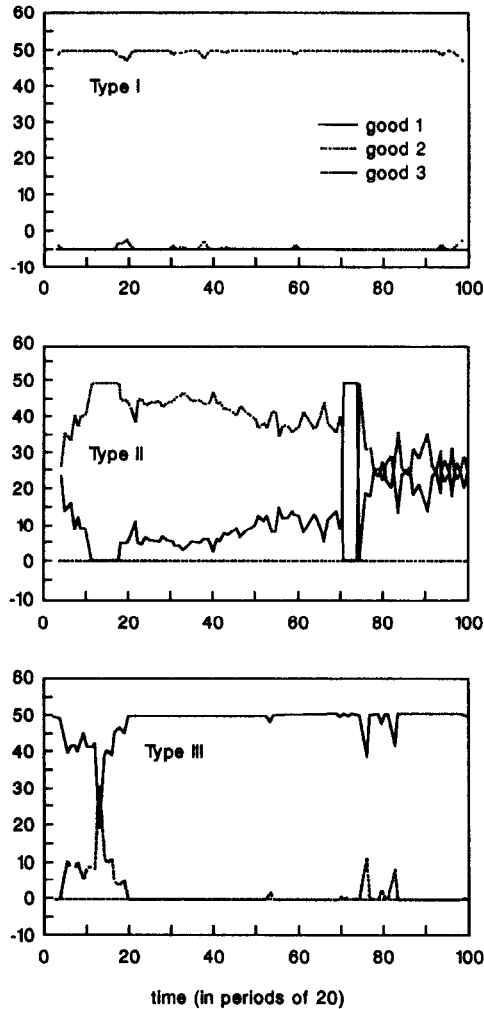
Fig. 6. Distribution of holdings for Economy A1.2.

frequency with which agent I brings good 3 into the state for the consumption decision is negligible, so that being wrong is not costly.

Economy A1.2 is identical to Economy A1.1, except that we use a randomly generated list of rules initially, and rely on a genetic algorithm to inject new rules into the classifier's system. Economy A1.2 thus provides our first test for the genetic algorithm of section 6.[12] It is important to notice that while the size

[12] More standard Holland algorithms were first tried without much success, prompting us to produce the modified algorithm described in section 6.

of the exchange and consumption classifier systems is 72 and 12, respectively, the set of possible classifiers is $3^6 \times 2 = 1458$ and $3^3 \times 2 = 54$, respectively. At any one time, only a small subset of possible classifiers is in the system. The results for Economy A1.2 are summarized in table 7 and fig. 6.

As we can see from fig. 6 and table 7, the distribution of holdings for agents of type I and III rapidly converge to their equilibrium levels. Type II agents only hold goods 1 and 3 as predicted but it takes longer to converge to the equilibrium values. The exchange decisions are the correct ones (even in period 1000), except for a couple of events of very small probability (type I agents trade good 2 for 3 and type III agents trade good 1 for 2).

Consumption decisions are basically correct, except that II consumes 3 and III consumes 1. Overeating has been a minor problem in several of our economies. In most such cases that we have encountered, such wrong behavior has been associated with low-frequency states for which it is difficult for the classifier system to acquire the experience required to reward better classifiers and 'right' actions. In our economies there are no production costs, and it is difficult for classifiers with low-frequency states to gain the adequate strength.

In summary, both with complete enumeration and without it, our simulations of Economy A1 seem to be converging to the stationary equilibrium, although this convergence is lower when initial classifiers are randomly generated and new classifiers are synthesized.

Economy A2

Economy A2 differs from Economy A1 only in that agents receive 500 utils from consuming their desired good instead of 100. With this change of parameters, for high enough discount factors the unique stationary equilibrium of Kiyotaki and Wright's economy is the so-called speculative equilibrium previously described. The probabilities that characterize the speculative equilibrium are summarized in table 8. Our simulation is summarized in table 9.

Table 9 depicts a pattern of holdings characteristic of a fundamental equilibrium, not a speculative one. Compare tables 9b and 9c with the theoretical predictions of tables 8b–8c and tables 5b–5c. Recall that a crucial difference between a speculative and a fundamental equilibrium is that in the former, type I agents are willing to exchange good 2 for good 3, while in the latter they are not. Table 9d shows that type I agents *are* willing to exchange good 2 for 3. That is, $\hat{\pi}_i^e(jk|2) = (1,0,1)$ for $i = I$. This is to be compared with the situation for economy A1, in which $\hat{\pi}_i^e(jk|2) = (1,0,0)$ (see table 5d or 6d). Then how is it that the distribution of holding patterns fails to support a speculative equilibrium? The answer is to be found in the consumption classifiers of type I agents, which are too general (have too many # signs) to

Table 8

$\pi_j^h(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	0.707	0.293
$i = 2$	0.586	0	0.414
$i = 3$	1	0	0

(b) The probability that i holds j in the speculative equilibrium of Economy A2.

$\pi_i^e(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0, 0, 0)$	$(0.138, [0, 0.167], 0.097)$	$(0.097, 0, [0, 0.029])$
$i = 2$	$([0, 0.269], 0.138, 0.057)$	$(0, 0, 0)$	$(0.138, 0.098, [0, 098])$
$i = 3$	$([0, 0.528], 0, 0.235)$	$(0, 0, 0)$	$(0, 0, 0)$

(c) The probability that i hold j, meets k, and trades in the speculative equilibrium of Economy A2.

$\hat{\pi}_i^e(jk \mid j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$([0, 1], 0, 0)$?	$(1, [0, 1], 1)$	$(1, 0, [0, 1])$
$i = 2$	$([0, 1], 1, 0)$	$(0, [0, 1], 0$?	$(1, 1, [0, 1])$
$i = 3$	$([0, 1], 0, 1)$	$(1, [0, 1], 1)$?	$(0, 0, [0, 1])$?

(d) Speculative equilibrium exchange strategies for Economy A2.

distinguish among goods stored.[13] To support the speculative equilibrium, it is necessary that the winning consumption classifiers be specific enough to permit distinguishing sufficiently among the goods stored so that the exchange classifiers can be properly compensated. It happens that the winning classifiers embody levels of generality that result in overconsumption of good 3 and a failure to transmit the information from the consumption classifier to the exchange classifier that would be required to support a speculative equilibrium.

We shall only briefly summarize the results for Economy A2.2 with random generation of initial classifiers, and refer the reader to our working paper for tables giving the numerical results. After 1000 iterations the economy had not converged to a steady state. Trading patterns were closer to the fundamental equilibria than to the speculative equilibrium in period 1000. (The reverse was true in period 500.) However, the classifier system of Type I agents did distinguish between storage of different good for consumption, and the classifier $c_{3,0}^1$ was slowly gaining strength over time. Nevertheless, our simulations provided no support for a hope that the economy would converge to the speculative equilibrium if it were to last longer.

While the results for Economy A2 are fairly inconclusive, they illustrate two interesting points. First, they expose some deficiencies of the genetic algorithm

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[13]To conserve space, these classifiers are not reported here. They are contained in a longer version of this paper which is available from the authors as a working paper.

Table 9

No. of agents	$A_1 = 50$
No. of classifiers	$E_a = 72$, $C_a = 12$
Storage costs	$s_1 = 0.1$, $s_2 = 1$, $s_3 = 20$
Utility	$u_i = 500$, $\forall i$
Initial strengths	$S^a_{e\eta_c(0)} = 0$, $S^a_{c\eta_c(0)} = 0$
Bids	$b_{11} = 0.025$, $b_{12} = 0.025$, $b_{21} = 0.25$, $b_{22} = 0.25$

(a) Parameter values used for Economy A2.1.

$\pi^h_{i500}(j)$	$j = 1$	$j = 2$	$j = 3$	$\pi^h_{i1000}(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	1	0	$i = 1$	0	1	0
$i = 2$	0.504	0	0.496	$i = 2$	0.466	0	0.534
$i = 3$	1	0	0	$i = 3$	1	0	0

(b) The frequency with which i holds j at $t = 500$ and $t = 1000$, $\pi^h_{it}(j)$, for Economy A2.1.

$\pi^e_{i500}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	(0,0,0)	(0.16,0.17,0.15)	(0,0,0)
$i = 2$	(0,0.16,0)	(0,0,0)	(0.17,0.15,0.04)
$i = 3$	(0.34,0,0.17)	(0,0,0)	(0,0,0)

$\pi^e_{i1000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	(0,0,0)	(0.18,0.13,0.19)	(0,0,0)
$i = 2$	(0,0.18,0)	(0,0,0)	(0.18,0.19,0.04)
$i = 3$	(0.34,0,0.18)	(0,0,0)	(0,0,0)

(c) The frequency with which i holds j, meets someone with k, and trades at $t = 500$ and $t = 1000$, $\pi^e_{it}(jk)$, for Economy A2.1.

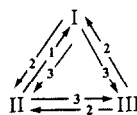| $\hat{\pi}^e_{i500}(jk\|j)$	$j = 1$	$j = 2$	$j = 3$	$\hat{\pi}^e_{i1000}(jk\|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	(0,0,0)	(1,0,1)	(0,0,0)	$i = 1$	(0,0,0)	(1,0,1)	(0,0,0)
$i = 2$	(0,1,0)	(0,1,0)	(1,1,0)	$i = 2$	(0,1,0)	(0,1,0)	(1,1,0)
$i = 3$	(1,0,1)	(1,1,1)	(1,0,1)	$i = 3$	(1,0,1)	(1,1,1)	(1,0,1)

(d) Actions of winning classifiers for i with condition (j, k), $\hat{\pi}^e_{it}(jk\|j)$, at $t = 500$ and $t = 1000$, for Economy A2.1.

which we have used (we return to this in section 8). Second, they raise the issue of whether our artificially intelligent agents are too impatient.

Patience requires experience. The transfer system inside the classifier system is designed to converge to a set of long-run average strengths. In the limit, the artificially intelligent agents should behave as long-run average payoff maximizers, since the steady-state strengths weight payoffs by their relative frequencies. It takes time, however, for optimal rules to achieve the desired strengths. The behavior of our artificially intelligent agents can be very myopic at the beginning. In economies, such as Economy A2, in which the nature of

(Fundamental Equilibrium)

(Speculative Equilibrium)

Fig. 7. Exchange pattern in Economy B.

the equilibrium changes with the discount rate, this early myopia might have a perverse effect in diverting the economy towards a low discount-factor stationary equilibrium, such as the fundamental equilibrium.[14] The underlying algorithm has to provide enough experimentation to avoid an early 'lock in'.[15] The present algorithm seems defective in that it has too little experimentation to support the speculative equilibrium even in the long simulations we have run.

Economy B

Stability tests based on classifier systems are potentially useful as selection criteria in economies with multiple equilibria. While Economy A has a unique equilibrium, Economy B has two equilibria if the discount factor is not too low.

Economy B has a different production pattern than Economy A (I produces 3, II produces 1, and III produces 2). For the specified parameters (see table 4) two stationary equilibria are possible whose trading patterns are described by fig. 7. In the fundamental equilibrium, goods 1 and 2 serve as media of exchange, while in the speculative equilibrium goods 2 and 3 are the media of exchange [see Kiyotaki and Wright (1989)]. Tables 10 and 11 summarize the trading patterns for the fundamental and the speculative equilibrium, respectively.

[14]A fundamental equilibrium exists for Economy A2 only if the discount factor is sufficiently low.

[15]Marimon and Miller (1989) report that in their experiments with the genetic algorithm 25 out of 30 runs of Economy A2 converged to the speculative equilibrium.

Table 10

$\pi_i^h(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	0.293	0.707
$i = 2$	1	0	0
$i = 3$	0.586	0.414	0

(b) The probability that i holds j in the fundamental equilibrium of Economy B.

$\pi_i^e(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.098,[0,0.069],0)$	$(0.138,0.098,[0,0.167])$
$i = 2$	$([0,0.529],0.236,0)$	$(0,0,0)$	$(0,0,0)$
$i = 3$	$([0,0.310],0,0.138)$	$(0.138,[0,0.098],0.098)$	$(0,0,0)$

(c) The probability that i hold j, meets k, and trades in the fundamental equilibrium of Economy B.

$\hat{\pi}_i^e(jk\mid j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$([0,1],0,0)?$	$(1,[0,1],0)$	$(1,1,[0,1])$
$i = 2$	$([0,1],1,0)$	$(0,[0,1],0)?$	$(1,1,[0,1])?$
$i = 3$	$([0,1],0,1)$	$(1,[0,1],1)$	$(0,0,[0,1])?$

(d) Fundamental equilibrium exchange strategies for Economy B.

Table 11

$\pi_i^h(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	0.586	0.414
$i = 2$	0.707	0	0.293
$i = 3$	0	1	0

(b) The probability that i holds j in the speculative equilibrium of Economy B.

$\pi_i^e(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.138,[0,0.310],0)$	$(0.098,0.138,[0,0.098])$
$i = 2$	$([0,0.167],0.138,0.098)$	$(0,0,0)$	$(0,0.098,[0,0.069])$
$i = 3$	$(0,0,0)$	$(0,[0,0.529],0.138)$	$(0,0,0)$

(c) The probability that i holds j, meets k, and trades in the speculative equilibrium of Economy B.

$\hat{\pi}_i^e(jk\mid j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$([0,1],0,0)?$	$(1,[0,1],0)$	$(1,1,[0,1])$
$i = 2$	$([0,1],1,1)$	$(0,[0,1],0)?$	$(0,1,[0,1])$
$i = 3$	$([0,1],0,1)?$	$(0,[0,1],1)$	$(0,0,[0,1])?$

(d) Speculative equilibrium exchange strategies for Economy B.

Table 12

No. of agents	$A_1 = 50$
No. of classifiers	$E_a = 72,\ C_a = 12$
Storage costs	$s_1 = 1,\ s_2 = 4,\ s_3 = 9$
Utility	$u_i = 100,\ \forall i$
Initial strengths	$S^a_{e\eta_e(0)} = 0,\ S^a_{c\eta_c(0)} = 0$
Bids	$b_{11} = 0.25,\ b_{12} = 0.25,\ b_{21} = 0.25,\ b_{22} = 0.25$

(a) Parameter values used for Economy B.1.

$\pi^h_{i5000}(j)$	$j = 1$	$j = 2$	$j = 3$	$\pi^h_{i1000}(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	0.464	0.536	$i = 1$	0	0.28	0.72
$i = 2$	0.994	0	0.006	$i = 2$	0.994	0	0.006
$i = 3$	0	1	0	$i = 3$	0.526	0.474	0

(b) The frequency with which i holds j at $t = 500$ and $t = 1000$, $\pi^h_{it}(j)$, for Economy B.1.

$\pi^e_{i500}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.16,0.05,0.01)$	$(0.01,0.19,0.04)$
$i = 2$	$(0,0.16,0.01)$	$(0,0,0)$	$(0,0,0)$
$i = 3$	$(0,0,0)$	$(0,0,0.18)$	$(0,0,0)$

$\pi^e_{i1000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(0,0,0)$	$(0.08,0.02,0.01)$	$(0.15,0.11,0.07)$
$i = 2$	$(0.41,0.25,0.01)$	$(0,0,0)$	$(0,0,0)$
$i = 3$	$(0.11,0.01,0.15)$	$(0.18,0.03,0.10)$	$(0,0,0)$

(c) The frequency with which i holds j, meets someone with k, and trades at $t = 500$ and $t = 1000$, $\pi^e_{it}(jk)$, for Economy B.1.

$\hat{\pi}^e_{i500}(jk|j)$	$j = 1$	$j = 2$	$j = 3$	$\hat{\pi}^e_{i1000}(jk|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(1,0,0)$	$(1,1,1)$	$(1,1,1)$	$i = 1$	$(1,0,0)$	$(1,0,0)$	$(1,1,1)$
$i = 2$	$(0,1,0)$	$(1,1,0)$	$(1,1,1)$	$i = 2$	$(1,1,1)$	$(1,1,0)$	$(1,1,0)$
$i = 3$	$(0,0,1)$	$(0,0,1)$	$(0,0,0)$	$i = 3$	$(1,0,1)$	$(1,1,1)$	$(0,0,0)$

(d) Actions of winning classifiers for i with condition (j, k), $\hat{\pi}^e_{it}(jk|j)$, for Economy B.1.

$\pi^c_{i1000}(j|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	1	0.07	0.508
$i = 2$	0.009	1	0.998
$i = 3$	0.050	0.632	1

(e) The frequency with which i consumes j at t given j is held after the trading period, $\pi^c_{tt}(j|j)$, for Economy B.1.

Table 13

No. of agents	$A_1 = 50$
No. of classifiers	$E_a = 72, C_a = 12$
Storage costs	$s_1 = 1, s_2 = 4, s_3 = 9$
Utility	$u_i = 100, \forall i$
Initial strengths	$S^a_{e\eta_e(0)} = 0, S^a_{c\eta_c(0)} = 0$
Bids	$b_{11} = 0.025, b_{12} = 0.025, b_{21} = 0.25, b_{22} = 0.25$
Specialization[1]	$f_s(t) = 1/2\sqrt{t}, p_s = 0.01$
Generalization[2]	$f_g(t) = 1/2\sqrt{t}, p_1 = 0.2, p_2 = 0.7, p_3 = 0.2, p_4 = 0.5,$
	$S = 0, N_e = 8, N_c = 4$

[1]See footnote 1 of table 7.
[2]See footnote 2 of table 7.

(a) Parameter values used for Economy B.2.

$\pi^h_{i1000}(j)$	$j = 1$	$j = 2$	$j = 3$	$\pi^h_{i2000}(j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	0	0.434	0.566	$i = 1$	0	0.354	0.646
$i = 2$	0.89	0	0.11	$i = 2$	0.996	0	0.004
$i = 3$	0.088	0.912	0	$i = 3$	0.268	0.732	0

(b) The frequency with which i holds j at $t = 1000$ and $t = 2000$, $\pi^h_{it}(j)$, for Economy B.2.

$\pi^e_{i1000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	(0,0,0)	(0.14,0.08,0.02)	(0.18,0.18,0.05)
$i = 2$	(0.14,0.41,0.19)	(0,0,0)	(0.03,0.05,0.1)
$i = 3$	(0.01,0,0.02)	(0.26,0.16,0.20)	(0,0,0)

$\pi^e_{i2000}(jk)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	(0,0,0)	(0.14,0.04,0.01)	(0.06,0.16,0.07)
$i = 2$	(0.13,0.36,0)	(0,0,0)	(0,0,0)
$i = 3$	(0.04,0,0.06)	(0.22,0.12,0.16)	(0,0,0)

(c) The frequency with which i holds j, meets someone with k, and trades at $t = 1000$ and $t = 2000$, $\pi^e_{it}(jk)$, for Economy B.2.

$\hat{\pi}^e_{i1000}(jk|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(-,-,-)$	(1,1,1)	(1,1,0)
$i = 2$	(1,1,0)	$(-,-,-)$	(1,1,1)
$i = 3$	(1,0,1)	(0,1,1)	$(-,-,1)$

$\hat{\pi}^e_{i2000}(jk|j)$	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$(-,-,-)$	(1,1,0)	(1,1,0)
$i = 2$	(0,1,1)	$(-,-,-)$	(1,1,1)
$i = 3$	(1,0,1)	(1,0,1)	$(1,-,1)$

(d) Actions of winning classifiers for i with condition (j, k), $\hat{\pi}^e_{it}(jk|j)$, for Economy B.2.

We simulated Economy B. Table 12 summarizes the results for Economy B.1 with complete enumeration of rules. Table 13 summarizes the results for Economy B.2 with random initial classifiers.

Economy B.1 displays an interesting pattern of evolution. At iteration 500 the distribution of holdings and, especially, the trading patterns correspond to the speculative equilibrium (except in some very low probability events, as II's unwillingness to trade good 1 for good 2). However, the economy moves away from this state and by iteration 1000 has practically converged to the fundamental equilibrium (with the same exception now of reverse sign but still of low probability).

Economy B.2 with random initial classifiers had not converged after 2000 periods. Nevertheless, as can be seen from table 13, the economy seems to be moving towards the fundamental equilibrium. In particular, with the exception of a zero-frequency event (II is willing to trade 1 for 3) the trading patterns correspond to the fundamental equilibrium.

These two simulations for Economy B provide examples in which the classifier systems seem to select the fundamental equilibrium over the speculative equilibrium.[16] Furthermore, the results for Economy B.1 indicate that this is not been the result of myopic behavior.

Economy C

In Economy C a new good, good 0, with the characteristics of 'fiat money' is introduced. No agent derives utility from consuming good 0. There are no storage costs associated with good C, although storing it uses up the storage capacity of each agent.

Fiat money is introduced into the system by forcing some agents to store good 0 in period 0. In particular, 48 units of good 0 are randomly allocated to 48 agents. To avoid fluctuations in the quantity of money, agents are not allowed to consume good 0. A modified setup, not pursued here, would give agents the opportunity to learn not to eat money.

The production technologies are as in Economy A (see table 4). Table 14 provides the probabilities that characterize the unique (fundamental) equilibrium of Economy C. The equilibrium trading patterns are depicted in fig. 8.

Table 15 describes the results for Economy C with random generation of initial classifiers. The exchange classifier systems have 150 classifiers, while the population of possible classifiers has 13,122 strings. The economy converges remarkably fast to the fundamental equilibrium and, with minor fluctuations, the distribution of holdings remains stationary. The final exchange actions are

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[16]In Marimon and Miller (1989) 30 out of 30 experiments resulted in convergence to the fundamental equilibrium.

Table 14

$\pi_i^h(j)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	0	0.26	0.62
$j = 2$	0.74	0	0
$j = 3$	0	0.42	0
$j = 0$	0.26	0.32	0.38

(b) The probability that i holds j in the fundamental equilibrium of Economy C.

$\pi_i^e(jk)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	(0,0,0,0)	([0,0.08],0.064,0,0.023)	([0,0.18],0,0.087,0.034)
$j = 2$	(0.064,[0,0.08],0,0.079)	(0,0,0,0)	(0,0,0,0)
$j = 3$	(0,0,0,0)	(0.087,0,[0,0.06],0.053)	(0,0,0,0)
$j = 0$	(0.076,0,0,[0,0.08])	(0,0.079,0,[0,0.1])	(0,0,0.053,[0,0.12])

(c) The probability that i holds j, meets k, and trades in the fundamental equilibrium of Economy C.

$\hat{\pi}_i^e(jk|j)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$(-,0,0,0)$?	$(-,1,0,1)$	$(-,0,1,1)$
$j = 2$	$(1,-,0,1)$	$(0,-,0,0)$?	$(1,-,1,1)$?
$j = 3$	$(1,1,-,1)$?	$(1,1,-,1)$	$(0,0,-,0)$?
$j = 0$	$(1,0,0,-)$	$(0,1,0,-)$	$(0,0,1,-)$

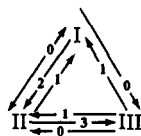(d) Fundamental equilibrium exchange strategies for Economy C.

Fig. 8. Exchange pattern in Economy C.

the correct ones with only two exceptions, each one occurring with a frequency of 0.002 in the last ten periods (agents of type II give up good 1 for good 3, and agents of type III do not accept good 0 for good 1; see tables 15d and 15c).

Given the complexity of the exchange patterns with fiat money, the results for Economy C indicate the ability of our artificially intelligent agents to set up complex social arrangements like fiat money.[17]

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[17]A version of Economy C with the storage costs of Economy A did not converge, probably because the low cost of storing good 1 (0.1) made this good too good a substitute for fiat money. These results for Economy C have been successfully replicated in Marimon and Miller (1989).

Table 15

No. of agents	$A_1 = 50$
No. of classifiers	$E_a = 150$, $C_a = 20$
Storage costs	$s_1 = 9$, $s_2 = 14$, $s_3 = 29$, $s_0 = 0$
Utility	$u_i = 100$, $\forall i$
Initial strengths	$S^a_{e\eta_e(0)} = 0$, $S^a_{c\eta_c(0)} = 0$
Bids	$b_{11} = 0.025$, $b_{12} = 0.025$, $b_{21} = 0.025$, $b_{22} = 0.25$
Specialization[1]	$f_s(t) = 1/2\sqrt{t}$, $p_s = 0.01$
Generalization[2]	$f_g(t) = 1/2\sqrt{t}$, $p_1 = 0.2$, $p_2 = 0.7$, $p_3 = 0.2$, $p_4 = 0.5$, $S = 0$, $N_e = 8$, $N_c = 4$

[1]See footnote 1 of table 7.

[2]See footnote 2 of table 7.

(a) Parameter values used for Economy C.

$\pi^h_{i750}(j)$	$i = 1$	$i = 2$	$i = 3$	$\pi^h_{i1250}(j)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	0	0.18	0.58	$j = 1$	0	0.18	0.77
$j = 2$	0.68	0	0.01	$j = 2$	0.54	0	0.01
$j = 3$	0	0.58	0	$j = 3$	0	0.53	0
$j = 0$	0.32	0.23	0.41	$j = 0$	0.46	0.28	0.21

(b) The frequency with which i holds j at $t = 750$ and $t = 1250$, $\pi^h_{ii}(j)$, for Economy C.

$\pi^e_{i750}(jk)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$(0,0,0,0)$	$(0,0.052,0.004,0.008)$	$(0.008,0.008,0.112,0.06)$
$j = 2$	$(0.06,0,0,0.062)$	$(0,0,0,0)$	$(0,0,0.002,0.004)$
$j = 3$	$(0,0,0,0)$	$(0.116,,0.002,0.052,0.048)$	$(0,0,0,0)$
$j = 0$	$(0.064,0,0,0)$	$(0,0.058,0,0)$	$(0.004,0.008,0.048,0.044)$

(c-i) The frequency with which i holds j, meets k, and trades at $t = 750$, $\pi^e_{ii}(jk)$, for Economy C.

$\pi^e_{i1250}(jk)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$(0,0,0,0)$	$(0.016,0.028,0.002,0.028)$	$(0,0.016,0.028,0.002)$
$j = 2$	$(0.044,0,0,0.066)$	$(0,0,0,0)$	$(0,0,0,0)$
$j = 3$	$(0,0,0,0)$	$(0.014,0,0.044,0.044)$	$(0,0,0,0)$
$j = 0$	$(0.06,0,0,0)$	$(0,0.066,0,0)$	$(0,0,0.044,0)$

(c-ii) The frequency with which i holds j, meets k, and trades at $t = 1250$, $\pi^e_{ii}(jk)$, for Economy C.

$\hat{\pi}^e_{i750}(jk|j)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$(-,-,-,-)$	$(1,1,0,1)$	$(0,0,1,0)$
$j = 2$	$(1,0,0,1)$	$(-,-,-,-)$	$(0,0,1,1)$
$j = 3$	$(0,0,0,0)$	$(1,1,0,1)$	$(0,0,0,1)$
$j = 0$	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,1)$

$\hat{\pi}^e_{i1250}(jk|j)$	$i = 1$	$i = 2$	$i = 3$
$j = 1$	$(-,-,-,-)$	$(1,1,1,1)$	$(0,0,1,0)$
$j = 2$	$(1,0,0,1)$	$(-,-,-,-)$	$(0,0,0,0)$
$j = 3$	$(0,1,0,0)$	$(1,1,0,1)$	$(-,-,-,-)$
$j = 0$	$(1,0,0,0)$	$(0,1,0,0)$	$(0,0,1,0)$

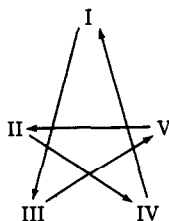(d) Actions of winning classifiers for i with condition (j, k), $\hat{\pi}^e_{ii}(jk|j)$, for Economy C.

Fig. 9. Production patterns in Economy D; an arrow shows that the agent at the origin of the arrow produces the good at the point of the arrow.

Economy D

In Economy D we enhance complexity by considering five goods and five types. The production technologies are described by fig. 9. That is, type I produces good 3, and so on. As before, each type of agent only derives utility from consuming the good of the same number. Storage costs are ranked in increasing order (see table 4). For this economy we do not start with any characterization of a stationary equilibrium. With enough work, one could obtain an analytic solution for the fundamental equilibrium for such economy, but here our purpose is to let our artificially intelligent agents suggest to us what an equilibrium might be. Given this suggestion, we could then presumably verify whether it is an equilibrium. Our simulation is reported in table 16.

From the results in table 16, we can see that the trading patterns nearly seem to describe a fundamental equilibrium in which agents are only willing to trade for commodities of lower cost than the one currently in storage, except that they always accept the commodity of their type. These fundamental trading patterns are shown in fig. 10. Some speculative moves can be detected. An example of this is that agents of type II accept (from I) good 3 for good 1 in order to trade 3 for 2 with type III. We do not pursue here a full analysis of partially speculative equilibria for this economy.

8. Conclusions

The work described above is presented as the first steps of our project to use the classifier systems of Holland to model learning in multi-agent environments. Classifier systems have previously been applied to solve some complex optimization problems [see *Machine Learning* (1988)]. Our application has involved some extensions to handle the fact that ours is a multi-agent environment in which agents are solving Markovian dynamic decision problems.

The work described in this paper has accomplished two objectives that are parts of our broader project. First, our simulations have demonstrated by

Table 16

$A_1 = 50$
$E_a = 180, C_a = 20$
$s_1 = 1, s_2 = 4, s_3 = 9$
$u_t = 200, \text{Ai}$
$S^a_{en_e(0)} = 0, S^a_{cn_e(0)} = 0$
$b_{11} = 0.025, b_{12} = 0.025, b_{21} = 0.25, b_{22} = 0.25$
$f_s(t) = 1/2\sqrt{t}, p_s = 0.01$
$f_g(t) = 1/2\sqrt{t}, p_1 = 0.2, p_2 = 0.7, p_3 = 0.2, p_4 = 0.5.$
$\mathbf{S} = 0, N_e = 8, N_c = 4$

No. of agents	
No. of classifiers	
Storage costs	
Utility	
Initial strengths	
Bids	
Specialization[1]	
Generalization[2]	

[1]See footnote 1 of table 7.
[2]See footnote 2 of table 7.

(a) Parameter values used for Economy D.

$\pi^h_{i500}(j)$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0	0.336	0.626	0.032	0.006
$i = 2$	0.322	0	0.038	0.614	0.026
$i = 3$	0.118	0.180	0	0.054	0.648
$i = 4$	0.922	0.044	0.002	0	0.032
$i = 5$	0.210	0.720	0.038	0.032	0

(b-i) The frequency with which i holds j at $t = 500$, $\pi^h_{it}(j)$, for Economy D.

$\pi^h_{i1750}(j)$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0	0.832	0.158	0.004	0.006
$i = 2$	0.096	0	0.004	0.870	0.030
$i = 3$	0.048	0.418	0	0.052	0.482
$i = 4$	0.988	0.008	0.002	0	0.002
$i = 5$	0	0.974	0.024	0.002	0

(b-ii) The frequency with which i holds j at $t = 1750$, $\pi^h_{it}(j)$, for Economy D.

$\pi^e_{i500}(jk)$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	(0,0,0,0,0)	(0.03,0.02,0.01,0.01,0)	(0.06,0.05,0.03,0,0)	(0.01,0,0,0,0)	(0,0,0,0,0)
$i = 2$	(0.08,0.11,0.04,0.01,0.02)	(0,0,0,0,0)	(0,0,0,0,0)	(0.17,0.04,0.01,0.03,0.03)	(0,0,0,0,0)
$i = 3$	(0.04,0,0.02,0.02,0)	(0.03,0,0.01,0.02,0.01)	(0,0,0,0,0)	(0.01,0,0,0.01,0)	(0.09,0.09,0.01,0.03,0.03)
$i = 4$	(0.24,0.04,0,0.14,0.03)	(0,0,0,0.01)	(0,0,0,0,0)	(0,0,0,0)	(0,0,0,0,0)
$i = 5$	(0.03,0,0.03,0.03)	(0.09,0.02,0.03,0,0.08)	(0,0,0.01,0.01)	(0.01,0,0,0,0)	(0,0,0,0,0)

(c-i) The frequency with which i holds j, meets someone with k, and trades, $\pi^e_{i500}(jk)$, for Economy D.

$\pi^e_{i1750}(jk)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	(0,0,0,0,0)	(0.03,0.17,0.01,0.01,0)	(0,0.05,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)
$i=2$	(0.01,0.04,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0.18,0.04,0,0.10,0.03)	(0,0,0,0,0)
$i=3$	(0.01,0.01,0,0.01,0)	(0.01,0.05,0,0.04,0.02)	(0,0,0,0,0)	(0.01,0,0,0,0)	(0,0.1,0,0.03,0.03)
$i=4$	(0.09,0.03,0,0.19,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)
$i=5$	(0,0,0,0,0)	(0.03,0.26,0.04,0,0.07)	(0.01,0,0,0,0)	(0,0,0,0,0)	(0,0,0,0,0)

(c-ii) The frequency with which holds j, meets someone with k, and trades, $\pi^e_{i1750}(jk)$, for Economy D.

$\hat\pi^e_{i500}(jk)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	(1 - - - 0)	(11100)	(10100)	(1111 -)	(0 - - 01)
$i=2$	(11110)	(111 - -)	(11100)	(11100)	(1111 -)
$i=3$	(10110)	(10111)	(1 - 11 -)	(1 - 111)	(11110)
$i=4$	(10011)	(1 - 011)	(1 - 11 -)	(1 - 11 -)	(0111 -)
$i=5$	(10011)	(10001)	(10011)	(11111)	(- - - - 1)

(d-i) Actions of winning classifiers i with condition (j, k), $\hat\pi^e_{i500}(jk|j)$, for Economy D.

$\hat\pi^e_{i1750}(jk)$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	(1 - - - 0)	(11010)	(11000)	(11111)	(01110)
$i=2$	(11100)	(- - - - -)	(110 - 1)	(11100)	(11111)
$i=3$	(1011 -)	(01100)	(- - - - -)	(0010 -)	(11110)
$i=4$	(00010)	(10110)	(00010)	(- - - - -)	(11111)
$i=5$	(10 - 01)	(11101)	(01011)	(10 - 11)	(- - - - -)

(d-ii) Actions of winning classifiers for i with condition (j, k), $\hat\pi^e_{i1750}(jk|j)$, for Economy D.
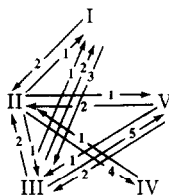
Fig. 10. Exchange patterns in Economy D, exhibited by classifier systems.

example that multi-agent systems of classifiers can exhibit interesting behavior, and can eventually learn to play Nash–Markov equilibria. Second, the process of making classifier systems cope with the Kiyotaki–Wright environment has prompted us to formalize a variety of concepts and definitions that our subsequent work shall build on. These definitions will facilitate formal analyses of systems of multi-agent classifier systems, which we intend to pursue.

Among the unfinished aspects which we leave for future papers, two interrelated ones are quite important. First, our experiments with the genetic-type algorithms in our incomplete-enumeration classifier systems have convinced us that existing algorithms can be improved. These improvements, which will increase the amount of experimentation that is done at the 'right times', will have the effect of improving the capacity of the classifier systems to settle upon optimal outcomes more rapidly, and to avoid becoming locked in to suboptimal patterns of interaction. The improved algorithms will also help with a second major piece of unfinished business, which is to develop analytical results on the convergence of classifier systems. The new algorithms will be set up in such a way that we can apply the stochastic approximation approach to convergence analysis which we alluded to in section 5.

References

Arthur, Brian, 1989, The dynamics of classifier competitions, Mimeo., March.

Axelrod, Robert, 1987, The evolution of strategies in the iterated prisoner's dilemma, in: Lawrence Davis, ed., Genetic algorithms and simulated annealing (Morgan Kaufman, Los Altos, CA).

Bray, Margaret, 1982, Learning, estimation, and stability of rational expectations, Journal of Economic Theory 26, 318–339.

Bray, M.M. and N.E. Savin, 1986, Rational expectations equilibria, learning and model specification, Econometrica 54, 1129–1160.

Fourgeaud, C., C. Gourieroux, and J. Pradel, 1986, Learning procedure and convergence to rationality, Econometrica 54, 845–868.

Goldberg, D.E., 1989, Genetic algorithms in search optimization and machine learning (Addison-Wesley, Menlo Park, CA).

Grefenstette, J.J., 1988, Credit assignment in rule discovery systems based on genetic algorithms, Machine Learning 3, 225–245.

Holland, J.H., 1986, Escaping brittleness: The possibilities of general-purpose learning algorithms applied to parallel rule-based systems, in: R.S. Michalski, J.G. Carbonell, and T.M. Mitchell, eds., Machine learning: An artificial intelligence approach, Vol. 2 (Morgan Kaufmann, Los Altos, CA).

Holland, J.H., 1975, Adaptation in natural and artificial systems (University of Michigan Press, Ann Arbor, MI).

Holland, J.H., K.J. Holyoak, R.E. Nisbett, and P.R. Thagard, 1986, Induction: Processes of inference, learning, and discovery (MIT Press, Cambridge, MA).

Kiyotaki, N. and R. Wright, 1989, On money as a medium of exchange, Journal of Political Economy 97, 927–954.

Knez, Peter and R. Litterman, 1989, Description of results in telephone conversations, May–July.

Ljung, L. and T. Soderstrom, 1983, Theory and practice of recursive identification (MIT Press, Cambridge, MA).

Machine Learning, 1988, Special issue on genetic algorithms 3, no. 2/3.

Marcet, Albert and T.J. Sargent, 1989a, Convergence of least squares learning mechanisms in self referential linear stochastic models, Journal of Economic Theory 48, 337–368.

Marcet, Albert and T.J. Sargent, 1989b, Convergence of least squares learning in environments with private information and hidden state variables, Journal of Political Economy 97, 1306–1322.

Marimon, R., 1989, A genetic algorithm for classifier systems, forthcoming.

Marimon, R. and J. Miller, 1989, Money as a medium of exchange in an economy with genetically reproduced decision rules, forthcoming.

Marimon, R.E., E. McGrattan, and T.J. Sargent, 1989, Money as a medium of exchange in an economy with artificially intelligent agents, Working paper, Oct. (Hoover Institution, Stanford, CA).

Simon, Carl, 1989, Talk given at Santa Fe Institute, March.

Wilson, S.W., 1987, Classifier systems and the animat problem, Machine Learning 2, 199–228.