EXTENSIONS OF THE UNIVERSAL COXETER GROUP BY $\mathbb{Z}$

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Abstract

Geometric group theorists have long been interested in F_n-by-$\mathbb{Z}$ groups, or extensions of the free group by $\mathbb{Z}$. The universal Coxeter group on n generators, denoted W_n, is virtually free, and W_n-by-$\mathbb{Z}$ groups are virtually free-by-$\mathbb{Z}$. However, the automorphism group of W_n is less complicated than that of a free group. The goal of this thesis is to explore the geometric properties of W_n-by-$\mathbb{Z}$ groups, specifically in rank three. We show that $W_3 \rtimes_\phi \mathbb{Z}$ is CAT(0) for every $\phi \in \mathrm{Aut}(W_3)$. Lastly, we discuss some progress toward identifying when $W_n \rtimes_\phi \mathbb{Z}$ is hyperbolic.

To my parents, whose unconventional parenting instilled confidence and independence.

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Extensions of the Universal Coxeter Group by $\mathbb{Z}$

Chapter 1

Introduction

Coxeter groups are a well-known and well-studied class of groups in geometric group theory. In the simplest example, the universal Coxeter group, denoted W_n, is generated by n involutions with no relations between them. This group is virtually free and has interesting combinatorial properties. One such property is that W_n contains an index two characteristic copy of a non-abelian free group, F_{n-1}, of rank $n-1$. This tells us that any automorphism of W_n is also an automorphism of F_{n-1}. This will allow us to use the theory of automorphisms of free groups to help us study W_n-by-$\mathbb{Z}$ groups.

Free-by-$\mathbb{Z}$ groups are well-studied, but there are still open questions about their geometry. For example, given $\phi \in \mathrm{Aut}(F_n)$, $n \geq 3$, can we determine whether $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic from the algebraic properties of ϕ? Brinkmann used train track theory to prove that $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if no power of ϕ preserves a conjugacy class [10]. Equivalently, $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if it does not contain a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. Brinkmann proved that there exists an algorithm that inputs $u, v \in F_n$ and outputs whether there exists a power p such $\phi^p(u)$ and v are conjugate [11]. In general, it is unknown how to check whether or not a power of ϕ preserves any conjugacy class.

As noted above, the universal Coxeter group W_n contains F_{n-1} as a subgroup of index two. Furthermore, every W_n-by-$\mathbb{Z}$ contains an index two F_{n-1}-by-$\mathbb{Z}$. The property of being hyperbolic passes to finite extensions, so W_n-by-$\mathbb{Z}$ is hyperbolic if and only if its F_{n-1}-by-$\mathbb{Z}$ subgroup is hyperbolic.

The original idea for this thesis was: Do the interesting combinatorics in W_n make it possible to determine when a power of an automorphism preserves a conjugacy class - i.e. is it "easier" to determine when an automorphism of W_n preserves a conjugacy class than it is for F_n? Unfortunately, I hit several roadblocks in exploring this idea. Some progress about this question and future directions are included at the end of this thesis. We include a specific example of a hyperbolic $W_4 \rtimes_\phi \mathbb{Z}$.

Every F_2-by-$\mathbb{Z}$ is CAT(0), but not hyperbolic. These groups are not hyperbolic because every automorphism of F_2 preserves the conjugacy class of the commutator of F_2. As a result, there are no hyperbolic W_3-by- $\mathbb{Z}$ groups.

If a group G acts geometrically on a CAT(0) space X, then there is a homomorphism from G into $\text{Isom}(X)$. We say G is *faithfully* CAT(0) if that homomorphism is injective. Now, if G' is a finite extension of G, then one can ask whether the action of G on X extends to a faithful action of G' on X. More generally, one can ask whether G' is also (faithfully) a CAT(0) group - i.e. it is possible that a different space is needed for G'. In general, it is still unknown if the property of being (faithfully) CAT(0) passes to finite extensions in this sense.

With this in mind, the new question for this thesis became:

Question: If $F_2 \rtimes_\phi \mathbb{Z}$ acts geometrically on a CAT(0) space X, can we extend the action to a faithful geometric action $W_3 \rtimes_\phi \mathbb{Z} \curvearrowright X$?

The Main theorem of this thesis answers this question in the affirmative.

Theorem 5.1.3 *For every automorphism ϕ of W_3, $W_3 \rtimes_\phi \mathbb{Z}$ is CAT(0).*

The automorphisms of F_2 can be categorized into three types (parabolic, hyperbolic, and elliptic) depending on their image in $GL(2,\mathbb{Z})$ (see section 4.1 for details). For the parabolic and hyperbolic automorphisms of F_2, we show that the action of $F_2 \rtimes_\phi \mathbb{Z}$ can be extended to a faithful action by $W_3 \rtimes_\phi \mathbb{Z}$. For the elliptic automorphisms of F_2, we use a slightly different CAT(0) space for W_3. As a result, we prove all $W_3 \rtimes_\phi \mathbb{Z}$ groups are CAT(0).

The inspiration for extending a geometric action to a finite extension comes from Piggott, Ruane, and Walsh [36]. They showed that $\text{Aut}(F_2) \cong B_4/Z_4 \rtimes \mathbb{Z}_2$ is CAT(0) by extending the action of B_4/Z_4 on a CAT(0) space by an order two isometry. The authors emphasize that a group is CAT(0) if it acts geometrically *and faithfully* on a CAT(0) space. To extend $G \curvearrowright X$ to $G \rtimes \mathbb{Z}_2 \curvearrowright X$, this amounts to finding an order two isometry such that $G \rtimes \mathbb{Z}_2 \hookrightarrow \text{Isom}(X)$ is injective.

Chapter 2

Automorphism Group of the Universal Coxeter Group

In this chapter, we discuss the structure of $\mathrm{Aut}(W_n)$. We will also see that this group is much simpler than $\mathrm{Aut}(F_n)$. The latter contains transvections, and the former does not.

Let $a_1, \ldots, a_n$ be the generators of W_n. We define W_n using the following group presentation:

$$W_n = \langle a_1, \ldots, a_n | a_1^2, \ldots, a_n^2 \rangle$$

The algebraic structure of $\mathrm{Aut}(W_n)$ comes from the following theorem of J. Tits.

Theorem 2.0.1 (Tits 1988) $Aut(W_n) = Aut^{\circ}(W_n) \rtimes \Sigma_n$

Σ_n is the symmetric group on n letters, or the group of permutations of the generators $\{a_1, \ldots, a_n\}$. $\mathrm{Aut}^{\circ}(W_n)$ is the group of automorphisms that send every generator to a conjugate of itself: $\phi(a_i) = w a_i w^{-1}$. It is generated by elementary partial conjugations [30]. An elementary partial conjugation χ_{ij} is "a_i conjugates a_j": $\chi_{ij}(a_j) = a_i a_j a_i$. The other generators are fixed.

$$\chi_{ij}(a_k) = \begin{cases} a_k, & k \neq j \\ a_i a_k a_i, & k = j \end{cases}$$

Since Σ_n is a finite group, $\mathrm{Aut}^{\circ}(W_n)$ is finite index in $\mathrm{Aut}(W_n)$.

We can see that every automorphism of W_n sends each generator to a conjugate of a generator: $\phi(a_i) = w a_j w^{-1}$. The inverse of any word in W_3 is the word read backwards: $w^{-1} = \overline{w}$, so $\phi(a_i)$ is a palindrome.

Define $\mathrm{Out}^{\circ}(W_n)$ by the short exact sequence below. Gutierrez and Kaul proved that the short exact sequence splits, and therefore $\mathrm{Aut}^\circ(W_n) = \mathrm{Inn}(W_n) \rtimes \mathrm{Out}^\circ(W_n)$ [22].

$$1 \longrightarrow \mathrm{Inn}(W_n) \longrightarrow \mathrm{Aut}^\circ(W_n) \longrightarrow \mathrm{Out}^\circ(W_n) \longrightarrow 1$$

In summary, we can say the following:

$$\mathrm{Aut}(W_n) = [\mathrm{Inn}(W_n) \rtimes \mathrm{Out}^\circ(W_n)] \rtimes \Sigma_n$$

Since the center of W_n is trivial, $\mathrm{Inn}(W_n)$ is isomorphic to W_n, we can see that $\mathrm{Aut}(W_n)$ can be written as:

$$\mathrm{Aut}(W_n) = [(W_n) \rtimes \mathrm{Out}^\circ(W_n)] \rtimes \Sigma_n$$

There are no transvections in $\mathrm{Aut}(W_n)$ because the map that sends a generator to a product of two generators is not an automorphism of W_n. This makes the automorphism group simpler than that of F_{n-1}.

We end this chapter with a specific example of an automorphism of W_4 Suppose $\psi \in \mathrm{Aut}(W_4)$ where W_4 is generated by $\{a, b, c, d\}$. We can write ψ as a composition of partial conjugations and a permutation of the generators:

$$\psi = \chi_{a,\{bc\}} \circ \chi_{d,(bc)} \circ \sigma_{(bdc)}$$

The partial conjugation $\chi_{a,\{bc\}}$ is "a conjugates b and c." A partial conjugation is a product of elementary partial conjugations: $\chi_{a,\{bc\}} = \chi_{a,b} \circ \chi_{a,c}$.

$$\psi(a) = a$$

$$\psi(b) = d$$

$$\psi(c) = dabad$$

$$\psi(d) = dacad$$

We will show in chapter 6 that $W_4 \rtimes_\psi \mathbb{Z}$ is hyperbolic.

Chapter 3

Relationship Between W_n and F_{n-1}

In this chapter, we will discuss the relationship between W_n and F_{n-1} that we will use in the rest of the thesis. Specifically, we will establish the following facts:

1. $W_n = F_{n-1} \rtimes_\tau \mathbb{Z}_2$
2. F_{n-1} is a characteristic subgroup of W_n
3. $\mathrm{Aut}(W_3) \cong \mathrm{Aut}(F_2)$
4. For any $\phi \in \mathrm{Aut}(W_3)$, $F_2 \rtimes_\phi \mathbb{Z}$ is an index two subgroup of $W_3 \rtimes_\phi \mathbb{Z}$.

3.1 F_{n-1} is an index two characteristic subgroup of W_n

The Universal Coxeter group has presentation $W_n = \langle a_1, \ldots, a_n | a_i^2 \rangle$. Let E_n be the subgroup of even length words.

Lemma 3.1.1 *E_n is an index two subgroup of W_n.*

Proof: There are two cosets in W_n/E_n: the even coset and the odd coset. For an odd length word $w \in W_n$, $w = a_1(a_1 w)$ where $a_1 w \in E_n$. Therefore, $\{1, a_1\}$ are coset representatives for W_n/E_n. □

Lemma 3.1.2 *E_n is a characteristic subgroup of W_n.*

Proof: Fix an automorphism $\phi \in \mathrm{Aut}(W_n)$. We will show that ϕ preserves parity of word length.

Assume $w \in W_n$ has length $|w| = r$. Then $w = s_1 \ldots s_r$ where $s_i \in \{a_1, \ldots, a_n\}$ and $\phi(w) = \phi(s_1) \ldots \phi(s_r)$.

For a generator a_i of W_n, $\phi(a_i)$ is a palindrome of some generator a_j. Therefore, $\phi(a_i)$ has odd length.

As a result, $\phi(w)$ is the product of r odd length words, and $|\phi(w)|$ and r have the same parity. All relations in W_n are length two, so reducing preserves parity. □

E_n is freely generated by $\{a_1a_2, \ldots, a_1a_n\}$. Let F_{n-1} be the free group with $n-1$ generators $\{x_1, \cdots, x_{n-1}\}$. The map $E_n \longrightarrow F_{n-1}$ given by $a_1a_i \mapsto x_{i-1}$ is an isomorphism. Define $\tau \in \mathrm{Aut}(F_{n-1})$ by $\tau(x_i) = x_i^{-1}$. Then $W_n \cong F_{n-1} \rtimes_\tau \mathbb{Z}_2$ where $\mathbb{Z}_2 = \langle a_1 \rangle$.

3.2 The Homomorphism $\rho_n : \mathbf{Aut}(W_n) \longrightarrow \mathbf{Aut}(F_{n-1})$

For a characteristic subgroup $H \leq G$, let the map $\rho : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(H)$ be determined by restricting to H. Then, ρ is a homomorphism.

Lemma 3.2.1 (Rose 1975) *Suppose H is a characteristic subgroup of G and ρ : $Aut(G) \to Aut(H)$ is determined by restricting to H. If the centralizer $C_G(H) = 1$, then ρ is injective.*

Proof: If $C_G(H) = 1$, then $Z(G) = 1$, and $G \cong \mathrm{Inn}(G)$. Identify H as a subgroup of $\mathrm{Inn}(G)$ by identifying h with conjugation by h. Then $C_{\mathrm{Inn}(G)}(H) = 1$.

We claim: if $Z(G) = 1$, then $C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = 1$. Suppose $\phi \in C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$. Then for all $\gamma_x \in \mathrm{Inn}(G)$ and $g \in G$, $\phi \circ \gamma_x(g) = \gamma_x \circ \phi(g)$. Rearrange to see $[x^{-1}\phi(x)]\phi(g) = \phi(g)[x^{-1}\phi(x)]$. Therefore, $x^{-1}\phi(x) \in Z(G)$. Since $Z(G) = 1$, $\phi(x) = x$ for all $x \in G$. Hence $\phi = Id$.

By a result of H. Weilandt, $H \leq \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$ and $C_{\mathrm{Inn}(G)}(H) = 1 = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$ implies $C_{\mathrm{Aut}(G)}(H) = 1$.

Now we show that $\mathrm{Ker}(\rho) = Id_G$. Suppose $\rho(\phi) = id_H$. For all $h \in H$ and $g \in G$:

$$\phi \circ \gamma_h(g) = \phi(hgh^{-1}) = h\phi(g)h^{-1} = \gamma_h \circ \phi(g)$$

Therefore, $\phi \in C_{\mathrm{Aut}(G)}(H)$ and we are done. □

Define $\rho_n : \mathrm{Aut}(W_n) \longrightarrow \mathrm{Aut}(F_{n-1})$ by $\rho_n(\phi) = \phi|_{E_n}$, and then identify E_n with F_{n-1}.

Corollary 3.2.2 *For $n \geq 3$, $\rho_n : Aut(W_n) \longrightarrow Aut(F_{n-1})$ is injective.*

Proof: For $n \geq 3$, the centralizer of F_{n-1} in W_n is trivial. □

Here, we explain injectivity for ρ_n. Assume ϕ is the identity on E_n. For a generator a_i of W_n, $\phi(a_i)$ is an odd length palindrome, call it p_i. Thus, $\phi(a_1a_2) = p_1p_2 = a_1a_2$.

Lemma 3.2.3 (Piggott-Ruane) *Let $p, q \in W_n$ be palindromes of odd length such that $pq = a_ia_j$ for some $i \neq j$. Then $p, q \in \langle a_i, a_j \rangle$.*

By the lemma above, $p_1, p_2 \in \langle a_1, a_2 \rangle$. Similarly, $\phi(a_1a_3) = p_1p_3 = a_1a_3$ implies $p_1 \in \langle a_1, a_3 \rangle$. Therefore, $p_1 = a_1$. Now we have $\phi(a_1) = a_1$ and $\phi|_{E_n} = Id_{E_n}$. It follows that $\phi = Id_{W_n}$.

ρ_n : $\mathbf{Aut}(W_n) \longrightarrow \mathbf{Aut}(F_{n-1})$ is Surjective, $n \leq 3$

For $n = 2, 3$, ρ_n is surjective because its image contains the generators of $\mathrm{Aut}(F_{n-1})$. The automorphism group of $F_2 = \langle x_1, x_2 \rangle$ is generated by $\{\alpha_1, \alpha_2, \beta_{12}, \beta_{21}\}$ where $\alpha_i(x_i) = x_i^{-1}$, $\beta_{ij}(x_i) = x_ix_j$, and the other generator is fixed [32]. Below, we show that all four generators are in the image of $\rho_3 : \mathrm{Aut}(W_3) \longrightarrow \mathrm{Aut}(F_2)$. The automorphism χ_{ij} of W_n is given by $\chi_{ij}(a_j) = a_ia_ja_i$. The other generators are fixed.

$$\chi_{12}(a_1a_2) = a_1a_1a_2a_1 = a_2a_1 \implies \rho_3(\chi_{12})(x_1) = x_1^{-1} = \alpha_1(x_1)$$

One can also verify:

$$\rho_3(\chi_{12}) = \alpha_1$$

$$\rho_3(\chi_{13}) = \alpha_2$$

$$\rho_3(\chi_{12} \circ \chi_{3\{12\}} \circ \sigma_{13}) = \beta_{12}$$

$$\rho_3(\chi_{13} \circ \chi_{2\{13\}} \circ \sigma_{12}) = \beta_{21}$$

3.3 $W_n \rtimes_\phi \mathbb{Z}$ is a finite extension of $F_{n-1} \rtimes_\phi \mathbb{Z}$

Proposition 3.3.1 *The following groups are isomorphic for any $\phi \in Aut(W_n)$.*

1. $W_n \rtimes_\phi \mathbb{Z}$
2. $(F_{n-1} \rtimes_\tau \mathbb{Z}_2) \rtimes_\phi \mathbb{Z}$
3. $(F_{n-1} \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$ *for some* $\hat{\tau} \in Aut(F_{n-1} \rtimes_\phi \mathbb{Z})$

We established $W_n \cong F_{n-1} \rtimes_\tau \mathbb{Z}_2$, and therefore (1) and (2) are isomorphic.

Proposition 3.3.2 *The short exact sequence splits.*

$$1 \longrightarrow F_{n-1} \rtimes_\phi \mathbb{Z} \xrightarrow{\iota} W_n \rtimes_\phi \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 1$$

The proposition above gives us $W_n \rtimes_\phi \mathbb{Z} \cong (F_{n-1} \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$ for some $\hat{\tau} \in \mathrm{Aut}(F_{n-1} \rtimes_\phi \mathbb{Z})$. We will show that $\hat{\tau}$ depends on $\phi \in \mathrm{Aut}(W_n)$.

Our goal for this section is to establish the following:

- $W_3/F_2 \cong W_3 \rtimes_\phi \mathbb{Z}/F_2 \rtimes_\phi \mathbb{Z}$
- $W_3 \rtimes_\phi \mathbb{Z} \cong (F_2 \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$ because the short exact sequence splits
- Define $\hat{\tau} \in \mathrm{Aut}(F_2 \rtimes_\phi \mathbb{Z})$

We rely on theorem 3.3 in Keith Conrad's notes for this section [14].

In order for the short exact sequence to make sense, we need $F_{n-1} \rtimes_\phi \mathbb{Z}$ to be normal in in $W_n \rtimes_\phi \mathbb{Z}$. There are two ways we can show this. First, the lemma below implies $[W_n \rtimes_\phi \mathbb{Z} : F_{n-1} \rtimes_\phi \mathbb{Z}] = [W_n : F_{n-1}] = 2$.

Lemma 3.3.3 *Let H be a characteristic subgroup of G. If $[G : H] = k$, then $[G \rtimes_f \mathbb{Z} : H \rtimes_f \mathbb{Z}] = k$.*

Proof: Every element in $G \rtimes_\phi \mathbb{Z}$ can be written as gt^p for some $g \in G$ and $p \in \mathbb{Z}$. Assume $\{g_1, \cdots, g_k\}$ are left coset representatives of G/H. Then $g = g_i h$ for some $h \in H$ and $i \in \{1, \cdots, k\}$. Now, every element in $G \rtimes_\phi \mathbb{Z}$ can be written as $(g_i t^0)(ht^p)$

where $ht^p \in H \rtimes_\phi \mathbb{Z}$ and $g_i t^0 \in G \rtimes_\phi \mathbb{Z}$. Therefore, $\{a_1 t^0, \ldots, a_k t^0\}$ is a set of coset representatives for $G \rtimes_\phi \mathbb{Z}/H \rtimes_\phi \mathbb{Z}$. □

The lemma below gives us another way to show $F_{n-1} \rtimes_\phi \mathbb{Z} \trianglelefteq W_n \rtimes_\phi \mathbb{Z}$. In our case, there are only two cosets $\{F_{n-1}, a_1 F_{n-1}\}$, and F_{n-1} is characteristic. Thus, the cosets are preserved by any $\phi \in \mathrm{Aut}(W_n)$.

Lemma 3.3.4 *Assume $H \trianglelefteq G$. If $\phi \in Aut(G)$ preserves the cosets of H in G (i.e. g and $\phi(g)$ are in the same coset), then $H \rtimes_\phi \mathbb{Z} \trianglelefteq G \rtimes_\phi \mathbb{Z}$.*

Proof: Fix $gt^p \in G \rtimes_\phi \mathbb{Z}$ and $ht^q \in H \rtimes_\phi \mathbb{Z}$. We use the relation $tgt^{-1} = \phi(g)$ to move t to the right.

$$\begin{aligned}(gt^p)(ht^q)(gt^p)^{-1} &= gt^p ht^q t^{-p} g^{-1} \\ &= g\phi^p(h) t^q g^{-1} \\ &= g\phi^p(h)\phi^q(g^{-1}) t^q\end{aligned}$$

We will show $g\phi^p(h)\phi^q(g^{-1}) \in H$. Let a be a coset representative of G/H such that $g, \phi^q(g) \in aH$. There exists $k, \ell \in H$ such that $g = ak$ and $\phi^q(g) = a\ell$.

$$\begin{aligned}g\phi^p(h)\phi^q(g^{-1}) &= ak\phi^p(h)(a\ell)^{-1} \\ &= ak\phi^p(h)\ell^{-1}a^{-1} \\ &= ah'a^{-1}\end{aligned}$$

Let $k\phi^p(h)\ell^{-1} = h' \in H$. Since H is normal in G, $ah'a^{-1} \in H$. □

Under the hypotheses in the lemma above, the bijection $f : G \rtimes_\phi \mathbb{Z}/H \rtimes_\phi \mathbb{Z} \longrightarrow G/H$ given by $f : [gt^p] \mapsto [g]$ is a homomorphism. Fix $gt^p, xt^q \in G \rtimes_\phi \mathbb{Z}$ and let $[gt^p], [xt^q]$ be their corresponding cosets in $G \rtimes_\phi \mathbb{Z}/H \rtimes_\phi \mathbb{Z}$.

$$\begin{aligned}f([gt^p][xt^q]) &= f([g\phi^p(x)t^{p+q}]) \\ &= [g\phi^p(x)]\end{aligned}$$

$$= [g][\phi^p(x)]$$

If ϕ preserves the cosets of H in G, then x and $\phi^p(x)$ belong to the same coset: $[x] = [\phi^p(x)]$.

$$\begin{aligned} f([gt^p][xt^q]) &= [g][\phi^p(x)] \\ &= [g][x] \\ &= f([gt^p])f([xt^q]) \end{aligned}$$

It is easy to see that the inverse bijection $f^{-1} : G/H \longrightarrow G \rtimes_\phi \mathbb{Z}/H \rtimes_\phi \mathbb{Z}$ is also a homomorphism. So, when ϕ preserves the cosets of H, $G/H \cong G \rtimes_\phi \mathbb{Z}/H \rtimes_\phi \mathbb{Z}$. We use this in the next lemma.

Lemma 3.3.5 *Let $H \trianglelefteq G$ and consider the usual short exact sequence (ι is inclusion π is projection onto G/H).*

$$1 \longrightarrow H \xrightarrow{\iota} G \xrightarrow{\pi} K \longrightarrow 1$$

Let $\phi \in Aut(G)$. If ϕ preserves the cosets of H in G, then there is a short exact sequence:

$$1 \longrightarrow H \rtimes_\phi \mathbb{Z} \xrightarrow{\iota} G \rtimes_\phi \mathbb{Z} \xrightarrow{\hat{\pi}} \hat{K} \longrightarrow 1$$

If the first sequence splits, then the second sequence splits.
Proof:

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\iota"] & G \arrow[r, "\pi"] & K \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & H \rtimes_\phi \mathbb{Z} \arrow[r, "\hat{\iota}"] & G \rtimes_\phi \mathbb{Z} \arrow[r, "\hat{\pi}"] \arrow[u] & \hat{K} \arrow[r] & 1 \end{tikzcd}$$

Define $\hat{\pi} : G \rtimes_\phi \mathbb{Z} \longrightarrow \hat{K}$ by the commutative diagram above. $\hat{\pi}$ is a composition of epimorphisms, and is therefore an epimorphism.

If the first short exact sequence splits, then there is a homomorphism $r : K \longrightarrow G$ such that $\pi(r(k)) = k$ for all $k \in K$. Define $\hat{r} : \hat{K} \longrightarrow G \rtimes_\phi \mathbb{Z}$ by $\hat{r}(k) = r(k)t^0$. Clearly,

$\hat{r}$ is a homomorphism if r is a homomorphism.

$$\begin{tikzcd} G/H \arrow[r, "r"] \arrow[d, "\cong"] & G \arrow[d, "\iota"] \\ G \rtimes_\phi \mathbb{Z}/H \rtimes_\phi \mathbb{Z} \arrow[r, "\hat{r}"] & G \rtimes_\phi \mathbb{Z} \end{tikzcd}$$

Since $\pi(r(k)) = k$, we have $\hat{\pi}(\hat{r}(k)) = \hat{\pi}(r(k)t^0) = \pi(r(k)) = k$. Therefore, the second short exact sequence splits. □

If a short exact sequence splits, then we get a semi-direct product. See, for example, Theorem 3.3 in Keith Conrad's notes [14]. The proof of the theorem tells us how to build the semi-direct product. There is a homomorphism $\hat{\tau} \in Aut(F_{n-1} \rtimes_\phi \mathbb{Z})$ and an isomorphism $\theta : W_n \rtimes_\phi \mathbb{Z} \longrightarrow (F_{n-1} \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$ such that the diagram commutes:

$$\begin{tikzcd} 1 \arrow[r] & F_{n-1} \rtimes_\phi \mathbb{Z} \arrow[r, "\iota"] \arrow[d, "id"] & W_n \rtimes_\phi \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\theta"] & \mathbb{Z}_2 \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & F_{n-1} \rtimes_\phi \mathbb{Z} \arrow[r, "\iota"] & (F_{n-1} \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2 \arrow[r, "\pi"] & \mathbb{Z}_2 \arrow[r] & 1 \end{tikzcd}$$

Assume $\mathbb{Z}_2 \cong \langle a \rangle$. To find $\hat{\tau}$, the proof tells us $\iota(\tau(ht^p)) = \hat{r}(a)\iota(ht^p)\hat{r}(a)$. In our case, ι is inclusion and $\hat{r}(a) = at^0$, so this translates to $\hat{\tau}(ht^p) = (at^0)(ht^p)(at^0) = aht^pa$. In $W_n \rtimes_\phi \mathbb{Z}$, $t^pat^{-p} = \phi^p(a)$. Therefore, $\hat{\tau}(ht^p) = ah\phi^p(a)t^p$.

For a generator x_i of F_{n-1}, $\hat{\tau}(x_it^0) = ax_iat^0$. Remember, in $W_n \cong F_{n-1} \rtimes_\tau \langle a \rangle$, $ax_ia = \tau(x_i) = x_i^{-1}$. Thus, $\hat{\tau}(x_it^0) = x_i^{-1}t^0$.

The image of $\mathbb{Z} = \langle t \rangle$ under $\hat{\tau}$ will depend on $\phi \in \mathrm{Aut}(W_n)$: $\hat{\tau}(t) = a\phi(a)t$

3.4 W_3 and F_2

Every F_2-by-$\mathbb{Z}$ is CAT(0), but this is not the case for higher rank free groups. For example, Gersten provided an example of a non-CAT(0) F_3-by-$\mathbb{Z}$ [19]. We therefore, focus our attention to F_2 and W_3.

We have established that $\mathrm{Aut}(W_3) \cong \mathrm{Aut}(F_2)$. In this thesis, we call the restriction isomorphism $R : \mathrm{Aut}(W_3) \longrightarrow \mathrm{Aut}(F_2)$. Its inverse is the extension isomorphism

$E : \mathrm{Aut}(F_2) \longrightarrow \mathrm{Aut}(E_3)$. For every $\phi \in \mathrm{Aut}(F_2)$, there is a unique extension of ϕ to W_3. When it is helpful to distinguish automorphisms of W_3 from those of F_2, we use the notation $\hat{\phi}$ for an automorphism of W_3, and ϕ for its restriction to F_2.

$$R(\hat{\phi}) = \phi \in \mathrm{Aut}(F_2)$$

$$E(\phi) = \hat{\phi} \in \mathrm{Aut}(W_3)$$

Since E is a homomorphism, $E(\phi^k) = E(\phi)^k$ or $\widehat{\phi^k} = (\hat{\phi})^k$. This is a handy fact when extrapolating from $F_2 \rtimes \mathbb{Z}$ to $W_3 \rtimes \mathbb{Z}$.

$$\begin{tikzcd}
F_2 \rtimes_{\phi^k} \mathbb{Z} \arrow[r, hook, "\text{index 2}"] \arrow[d, hook, "\text{index k}"] & W_3 \rtimes_{\widehat{\phi^k}} \mathbb{Z} = W_3 \rtimes_{(\hat{\phi})^k} \mathbb{Z} \arrow[d, hook, "\text{index k}"] \\
F_2 \rtimes_{\phi} \mathbb{Z} \arrow[r, hook, "\text{index 2}"] & W_3 \rtimes_{\hat{\phi}} \mathbb{Z}
\end{tikzcd}$$

Chapter 4

F_2-by-$\mathbb{Z}$ is CAT(0)

In this chapter, we will review what is known about F_2-by-$\mathbb{Z}$ groups. For every $\phi \in \mathrm{Aut}(F_2)$, $F_2 \rtimes_\phi \mathbb{Z}$ is CAT(0). Thomas Brady showed that every F_2-by-$\mathbb{Z}$ that is not virtually $F_2 \times \mathbb{Z}$ is the fundamental group of a CAT(0) 2-complex made from equilateral triangles [8]. Button and Kropheller showed that every F_2-by-$\mathbb{Z}$ acts geometrically on a CAT(0) square complex [29].

Gersten provided a well-known non-example for rank three [19]. His $F_3 \rtimes_\phi \mathbb{Z}$ group cannot be the subgroup of a CAT(0) group. Lyman, in contrast, avoided the Gersten roadblock to provide new families of CAT(0) $F_n \rtimes_\phi \mathbb{Z}$ groups without restriction on rank n [31]. Her examples lead to some virtually CAT(0) $W_n \rtimes_\phi \mathbb{Z}$ groups that we discuss in section 5.1.

Hagen and Wise showed that hyperbolic free-by-$\mathbb{Z}$ groups act geometrically on CAT(0) cube complexes [25]. Recall that $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if ϕ is atoroidal, i.e. no power of ϕ fixes the conjugacy class of a nontrivial element [5].

4.1 F_2-by-$\mathbb{Z}$ by Cases

For an arbitrary group G, the isomorphism class of $G \rtimes_\phi \mathbb{Z}$ depends only on the outer automorphism class of $\phi^{\pm}$. Nielsen proved that $\mathrm{Out}(F_2) \cong GL(2, \mathbb{Z})$. Thus, we break $\{F_2 \rtimes_\phi \mathbb{Z}\}$ into classes based on the image of ϕ in $GL(2, \mathbb{Z})$.

In the first class, $\phi \in \mathrm{Inn}(F_2)$, and its image in $GL(2, \mathbb{Z})$ is the identity. The remaining three classes of automorphisms correspond to the elliptic, parabolic, and hyperbolic matrices of $GL(2, \mathbb{Z})$.

The isomorphism class of $F_n \rtimes_\phi \mathbb{Z}$ depends only on the outer automorphism *conjugacy* class [7]. For F_2, this exactly describes the isomorphism class.

G	$[\phi] = [\psi^{\pm}]$ in $\mathrm{Out}(G) \implies G \rtimes_{\phi} \mathbb{Z} \cong G \rtimes_{\psi} \mathbb{Z}$
F_n	$[\phi]$ is conjugate to $[\psi^{\pm}]$ in $\mathrm{Out}(F_n) \implies F_n \rtimes_{\phi} \mathbb{Z} \cong F_n \rtimes_{\psi} \mathbb{Z}$
F_2	$[\phi]$ is conjugate to $[\psi^{\pm}]$ in $\mathrm{Out}(F_2) \iff F_2 \rtimes_{\phi} \mathbb{Z} \cong F_2 \rtimes_{\psi} \mathbb{Z}$

A finite order matrix $A \neq I$ in $GL(2,\mathbb{Z})$ is elliptic. An infinite order matrix with eigenvalues in $\{1,-1\}$ is parabolic. Lastly, a matrix whose eigenvalues are not roots of unity is hyperbolic.

In order for $A \in GL(2,\mathbb{Z})$ and its inverse to have integer entries, $\det(A) = \pm 1$. The set of matrices with determinant 1 is an index two subgroup $SL(2,\mathbb{Z}) \leq GL(2,\mathbb{Z})$. If $A \in SL(2,\mathbb{Z}) - \{I\}$, then we can classify A by the magnitude of its trace. A is elliptic, parabolic, or hyperbolic if $|\mathrm{tr}(A)|$ is less than 2, equal to 2, or greater than 2, respectively. For an invertible $A \in GL(2,\mathbb{Z})$, $\mathrm{tr}(A^{-1}) = \det(A)\mathrm{tr}(A)$. If $A \in SL(2,\mathbb{Z})$ and A and B are conjugate or conjugate inverse, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

	Eigenvalues	**Example**	$A \in SL(2,\mathbb{Z})$ $\lvert\mathbf{tr}(A)\rvert$
Elliptic	$\lambda_1 = \lambda_2 = -1$ or $\lambda_1 = \lambda_2^* \neq 1$	$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$	< 2
Parabolic	$\lambda_1 = \lambda_2 = \pm 1$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	2
Hyperbolic	$\lvert\lambda_1\rvert = \lvert\frac{1}{\lambda_2}\rvert > 1$	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	> 2

Unlike higher rank free groups, every automorphism of F_2 is induced by a homeomorphism of the once punctured torus [6]. As a result, every $F_2 \rtimes_{\phi} \mathbb{Z}$ is the fundamental group of a mapping torus of the once punctured torus. Let us break this down.

A homeomorphism f of the once-punctured torus T° induces an automorphism $f_* : \pi_1(T^{\circ}) \longrightarrow \pi_1(T^{\circ})$. The fundamental group $\pi_1(T^{\circ}) \cong F_2$, and so $f_* \in \mathrm{Aut}(F_2)$. For each homeomorphism f of the once-punctured torus, we can build a mapping torus $M_f = T^{\circ} \times [0,1]/(x,0) \sim (f(x),1)$. The fundamental group of the mapping torus is $F_2 \rtimes_{f_*} \mathbb{Z}$. The mapping class group of the once punctured torus is $MCG(T^\circ) \cong GL(2,\mathbb{Z})$, which is isomorphic to $\mathrm{Out}(F_2)$. The isotopy class $[f] \in MCG(T^\circ)$ corresponds to the outer automorphism class $[f_*] \in \mathrm{Out}(F_2)$.

$$
\begin{tikzcd}
\boxed{[f] \text{ conjugate to } [g] \text{ in } \mathrm{MCG}(T^\circ)} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \boxed{M_f \text{ homeomorphic to } M_g} \arrow[d, leftrightarrow] \\
\boxed{[f_*] \text{ conjugate to } [g_*] \text{ in } \mathrm{Out}(F_2)} \arrow[r, leftrightarrow] & \boxed{F_2 \rtimes_{f_*} \mathbb{Z} \cong F_2 \rtimes_{g_*} \mathbb{Z}}
\end{tikzcd}
$$

Assuming $F_2 = \langle a, b \rangle$, the boundary loop of T° corresponds to the commutator $[a, b]$. A homeomorphism $f : T^\circ \to T^\circ$ must preserve the boundary. Therefore, the induced automorphism $f_* : \pi_1(T^\circ) \to \pi_1(T^\circ)$ sends $[a, b]$ to a conjugate of $[a, b]^{\pm}$. After composing with an inner automorphism (which does not change the isomorphism class of $F_2 \rtimes_{f_*} \mathbb{Z}$), $f_*([a, b]) = [a, b]$ or $f_*([a, b]) = [a, b]^{-1}$. In the former case, $\langle [a, b], t \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, and in the latter case, $\langle [a, b], t^2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore, no extension of F_2 by $\mathbb{Z}$ is hyperbolic. The subgroup $\langle [a, b], t^p \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ corresponds to a torus cusp in the mapping torus M_f.

4.2 Parabolic Automorphisms of F_2

An automorphism of F_2 is parabolic if it has trace ± 2. Every $A \in GL(2,\mathbb{Z})$ with trace 2 (respectively -2) is conjugate in $GL(2,\mathbb{Z})$ to $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (respectively $\begin{bmatrix} -1 & k \\ 0 & -1 \end{bmatrix}$) for some $k \in \mathbb{Z}$ [7].

As mentioned above, ϕ and ψ are conjugate or conjugate inverse in $\mathrm{Out}(F_2)$ if and only if $F_2 \rtimes_\phi \mathbb{Z} \cong F_2 \rtimes_\psi \mathbb{Z}$. Therefore, we only need to handle the cases $\pm \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ for $k \in \mathbb{Z}$.

Note that $\pm\begin{bmatrix}1 & k \\ 0 & 1\end{bmatrix}$ and $\pm\begin{bmatrix}1 & -k \\ 0 & 1\end{bmatrix}$ are conjugate in $GL(2,\mathbb{Z})$ so we may assume $k > 0$.

$$\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & k \\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -k \\ 0 & 1\end{bmatrix}$$

Let $\psi \in \mathrm{Aut}(F_2)$ be given by $\psi(x) = x$ and $\psi(y) = xy$. We use ψ_{ab} to denote the image of ψ in $GL(2,\mathbb{Z})$.

$$\psi_{ab} = \begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix}$$

$$\psi_{ab}^{k} = \begin{bmatrix}1 & k \\ 0 & 1\end{bmatrix}$$

$$\psi_{ab}^{-k} = \begin{bmatrix}1 & -k \\ 0 & 1\end{bmatrix}$$

If $\phi \in \mathrm{Aut}(F_2)$ has trace 2, then ϕ is conjugate in $GL(2,\mathbb{Z})$ to ψ^k or $(\psi^k)^{-1}$ for some $k \geq 0$.

$$\phi_{ab} \sim \begin{bmatrix}1 & k \\ 0 & 1\end{bmatrix} \text{ or } \phi_{ab} \sim \begin{bmatrix}1 & -k \\ 0 & 1\end{bmatrix} \text{ for } k > 0$$

Either way, $F_2 \rtimes_\phi \mathbb{Z} \cong F_2 \rtimes_{\psi^k} \mathbb{Z}$.

Lemma 4.2.1 *For a group G with an infinite order $\phi \in Aut(G)$ and $k > 0$, $G \rtimes_{\phi^k} \mathbb{Z}$ is an index k subgroup of $G \rtimes_\phi \mathbb{Z}$.*

Proof: Define a map $f : G_{\phi^k} \to G_\phi$ by $gt^n \mapsto gt^{nk}$. First, we check that f is a homomorphism:

$$f(gt^n)f(ht^m) = (gt^{kn})(ht^{km}) = g\phi^{kn}(h)t^{k(n+m)}$$

$$f[(gt^n)(ht^m)] = f(g\phi^{kn}(h)t^{n+m}) = g\phi^{kn}(h)t^{k(n+m)}$$

Next, we show that the kernel of f is trivial.

$$f(gt^n) = gt^{kn} = 1 \implies g = 1, n = 0 \implies gt^n = 1$$

Lastly, we show that the image of f has index k in G_ϕ. Fix $gt^p \in G_\phi$. By the division algorithm (for dividing p by k), $p = kq + r$ with $0 \leq r < k$ and $q \in \mathbb{Z}$.

$$\begin{aligned} gt^p &= gt^{kq+r} \\ &= (gt^{kq})(t^r), 0 \leq r < k \end{aligned}$$

$gt^{kq} \in \mathrm{im} f$ because $f(gt^q) = gt^{kq}$. Therefore, $\{t^r | 0 \leq r < k\}$ is a set of right coset representatives for $G_\phi / \mathrm{im} f$. □

By lemma 4.2.1, $F_2 \rtimes_{\psi^k} \mathbb{Z}$ is an index k subgroup of $F_2 \rtimes_\psi \mathbb{Z}$. Therefore, if $F_2 \rtimes_\psi \mathbb{Z}$ acts geometrically on a space X, then $F_2 \rtimes_{\psi^k} \mathbb{Z}$ acts geometrically on X.

4.2.1 $F_2 \rtimes_\psi \mathbb{Z}$ acts geometrically on a CAT(0) square complex

In this section, we show that $F_2 \rtimes_\psi \mathbb{Z}$ acts geometrically on a CAT(0) square complex X. It follows that $F_2 \rtimes_{\psi^k} \mathbb{Z} \cong F_2 \rtimes_\phi \mathbb{Z}$ acts geometrically on X for every $\phi \in \mathrm{Aut}(F_2)$ with trace two.

Lemma 4.2.2 *$F_2 \rtimes_\psi \mathbb{Z}$ is isomorphic to an HNN extension $(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$.*

Proof: We use the presentation $F_2 \rtimes_\psi \mathbb{Z} = \langle x, y, t | txt^{-1} = x, tyt^{-1} = xy \rangle$. We can rearrange the relation $tyt^{-1} = xy$ into $y^{-1}(x^{-1}t)y = t$. Then, by the change of variables $\alpha = x^{-1}t$ and $\beta = t$, $F_2 \rtimes_\psi \mathbb{Z} \cong \langle \alpha, \beta, y | [\alpha, \beta], y^{-1}\alpha y = \beta \rangle$. □

The HNN extension $(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}} \cong \langle \alpha, \beta, y | [\alpha, \beta], y^{-1}\alpha y = \beta \rangle$ has stable letter y and $\mathbb{Z} \oplus \mathbb{Z}$ generated by α and β. Let X be the Cayley complex of the HNN extension.

X is a tree of planes. Each plane has strips coming up from its horizontal axes. A strip is an isometric embedding $[0,1] \times \mathbb{R} \to X$. Figure 5.2 shows the base plane (containing v_1) and a vertical strip coming out of the $\beta = 2$ axis. The strip comes from the relation $\alpha = y\beta y^{-1}$. The top of the strip is the $\alpha = 0$ axis in a neighboring plane (red axes are glued to blue axes). Every plane has $\mathbb{Z}$ neighboring planes glued above.

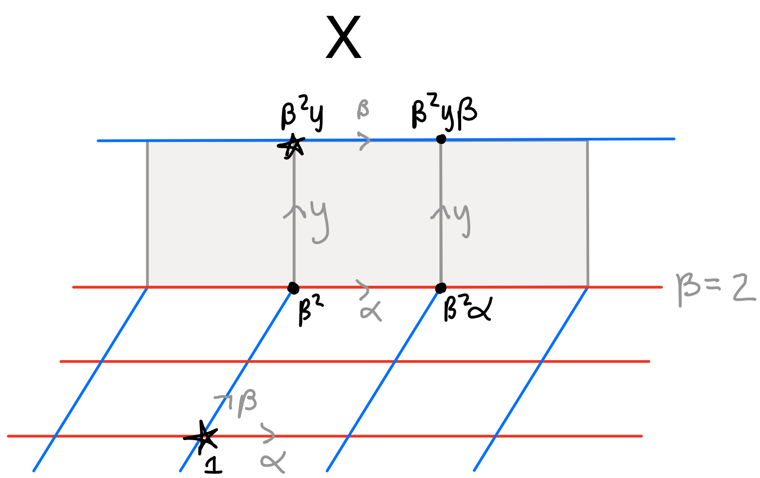

Figure 4.1

Similarly, every plane has strips glued beneath the plane along blue vertical $\alpha = n$ axes. The bottom of such a strip is the red $\beta = 0$ axis in the neighboring plane.

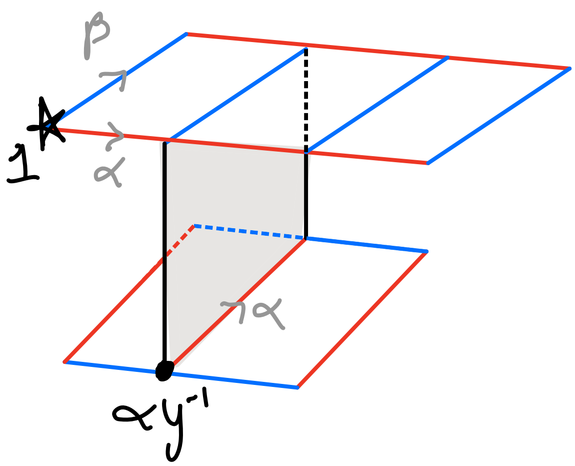

Figure 4.2: Two planes connected by a strip. The top plane is the base plane. The strip is glued to its $\beta = 1$ axis.

We claim that X is CAT(0). If a metric space is simply connected and locally CAT(0), then it is CAT(0). A cube complex is locally CAT(0) if and only if every vertex link is a CAT(1) space [21]. A vertex link is CAT(1) if and only if it is a flag complex [21]. A flag complex is a simplicial complex with no "missing simplices." For every complete graph K_n in the 1-skeleton of a flag complex, there is an n-simplex glued to the 1-skeleton.

In a square complex, the vertex links are graphs. The metric link of a vertex in a square complex is CAT(1) if and only every cycle has length at least 2π [21].

Every vertex in X has the same link. Each vertex in X has six incident edges labeled $\{\alpha, \alpha^{-1}, \beta, \beta^{-1}, y, y^{-1}\}$. Figure 4.2.1 shows some of the edges leaving v. There is a strip glued under the white plane, perpendicular to the gray strip above the plane.

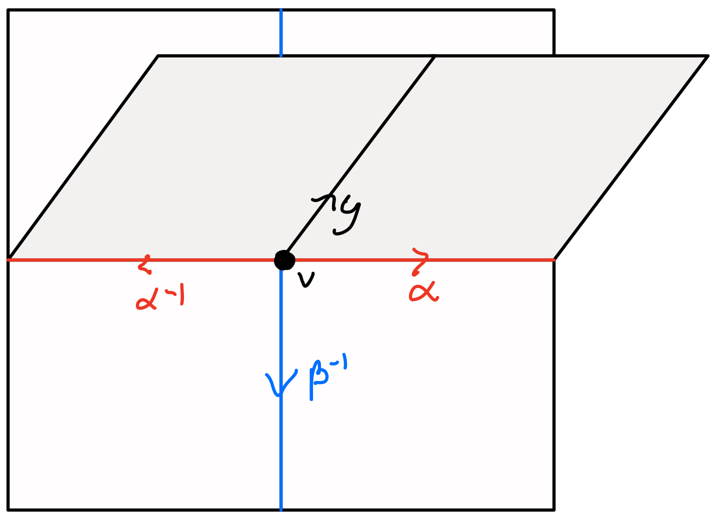

Figure 4.3

The link of v has six vertices and eight edges, corresponding to the 1-cells and 2-cells containing v, respectively.

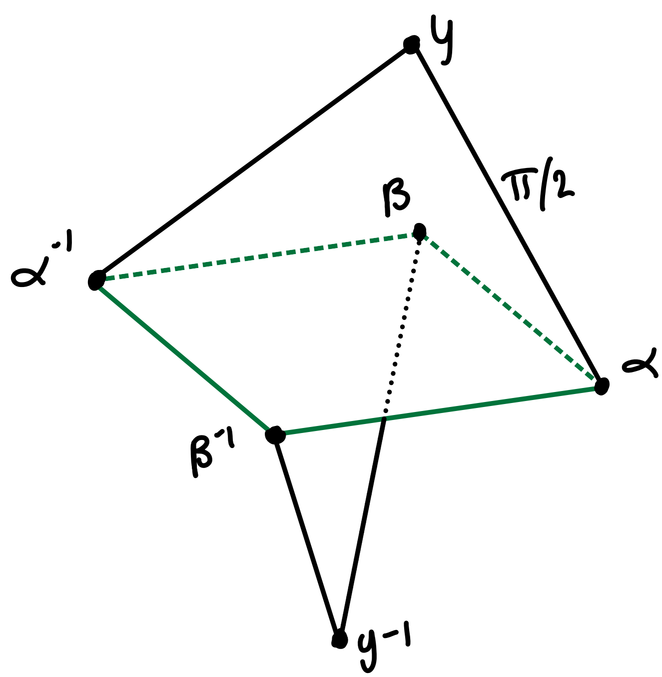

Figure 4.4

The length of every edge in the metric link of a square complex is $\pi/2$. Every loop in $lk(v)$ has length $\geq 2\pi$ (at least four edges). Therefore, X is CAT(0).

4.3 Elliptic Automorphisms of F_2

An elliptic matrix $A \in GL(2,\mathbb{Z})$ has trace 0, 1, or -1 and order $2, 3, 4$, or 6. If $A^2 = I$, then $A = -I$ or A is conjugate to:

$$\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$$

The results in this section hold for higher rank free groups as well. Throughout this section, ϕ is an automorphism of F_n such that $[\phi]$ is order m in $\mathrm{Out}(F_n)$.

Culler, Khramtsov, and Zimmerman independently proved the following realization theorem [16] [43].

Theorem 4.3.1 (Culler 1984, Khramtsov 1985, Zimmerman 1996) *Let H be a finite subgroup of Out(F_n). Then there exists a finite connected graph Γ, with*

$\pi_1(\Gamma) = F_n$, and an action of H on Γ inducing on its fundamental group the given action of H on F_n.

The proof of the realization theorem uses $\mathrm{Inn}(F_n) \rtimes \langle[\phi]\rangle \cong F_n \rtimes_{[\phi]} \mathbb{Z}_m$. Note that ϕ may be infinite order in $\mathrm{Aut}(F_n)$, but $[\phi] \in \mathrm{Out}(F_n)$ is finite order. There are two ways to think about $F_n \rtimes_{[\phi]} \mathbb{Z}_m$.

1. $[\phi]$ acts on the conjugacy classes of F_n: $[\phi] \cdot [w] = [\phi(w)]$.
2. ϕ is in the same outer class as some ψ with order m in $\mathrm{Aut}(F_n)$ and $F_n \rtimes_{[\phi]} \mathbb{Z}_m \cong F_n \rtimes_\psi \mathbb{Z}_m$.

Lemma 4.3.2 *Assume $\phi \in Aut(G)$ has order m in $Out(G)$ (i.e. $\phi^m \in Inn(G)$). Then there exists ψ in the same outer class as ϕ such that ψ has order m in $Aut(G)$.*

Proof: Let $\pi : \mathrm{Aut}(G) \to \mathrm{Out(G)}$ be the projection homomorphism.

$$\pi(\phi^m) = [\phi^m] = [\mathrm{Id}_G]$$
$$\pi(\phi)^m = [\phi]^m$$

The former outer class contains the inner automorphisms (including the identity Id_G). The latter outer class contains every ψ^m such that $[\psi] = [\phi]$. Since these two outer classes agree, there exists $\psi \in [\phi]$ such that $\psi^m = \mathrm{Id}_G$. □

In our application of the realization theorem, $\langle[\phi]\rangle$ is a finite subgroup of $\mathrm{Out}(F_n)$. By the realization theorem, there exists a graph Γ with fundamental group F_n and an action of $[\phi]$ on Γ that realizes the action of $[\phi]$ on F_n. Since Γ is a finite graph, its universal cover is a locally finite tree T. We can lift the action $F_n \rtimes_{[\phi]} \mathbb{Z}_m \curvearrowright \Gamma$ to a properly discontinuous action $F_n \rtimes_{[\phi]} \mathbb{Z}_m \curvearrowright T$. The action $F_n \curvearrowright T$ is free with quotient Γ.

Theorem 4.3.3 *If $G \rtimes_\psi \mathbb{Z}_m$ acts faithfully and geometrically on a metric space X, then $G \rtimes_\psi \mathbb{Z}$ acts faithfully and geometrically on $X \times \mathbb{R}$.*

By the realization theorem, $F_n \rtimes_\psi \mathbb{Z}_m \cong F_n \rtimes_{[\phi]} \mathbb{Z}_m$ acts geometrically on a tree T. By theorem 4.3.3, $F_n \rtimes_\psi \mathbb{Z} \cong F_n \rtimes_\phi \mathbb{Z}$ acts geometrically on $T \times \mathbb{R}$. Since the product of CAT(0) spaces is CAT(0), this shows that $F_n \rtimes_\phi \mathbb{Z}$ is CAT(0) for any $\phi \in \mathrm{Aut}(F_n)$ with finite order in $\mathrm{Out}(F_n)$. The rest of this section is dedicated to proving theorem 4.3.3.

In theorem 4.3.3, let $G \rtimes_\psi \mathbb{Z} \curvearrowright X \times \mathbb{R}$ by $gt^n \cdot (x, y) = (gs^n \cdot x, y + n)$ where $gs^n \in G \rtimes_\psi \mathbb{Z}_m$.

First, we check that the algebra of $G \rtimes_\psi \mathbb{Z}$ matches the geometry. For all $g \in G$, $tgt^{-1} = \psi(g)$ in $G \rtimes_\psi \mathbb{Z}$. Fix a point $(x, y) \in X \times Y$.

$$\begin{aligned}
tgt^{-1} \cdot (x, y) &= tg \cdot (s^{-1} \cdot x, y - 1) \\
&= t \cdot (gs^{-1} \cdot x, y - 1) \\
&= (sgs^{-1} \cdot x, y) \\
&= (\psi(g) \cdot x, y) \text{ because } sws^{-1} = \psi(g) \in G \rtimes_\psi \mathbb{Z}_m \\
&= \psi(g) \cdot (x, y)
\end{aligned}$$

Lemma 4.3.4 *$G \rtimes_\psi \mathbb{Z}$ acts by isometries on $X \times \mathbb{R}$.*

The distance between two points $(x, y), (x', y') \in X \times \mathbb{R}$, is $d(x, x') + |y' - y|$ because we have to travel along edges.

Proof: Fix $gt^n \in G \rtimes_\psi \mathbb{Z}$.

$$\begin{aligned}
d[gt^n \cdot (x, y), gt^n \cdot (x', y')] &= d[(gs^n \cdot x, y + n), (gs^n \cdot x', y' + n)] \\
&= d(gs^n \cdot x, gs^n \cdot x') + \left|y' - y\right| \\
&= d(x, x') + \left|y' - y\right|
\end{aligned}$$

Since $gs^n \in W_3 \rtimes_\psi \mathbb{Z}_m$ is an isometry of X, $d(gs^n \cdot x, gs^n \cdot x') = d(x, x')$. □

Lemma 4.3.5 *$G \rtimes_\psi \mathbb{Z}$ acts cocompactly on $X \times \mathbb{R}$.*

Proof: Let $D \subset X$ be a compact fundamental domain for $G \rtimes_\psi \mathbb{Z}_m \curvearrowright X$. We claim that $D \times [0, m - 1]$ is a fundamental domain for $G \rtimes_\psi \mathbb{Z} \curvearrowright X \times \mathbb{R}$.

Fix a vertex $(v,n) \in X \times \mathbb{R}$, where v is a vertex in X and n is an integer. We will show that (v,n) is in the same orbit as a vertex in $D \times [0, m-1]$.

First, let $t^{-n} \in G \rtimes_\psi \mathbb{Z}$ act on (v,n).

$$(v,n) \xrightarrow{t^{-n}} (s^{-n} \cdot v, 0) \in X \times \{0\}$$

Since $s^{-n} \cdot v$ is a vertex in X, there exists $gs^p \in W_3 \rtimes_\phi \mathbb{Z}_m$ such that $gs^p \cdot (s^{-n} \cdot v) \in D$.

$$(v,n) \xrightarrow{t^{-n}} (s^{-n} \cdot v, 0) \xrightarrow{gt^p} (gs^p s^{-n} \cdot v, p) = (d,p)$$
$$X \times \mathbb{R} \to X \times \{0\} \to D \times \{p\}$$

Therefore, an arbitrary vertex $(v,n) \in X \times \mathbb{R}$ is in the same orbit as some $(d,p) \in D \times [0, m-1]$. $\square$

Lemma 4.3.6 *The action $G \rtimes_\psi \mathbb{Z} \curvearrowright X \times \mathbb{R}$ is properly discontinuous.*

Proof: Suppose, for sake of contradiction, there is a compact $K \subset X \times \mathbb{R}$ and an infinite set $\{\gamma_n\} \subset G \rtimes_\psi \mathbb{Z}$ such that $\gamma_n \cdot K \cap K \neq \varnothing$. Let K_1 and K_2 be the projections of K to X and $\mathbb{R}$, respectively. Since $K \subset K_1 \times K_2$, $\gamma_n \cdot (K_1 \times K_2) \cap (K_1 \times K_2) \neq \varnothing$. We can write each γ_n as $g_n t^{p_n}$ for some $g_n \in G$ and $p_n \in \mathbb{Z}$.

$$K_1 \times K_2 = \{(x,y) | x \in K_1, y \in K_2\}$$
$$\gamma_n \cdot (K_1 \times K_2) = \{(g_n s^{p_n} \cdot x, y + p_n) : x \in K_1, y \in K_2\}$$

Fix n. By assumption, there exists (x', y') in the intersection of the sets above.

$$(x', y') \in K_1 \times K_2 \implies y' \in K_2$$
$$(x', y') \in \gamma_n \cdot (K_1 \times K_2) \implies y' = y + p_n \text{ for some } y \in K_2 \implies y' - p_n \in K_2$$

$K_2 \subset [a,b] \subset \mathbb{R}$. Since y' and $y' - p_n$ are in K_2, $|p_n| < b - a$.

There are finitely many integers between a and b, so there exists p such that $p_n = p$ for infinitely many n. Now we have an infinite set $\{g_n t^p\}$ such that $g_n t^p \cdot$

$(K_1 \times K_2) \cap (K_1 \times K_2) \neq \emptyset$. Fix n. Assume (x', y') is in the intersection.

$$(x', y') \in K_1 \times K_2 \implies x' \in K_1$$
$$(x', y') \in g_n t^p \cdot (K_1 \times K_2) \implies x' \in g_n s^p \cdot K_1$$

Therefore, $x' \in g_n s^p \cdot K_1 \cap K_1$. Now we have an infinite set $\{g_n s^p\} \subset G \rtimes_\phi \mathbb{Z}_m$ such that $g_n s^p \cdot K_1 \cap K_1 \neq \emptyset$. But, $G \rtimes_\phi \mathbb{Z}_m \curvearrowright X$ cocompactly so we have a contradiction.□

4.4 Hyperbolic Automorphisms of F_2

We call $\phi \in \mathrm{Aut}(F_2)$ hyperbolic if $\phi_{ab} \in GL(2, \mathbb{Z})$ has two real eigenvalues, λ and $\pm 1/\lambda$, such that $|\lambda| > 1$. Notice, ϕ_{ab} does not have any roots of unity as eigenvalues. Roughly speaking, all of the $\mathbb{Z} \oplus \mathbb{Z}$ subgroups in $F_2 \rtimes_\phi \mathbb{Z}$ come from the commutator $[x, y]$ and $\mathbb{Z}$. Recall from section 4.1, either $\langle [x, y], t\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ or $\langle [x, y], t^2\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. This $\mathbb{Z} \oplus \mathbb{Z}$ subgroup generates the torus cusp in the mapping torus of the once-punctured torus M_ϕ. For a hyperbolic $\phi \in \mathrm{Aut}(F_2)$, there are no other sources of $\mathbb{Z} \oplus \mathbb{Z}$ subgroups. All of the non-hyperbolic parts of the mapping torus are relegated to the torus cusp.

By Thurston's Hyperbolization Theorem, the mapping torus M_ϕ of a hyperbolic $\phi \in \mathrm{Aut}(F_2)$ admits a finite volume, complete hyperbolic metric. The fundamental group $\pi_1(M_\phi) \cong F_2 \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to the cusp.

By Mostow-Prasad Rigidity, the hyperbolic structure on M_ϕ is unique. The hyperbolic structure on M_ϕ corresponds to a discrete faithful representation $\pi_1(M_\phi) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$, i.e. $F_2 \rtimes_\phi \mathbb{Z} \hookrightarrow \mathrm{Isom}(\mathbb{H}^3)$. Thus, $F_2 \rtimes_\phi \mathbb{Z} \curvearrowright \mathbb{H}^3$ properly discontinuously by isometries, but not cocompactly, because the quotient M_ϕ has a cusp. However, $F_2 \rtimes_\phi \mathbb{Z}$ acts on truncated hyperbolic space geometrically. We construct truncated hyperbolic space by removing a set of disjoint open horoballs about the parabolic fixed points of $F_2 \rtimes_\phi \mathbb{Z}$. The commutator $[x, y]$ and t are parabolic isometries of $\mathbb{H}^3$ with the same fixed point. The horoballs are indexed by conjugates of the commutator. Truncated hyperbolic space is a complete CAT(0) space [9]. Therefore,

$F_2 \rtimes_\phi \mathbb{Z}$ with hyperbolic ϕ is CAT(0).

Chapter 5

W_3-by-$\mathbb{Z}$ is CAT(0)

In this chapter, we prove the main theorem of this thesis. Specifically, in Theorem 5.1.3, we prove that all W_3-by-$\mathbb{Z}$ groups are CAT(0). We do this by considering each of the cases for for F_2-by-$\mathbb{Z}$ groups discussed in Chapter 4. In the parabolic and hyperbolic cases, we show the action on F_2-by-$\mathbb{Z}$ extends to the appropriate W_3-by-$\mathbb{Z}$ group and in the elliptic case, we construct a different space for the corresponding W_3-by-$\mathbb{Z}$ group.

5.1 W_n-by-$\mathbb{Z}$

In general, it is not known when W_n-by-$\mathbb{Z}$ is CAT(0).There are no known non-CAT(0) examples, and Kim Ruane conjectures that all W_n-by-$\mathbb{Z}$ are CAT(0). Gersten provided an automorphism θ of F_3 such that $F_3 \rtimes_\theta \mathbb{Z}$ cannot be the subgroup of a CAT(0) group [19].

$$\theta(x) = x$$

$$\theta(y) = yx$$

$$\theta(z) = zx^2$$

This automorphism is not in the image of $\rho_3 : \mathrm{Aut}(W_4) \longrightarrow \mathrm{Aut}(F_3)$, i.e. $F_3 \rtimes_\theta \mathbb{Z}$ is not the subgroup of a W_4-by-$\mathbb{Z}$. There is hope that all of the "bad" automorphisms in $\mathrm{Aut}(F_{n-1})$ (those that preclude non-positive curvature) do not live inside $\mathrm{Aut}(W_n)$. Rylee Lyman proved the following [31]:

Theorem 5.1.1 (Lyman 2023) *Let A be a finite group. Let $W_n = A * \cdots * A$ denote the free product of n copies of A, and let $\phi : W_n \longrightarrow W_n$ be a polynomially-growing automorphism. There exists an integer $k \geq 1$ such that the mapping torus of ϕ^k acts geometrically on a CAT(0) 2-complex.*

In other terms, if $\phi \in \mathrm{Aut}(W_n)$ is polynomially growing, then $W_n \rtimes_\phi \mathbb{Z}$ is virtually CAT(0). An automorphism $\phi : W_n \to W_n$ is polynomially growing if $f_w(k) = |\phi^k(w)|$ grows at most polynomially in k for all $w \in W_n$. Lyman used the theorem above to show that certain free-by-$\mathbb{Z}$ groups are CAT(0). The group of automorphisms of W_n that fix a generator is isomorphic to the group of palindromic automorphisms of F_n. Note that Gersten's non-example is exponentially growing.

Theorem 5.1.2 (Lyman 2023) *Let $\phi : F_n \longrightarrow F_n$ be a polynomially-growing, palindromic automorphism. There exists an integer $k \geq 1$ such that the mapping torus of ϕ^k acts geometrically on a CAT(0) 2-complex.*

In contrast, this chapter starts with $F_2 \rtimes \mathbb{Z}$ and extends to $W_3 \rtimes \mathbb{Z}$. We prove the following theorem:

Theorem 5.1.3 *For every automorphism ϕ of W_3, $W_3 \rtimes_\phi \mathbb{Z}$ is CAT(0).*

5.2 W_3-by-$\mathbb{Z}$

Let $W_3 = \langle a, b, c | a^2, b^2, c^2 \rangle$ and $F_2 = \langle x, y \rangle$. We identify F_2 as a subgroup of W_3 by $x \mapsto ab$ and $y \mapsto ac$.

$$F_2 \rtimes_\tau \langle a \rangle \cong W_3$$

We classify the automorphisms in W_3 based on their restriction to F_2.

Aut(W_3)	**Aut**(F_2)	$GL(2,\mathbb{Z})$
id_{W_3}	id_{F_2}	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Conjugation by a	τ	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Elliptic	Elliptic	$\lvert\mathrm{tr}\rvert < 2$
Parabolic	Parabolic	$\lvert\mathrm{tr}\rvert = 2$
Hyperbolic	Hyperbolic	$\lvert\mathrm{tr}\rvert > 2$

5.3 Inner Automorphisms of W_3

For any inner automorphism $\phi \in \mathrm{Aut}(W_3)$, $W_3 \rtimes_\phi \mathbb{Z} \cong W_3 \times \mathbb{Z}$. W_3 acts on the trivalent tree T_3 where a, b, and c are reflections across half edges. $W_3 \times \mathbb{Z}$ acts on $T_3 \times \mathbb{R}$.

To introduce action extensions, we argue that we can extend the action of $F_2 \times \mathbb{Z}$ on $T_4 \times \mathbb{R}$ to $W_3 \times \mathbb{Z}$. Let τ be a π rotation of T_4. Then $F_2 \rtimes_\tau \mathbb{Z}_2 \curvearrowright T_4$ such that F_2 acts by left multiplication and a (the generator of $\mathbb{Z}_2$) acts by τ. Now we have $W_3 \curvearrowright T_4$. The quotient is compact because it is “smaller” than that of $F_2 \curvearrowright T_4$. The vertex stabilizers are finite so the action is properly discontinuous. We can extend the action to a geometric action $W_3 \times \mathbb{Z} \curvearrowright T_4 \times \mathbb{R}$.

5.4 Parabolic Automorphisms of W_3

In section 4.2, we showed that we only need consider a particular automorphism $\psi \in \mathrm{Aut}(F_2)$. For every automorphism $\phi \in \mathrm{Aut}(F_2)$ with trace 2, $F_2 \rtimes_\phi \mathbb{Z} \cong F_2 \rtimes_{\psi^k} \mathbb{Z}$ for some $k > 0$. Then, we showed that $F_2 \rtimes_\psi \mathbb{Z}$ is isomorphic to an HNN extension whose Cayley complex is CAT(0). In this section, we extend the action of $F_2 \rtimes_\psi \mathbb{Z} \curvearrowright X$ to $W_3 \rtimes_{\hat{\psi}} \mathbb{Z} \curvearrowright X$. First, we need to argue that all W_3 parabolic cases stem from $\hat{\psi}$, the extension of ψ to W_3.

We call an automorphism of W_3 parabolic if its restriction to F_2 is parabolic. Remember, a parabolic $\phi \in Aut(F_2)$ has trace ± 2.

5.4.1 Part I

Let $\hat{\psi}$ be the extension of $\psi \in \mathrm{Aut}(F_2)$ to W_3.

$$\psi(x) = x, \psi(y) = xy$$

$$\psi_{ab} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\psi_{ab}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, k \in \mathbb{Z}$$

If $W_3 \rtimes_{\hat{\psi}} \mathbb{Z}$ is CAT(0), then for any parabolic $\hat{\phi}$, $W_3 \rtimes_{\hat{\phi}} \mathbb{Z}$ is CAT(0). Here is a diagram of the logic:

$$\begin{tikzcd}
\boxed{W_3 \rtimes_{\hat{\psi}} \mathbb{Z} \text{ is CAT(0)}} \arrow[d] \arrow[d, bend left, dashed, "W_3 \rtimes_{(\hat{\psi})^k} \mathbb{Z} \text{ is finite index in } W_3 \rtimes_{\hat{\psi}} \mathbb{Z}"] \\
\boxed{W_3 \rtimes_{(\hat{\psi})^k} \mathbb{Z} \text{ is CAT(0)}} \arrow[d] \arrow[d, bend left, dashed, "(\hat{\psi})^k = \widehat{\psi^k}"] \\
\boxed{W_3 \rtimes_{\widehat{\psi^k}} \mathbb{Z} \text{ is CAT(0)}} \arrow[d] \arrow[d, bend left, dashed, "W_3 \rtimes_{\widehat{\psi^k}} \mathbb{Z} \cong (F_2 \rtimes_\tau \mathbb{Z}_2) \rtimes_{\widehat{\psi^k}} \mathbb{Z} \cong (F_2 \rtimes_{\psi^k} \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2"] \\
\boxed{(F_2 \rtimes_{\psi^k} \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2 \text{ is CAT(0)}} \arrow[d] \arrow[d, bend left, dashed, "\text{isomorphic groups because } \psi^k \text{ and } \phi \text{ are conjugate in } Out(F_2)"] \\
\boxed{(F_2 \rtimes_{\phi} \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2 \text{ is CAT(0)}} \arrow[d] \arrow[d, bend left, dashed, "(F_2 \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2 \cong (F_2 \rtimes_\tau \mathbb{Z}_2) \rtimes_{\hat{\phi}} \mathbb{Z}"] \\
\boxed{W_3 \rtimes_{\hat{\phi}} \mathbb{Z} \text{ is CAT(0)}}
\end{tikzcd}$$

The first arrow follows from the following lemma.

The second arrow follows from a fact discussed earlier. The extension isomorphism $E : Aut(F_2) \to Aut(W_3)$ extends automorphisms from F_2 to W_3. Since E is a homomorphism, $E(\psi)^k = E(\psi^k)$. In our notation, $(\hat{\psi})^k = \widehat{\psi^k}$.

The third arrow and last arrow are from proposition 3.3.1: $(F_2 \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2 \cong W_3 \rtimes_{\hat{\phi}} \mathbb{Z}$.

The fourth arrow was established in section 4.2: every $\phi \in \mathrm{Aut}(F_2)$ with trace 2 is conjugate in $GL(2,\mathbb{Z})$ to ψ^k or ψ^{-k} for some $k > 0$. The conjugacy classes of $\phi^{\pm}$ characterize the isomorphism class of $F_2 \rtimes_\phi \mathbb{Z}$ (see section 4.1). Therefore, $F_2 \rtimes_\phi \mathbb{Z} \cong F_2 \rtimes_{\psi^k} \mathbb{Z}$.

By the above logic, if $W_3 \rtimes_{\hat{\psi}} \mathbb{Z}$ is CAT(0), then $W_3 \rtimes_{\hat{\phi}} \mathbb{Z}$ is CAT(0) where $\hat{\phi}$ is any automorphism of W_3 whose image in $GL(2,\mathbb{Z})$ has trace 2.

5.4.2 Part II

In this section, we prove that $W_3 \rtimes_\psi \mathbb{Z}$ is CAT(0) where $\psi|_{F_2}$ is given by $\psi(x) = x$ and $\psi(y) = xy$.

$$\psi_{ab} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

In section 4.2, we showed that $F_2 \rtimes_\psi \mathbb{Z}$ is isomorphic to an HNN extension $(\mathbb{Z} \oplus \mathbb{Z})*_\mathbb{Z}$. The Cayley complex X of the HNN extension is a CAT(0) square complex.

Theorem 5.4.1 *There is a geometric action $W_3 \rtimes_\psi \mathbb{Z} \curvearrowright X$ that extends the action $F_2 \rtimes_\psi \mathbb{Z} \curvearrowright X$. The action is faithful.*

Since $W_3 \rtimes_\psi \mathbb{Z} \cong (F_2 \rtimes_\psi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$, we need to find an order two isometry $\hat{\tau}$ of X that is compatible with the algebra.

First, we need to understand the algebra. Recall $\hat{\tau} \in \mathrm{Aut}(F_2 \rtimes_\psi \mathbb{Z})$:

$$\hat{\tau}(x) = x^{-1}, \hat{\tau}(y) = y^{-1}, \hat{\tau}(t) = a\psi(a)t$$

To find $\psi(a)$, we extend $\psi \in Aut(F_2)$ to W_3. Remember, this extension is unique. We have the right extension as long as its restriction to F_2 is ψ.

$$\psi(x) = x \implies \psi(ab) = ab$$

$$\psi(y) = xy \implies \psi(ac) = abac$$

$$\psi(a) = aba, \psi(b) = a, \psi(c) = c$$

Therefore,

$$\hat{\tau}(t) = a\hat{\phi}(a)t = aabat = bat = x^{-1}t$$

.

Use the following change of variables: $\alpha = x^{-1}t$ and $\beta = t$. Then $(\mathbb{Z} \oplus \mathbb{Z})*_\mathbb{Z} =$

$\langle \alpha, \beta, y | [\alpha, \beta] = 1, y\beta y^{-1} = \alpha \rangle$.

$$\hat{\tau}(\alpha) = \beta$$

$$\hat{\tau}(\beta) = \alpha$$

$$\hat{\tau}(y) = y^{-1}$$

By abuse of notation, we call the elements of $(\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}}$ and their images in Isom(X) by the same name. We need to find a non-trivial order two $\hat{\tau} \in$ Isom(X) such that $\hat{\tau}(\alpha)\hat{\tau} = \beta$, $\hat{\tau}(\beta)\hat{\tau} = \alpha$, and $\hat{\tau}(y)\hat{\tau} = y^{-1}$.

Define a map on the 0-skeleton of the CAT(0) square complex by $v_g \mapsto v_{\hat{\tau}(g)}$. We will show that this map extends to an isometry of X such that $(\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}} \rtimes \langle \hat{\tau} \rangle$ is a cocompact and properly discontinuous subgroup of Isom(X).

Geometric Description of the Action

Before launching into a technical proof, we discuss how $\hat{\tau}$ acts on X. Recall from section 4.2.1 that X is a tree of planes. Neighboring planes are connected by strips ($I \times \mathbb{R}$). See section 4.2.1 for more information. Loosely speaking, $\hat{\tau}$ interchanges each plane with a plane on the other side of the origin. Within each plane, $\hat{\tau}$ reflects across the 45° line. X is the Cayley 2-Complex of an HNN extension, so it has a Bass-Serre tree. We use the Bass-Serre tree to track which planes are interchanged by $\hat{\tau}$.

First, lets look at the $\langle \alpha, \beta \rangle$ plane containing the identity vertex v_1. $\hat{\tau}$ sends α to β, and vice versa. Therefore, in the base plane, $\hat{\tau}$ is a reflection across the 45° line.

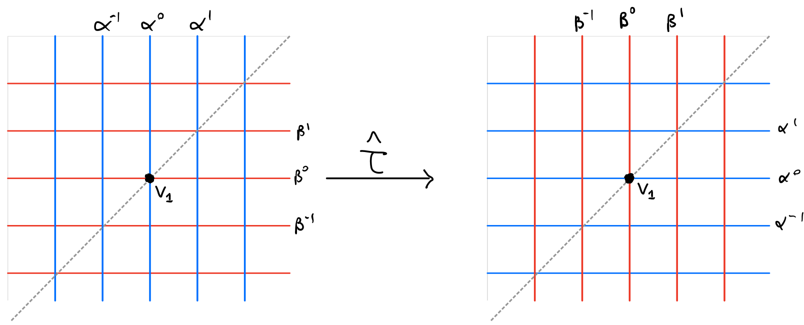

Figure 5.1

The base plane is the only plane left invariant under $\hat{\tau}$. We establish a normal form so that we can write down a unique path from the origin to a plane. Then, we describe how $\hat{\tau}$ acts on the path.

Overview: There is a unique normal form for every $g \in (\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$. There is a unique "normal form path" in X from v_1 to v_g. There is a unique path in the Bass-Serre tree T from the base plane to the plane containing v_g. The edges in T are the strips that take us from one plane to the next. $\hat{\tau}$ descends to an isometry of T.

A word in $(\mathbb{Z}\oplus\mathbb{Z}){*_\mathbb{Z}} = \langle \alpha, \beta, y | [\alpha, \beta] = 1, y\beta y^{-1} = \alpha \rangle$ can be written $g_1 y^{\epsilon_1} \dots g_n y^{\epsilon_n} w$ for $g_i, w \in \langle \alpha, \beta \rangle$ and $\epsilon_i = \pm 1$. We use the convention that we act left to right, i.e. $gh \cdot x$ is g acts and then h acts. We can "pinch" a word by replacing $y\beta y^{-1}$ with α and $y^{-1}\alpha y$ with β. Thus, we stipulate that, in normal form, $\beta^{\pm}$ is to the left of y and $\alpha^{\pm}$ is to the left of y^{-1}. Still, we have a choice between $\alpha\beta y$ and $\beta y \beta$, for example. To make the normal form unique, we require g_i to be in $\langle \alpha \rangle$ or $\langle \beta \rangle$. An HNN extension $G_{*\mathbb{Z}}$ has a normal form for each choice of coset representatives for G/A and G/B. We choose $\{\beta^n\}$ and $\{\alpha^n\}$ to represent $\mathbb{Z} \oplus \mathbb{Z}/\langle \alpha \rangle$ and $\mathbb{Z} \oplus \mathbb{Z}/\langle \beta \rangle$, respectively.

Geometrically, w is a path in the destination plane. Choosing $\beta y \beta$ over $\alpha\beta y$ means we first navigate to the appropriate strip, then go up, then travel in the final plane.

There is only one strip connecting any two neighboring planes P and P'. The bottom of the strip is a red axis ($\beta = n$) in P, and the top of the strip is a blue axis

$(\alpha = 0)$ in P'. Figure 5.2 shows a strip above the $\beta = 2$ axis in the base plane.

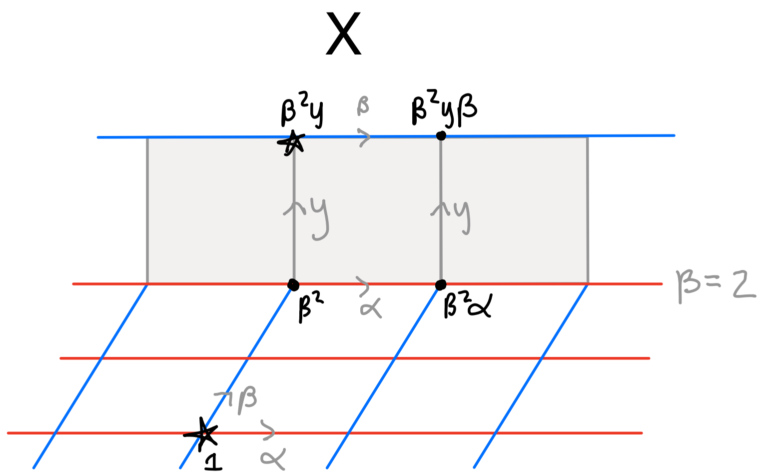

Figure 5.2

To reiterate, $g_i y^{\epsilon_i} = \beta^n y$ tells us to navigate to the $\beta = n$ axis then go up. Alternatively, $g_i y^{\epsilon_i} = \alpha^n y^{-1}$ tells us to travel to the $\alpha = n$ axis before going down.

Build the Bass-Serre tree T by collapsing each $\langle \alpha, \beta \rangle$ plane in X to a vertex. The vertical strips in X collapse to edges. Each vertex in T has valence $2\mathbb{Z}$, representing the $\mathbb{Z}$ planes glued to strips above and the $\mathbb{Z}$ planes glued to strips below.

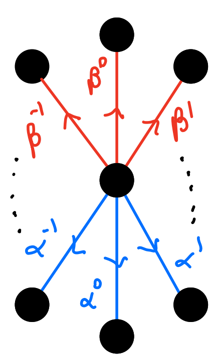

Figure 5.3

Each plane has $\mathbb{Z}$ strips glued above along the $\beta = n$ horizontal axes. The corresponding edges in the Bass-Serre tree are labeled by $\{\beta^n\}$. Figure 5.4 shows the same strip in X shown above, along with its corresponding edge in T.

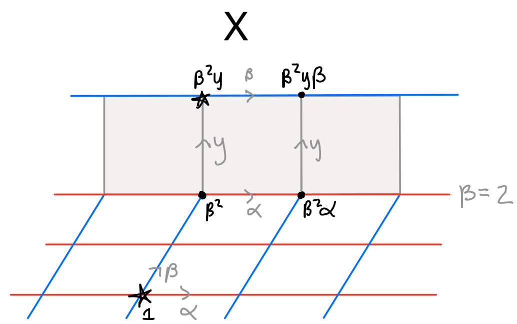

Figure 5.4

The graph of groups is a loop with vertex group $\mathbb{Z} \oplus \mathbb{Z}$ and edge group $\mathbb{Z}$. The stabilizer of a vertex in T (or a plane in X) is a conjugate of $\langle \alpha, \beta \rangle$. The stabilizer of an edge in T (or a strip in X) is a conjugate of $\langle \alpha \rangle$.

Let $g = g_1 y^{\epsilon_1} \dots g_n y^{\epsilon_n} w$ in normal form. Let $h = g_1 y^{\epsilon_1} \dots g_n y^{\epsilon_n}$ (so that $g = hw$). The plane in X containing v_g is stabilized by $h\langle \alpha, \beta \rangle h^{-1}$ (we multiply on the left). As such, the corresponding vertex in T is labeled h. The unique edge path in T from 1 to h is $[g_1, \dots, g_n]$. The vertices along the way are $[1, g_1 y^{\epsilon_1}, g_1 y^{\epsilon_1} g_2 y^{\epsilon_2}, \dots, h]$.

Consider $g = (\beta y)^3 \alpha y^{-1} \alpha^2 \beta$. Figure 5.5 shows a schematic of the path in X from v_1 to v_g. Axes highlighted in the same color are attached by a strip. Figure 5.5 shows the corresponding path in T. Per our normal form, $g_1 y^{\epsilon_1} = g_2 y^{\epsilon_2} = g_3 y^{\epsilon_3} = \beta y$, $g_4 y^{\epsilon_4} = \alpha y^{-1}$, and $w = \alpha^2 \beta$. The edge path in T from the base plane to the destination plane is $[\beta^1, \beta^1, \beta^1, \alpha^1]$. The yellow strip is stabilized by $\langle \alpha \rangle$. The green strip is stabilized by $(\beta y)^3 \alpha \langle \beta \rangle \alpha^{-1} (\beta y)^{-3}$. Using the relation $\langle \beta \rangle = y^{-1} \langle \alpha \rangle y$, the green strip is stabilized by $(\beta y)^3 \alpha y^{-1} \langle \alpha \rangle y \alpha^{-1} (\beta y)^{-3}$. Edges are oriented away from the origin.

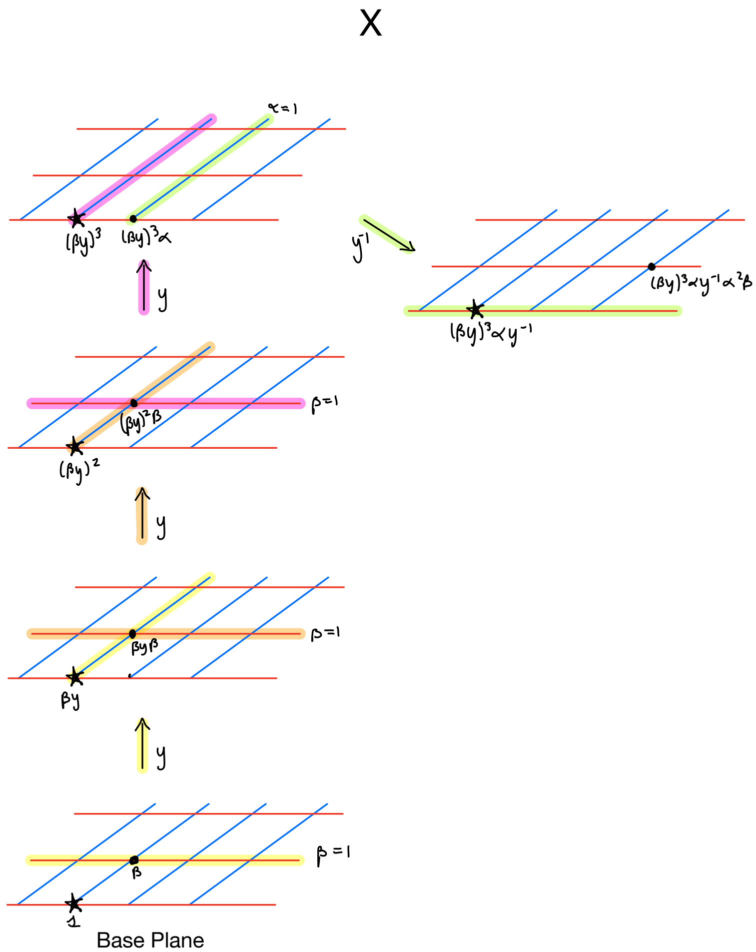

Figure 5.5

Figure 5.6 shows the paths in T corresponding to g and $\hat{\tau}(g)$. For an edge path $[g_1, \ldots, g_n]$ in T, $\hat{\tau} \cdot [g_1, \ldots, g_n] = [\hat{\tau}(g_1), \ldots, \hat{\tau}(g_n)]$. In other terms, the vertex h in T (the plane stabilized by $h\langle\alpha, \beta\rangle h^{-1}$) gets sent to $\hat{\tau}(h)$ (the plane stabilized by $\hat{\tau}(h)\langle\alpha, \beta\rangle\hat{\tau}(h^{-1})$). Within the h-plane, we get to g by the geodesic path w. Within the $\hat{\tau}(h)$-plane, we get to $\hat{\tau}(g)$ by w reflected across the 45° line.

In our example, the edge path to h is $[\beta^1, \beta^1, \beta^1, \alpha^1]$. The edge path to $\hat{\tau}(h)$ is $[\alpha^1, \alpha^1, \alpha^1, \beta^1]$. The path in the h-plane to g is $w = \alpha^2\beta$. The path in the $\hat{\tau}(h)$ plane to $\hat{\tau}(g)$ is $\hat{\tau}(w) = \beta^2\alpha$

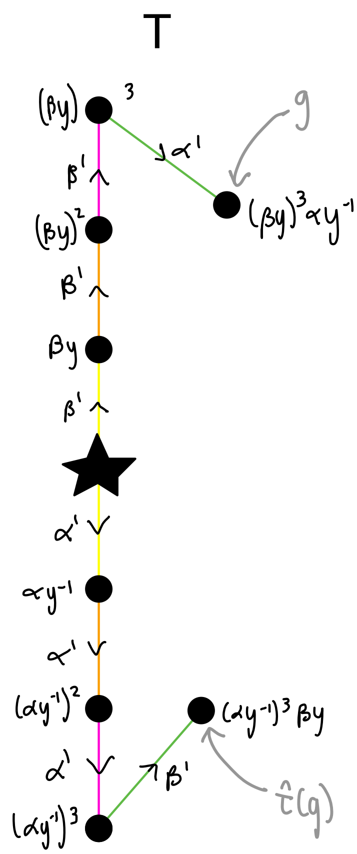

Figure 5.6

We include the path in X from to $\hat{\tau}(g)$ as well:

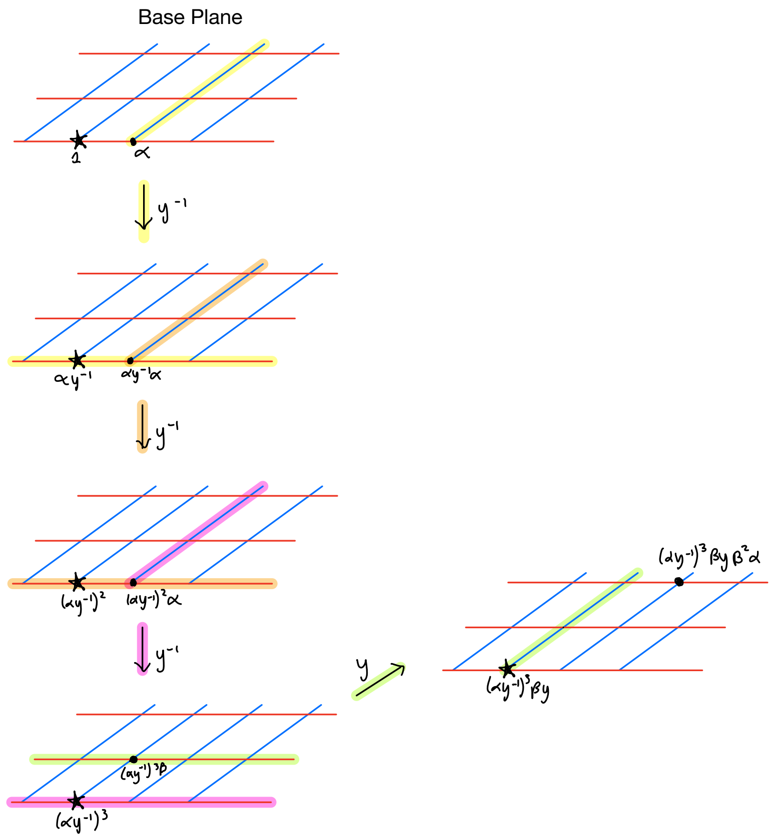

Figure 5.7

Proof

In this section, we prove that the map $v_g \mapsto v_{\hat{\tau}(g)}$ extends to an isometry of X. Then, we show that the isometry is compatible with the algebra so that $(\mathbb{Z}\oplus\mathbb{Z})*_{\mathbb{Z}}\rtimes\langle\hat{\tau}\rangle$ is a subgroup of Isom(X). Finally, we argue that the subgroup is cocompact and properly discontinuous.

This map $v_g \mapsto v_{\hat{\tau}(g)}$ is a bijection of the vertices because $\hat{\tau}$ is an automorphism of $F_2 \rtimes_\psi \mathbb{Z} \cong (\mathbb{Z}\oplus\mathbb{Z})*_{\mathbb{Z}}$. We need to prove that it respects adjacency.

By design, the geometry of X is determined by its 1-skeleton: we add a 2-cell everywhere we see the 1-skeleton of a 2-cell. Therefore, if a map on the vertex set respects adjacency, then it extends to an isometry of X. As an automorphism, $\hat{\tau}$ permutes the set of edge labels $\{\alpha, \beta, y\}^{\pm}$. More precisely, v_{g_1} and v_{g_2} are connected by an edge of unit length if and only if there exists an element $\ell \in \{\alpha, \beta, y\}^{\pm}$ such that $g_2^{-1}g_1 = \ell$.

$$g_1\ell = g_2 \implies \hat{\tau}(g_1)\hat{\tau}(\ell) = \hat{\tau}(g_2)$$

If g_1 and g_2 are connected by an edge labeled ℓ, then $\hat{\tau}(g_1)$ and $\hat{\tau}(g_2)$ are connected by an edge labeled $\hat{\tau}(\ell)$.

Now, we check that the geometry matches the algebra. Fix $g \in (\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}}$. We want to show that isometries $\hat{\tau}g\hat{\tau}$ and $\hat{\tau}(g)$ agree on all vertices. Fix a vertex v_h where $h \in (\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}}$.

$$\hat{\tau}g\hat{\tau} \cdot v_h = \hat{\tau}g \cdot v_{\hat{\tau}(h)} = \hat{\tau} \cdot v_{g\hat{\tau}(h)} = v_{\hat{\tau}[g\hat{\tau}(h)]} = v_{\hat{\tau}(g)h} = \hat{\tau}(g) \cdot v_h$$

Therefore, $(\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}} \rtimes \langle\hat{\tau}\rangle$ is a subgroup of $\mathrm{Isom}(X)$.

The extended action is cocompact because the original action is cocompact- there is one orbit of vertices for a group acting on its Cayley complex. Lastly, the extended action is properly discontinuous by the lemma below.

Lemma 5.4.2 *If a group G acts properly discontinuously on a metric space X, then any finite split extension of G acts properly discontinuously on X.*

Proof: Let F be a finite group. G' is a split extension of G by F iff $G' \cong G \rtimes F$. Suppose, for sake of contradiction, $\exists$ compact $K \subset X$ and an infinite set $\{g_n f_n\} \subset G'$ such that $(g_n f_n \cdot K) \cap K \neq \varnothing$ for all n. Since F is finite, there exists $f \in F$ such that $f_n = f$ for infinitely many n. Pass to the subset $\{g_n f\}$.

We assumed $(g_n f) \cdot K \cap K \neq \varnothing$. Equivalently, $g_n \cdot (f \cdot K) \cap K \neq \varnothing$. Let C be a compact set containing $f \cdot K$ and K. Then, $g_n \cdot C \cap C \neq \varnothing$. But then G does not act properly discontinuously on X. $\Rightarrow\Leftarrow$ $\square$

Now, we have shown that $(\mathbb{Z}\oplus\mathbb{Z}) *_{\mathbb{Z}} \rtimes \langle\hat{\tau}\rangle$ acts faithfully and geometrically on X. Therefore, $W_3 \rtimes_{\hat{\psi}} \mathbb{Z}$ is CAT(0).

$$(\mathbb{Z} \oplus \mathbb{Z}) *_{\mathbb{Z}} \rtimes \langle\hat{\tau}\rangle \cong (F_2 \rtimes_{\psi} \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2 \cong W_3 \rtimes_{\hat{\psi}} \mathbb{Z}$$

5.5 Elliptic Automorphisms of W_3

We call an automorphism of W_3 elliptic if its restriction to F_2 is elliptic. An automorphism of F_2 is elliptic if it is finite order in $\mathrm{Out}(F_2)$. An automorphism ϕ of W_3 is elliptic if there is an integer $m \geq 1$ such that ϕ^m is conjugation by an even length word.

The image of $\mathrm{Inn}(W_3)$ in $GL(2,\mathbb{Z})$ is $\pm I$. Let γ_a be conjugation by $a \in W_3$. The image of γ_a in $\mathrm{Aut}(F_2)$ is τ, and the image of τ in $GL(2,\mathbb{Z})$ is $-I$.

$$\mathrm{Inn}(W_3) \cong \mathrm{Inn}(F_2) \rtimes \langle \tau \rangle$$

The two cosets in $\mathrm{Inn}(W_3)/\mathrm{Inn}(F_2)$ are: {conjugation by even length words} and {conjugation by odd length words}. If $\phi^m \in \mathrm{Inn}(F_2)$, then the extension of ϕ^m to W_3 is conjugation by an even length word.

The results in this section apply beyond elliptic $\phi \in \mathrm{Aut}(W_3)$ and beyond rank three. Throughout this section, $\phi \in \mathrm{Aut}(W_n)$ is order m in $\mathrm{Out}(W_n)$.

Theorem 5.5.1 *If $\phi \in Aut(W_n)$ is finite order in $Out(W_n)$, then there is a tree T such that $W_n \rtimes_\phi \mathbb{Z} \curvearrowright T \times \mathbb{R}$ geometrically.*

Here is the outline of the proof:

$$
\begin{tikzcd}
\boxed{W_n \rtimes_\psi \mathbb{Z}_m \text{ is virtually free}} \arrow[d] \arrow[d, bend left=60, dashed, "\text{Karass-Pietrewski-Solitar}"] \\
\boxed{W_n \rtimes_\psi \mathbb{Z}_m \curvearrowright T \text{ geometrically, } T \text{ is a tree}} \arrow[d] \\
\boxed{W_n \rtimes_\psi \mathbb{Z} \curvearrowright T \times \mathbb{R} \text{ geometrically}} \arrow[d] \\
\boxed{W_n \rtimes_\psi \mathbb{Z} \text{ is CAT(0)}} \arrow[d] \arrow[d, bend left=60, dashed, "{[\psi] = [\phi] \implies W_n \rtimes_\psi \mathbb{Z} \cong W_n \rtimes_\phi \mathbb{Z}}"] \\
\boxed{W_n \rtimes_\phi \mathbb{Z} \text{ is CAT(0)}}
\end{tikzcd}
$$

Use lemma 4.3.2 to choose a finite order $\psi \in \mathrm{Aut}(W_n)$ such that ψ and ϕ are in the same outer automorphism class. Then, $W_n \rtimes_\phi \mathbb{Z} \cong W_n \rtimes_\psi \mathbb{Z}$.

The elliptic case for F_n relied on the fact that $F_n \rtimes_\psi \mathbb{Z}_m$ is virtually free. Here, $W_n \rtimes_\psi \mathbb{Z}_m \cong (F_n \rtimes_\tau \mathbb{Z}_2) \rtimes_\psi \mathbb{Z}_m$ is also virtually free.

Theorem 5.5.2 (Karrass-Pietrowski-Solitar 1973) *A finitely generated group G is a finite extension of a free group if and only if G is the fundamental group of a finite graph of groups with finite edge and vertex groups.*

By Karrass-Pietrowski-Solitar, $W_n \rtimes_\psi \mathbb{Z}_m$ is the fundamental group of a finite graph with finite edge and vertex groups. In the language of Bass-Serre theory, $W_n \rtimes_\psi \mathbb{Z}_m$ acts on the universal covering tree T of the graph, with finite edge and vertex stabilizers. The tree is locally finite because it is the universal cover of a finite graph. An action on a locally finite tree with finite vertex stabilizers is properly discontinuous.

The subgroup $F_{n-1} \leq W_n \rtimes_\psi \mathbb{Z}_m$ acts freely on T, and so $\pi_1(T/F_{n-1}) \cong F_{n-1}$. Therefore, the quotient $T/W_n \rtimes_\psi \mathbb{Z}_m$ is compact and we have a geometric action $W_n \rtimes_\psi \mathbb{Z}_m \curvearrowright T$.

By theorem 4.3.3, we can extend the action $W_n \rtimes_\psi \mathbb{Z}_m \curvearrowright T$ to a geometric action

$W_n \rtimes_\psi \mathbb{Z} \curvearrowright T \times \mathbb{R}$. Assume $\mathbb{Z}_m = \langle s \rangle$ and $\mathbb{Z} = \langle t \rangle$. The generator t of $\mathbb{Z}$ is an isometry of X (namely, s) composed with a translation up the $\mathbb{R}$ axis. The W_3 subgroup of $W_3 \rtimes_\psi \mathbb{Z}$ acts on X as it previously did, and fixes the $\mathbb{R}$ axis. For an arbitrary $wt^n \in W_3 \rtimes_\psi \mathbb{Z}$ and $(x,y) \in X \times \mathbb{R}$, let $wt^n \cdot (x,y) = (ws^n \cdot x, y+n)$ where $ws^n \in W_3 \rtimes_\psi \mathbb{Z}_m$. This means that $w \cdot (x,y) = (w \cdot x, y)$ and $t \cdot (x,y) = (s \cdot x, y+1)$, as desired. Notice that t^m is the identity on X, and a translation up $\mathbb{R}$: $t^m \cdot (x,y) = (x, y+m)$.

5.6 Hyperbolic Automorphisms of W_3

We call $\phi \in \mathrm{Aut}(W_3)$ hyperbolic if its restriction to F_2 is hyperbolic. As an example, consider the figure eight knot complement.

$$\phi_{ab} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

If $\phi(x) = xyx$ and $\phi(y) = xy$, then we extend ϕ to W_3 as follows:

$$\phi(a) = (aba)c(aba)$$
$$\phi(b) = a$$
$$\phi(c) = aba$$

For a hyperbolic $\phi \in \mathrm{Aut}(F_2)$, $F_2 \rtimes_\phi \mathbb{Z}$ is the fundamental group of a finite volume hyperbolic 3-manifold. Let $M = \mathbb{H}^3/\Gamma$ where $F_2 \rtimes_\phi \mathbb{Z} \cong \Gamma \leq \mathrm{Isom}(\mathbb{H}^3)$. Γ acts geometrically on truncated hyperbolic space, which is CAT(0) [9]. See section 4.4 for more details.

We claim that we can extend the action of $F_2 \rtimes_\phi \mathbb{Z}$ on truncated hyperbolic space faithfully to $W_3 \rtimes_\phi \mathbb{Z}$. Remember, $W_3 \rtimes_\phi \mathbb{Z} \cong (F_2 \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$ so we need to find an order two isometry of $\mathbb{H}^3$ that preserves the missing horoballs and respects the algebra.

Lemma 5.6.1 *The action of $F_2 \rtimes_\phi \mathbb{Z}$ on truncated hyperbolic space can be extended faithfully to $(F_2 \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$.*

Mostow-Prasad Rigidity

The lemma is a consequence of Mostow-Prasad Rigidity. Mostow proved that the geometry of a compact hyperbolic manifold of dimension at least three is determined by its fundamental group [34]. Prasad extended the result to finite volume hyperbolic manifolds [37]. We can state the result geometrically or algebraically:

Theorem 5.6.2 (Mostow-Prasad Rigidity Geometric Version) *Let M_1 and M_2 be complete, finite-volume hyperbolic n-manifolds, $n \geq 3$. If $f : M_1 \longrightarrow M_2$ is a homotopy equivalence, then there is an isometry $F : M_1 \longrightarrow M_2$ homotopic to f.*

Theorem 5.6.3 (Mostow-Prasad Rigidity Algebraic Version) *Let M_1 and M_2 be complete, finite-volume hyperbolic n-manifolds, $n \geq 3$. Given an isomorphism $\theta : \pi_1(M_1) \longrightarrow \pi_1(M_2)$, there is an isometry $g \in Isom(\mathbb{H}^3)$ such that $\theta(\gamma) = g\gamma g^{-1}$ for all $\gamma \in \pi_1(M_1)$.*

In short, Mostow Rigidity allows us to reverse the following implications:

$$\text{Isometric} \Rightarrow \text{Homeomorphic} \Rightarrow \text{Homotopy equivalent} \Rightarrow \text{Isomorphic } \pi_1$$

The diagram below illustrates why the geometric and algebraic statements are equivalent. The vertical arrows are explained below the diagram.

$$\begin{array}{ccc}
f : M_1 \xrightarrow{h.e.} M_2 & \xRightarrow{\quad\text{Geo}\quad} & F : M_1 \xrightarrow{isom} M_2 \\
\Updownarrow * & & \Updownarrow ** \\
f_* : \pi_1(M_1) \xrightarrow{\cong} \pi_1(M_2) & \xRightarrow{\text{Alg}} & \tilde{F}\pi_1(M_1)\tilde{F}^{-1} = f_*(\pi_1(M_1))
\end{array}$$

$(*) \Rightarrow$: By a basic fact of algebraic topology, a homotopy equivalence induces an isomorphism on fundamental groups. The isomorphism is given by $f_*([\ell]) = [f \circ \ell]$ where $f : M_1 \longrightarrow M_2$ is a homotopy equivalence.

$(**) \Rightarrow$: An isometry $F : M_1 \longrightarrow M_2$ induces an isomorphism $F_* : \pi_1(M_1) \longrightarrow \pi_1(M_2)$. Additionally, F lifts to an isometry of the universal cover $\tilde{F} : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ that is equivariant with respect to the actions of $\pi_1(M_i) \cong \Gamma_i$ on $\mathbb{H}^3$: for all $\gamma \in \Gamma_1$, $\tilde{F} \circ \gamma = F_*(\gamma) \circ \tilde{F}$. Therefore, $\tilde{F}\Gamma_1\tilde{F}^{-1} = F_*(\Gamma_1)$.

$(**) \Leftarrow$: Going the other way, we have an isometry $g : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ such that $g \circ \gamma = \theta(\gamma) \circ g$ for all $\gamma \in \Gamma_1$. The isometry g descends to the quotient by sending $[x]$ in $\mathbb{H}^3/\Gamma_1$ to $[g(x)]$ in $\mathbb{H}^3/\Gamma_2$.

$$\begin{tikzcd} \mathbb{H}^3 \arrow[r, "g"] \arrow[d, "\Gamma_1"] & \mathbb{H}^3 \arrow[d, "\Gamma_2"] \\ M_1 \arrow[r, "g"] & M_2 \end{tikzcd} \qquad \begin{tikzcd} \gamma \cdot x \arrow[r, "g"] \arrow[d, "\Gamma_1"] & g(\gamma \cdot x) = \theta(\gamma) \cdot g(x) \arrow[d, "\Gamma_2"] \\ {[x]} \arrow[r, "g"] & {[g(x)]} \end{tikzcd}$$

In the diagram above, we see that the map $g : M_1 \longrightarrow M_2$ is well defined.

Hyperbolic Case for W_3

Let Γ be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ that acts freely with finite covolume. Let θ be an automorphism of Γ. We care about the case in which $\Gamma \cong F_2 \rtimes_\phi \mathbb{Z}$ and $\theta = \hat{\tau}$. By Mostow-Prasad Rigidity, there exists $\tilde{g} \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\tilde{g}\gamma\tilde{g}^{-1} = \theta(\gamma)$ for all $\gamma \in \Gamma$. Therefore, $\Gamma \rtimes_\theta \langle\tilde{g}\rangle$ acts by isometries on $\mathbb{H}^3$, but we want to show it acts on truncated hyperbolic space.

As explained above, $\tilde{g}$ descends to an isometry $g : M \longrightarrow M$ where $M = \mathbb{H}^3/\Gamma$ is a cusped hyperbolic manifold. An isometry of a cusped manifold sends cusps to cusps. Correspondingly, $\tilde{g}$ leaves the set of Γ-parabolic fixed points in the universal cover invariant. More precisely, if a parabolic isometry γ fixes $p \in \partial\mathbb{H}^3$, then $\theta(\gamma)$ fixes $\tilde{g} \cdot p$ because $\theta(\gamma) = \tilde{g}\gamma\tilde{g}^{-1}$.

Recall, we build the truncated hyperbolic space on which Γ acts cocompactly by removing a Γ-equivariant set of disjoint horoballs about the parabolic fixed points. The isometry $\tilde{g}$ leaves the set of parabolic fixed points and their associated horoballs invariant. Thus, $\tilde{g}$ is an isometry of truncated hyperbolic space.

Now we have $\Gamma \rtimes_\theta \langle\tilde{g}\rangle \cong (F_2 \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$ acting by isometries on truncated hyperbolic space. The action is faithful because the isometry $\tilde{g}$ is not trivial- it conjugates γ to $\theta(\gamma)$. The action is properly discontinuous because $\Gamma \rtimes_\theta \langle\tilde{g}\rangle$ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. Of course it is cocompact, because it is the extension of a cocompact action.

Chapter 6

Hyperbolic W_n-by-$\mathbb{Z}$

In this Chapter, we summarize our investigation of the original question of whether one identify when a particular W_n-by-$\mathbb{Z}$ group is hyperbolic or not.

Brinkmann proved that $F_2 \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$ [10]. Dahmani, Krishna, and Mutanguha extended the results to any hyperbolic group [18].

Theorem 6.0.1 (Dahmani-Krishna-Mutanguha 2023) *Suppose G is a hyperbolic group. Then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$.*

There is a $\mathbb{Z}\oplus\mathbb{Z}$ subgroup in $W_n \rtimes_\phi \mathbb{Z}$ if and only if a power of ϕ fixes the conjugacy class of an infinite order element. If $\phi^k(w) = gwg^{-1}$ for an infinite order $w \in W_n$ and $k \geq 1$, then in $W_n \rtimes_\phi \mathbb{Z}$:

$$
\begin{aligned}
twt^{-1} &= \phi(w) \\
t^k w t^{-k} &= \phi^k(w) \\
t^k w t^{-k} &= gwg^{-1} \\
g^{-1}t^k w &= wg^{-1}t^k \\
\langle g^{-1}t^k, w\rangle &\cong \mathbb{Z} \oplus \mathbb{Z}
\end{aligned}
$$

There are no hyperbolic $W_3 \rtimes_\phi \mathbb{Z}$ because every automorphism of W_3 sends abc to a conjugate of $(abc)^{\pm}$. This is easy to see by checking the generators of $\mathrm{Aut}(W_3)$. If $\phi(abc)$ is conjugate to $(abc)^{-1}$, then $\phi^2(abc)$ is conjugate to abc, so we get a $\mathbb{Z} \oplus \mathbb{Z}$ in the mapping torus.

We also know that $W_3 \rtimes_\phi \mathbb{Z}$ is not hyperbolic because it contains a finite index $F_2 \rtimes_\phi \mathbb{Z}$. Every automorphism of F_2 sends the commutator $[x, y]$ to a conjugate of $[x, y]^{\pm}$. Therefore, $F_2 \rtimes_\phi \mathbb{Z}$ contains a $\mathbb{Z} \oplus \mathbb{Z}$.

6.0.1 Example of a Hyperbolic W_4-by-$\mathbb{Z}$ Group

There are hyperbolic W_4-by-$\mathbb{Z}$ groups. For example, we will explain why $W_4 \rtimes_\psi \mathbb{Z}$ is hyperbolic for the automorphism ψ below. Assume $W_4 = \langle a, b, c, d | a^2, b^2, c^2, d^2 \rangle$.

$$\psi(a) = a$$

$$\psi(b) = d$$

$$\psi(c) = dabad$$

$$\psi(d) = dacad$$

We can write ψ as composition of partial conjugations and a permutation of the generators:

$$\psi = \chi_{a,\{bc\}} \circ \chi_{d,(bc)} \circ \sigma_{(bdc)}$$

The partial conjugation $\chi_{a,\{bc\}}$ is "a conjugates b and c." A partial conjugation is a product of elementary partial conjugations: $\chi_{a,\{bc\}} = \chi_{a,b} \circ \chi_{a,c}$.

$$\chi_{a,\{bc\}}(a) = a$$

$$\chi_{a,\{bc\}}(b) = aba$$

$$\chi_{a,\{bc\}}(c) = aca$$

$$\chi_{a,\{bc\}}(d) = d$$

The work of Gersten-Stallings and Bestvina-Handel gives us a way to find $\phi \in \mathrm{Aut}(F_3)$ such that $F_3 \rtimes_\phi \mathbb{Z}$ is hyperbolic. Some of these automorphisms are in the image of $\rho_4 : \mathrm{Aut}(W_4) \longrightarrow \mathrm{Aut}(F_3)$. We established early on that $W_n \rtimes_\phi \mathbb{Z}$ has an index two copy of $F_{n-1} \rtimes_\phi \mathbb{Z}$, so one group is hyperbolic if and only if the other is hyperbolic.

For $\phi \in \mathrm{Aut}(F_n)$, the abelianization ϕ_{ab} is a matrix in $GL(n, \mathbb{Z})$. The following theorem immediately follows from the work of Gersten-Stallings and Bestvina-Handel [20] [5].

Theorem 6.0.2 *Let $\phi \in Aut(F_n)$, $n \geq 3$. If ϕ_{ab} is a PV matrix, then no power of ϕ*

preserves a conjugacy class in F_n.

A matrix is PV if it has determinant ±1, exactly one eigenvalue with magnitude greater than one, and all other eigenvalues with magnitude less than one. The leading eigenvalue is a Pisot-Vijayaraghavan number, hence the name. Adding in Brinkmann's work (which came later), we have the following theorem [10]:

Theorem 6.0.3 *Let* $\phi \in Aut(F_n)$, $n \geq 3$. *If* ϕ_{ab} *is a PV matrix, then* $F_n \rtimes_\phi \mathbb{Z}$ *is hyperbolic.*

Here is the outline of the theorem. Assume $\phi \in \mathrm{Aut}(F_n)$, $n \geq 3$:

Gersten-Stallings: ϕ_{ab} is PV $\implies$ ϕ^k is irreducible for all $k \geq 1$

Stallings: ϕ_{ab} is PV $\implies$ ϕ is not geometric (not induced by a homeomorphism)

Bestvina-Handel: ϕ^k is irreducible for all $k \geq 1$ and ϕ is not geometric $\implies$ No power of ϕ preserves a conjugacy class

Brinkmann: No power of ϕ preserves a conjugacy class $\iff$ $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic

The matrix below is PV. It has eigenvalues $\lambda_1 \approx 2.83, \lambda_2 \approx -.42 + .42i, \lambda_3 \approx -.42 - .42i$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Now, lets circle back to $\psi \in \mathrm{Aut}(W_4)$. Identify the even length words in W_4 with F_3 via $\langle ab, ac, ad \rangle = \langle x, y, z \rangle$. Then, ψ restricted to F_3 is as follows:

$$x = ab \mapsto ad = z$$

$$y = ac \mapsto adabad = zxz$$

$$z = ad \mapsto adacad = zyz$$

The abelianization of $\psi|_{F_3}$ is the PV matrix above. By the theorem, $F_3 \rtimes_\psi \mathbb{Z}$ is hyperbolic and, therefore, $W_4 \rtimes_\psi \mathbb{Z}$ is hyperbolic.

According to [27], Miller observed that the palindromic automorphisms of F_{n-1} are isomorphic to the automorphisms of W_n that fix a generator. An automorphism of F_{n-1} is palindromic if $\phi(x_i)$ is a palindromic word for every generator x_i. Notice that ψ above was a palindromic automorphism of F_3, and $\psi(a) = a \in W_4$.

The matrix $\phi_{ab} \in GL(n-1, \mathbb{Z})$ has a palindromic automorphism in its outer class if there is exactly one odd number in each column. Therefore, any PV matrix in $GL(n-1, \mathbb{Z})$ with exactly one odd number in each column produces a hyperbolic $F_{n-1} \rtimes_\phi \mathbb{Z}$, and a hyperbolic $W_n \rtimes_\phi \mathbb{Z}$.

The ultimate goal, however, is to use properties inherit to W_4 to identify hyperbolic W_4-by-$\mathbb{Z}$ groups.

6.0.2 W_n

We want to identify $\phi \in \mathrm{Aut}(W_n)$ such that $W_n \rtimes_\phi \mathbb{Z}$ is hyperbolic.

$$\mathrm{Aut}(W_n) = \mathrm{Aut}^\circ(W_n) \rtimes \Sigma_n$$

Σ_n is the subgroup of permutations of the generators. $\mathrm{Aut}^\circ(W_n)$ is generated by elementary partial conjugations χ_{ij}.

$$\chi_{ij}(a_j) = a_i a_j a_i$$

If $\phi \in \mathrm{Aut}^\circ(W_n)$ is the product of $n-2$ elementary partial conjugations, then it fixes at least two generators and its mapping torus is not hyperbolic. In the case of W_4, the product of three partial conjugations is not enough either.

Lemma 6.0.4 *If $\phi \in Aut(W_{n+1})$ is the product of n elementary partial conjugations, then at least one of the following is true:*

1. *There is an infinite order $w \in W_{n+1}$ such that $\phi(w) = w^\pm$.*

2. *ϕ preserves a subgroup $H \cong W_n$*

Here are two examples with $W_{n+1} = W_4$ to demonstrate the proof. The first example falls into case $2a$: ϕ restricts to the subgroup generated by $\{a_1, a_2, a_3\}$.

$$\phi = \chi_{13}\chi_{32}\chi_{21}$$

$$\phi(a_1) = a_2a_1a_2$$

$$\phi(a_2) = a_1a_3a_1a_2a_1a_3a_1$$

$$\phi(a_3) = a_1a_3a_1$$

$$\phi(a_4) = a_4$$

The next example falls into case $2b$: $\phi(a_2a_4) = (a_2a_4)^{-1}$

$$\phi = \chi_{13}\chi_{42}\chi_{21}$$

$$\phi(a_1) = a_2a_1a_2$$

$$\phi(a_2) = a_4a_2a_4$$

$$\phi(a_3) = a_1a_3a_1$$

$$\phi(a_4) = a_4$$

$$\phi(a_2a_4) = a_4a_2$$

Proof: Let ϕ be the product of n elementary partial conjugations.

Case 1: ϕ fixes two generators

Their product is an infinite order element fixed by ϕ.

Case 2: ϕ fixes exactly one generator

Without loss of generality, let $\phi = \chi_{i_n,n} \dots \chi_{i_1,1}$ so that ϕ fixes a_{n+1}. We know:

$$\phi(a_n) = a_{i_n}a_na_{i_n}$$

$$\phi(a_{n+1}) = a_{n+1}$$

Case 2a: If $\{a_{i_1}, \dots, a_{i_n}\} \subseteq \{a_1, \dots, a_n\}$, then ϕ restricts to $\langle a_1, \dots, a_n\rangle$.

Case 2b: Else, one of $\{a_{i_1}, \dots, a_{i_n}\}$ is a_{n+1}. Without loss of generality, assume

$a_{i_1} = a_{n+1}$.

$$\phi(a_1) = a_{n+1}a_1a_{n+1}$$

$$\phi(a_{n+1}) = a_{n+1}$$

$$\phi(a_1a_{n+1}) = a_{n+1}a_1 = (a_1a_{n+1})^{-1} \qquad \square$$

Corollary 6.0.5 *If $\phi \in Aut(W_4)$ is the product of three elementary partial conjugations, then $W_4 \rtimes_\phi \mathbb{Z}$ is not hyperbolic.*

A future goal is to understand when $\phi \in \mathrm{Aut}^\circ(W_n)$ fixes an infinite order element. Remember, $\mathrm{Aut}(W_n) = \mathrm{Aut}^\circ(W_n) \rtimes \Sigma_n$. The automorphisms in $\mathrm{Aut}^\circ(W_n)$ send each generator to a conjugate of itself (no permutation of the generators). We know $\phi \in \mathrm{Aut}^\circ(W_n)$ preserves the conjugacy class of w if and only if it is in the same outer class as an automorphism that fixes w. The next step is to understand when a power of ϕ fixes an infinite order element.

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