Extensions of the Universal Coxeter Group by $\mathbb{Z}$

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Geometric Group Theory

$G \curvearrowright X$

G = finitely generated infinite group

X = proper geodesic metric space

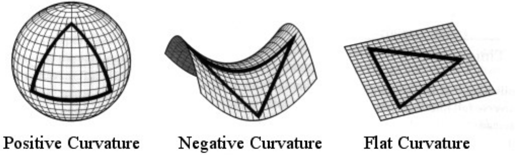
G acts on X by isometries

$G \curvearrowright X$ "nicely"

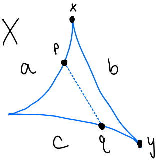
Algebra of $G \longleftrightarrow$ Geometry of X

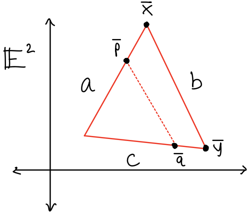
CAT(0) Geometry

Notion of non-positive curvature



CAT(0) Metric Space





- Both triangles have side lengths a, b, c
- X is CAT(0) if $d_X(p,q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q})$
- Example: Hyperbolic space $\mathbb{H}^n$
- Example: Trees (no other CAT(0) graphs)

CAT(0) Group

- $G \curvearrowright X$ faithfully and geometrically
- $G \hookrightarrow \mathrm{Isom}(X)$
- Example: W_n acts faithfully and geometrically on a tree

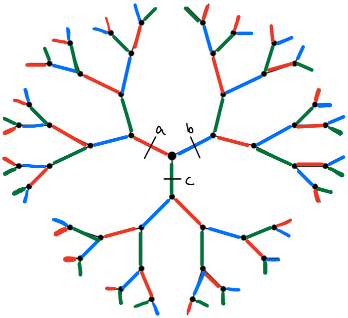
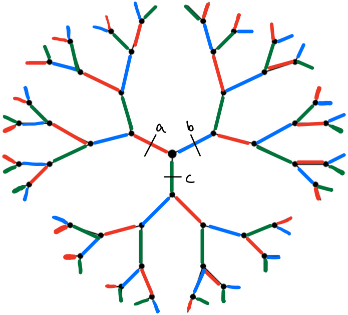


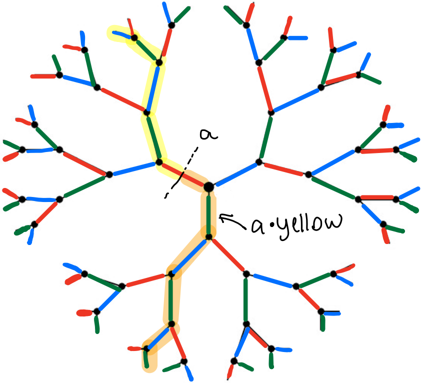
Figure: $W_3 \curvearrowright T_3$

Universal Coxeter Group: W_n

- $W_3 = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$
- $W_3 = \langle a, b, c | a^2, b^2, c^2 \rangle$
- $W_n = \langle a_1, \ldots, a_n | a_1^2, \ldots, a_n^2 \rangle$
- $W_3 \curvearrowright T_3$ faithfully and geometrically
- a, b, c are reflections



W_3 Acts Geometrically on T_3



W_3 Acts Geometrically on T_3

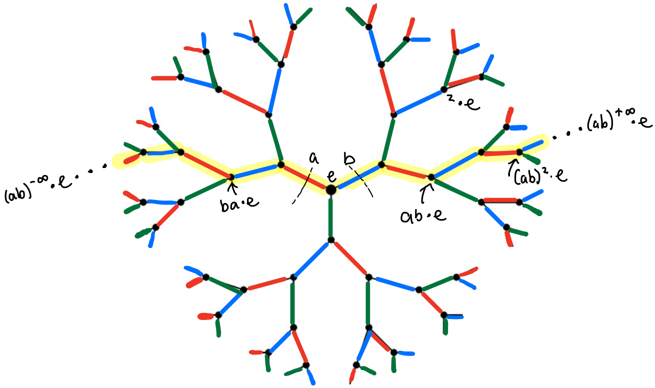
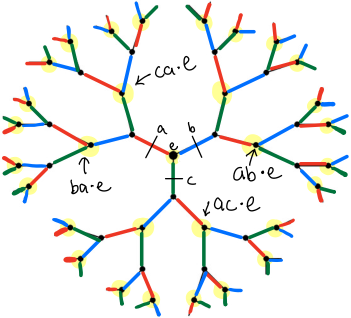


Figure: Translation axis for *ab*

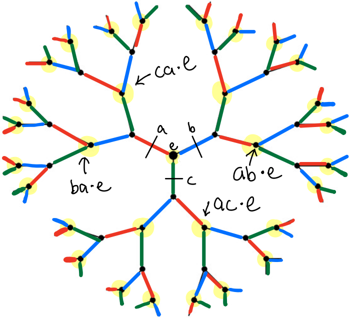
W_n is virtually free

- E_n = Subgroup of even length words



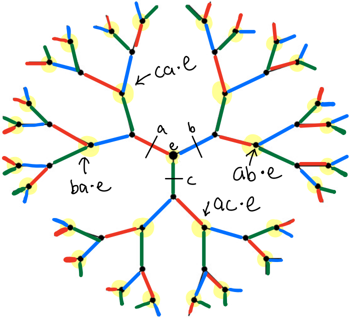
W_n is virtually free

- E_n = Subgroup of even length words
- $E_n = \langle a_1 a_2, \ldots, a_1 a_n \rangle \cong F_{n-1}$



W_n is virtually free

- E_n = Subgroup of even length words
- $E_n = \langle a_1 a_2, \ldots, a_1 a_n \rangle \cong F_{n-1}$
- Ex: $E_3 = \langle ab, ac \rangle \cong \langle x, y \rangle = F_2$



Where we are going

- $W_n \rtimes_\phi \mathbb{Z}$ is a finite extension of $F_{n-1} \rtimes_\phi \mathbb{Z}$
- Do the geometric properties of $F_{n-1} \rtimes_\phi \mathbb{Z}$ transfer to $W_n \rtimes_\phi \mathbb{Z}$?

Group Extension

$$1 \to H \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

- G is an extension of H
  - H injects into G
  - $\iota(H) \trianglelefteq G$
  - $Q \cong G/\iota(H)$
- G is a finite extension of H if Q is finite
- The short exact sequence splits if and only if $G \cong H \rtimes_\phi Q$ for some $\phi : Q \to \mathrm{Aut}(H)$

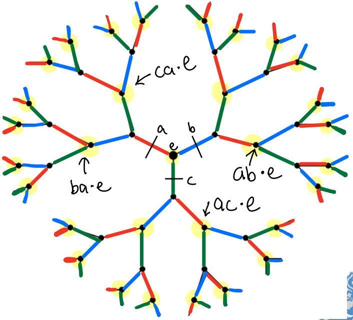
W_n is a finite extension of F_{n-1}

$$1 \to F_{n-1} \to W_n \to \mathbb{Z}_2 \to 1$$

F_{n-1} is index two
(and therefore normal) in W_n

$$W_n \cong F_{n-1} \rtimes_\tau \mathbb{Z}_2$$
$$\cong E_n \rtimes_\tau \langle a_1 \rangle$$

$$\tau(x_i) = x_i^{-1}$$



G is a Finite Extension of H

$$1 \to H \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

Example: $W_n \rtimes_\phi \mathbb{Z}$ is a finite extension of $F_{n-1} \rtimes_\phi \mathbb{Z}$

Fact

H is hyperbolic $\iff$ G is hyperbolic

Open Question

Suppose H is CAT(0). Is G CAT(0)?

Free-by-Cyclic Groups

$F_n \rtimes_\phi \mathbb{Z}$ ("free-by-cyclic") is an extension of F_n by the integers.

$G \cong F_{n-1} \rtimes_\phi \mathbb{Z}$

$1 \to F_{n-1} \xrightarrow{\iota} G \xrightarrow{\pi} \mathbb{Z} \to 1$

Free-by-cyclic groups:

1. Well-studied
2. Help us understand $\phi \in \mathrm{Aut}(F_n)$
3. Mapping tori

$W_n \rtimes_\phi \mathbb{Z}$ is a finite extension of $F_{n-1} \rtimes_\phi \mathbb{Z}$

- $F_{n-1} \leq W_n$ is characteristic: $\phi(F_{n-1}) = F_{n-1}$
- $F_{n-1} \rtimes_\phi \mathbb{Z} \leq W_n \rtimes_\phi \mathbb{Z}$, index two

$$1 \to F_{n-1} \rtimes_\phi \mathbb{Z} \xrightarrow{\iota} W_n \rtimes_\phi \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_2 \to 1$$

- Short exact sequence splits

$$W_n \rtimes_\phi \mathbb{Z} \cong (F_{n-1} \rtimes_\phi \mathbb{Z}) \rtimes_{\hat{\tau}} \mathbb{Z}_2$$

$$\hat{\tau}(x_i) = x_i^{-1}, \hat{\tau}(t) = a_1\phi(a_1)t$$

CAT(0) Free-by-cyclic and W_n-by-cyclic

$F_2 \rtimes_\phi \mathbb{Z}$	CAT(0) for every $\phi \in \mathrm{Aut}(F_2)$ (Tom Brady '94)
$W_3 \rtimes_\phi \mathbb{Z}$	CAT(0) for every $\phi \in \mathrm{Aut}(W_3)$ (this thesis)

$n \geq 4$

$F_{n-1} \rtimes_\phi \mathbb{Z}$	Non-examples (Gersten) and examples (Samuelson, Lyman)
$W_n \rtimes_{\phi^p} \mathbb{Z}$	Virtually CAT(0) Examples (Lyman)

Question (Piggott-Ruane)

Are all $W_n \rtimes_\phi \mathbb{Z}$ CAT(0)?

Objectives

Question

Can we extend the action of $F_2 \rtimes_\phi \mathbb{Z} \curvearrowright X$, X CAT(0) to $W_3 \rtimes_\phi \mathbb{Z} \curvearrowright X$ faithfully and geometrically?

Aut(W_n) is much "simpler" than Aut(F_n).

Question

Can we use the combinatorial properties of W_n to determine when $W_n \rtimes_\phi \mathbb{Z}$, $n \geq 4$ is hyperbolic?

Piggott-Ruane-Walsh '10

- Inspiration for extending action to finite extension

$$1 \to \mathrm{Inn}(B_4) \xrightarrow{\iota} \mathrm{Aut}(B_4) \xrightarrow{\pi} \mathbb{Z}_2 \to 1$$

- $\mathrm{Inn}(B_4)$ acts faithfully and geometrically on a CAT(0) 2-complex X (Brady '94, Crisp-Paoluzzi '05)
- There is an order two isometry of X that extends the action to a faithful geometric action $Aut(B_4) \curvearrowright X$
- Fun fact: $\mathrm{Aut}(B_4) \cong \mathrm{Aut}(F_2) \cong \mathrm{Aut}(W_3)$

$W_3 \rtimes_\phi \mathbb{Z}$ is CAT(0): Four Cases

$$\operatorname{Aut}(W_3) \underset{\cong}{\longrightarrow} \operatorname{Aut}(F_2) \xrightarrow{\pi} \operatorname{Out}(F_2) \cong GL(2,\mathbb{Z})$$

ϕ $\qquad$ $\phi|_{F_2}$ $\qquad$ Identity, elliptic, parabolic, or hyperbolic

1. $\operatorname{Inn}(W_3)$
2. $\phi|_{F_2}$ is elliptic
3. $\phi|_{F_2}$ is parabolic $\longrightarrow$ extend action
4. $\phi|_{F_2}$ is hyperbolic $\longrightarrow$ extend action

$GL(2, \mathbb{Z})$

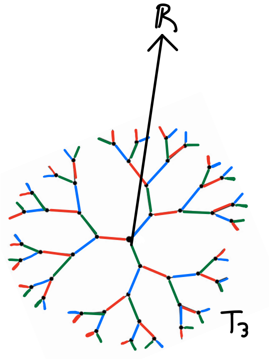
$A \in GL(2, \mathbb{Z})$	$\lvert \mathbf{tr}(A) \rvert$
Identity	2
Elliptic	< 2
Parabolic	2
Hyperbolic	> 2

Inn(W_3)

$\phi \in \mathrm{Inn}(W_3)$

$W_3 \rtimes_\phi \mathbb{Z} \curvearrowright T_3 \times \mathbb{R}$ faithfully and geometrically

1. $W_3 \rtimes_\phi \mathbb{Z} \cong W_3 \times \mathbb{Z}$: $[\phi] = [id]$
2. $W_3 \times \mathbb{Z} \curvearrowright T_3 \times \mathbb{R}$
3. Product of two CAT(0) spaces is CAT(0)



Elliptic: Finite order in Out(W_3)

- $\phi^p \in \mathrm{Inn}(W_3)$
- $\exists \psi \in [\phi]$ such that $\psi^p = \mathrm{Id}_{W_3}$
- $W_3 \rtimes_\phi \mathbb{Z} \cong W_3 \rtimes_\psi \mathbb{Z}$

Claim

$W_3 \rtimes_\psi \mathbb{Z}_p$ acts faithfully and geometrically on a tree T.

$W_3 \rtimes_\psi \mathbb{Z}$ acts faithfully and geometrically on $T \times \mathbb{R}$.

Elliptic: Finite order in Out(W_3)

Theorem (Karrass, Pietrowski, and Solitar '94)

G is a finite extension of a free group if and only if G acts on a locally finite tree T with finite edge and vertex stabilizers.

ψ is order p in Aut(F_2)

$W_3 \rtimes_\psi \mathbb{Z}_p$ is a finite extension of F_2

$W_3 \rtimes_\psi \mathbb{Z}_p \curvearrowright T$ with finite edge and vertex stabilizers

T/F_2 is a finite graph with $\pi_1(T/F_2) \cong F_2$

"Extend" the action to $W_3 \rtimes_\psi \mathbb{Z} \curvearrowright T \times \mathbb{R}$

Parabolic Automorphisms of F_2

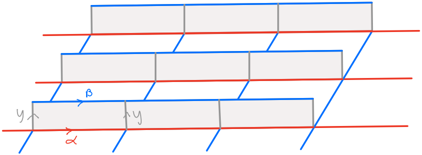
- $\phi|_{F_2}$ has trace ± 2
- We only need to consider $\phi|_{F_2}$ with abelianization:

$$\phi_{ab} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

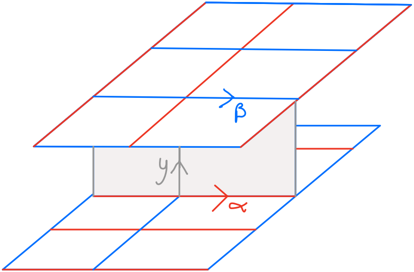
- $F_2 \rtimes_\phi \mathbb{Z} \cong (\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$
- $X =$ Cayley complex of $(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ is CAT(0)

Cayley Complex of $(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$

$$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}} \cong \langle \alpha, \beta, y | [\alpha, \beta], y\beta y^{-1} = \alpha \rangle$$

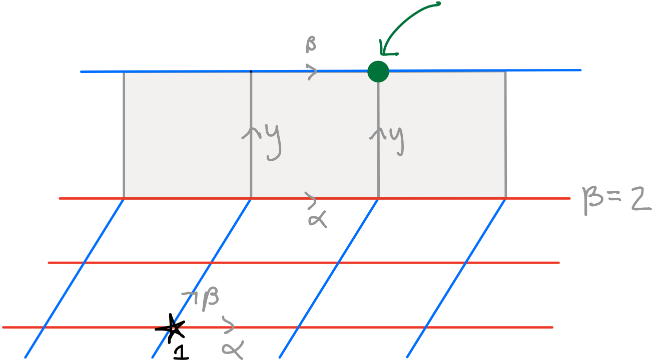


(a) Strips above red axes



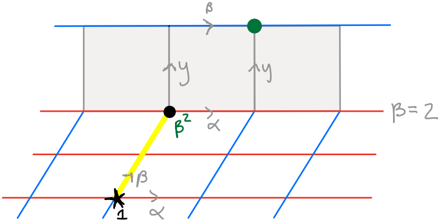
(b) Plane glued to top of each strip

$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ Normal form



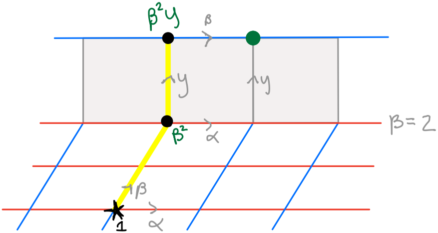
$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ Normal form

1. Move to appropriate strip
   - β^n to the left of y
   - α^n to the left of y^{-1}



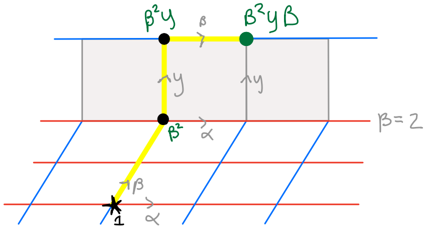
$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ Normal form

1. Move to appropriate strip
   - β^n to the left of y
   - α^n to the left of y^{-1}
2. y or y^{-1} to go up/down



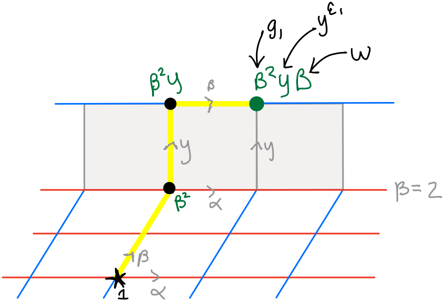
$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ Normal form

1. Move to appropriate strip
   - β^n to the left of y
   - α^n to the left of y^{-1}
2. y or y^{-1} to go up/down
3. Repeat until in "destination plane"
4. Path in "destination plane"

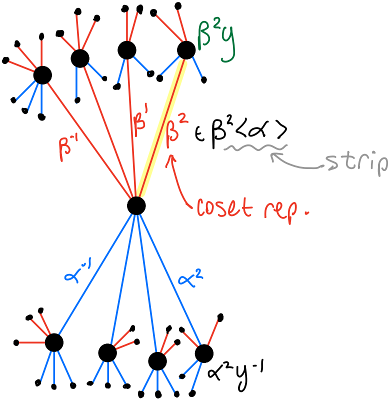


$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ Normal form: $g = hw$

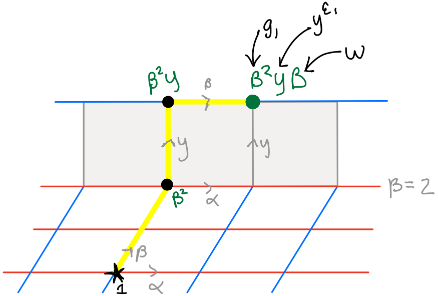
- $w \in \langle \alpha, \beta \rangle$ is path in destination plane
- $h = \prod_{i=1}^{n} g_i y^{\epsilon_i}$ is path in Bass-Serre tree
- $g_i y^{\epsilon_i} = \beta^n y$ or $\alpha^n y^{-1}$



$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}}$ Bass-Serre Tree



(a) Bass-Serre Tree



(b) Cayley Complex X

$W_3 \rtimes_\phi \mathbb{Z} \curvearrowright X$

$W_3 \rtimes_\phi \mathbb{Z} \cong (\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}} \rtimes_{\hat{\tau}} \mathbb{Z}_2$

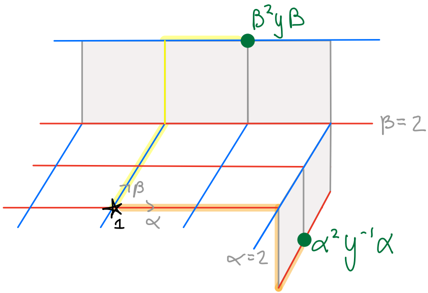
$(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}} \curvearrowright X$ by left multiplication

Extend the action by an order two isometry such that $(\mathbb{Z} \oplus \mathbb{Z})_{*\mathbb{Z}} \rtimes \langle \hat{\tau} \rangle \hookrightarrow \mathrm{Isom}(X)$

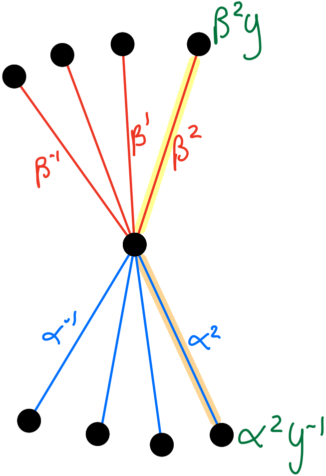
As an automorphism: $\hat{\tau}(\alpha) = \beta$, $\hat{\tau}(\beta) = \alpha$, $\hat{\tau}(y) = y^{-1}$

As an isometry of X: $\hat{\tau} : v_g \mapsto v_{\hat{\tau}(g)}$

$W_3 \rtimes_\psi \mathbb{Z} \curvearrowright X$



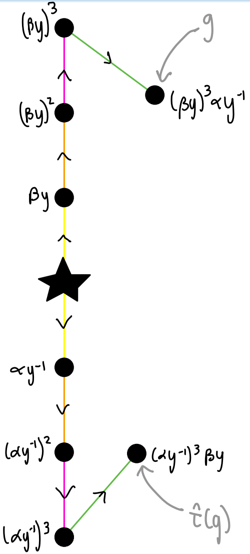
(a) X



(b) T

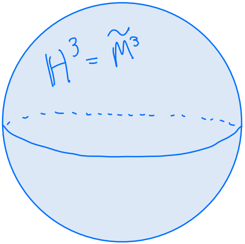
$W_3 \rtimes_\psi \mathbb{Z} \curvearrowright X$

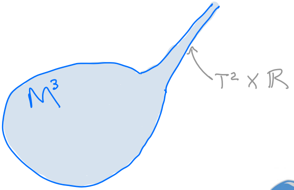
- g lives in plane $h = (\beta y)^3 \alpha y^{-1}$
- $\hat{\tau}(g)$ lives in plane $\hat{\tau}(h) = (\alpha y^{-1})^3 \beta y$



Hyperbolic Automorphisms of F_2

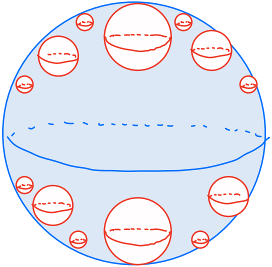
- $\phi|_{F_2}$ has eigenvalues $\lambda, \frac{1}{\lambda}$, $|\lambda| > 1$
- $F_2 \rtimes_\phi \mathbb{Z}$ is the fundamental group of a finite volume hyperbolic manifold with torus cusp
- Torus is generated by $\langle [x, y], t \rangle$



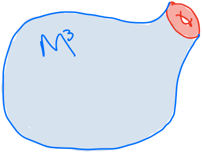


Hyperbolic Automorphisms of F_2

- $F_2 \rtimes_\phi \mathbb{Z}$ acts faithfully and geometrically on truncated hyperbolic space
- (Bridson and Haefliger) Truncated hyperbolic space is CAT(0)



Truncated Hyperbolic Space



Mostow-Prasad Rigidity

Theorem: Mostow-Prasad Rigidity

Let M_1 and M_2 be finite volume hyperbolic n-manifolds, $n \geq 3$. Any isomorphism $\theta : \pi_1(M_1) \longrightarrow \pi_1(M_2)$ is induced, up to conjugacy, by an isometry $f : M_1 \longrightarrow M_2$.

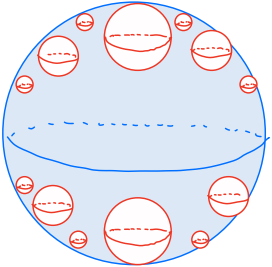
- Mostow proved the compact case in 1968
- Prasad extended to finite volume manifolds in 1973
- See board

Hyperbolic Automorphism of W_3

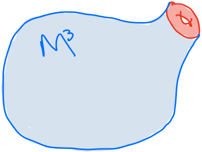
Want: $\pi_1(M) \rtimes_{\hat{\tau}} \langle \tilde{f} \rangle \hookrightarrow \mathrm{Isom}(X)$

Make sure $\tilde{f}$ is an isometry of

X = truncated hyperbolic space

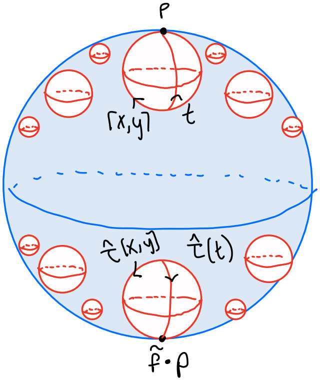


Truncated Hyperbolic Space



Hyperbolic Automorphism of W_3

- $\tilde{f} : \mathbb{H}^3 \to \mathbb{H}^3$ descends to quotient
- $f : \mathbb{H}^3/\pi_1(M) \longrightarrow \mathbb{H}^3/\hat{\tau}(\pi_1(M))$
- f : cusp $\longrightarrow$ cusp
- $\tilde{f}$ leaves the set horoballs invariant



Conclusion & Next Steps

Theorem

For every $\phi \in \mathrm{Aut}(W_3)$, $W_3 \rtimes_\phi \mathbb{Z}$ is CAT(0).

Original Question

There are no hyperbolic $W_3 \rtimes_\phi \mathbb{Z}$. Are there any hyperbolic $W_4 \rtimes_\phi \mathbb{Z}$?

Future directions: When is $W_n \rtimes_\phi \mathbb{Z}$ hyperbolic?

$$\mathrm{Aut}(W_n) = \mathrm{Aut}^\circ(W_n) \rtimes \Sigma_n$$

Σ_n = permutations of the generators $a_1, \ldots, a_n$

$\phi \in \mathrm{Aut}^\circ(W_n)$ sends every generator to a conjugate of itself

Generated by χ_{ij}: $\chi_{ij}(a_j) = a_i a_j a_i$

$W_3 \rtimes_\phi \mathbb{Z}$ is never hyperbolic

$\phi(abc)$ is a conjugate of $(abc)^{\pm} \to$ check the generators of $\mathrm{Aut}(W_3)$

$W_3 \rtimes_\phi \mathbb{Z}$ contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup

$\mathbb{Z} \oplus \mathbb{Z}$ subgroups

Theorem (Dahmani-Krishna-Mutanguha 2023)

Suppose G is a hyperbolic group. Then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$.

Brinkmann ($'00$) proved for F_2 using train track theory.

$F_2 \rtimes_\phi \mathbb{Z}$ does not contain a $\mathbb{Z} \oplus \mathbb{Z}$ if and only if ϕ is atoroidal.

Atoroidal: No power of ϕ preserves the conjugacy class of an infinite order element

Open Questions

1. When does $\phi \in \mathrm{Aut}^{\circ}(W_n), n \geq 4$ fix an infinite order element?
   Lemma: If $\phi \in \mathrm{Aut}^{\circ}(W_4)$ is the product of 3 elementary partial conjugations, then there is an infinite order w such that $\phi(w) = w^{\pm}$
2. When does a power of $\phi \in \mathrm{Aut}^{\circ}(W_n)$ fix an infinite order element?
3. When does a power of $\phi \in \mathrm{Aut}(W_n)$ preserve the conjugacy class of an infinite order element?

Example: $W_4 \rtimes_\psi \mathbb{Z}$ is Hyperbolic

By Gersten-Stallings and Bestvina-Handel, $F_3 \rtimes_{\psi|_{F_3}} \mathbb{Z}$ is hyperbolic.

$$\psi = \chi_{a,\{bc\}} \circ \chi_{d,(bc)} \circ \sigma_{(bdc)} \in \mathrm{Aut}(W_n)$$

$$\begin{aligned} \psi(a) &= a \\ \psi(b) &= d \\ \psi(c) &= dabad \\ \psi(d) &= dacad \end{aligned}$$

Can we come up with an example with $\phi \in \mathrm{Aut}^\circ(W_n)$?

Thank You!

Q & A