Likelihood

Statistical Inference

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Likelihood

- A common and fruitful approach to statistics is to assume that the data arises from a family of distributions indexed by a parameter that represents a useful summary of the distribution
- The likelihood of a collection of data is the joint density evaluated as a function of the parameters with the data fixed
- Likelihood analysis of data uses the likelihood to perform inference regarding the unknown parameter

Likelihood

Given a statistical probability mass function or density, say $f(x, \theta)$, where θ is an unknown parameter, the likelihood is f viewed as a function of θ for a fixed, observed value of x.

Interpretations of likelihoods

The likelihood has the following properties:

1. Ratios of likelihood values measure the relative evidence of one value of the unknown parameter to another.
2. Given a statistical model and observed data, all of the relevant information contained in the data regarding the unknown parameter is contained in the likelihood.
3. If $\{X_i\}$ are independent random variables, then their likelihoods multiply. That is, the likelihood of the parameters given all of the X_i is simply the product of the individual likelihoods.

Example

- Suppose that we flip a coin with success probability θ
- Recall that the mass function for x

$$f(x,\theta) = \theta^x (1-\theta)^{1-x} \quad \text{for} \quad \theta \in [0,1].$$

  where x is either 0 (Tails) or 1 (Heads)
- Suppose that the result is a head
- The likelihood is

$$\mathcal{L}(\theta,1) = \theta^1 (1-\theta)^{1-1} = \theta \quad \text{for} \quad \theta \in [0,1].$$

- Therefore, $\mathcal{L}(.5,1)/\mathcal{L}(.25,1) = 2$,
- There is twice as much evidence supporting the hypothesis that $\theta = .5$ to the hypothesis that $\theta = .25$

Example continued

- Suppose now that we flip our coin from the previous example 4 times and get the sequence 1, 0, 1, 1
- The likelihood is:

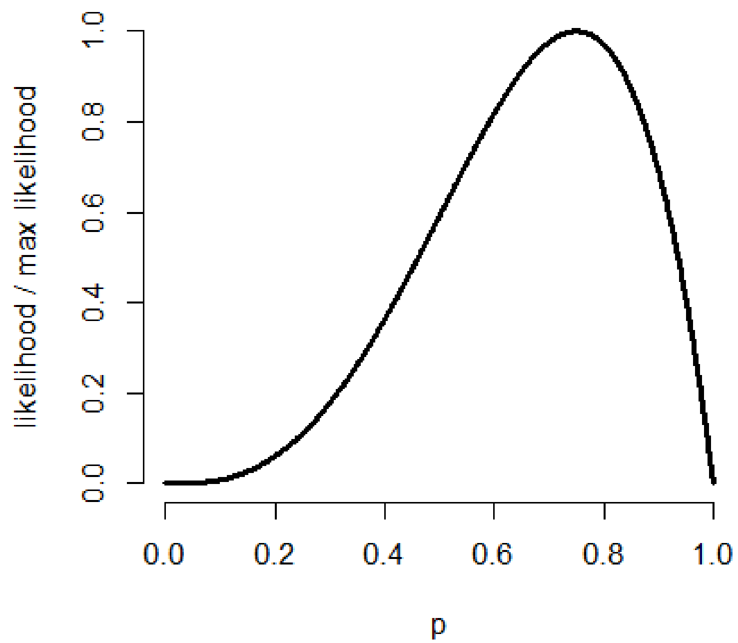
$$\begin{aligned}\mathcal{L}(\theta, 1, 0, 1, 1) &= \theta^1 (1-\theta)^{1-1} \theta^0 (1-\theta)^{1-0} \\ &\times \theta^1 (1-\theta)^{1-1} \theta^1 (1-\theta)^{1-1} \\ &= \theta^3 (1-\theta)^1\end{aligned}$$

- This likelihood only depends on the total number of heads and the total number of tails; we might write $\mathcal{L}(\theta, 1, 3)$ for shorthand
- Now consider $\mathcal{L}(.5, 1, 3)/\mathcal{L}(.25, 1, 3) = 5.33$
- There is over five times as much evidence supporting the hypothesis that $\theta = .5$ over that $\theta = .25$

Plotting likelihoods

- Generally, we want to consider all the values of θ between 0 and 1
- A likelihood plot displays θ by $\mathcal{L}(\theta, x)$
- Because the likelihood measures relative evidence, dividing the curve by its maximum value (or any other value for that matter) does not change its interpretation

```
pvals <- seq(0, 1, length = 1000)
plot(pvals, dbinom(3, 4, pvals) / dbinom(3, 4, 3/4), type = "l", frame = FALSE, lwd = 3, xlab = "p", yl
```



Maximum likelihood

- The value of θ where the curve reaches its maximum has a special meaning
- It is the value of θ that is most well supported by the data
- This point is called the maximum likelihood estimate (or MLE) of θ

$$MLE = \text{argmax}_\theta \mathcal{L}(\theta, x).$$

- Another interpretation of the MLE is that it is the value of θ that would make the data that we observed most probable

Some results

- $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ the MLE of μ is $\bar{X}$ and the ML of σ^2 is the biased sample variance estimate.
- If $X_1, \ldots, X_n \overset{iid}{\sim} Bernoulli(p)$ then the MLE of p is $\bar{X}$ (the sample proportion of 1s).
- If $X_i \overset{iid}{\sim} Binomial(n_i, p)$ then the MLE of p is $\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} n_i}$ (the sample proportion of 1s).
- If $X \overset{iid}{\sim} Poisson(\lambda t)$ then the MLE of λ is X/t.
- If $X_i \overset{iid}{\sim} Poisson(\lambda t_i)$ then the MLE of λ is $\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} t_i}$

Example

- You saw 5 failure events per 94 days of monitoring a nuclear pump.
- Assuming Poisson, plot the likelihood

```
lambda <- seq(0, .2, length = 1000)
likelihood <- dpois(5, 94 * lambda) / dpois(5, 5)
plot(lambda, likelihood, frame = FALSE, lwd = 3, type = "l", xlab = expression(lambda))
lines(rep(5/94, 2), 0 : 1, col = "red", lwd = 3)
lines(range(lambda[likelihood > 1/16]), rep(1/16, 2), lwd = 2)
lines(range(lambda[likelihood > 1/8]), rep(1/8, 2), lwd = 2)
```