# Power

## Statistical Inference

Brian Caffo, Jeff Leek, Roger Peng
Johns Hopkins Bloomberg School of Public Health

# Power

- Power is the probability of rejecting the null hypothesis when it is false
- Ergo, power (as it's name would suggest) is a good thing; you want more power
- A type II error (a bad thing, as its name would suggest) is failing to reject the null hypothesis when it's false; the probability of a type II error is usually called $\beta$
- Note Power $= 1 - \beta$

# Notes

- Consider our previous example involving RDI
- $H_0 : \mu = 30$ versus $H_a : \mu > 30$
- Then power is

$$P\left(\frac{\bar{X} - 30}{s/\sqrt{n}} > t_{1-\alpha,n-1} \mid \mu = \mu_a\right)$$

- Note that this is a function that depends on the specific value of $\mu_a$!
- Notice as $\mu_a$ approaches $30$ the power approaches $\alpha$

# Calculating power for Gaussian data

Assume that $n$ is large and that we know $\sigma$

$$
\begin{aligned}
1 - \beta &= P\left(\frac{\bar{X} - 30}{\sigma/\sqrt{n}} > z_{1-\alpha} \mid \mu = \mu_a\right) \\
&= P\left(\frac{\bar{X} - \mu_a + \mu_a - 30}{\sigma/\sqrt{n}} > z_{1-\alpha} \mid \mu = \mu_a\right) \\
&= P\left(\frac{\bar{X} - \mu_a}{\sigma/\sqrt{n}} > z_{1-\alpha} - \frac{\mu_a - 30}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right) \\
&= P\left(Z > z_{1-\alpha} - \frac{\mu_a - 30}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right)
\end{aligned}
$$

# Example continued

- Suppose that we wanted to detect a increase in mean RDI of at least 2 events / hour (above 30).
- Assume normality and that the sample in question will have a standard deviation of $4$;
- What would be the power if we took a sample size of $16$?
  - $Z_{1-\alpha} = 1.645$
  - $\frac{\mu_a - 30}{\sigma / \sqrt{n}} = 2 / (4 / \sqrt{16}) = 2$
  - $P(Z > 1.645 - 2) = P(Z > -0.355) = 64\%$

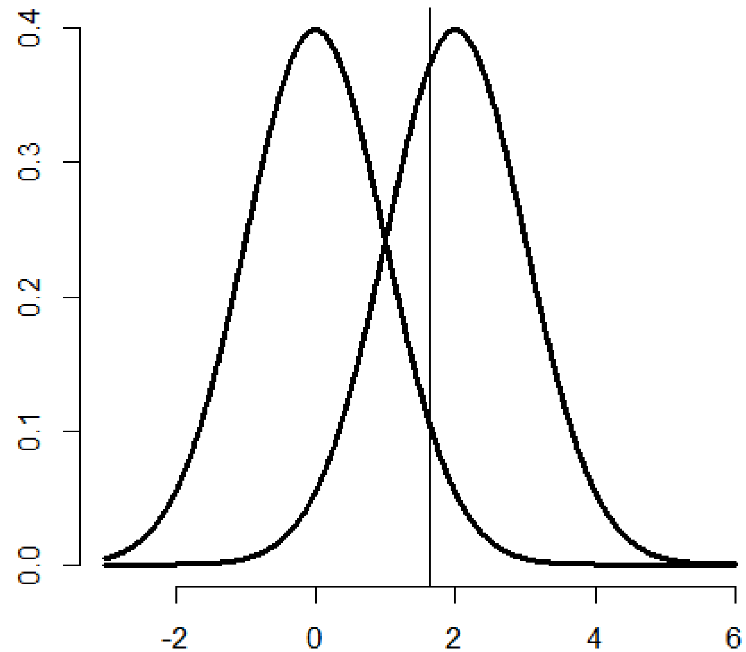
```
pnorm(-0.355, lower.tail = FALSE)
```

```
[1] 0.6387
```

# Note

- Consider $H_0 : \mu = \mu_0$ and $H_a : \mu > \mu_0$ with $\mu = \mu_a$ under $H_a$.
- Under $H_0$ the statistic $Z = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}$ is $N(0,1)$
- Under $H_a$ $Z$ is $N\left(\frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma}, 1\right)$
- We reject if $Z > Z_{1-\alpha}$

```
sigma <- 10; mu_0 = 0; mu_a = 2; n <- 100; alpha = .05
plot(c(-3, 6),c(0, dnorm(0)), type = "n", frame = false, xlab = "Z value", ylab = "")
xvals <- seq(-3, 6, length = 1000)
lines(xvals, dnorm(xvals), type = "l", lwd = 3)
lines(xvals, dnorm(xvals, mean = sqrt(n) * (mu_a - mu_0) / sigma), lwd =3)
abline(v = qnorm(1 - alpha))
```


0.0
0.1
0.2
0.3
0.4
-2
0
2
4
6

# Question

- When testing $H_a : \mu > \mu_0$, notice if power is $1 - \beta$, then

$$1 - \beta = P\left(Z > z_{1-\alpha} - \frac{\mu_a - \mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_a\right) = P(Z > z_\beta)$$

- This yields the equation

$$z_{1-\alpha} - \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma} = z_\beta$$

- Unknowns: $\mu_a$, $\sigma$, $n$, $\beta$
- Knowns: $\mu_0$, $\alpha$
- Specify any 3 of the unknowns and you can solve for the remainder

# Notes

- The calculation for $H_a : \mu < \mu_0$ is similar
- For $H_a : \mu \neq \mu_0$ calculate the one sided power using $\alpha / 2$ (this is only approximately right, it excludes the probability of getting a large TS in the opposite direction of the truth)
- Power goes up as $\alpha$ gets larger
- Power of a one sided test is greater than the power of the associated two sided test
- Power goes up as $\mu_1$ gets further away from $\mu_0$
- Power goes up as $n$ goes up
- Power doesn't need $\mu_a$, $\sigma$ and $n$, instead only $\frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma}$
  - The quantity $\frac{\mu_a - \mu_0}{\sigma}$ is called the effect size, the difference in the means in standard deviation units.
  - Being unit free, it has some hope of interpretability across settings

# T-test power

- Consider calculating power for a Gossett's $T$ test for our example
- The power is

$$P\left(\frac{\bar X - \mu_0}{S/\sqrt{n}} > t_{1-\alpha,n-1} ~|~ \mu = \mu_a\right)$$

- Calcuting this requires the non-central t distribution.
- `power.t.test` does this very well
  - Omit one of the arguments and it solves for it

# Example

```
power.t.test(n = 16, delta = 2 / 4, sd=1, type = "one.sample",  alt = "one.sided")$power
```

```
[1] 0.604
```

```
power.t.test(n = 16, delta = 2, sd=4, type = "one.sample",  alt = "one.sided")$power
```

```
[1] 0.604
```

```
power.t.test(n = 16, delta = 100, sd=200, type = "one.sample", alt = "one.sided")$power
```

```
[1] 0.604
```

# Example

```
power.t.test(power = .8, delta = 2 / 4, sd=1, type = "one.sample",  alt = "one.sided")$n
```

```
[1] 26.14
```

```
power.t.test(power = .8, delta = 2, sd=4, type = "one.sample",  alt = "one.sided")$n
```

```
[1] 26.14
```

```
power.t.test(power = .8, delta = 100, sd=200, type = "one.sample", alt = "one.sided")$n
```

```
[1] 26.14
```