Multivariable regression examples

Regression Models

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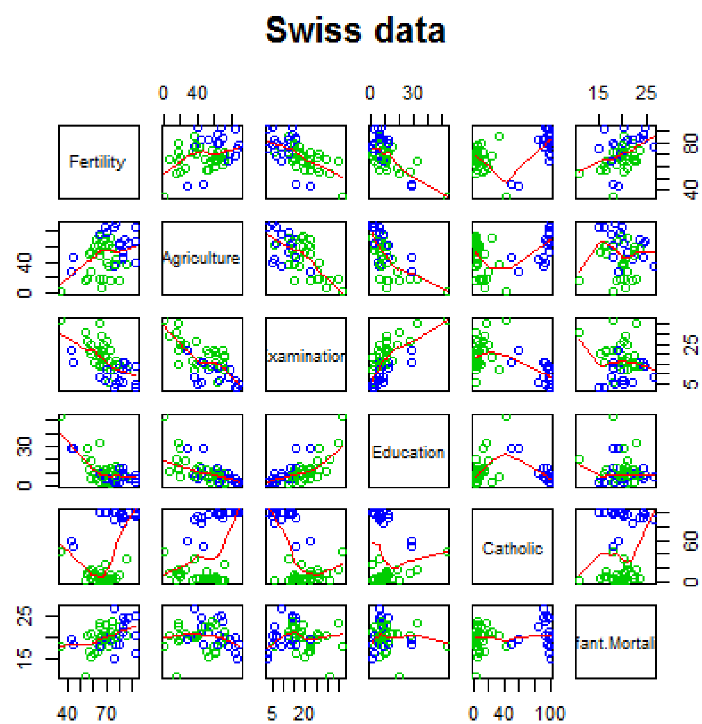
Swiss fertility data

```r
library(datasets); data(swiss); require(stats); require(graphics)
pairs(swiss, panel = panel.smooth, main = "Swiss data", col = 3 + (swiss$Catholic > 50))
```



?swiss

Description

Standardized fertility measure and socio-economic indicators for each of 47 French-speaking provinces of Switzerland at about 1888.

A data frame with 47 observations on 6 variables, each of which is in percent, i.e., in [0, 100].

- [,1] Fertility Ig, ‘ common standardized fertility measure’
- [,2] Agriculture % of males involved in agriculture as occupation
- [,3] Examination % draftees receiving highest mark on army examination
- [,4] Education % education beyond primary school for draftees.
- [,5] Catholic % ‘catholic’ (as opposed to ‘protestant’).
- [,6] Infant.Mortality live births who live less than 1 year.

All variables but ‘Fertility’ give proportions of the population.

Calling `lm`

```
summary(lm(Fertility ~ . , data = swiss))
```

```
                 Estimate Std. Error t value  Pr(>|t|)
(Intercept)       66.9152   10.70604   6.250 1.906e-07
Agriculture       -0.1721    0.07030  -2.448 1.873e-02
Examination       -0.2580    0.25388  -1.016 3.155e-01
Education         -0.8709    0.18303  -4.758 2.431e-05
Catholic           0.1041    0.03526   2.953 5.190e-03
Infant.Mortality   1.0770    0.38172   2.822 7.336e-03
```

Example interpretation

- Agriculture is expressed in percentages (0 - 100)
- Estimate is -0.1721.
- We estimate an expected 0.17 decrease in standardized fertility for every 1\% increase in percentage of males involved in agriculture in holding the remaining variables constant.
- The t-test for $H_0 : \beta_{Agri} = 0$ versus $H_a : \beta_{Agri} \neq 0$ is significant.
- Interestingly, the unadjusted estimate is

```
summary(lm(Fertility ~ Agriculture, data = swiss))$coefficients
```

```
            Estimate Std. Error t value  Pr(>|t|)
(Intercept)  60.3044    4.25126  14.185 3.216e-18
Agriculture   0.1942    0.07671   2.532 1.492e-02
```

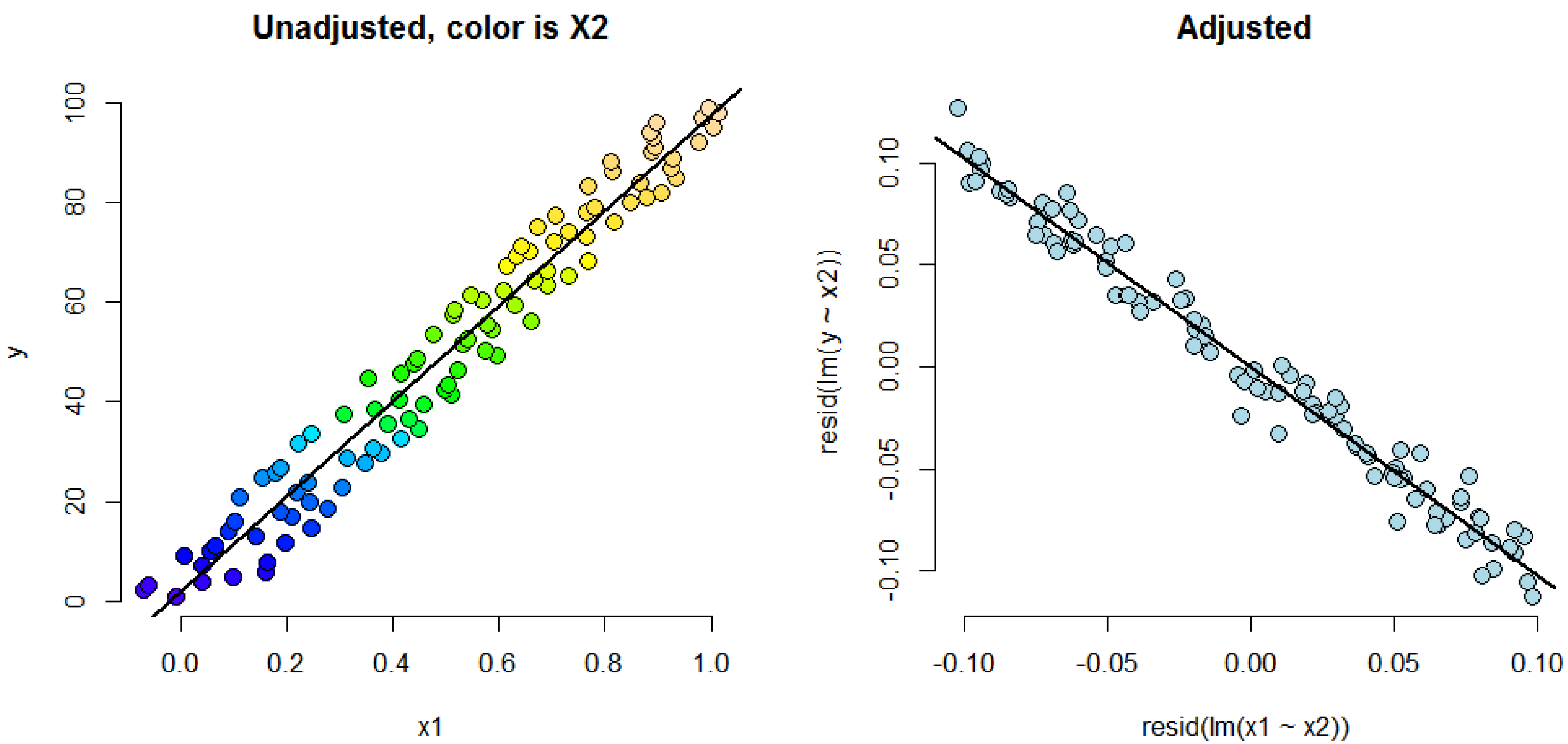
How can adjustment reverse the sign of an effect? Let's try a simulation.

```
n <- 100; x2 <- 1 : n; x1 <- .01 * x2 + runif(n, -.1, .1); y = -x1 + x2 + rnorm(n, sd = .01)
summary(lm(y ~ x1))$coef
```

```
            Estimate Std. Error t value  Pr(>|t|)
(Intercept)    1.618      1.200   1.349 1.806e-01
x1            95.854      2.058  46.579 1.153e-68
```

```
summary(lm(y ~ x1 + x2))$coef
```

```
               Estimate  Std. Error    t value   Pr(>|t|)
(Intercept)  0.0003683  0.0020141     0.1829  8.553e-01
x1          -1.0215256  0.0166372   -61.4001  1.922e-79
x2           1.0001909  0.0001681  5950.1818 1.369e-271
```


Unadjusted, color is X2
y
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0
x1
Adjusted
resid(lm(y ~ x2))
-0.10
-0.05
0.00
0.05
0.10
-0.10
-0.05
0.00
0.05
0.10
resid(lm(x1 ~ x2))

Back to this data set

- The sign reverses itself with the inclusion of Examination and Education, but of which are negatively correlated with Agriculture.
- The percent of males in the province working in agriculture is negatively related to educational attainment (correlation of -0.6395) and Education and Examination (correlation of 0.6984) are obviously measuring similar things.
  - Is the positive marginal an artifact for not having accounted for, say, Education level? (Education does have a stronger effect, by the way.)
- At the minimum, anyone claiming that provinces that are more agricultural have higher fertility rates would immediately be open to criticism.

What if we include an unnecessary variable?

z adds no new linear information, since it's a linear combination of variables already included. R just drops terms that are linear combinations of other terms.

```
z <- swiss$Agriculture + swiss$Education
lm(Fertility ~ . + z, data = swiss)
```

```

Call:
lm(formula = Fertility ~ . + z, data = swiss)

Coefficients:
     (Intercept)       Agriculture       Examination         Education          Catholic  
          66.915            -0.172            -0.258            -0.871             0.104  
Infant.Mortality                 z  
           1.077                NA  
```

Dummy variables are smart

- Consider the linear model

$$Y_i = \beta_0 + X_{i1}\beta_1 + \epsilon_i$$

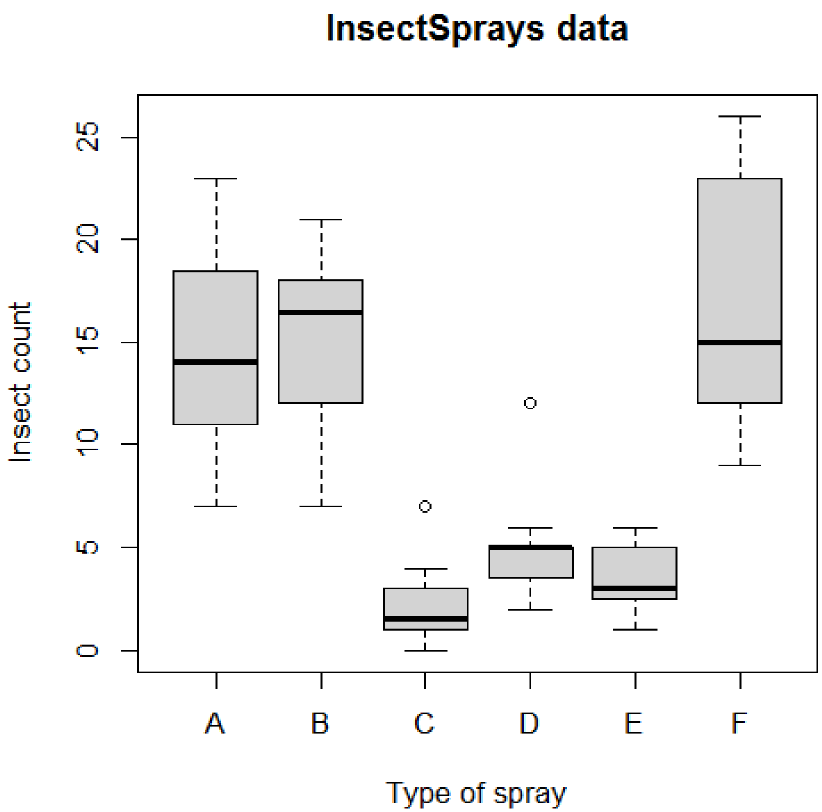
  where each X_{i1} is binary so that it is a 1 if measurement i is in a group and 0 otherwise. (Treated versus not in a clinical trial, for example.)

- Then for people in the group $E[Y_i] = \beta_0 + \beta_1$
- And for people not in the group $E[Y_i] = \beta_0$
- The LS fits work out to be $\hat\beta_0 + \hat\beta_1$ is the mean for those in the group and $\hat\beta_0$ is the mean for those not in the group.
- β_1 is interpretted as the increase or decrease in the mean comparing those in the group to those not.
- Note including a binary variable that is 1 for those not in the group would be redundant. It would create three parameters to describe two means.

More than 2 levels

- Consider a multilevel factor level. For didactic reasons, let's say a three level factor (example, US political party affiliation: Republican, Democrat, Independent)
- $Y_i = \beta_0 + X_{i1}\beta_1 + X_{i2}\beta_2 + \epsilon_i$.
- X_{i1} is 1 for Republicans and 0 otherwise.
- X_{i2} is 1 for Democrats and 0 otherwise.
- If i is Republican $E[Y_i] = \beta_0 + \beta_1$
- If i is Democrat $E[Y_i] = \beta_0 + \beta_2$.
- If i is Independent $E[Y_i] = \beta_0$.
- β_1 compares Republicans to Independents.
- β_2 compares Democrats to Independents.
- $\beta_1 - \beta_2$ compares Republicans to Democrats.
- (Choice of reference category changes the interpretation.)

Insect Sprays



Linear model fit, group A is the reference

```
summary(lm(count ~ spray, data = InsectSprays))$coef
```

```
            Estimate Std. Error t value  Pr(>|t|)
(Intercept)  14.5000      1.132 12.8074 1.471e-19
sprayB        0.8333      1.601  0.5205 6.045e-01
sprayC      -12.4167      1.601 -7.7550 7.267e-11
sprayD       -9.5833      1.601 -5.9854 9.817e-08
sprayE      -11.0000      1.601 -6.8702 2.754e-09
sprayF        2.1667      1.601  1.3532 1.806e-01
```

Hard coding the dummy variables

```
summary(lm(count ~
             I(1 * (spray == 'B')) + I(1 * (spray == 'C')) +
             I(1 * (spray == 'D')) + I(1 * (spray == 'E')) +
             I(1 * (spray == 'F'))
           , data = InsectSprays))$coef
```

```
                       Estimate Std. Error t value  Pr(>|t|)
(Intercept)             14.5000      1.132 12.8074 1.471e-19
I(1 * (spray == "B"))    0.8333      1.601  0.5205 6.045e-01
I(1 * (spray == "C")) -12.4167      1.601 -7.7550 7.267e-11
I(1 * (spray == "D"))  -9.5833      1.601 -5.9854 9.817e-08
I(1 * (spray == "E")) -11.0000      1.601 -6.8702 2.754e-09
I(1 * (spray == "F"))   2.1667      1.601  1.3532 1.806e-01
```

What if we include all 6?

```
lm(count ~
   I(1 * (spray == 'B')) + I(1 * (spray == 'C')) +
   I(1 * (spray == 'D')) + I(1 * (spray == 'E')) +
   I(1 * (spray == 'F')) + I(1 * (spray == 'A')), data = InsectSprays)
```

```
Call:
lm(formula = count ~ I(1 * (spray == "B")) + I(1 * (spray ==
    "C")) + I(1 * (spray == "D")) + I(1 * (spray == "E")) + I(1 *
    (spray == "F")) + I(1 * (spray == "A")), data = InsectSprays)

Coefficients:
          (Intercept)  I(1 * (spray == "B"))  I(1 * (spray == "C"))  I(1 * (spray == "D"))
               14.500                  0.833                -12.417                 -9.583
I(1 * (spray == "E"))  I(1 * (spray == "F"))  I(1 * (spray == "A"))
              -11.000                  2.167                     NA
```

What if we omit the intercept?

```
summary(lm(count ~ spray - 1, data = InsectSprays))$coef
```

```
       Estimate Std. Error t value  Pr(>|t|)
sprayA   14.500      1.132  12.807 1.471e-19
sprayB   15.333      1.132  13.543 1.002e-20
sprayC    2.083      1.132   1.840 7.024e-02
sprayD    4.917      1.132   4.343 4.953e-05
sprayE    3.500      1.132   3.091 2.917e-03
sprayF   16.667      1.132  14.721 1.573e-22
```

```
unique(ave(InsectSprays$count, InsectSprays$spray))
```

```
[1] 14.500 15.333  2.083  4.917  3.500 16.667
```

Summary

- If we treat Spray as a factor, R includes an intercept and omits the alphabetically first level of the factor.
  - All t-tests are for comparisons of Sprays versus Spray A.
  - Emprirical mean for A is the intercept.
  - Other group means are the itc plus their coefficient.
- If we omit an intercept, then it includes terms for all levels of the factor.
  - Group means are the coefficients.
  - Tests are tests of whether the groups are different than zero. (Are the expected counts zero for that spray.)
- If we want comparisons between, Spray B and C, say we could refit the model with C (or B) as the reference level.

Reordering the levels

```
spray2 <- relevel(InsectSprays$spray, "C")
summary(lm(count ~ spray2, data = InsectSprays))$coef
```

```
            Estimate Std. Error t value  Pr(>|t|)
(Intercept)    2.083      1.132  1.8401 7.024e-02
spray2A       12.417      1.601  7.7550 7.267e-11
spray2B       13.250      1.601  8.2755 8.510e-12
spray2D        2.833      1.601  1.7696 8.141e-02
spray2E        1.417      1.601  0.8848 3.795e-01
spray2F       14.583      1.601  9.1083 2.794e-13
```

Doing it manually

Equivalently

$$\mathrm{Var}(\hat\beta_B - \hat\beta_C) = \mathrm{Var}(\hat\beta_B) + \mathrm{Var}(\hat\beta_C) - 2\mathrm{Cov}(\hat\beta_B, \hat\beta_C)$$

```
fit <- lm(count ~ spray, data = InsectSprays) #A is ref
bbmbc <- coef(fit)[2] - coef(fit)[3] #B - C
temp <- summary(fit)
se <- temp$sigma * sqrt(temp$cov.unscaled[2, 2] + temp$cov.unscaled[3,3] - 2 *temp$cov.unscaled[2,3])
t <- (bbmbc) / se
p <- pt(-abs(t), df = fit$df)
out <- c(bbmbc, se, t, p)
names(out) <- c("B - C", "SE", "T", "P")
round(out, 3)
```

```
 B - C     SE      T      P
13.250  1.601  8.276  0.000
```

Other thoughts on this data

- Counts are bounded from below by 0, violates the assumption of normality of the errors.
  - Also there are counts near zero, so both the actual assumption and the intent of the assumption are violated.
- Variance does not appear to be constant.
- Perhaps taking logs of the counts would help.
  - There are 0 counts, so maybe log(Count + 1)
- Also, we'll cover Poisson GLMs for fitting count data.

Example - Millenium Development Goal 1

http://www.un.org/millenniumgoals/pdf/MDG_FS_1_EN.pdf

http://apps.who.int/gho/athena/data/GHO/WHOSIS_000008.csv?profile=text&filter=COUNTRY:;*SEX:*

WHO childhood hunger data

```
#download.file("http://apps.who.int/gho/athena/data/GHO/WHOSIS_000008.csv?profile=text&filter=COUNTRY:*
hunger <- read.csv("hunger.csv")
hunger <- hunger[hunger$Sex!="Both sexes",]
head(hunger)
```

```
                           Indicator Data.Source PUBLISH.STATES Year            WHO.region
1 Children aged <5 years underweight (%) NLIS_310044      Published 1986                Africa
2 Children aged <5 years underweight (%) NLIS_310233      Published 1990              Americas
3 Children aged <5 years underweight (%) NLIS_312902      Published 2005              Americas
5 Children aged <5 years underweight (%) NLIS_312522      Published 2002 Eastern Mediterranean
6 Children aged <5 years underweight (%) NLIS_312955      Published 2008                Africa
8 Children aged <5 years underweight (%) NLIS_312963      Published 2008                Africa
        Country    Sex Display.Value Numeric Low High Comments
1       Senegal   Male          19.3    19.3  NA   NA       NA
2      Paraguay   Male           2.2     2.2  NA   NA       NA
3     Nicaragua   Male           5.3     5.3  NA   NA       NA
5        Jordan Female           3.2     3.2  NA   NA       NA
6 Guinea-Bissau Female          17.0    17.0  NA   NA       NA
8         Ghana   Male          15.7    15.7  NA   NA       NA
```

Plot percent hungry versus time

```
lm1 <- lm(hunger$Numeric ~ hunger$Year)
plot(hunger$Year,hunger$Numeric,pch=19,col="blue")
```

Remember the linear model

$$Hu_i = b_0 + b_1 Y_i + e_i$$

b_0 = percent hungry at Year 0

b_1 = decrease in percent hungry per year

e_i = everything we didn't measure

Add the linear model

```
lm1 <- lm(hunger$Numeric ~ hunger$Year)
plot(hunger$Year,hunger$Numeric,pch=19,col="blue")
lines(hunger$Year,lm1$fitted,lwd=3,col="darkgrey")
```

Color by male/female

```
plot(hunger$Year,hunger$Numeric,pch=19)
points(hunger$Year,hunger$Numeric,pch=19,col=((hunger$Sex=="Male")*1+1))
```

Now two lines

$$HuF_i = bf_0 + bf_1 YF_i + ef_i$$

bf_0 = percent of girls hungry at Year 0

bf_1 = decrease in percent of girls hungry per year

ef_i = everything we didn't measure

$$HuM_i = bm_0 + bm_1 YM_i + em_i$$

bm_0 = percent of boys hungry at Year 0

bm_1 = decrease in percent of boys hungry per year

em_i = everything we didn't measure

Color by male/female

```
lmM <- lm(hunger$Numeric[hunger$Sex=="Male"] ~ hunger$Year[hunger$Sex=="Male"])
lmF <- lm(hunger$Numeric[hunger$Sex=="Female"] ~ hunger$Year[hunger$Sex=="Female"])
plot(hunger$Year,hunger$Numeric,pch=19)
points(hunger$Year,hunger$Numeric,pch=19,col=((hunger$Sex=="Male")*1+1))
lines(hunger$Year[hunger$Sex=="Male"],lmM$fitted,col="black",lwd=3)
lines(hunger$Year[hunger$Sex=="Female"],lmF$fitted,col="red",lwd=3)
```

Two lines, same slope

$$Hu_i = b_0 + b_1 \mathbb{1}(Sex_i = " \text{ Male } ") + b_2 Y_i + e_i^*$$

b_0 - percent hungry at year zero for females

$b_0 + b_1$ - percent hungry at year zero for males

b_2 - change in percent hungry (for either males or females) in one year

e_i^* - everything we didn't measure

Two lines, same slope in R

```
lmBoth <- lm(hunger$Numeric ~ hunger$Year + hunger$Sex)
plot(hunger$Year,hunger$Numeric,pch=19)
points(hunger$Year,hunger$Numeric,pch=19,col=((hunger$Sex=="Male")*1+1))
abline(c(lmBoth$coeff[1],lmBoth$coeff[2]),col="red",lwd=3)
abline(c(lmBoth$coeff[1] + lmBoth$coeff[3],lmBoth$coeff[2] ),col="black",lwd=3)
```

Two lines, different slopes (interactions)

$$Hu_i = b_0 + b_1 \mathbb{1}(Sex_i =" \text{ Male } ") + b_2 Y_i + b_3 \mathbb{1}(Sex_i =" \text{ Male } ") \times Y_i + e_i^+$$

b_0 - percent hungry at year zero for females

$b_0 + b_1$ - percent hungry at year zero for males

b_2 - change in percent hungry (females) in one year

$b_2 + b_3$ - change in percent hungry (males) in one year

e_i^+ - everything we didn't measure

Two lines, different slopes in R

```
lmBoth <- lm(hunger$Numeric ~ hunger$Year + hunger$Sex + hunger$Sex*hunger$Year)
plot(hunger$Year,hunger$Numeric,pch=19)
points(hunger$Year,hunger$Numeric,pch=19,col=((hunger$Sex=="Male")*1+1))
abline(c(lmBoth$coeff[1],lmBoth$coeff[2]),col="red",lwd=3)
abline(c(lmBoth$coeff[1] + lmBoth$coeff[3],lmBoth$coeff[2] +lmBoth$coeff[4]),col="black",lwd=3)
```

Two lines, different slopes in R

```
summary(lmBoth)
```

```
Call:
lm(formula = hunger$Numeric ~ hunger$Year + hunger$Sex + hunger$Sex *
    hunger$Year)

Residuals:
   Min     1Q Median     3Q    Max
-25.91 -11.25  -1.85   7.09  46.15

Coefficients:
                           Estimate Std. Error t value Pr(>|t|)
(Intercept)                603.5058   171.0552    3.53  0.00044 ***
hunger$Year                 -0.2934     0.0855   -3.43  0.00062 ***
hunger$SexMale              61.9477   241.9086    0.26  0.79795
hunger$Year:hunger$SexMale  -0.0300     0.1209   -0.25  0.80402
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 13.2 on 944 degrees of freedom
Multiple R-squared:  0.0318,	Adjusted R-squared:  0.0287
```

Interpretting a continuous interaction

$$E[Y_i|X_{1i} = x_1, X_{2i} = x_2] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$$

Holding X_2 constant we have

$$E[Y_i|X_{1i} = x_1 + 1, X_{2i} = x_2] - E[Y_i|X_{1i} = x_1, X_{2i} = x_2] = \beta_1 + \beta_3 x_2$$

And thus the expected change in Y per unit change in X_1 holding all else constant is not constant. β_1 is the slope when $x_2 = 0$. Note further that:

$$E[Y_i|X_{1i} = x_1 + 1, X_{2i} = x_2 + 1] - E[Y_i|X_{1i} = x_1, X_{2i} = x_2 + 1]$$

$$-E[Y_i|X_{1i} = x_1 + 1, X_{2i} = x_2] - E[Y_i|X_{1i} = x_1, X_{2i} = x_2]$$

$$= \beta_3$$

Thus, β_3 is the change in the expected change in Y per unit change in X_1, per unit change in X_2.

Or, the change in the slope relating X_1 and Y per unit change in X_2.

Example

$$Hu_i = b_0 + b_1 In_i + b_2 Y_i + b_3 In_i \times Y_i + e_i^+$$

b_0 - percent hungry at year zero for children with whose parents have no income

b_1 - change in percent hungry for each dollar of income in year zero

b_2 - change in percent hungry in one year for children whose parents have no income

b_3 - increased change in percent hungry by year for each dollar of income - e.g. if income is $10,000, then change in percent hungry in one year will be

$$b_2 + 1e4 \times b_3$$

e_i^+ - everything we didn't measure

Lot's of care/caution needed!