A trip to Asymptopia

Statistical Inference

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Asymptotics

- Asymptotics is the term for the behavior of statistics as the sample size (or some other relevant quantity) limits to infinity (or some other relevant number)
- (Asymptopia is my name for the land of asymptotics, where everything works out well and there's no messes. The land of infinite data is nice that way.)
- Asymptotics are incredibly useful for simple statistical inference and approximations
- (Not covered in this class) Asymptotics often lead to nice understanding of procedures
- Asymptotics generally give no assurances about finite sample performance
  - The kinds of asymptotics that do are orders of magnitude more difficult to work with
- Asymptotics form the basis for frequency interpretation of probabilities (the long run proportion of times an event occurs)
- To understand asymptotics, we need a very basic understanding of limits.

Numerical limits

- Imagine a sequence
  - $a_1 = .9$,
  - $a_2 = .99$,
  - $a_3 = .999$, ...
- Clearly this sequence converges to 1
- Definition of a limit: For any fixed distance we can find a point in the sequence so that the sequence is closer to the limit than that distance from that point on

Limits of random variables

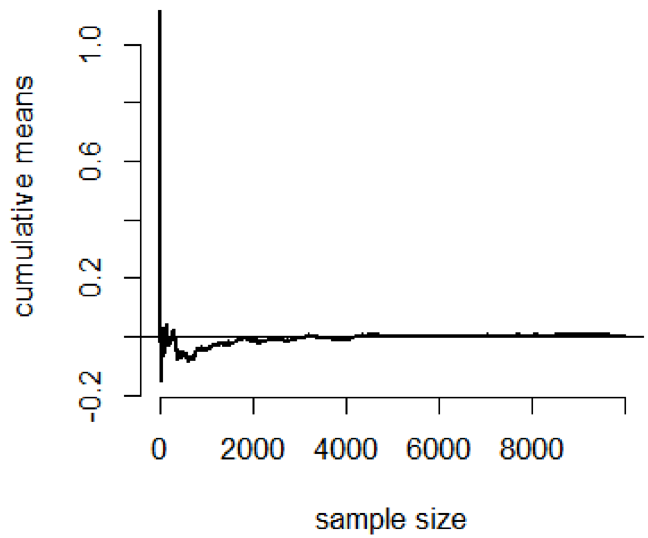
- The problem is harder for random variables
- Consider $\bar X_n$ the sample average of the first n of a collection of iid observations
  - Example $\bar X_n$ could be the average of the result of n coin flips (i.e. the sample proportion of heads)
- We say that $\bar X_n$ converges in probability to a limit if for any fixed distance the probability of $\bar X_n$ being closer (further away) than that distance from the limit converges to one (zero)

The Law of Large Numbers

- Establishing that a random sequence converges to a limit is hard
- Fortunately, we have a theorem that does all the work for us, called the Law of Large Numbers
- The law of large numbers states that if $X_1, \ldots X_n$ are iid from a population with mean μ and variance σ^2 then $\bar{X}_n$ converges in probability to μ
- (There are many variations on the LLN; we are using a particularly lazy version, my favorite kind of version)

Law of large numbers in action

```
n <- 10000; means <- cumsum(rnorm(n)) / (1  : n)
plot(1 : n, means, type = "l", lwd = 2,
     frame = FALSE, ylab = "cumulative means", xlab = "sample size")
abline(h = 0)
```



Discussion

- An estimator is consistent if it converges to what you want to estimate
  - Consistency is neither necessary nor sufficient for one estimator to be better than another
  - Typically, good estimators are consistent; it's not too much to ask that if we go to the trouble of collecting an infinite amount of data that we get the right answer
- The LLN basically states that the sample mean is consistent
- The sample variance and the sample standard deviation are consistent as well
- Recall also that the sample mean and the sample variance are unbiased as well
- (The sample standard deviation is biased, by the way)

The Central Limit Theorem

- The Central Limit Theorem (CLT) is one of the most important theorems in statistics
- For our purposes, the CLT states that the distribution of averages of iid variables, properly normalized, becomes that of a standard normal as the sample size increases
- The CLT applies in an endless variety of settings
- Let $X_1, \ldots, X_n$ be a collection of iid random variables with mean μ and variance σ^2
- Let $\bar X_n$ be their sample average
- Then $\frac{\bar X_n - \mu}{\sigma / \sqrt{n}}$ has a distribution like that of a standard normal for large n.
- Remember the form

$$\frac{\bar X_n - \mu}{\sigma / \sqrt{n}} = \frac{\mbox{Estimate} - \mbox{Mean of estimate}}{\mbox{Std. Err. of estimate}}.$$

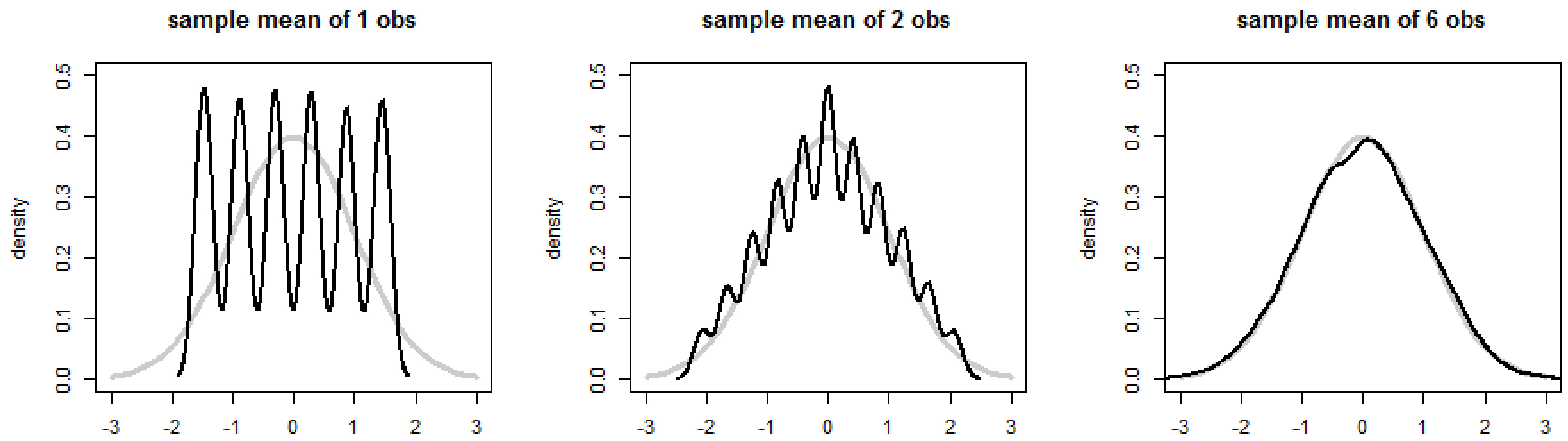
- Usually, replacing the standard error by its estimated value doesn't change the CLT

Example

- Simulate a standard normal random variable by rolling n (six sided)
- Let X_i be the outcome for die i
- Then note that $\mu = E[X_i] = 3.5$
- $Var(X_i) = 2.92$
- SE $\sqrt{2.92/n} = 1.71/\sqrt{n}$
- Standardized mean

$$\frac{\bar X_n - 3.5}{1.71/\sqrt{n}}$$

Simulation of mean of n dice

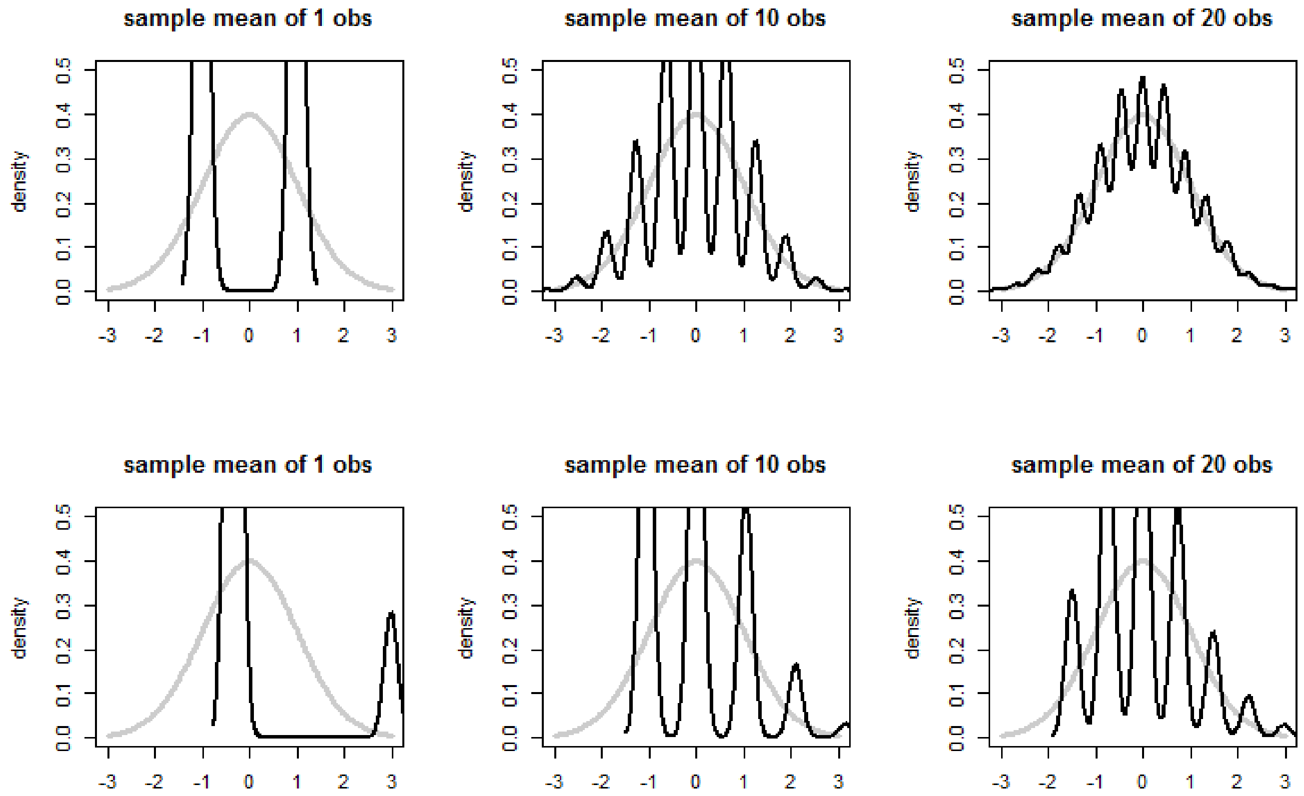


Coin CLT

- Let X_i be the 0 or 1 result of the i^{th} flip of a possibly unfair coin
  - The sample proportion, say $\hat p$, is the average of the coin flips
  - $E[X_i] = p$ and $Var(X_i) = p(1-p)$
  - Standard error of the mean is $\sqrt{p(1-p)/n}$
  - Then

$$\frac{\hat p - p}{\sqrt{p(1-p)/n}}$$

  will be approximately normally distributed


sample mean of 1 obs
sample mean of 10 obs
sample mean of 20 obs
sample mean of 1 obs
sample mean of 10 obs
sample mean of 20 obs
density
0.0 0.1 0.2 0.3 0.4 0.5
-3 -2 -1 0 1 2 3

CLT in practice

- In practice the CLT is mostly useful as an approximation

$$P\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) \approx \Phi(z).$$

- Recall 1.96 is a good approximation to the $.975^{th}$ quantile of the standard normal
- Consider

$$\begin{aligned} .95 &\approx P\left(-1.96 \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq 1.96\right) \\ &= P\left(\bar{X}_n + 1.96\sigma/\sqrt{n} \geq \mu \geq \bar{X}_n - 1.96\sigma/\sqrt{n}\right), \end{aligned}$$

Confidence intervals

- Therefore, according to the CLT, the probability that the random interval

$$\bar{X}_n \pm z_{1-\alpha/2}\,\sigma/\sqrt{n}$$

  contains μ is approximately 100$(1-\alpha)$%, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution

- This is called a $100(1-\alpha)$% confidence interval for μ

- We can replace the unknown σ with s

Give a confidence interval for the average height of sons

in Galton's data

```
library(UsingR);data(father.son); x <- father.son$sheight
(mean(x) + c(-1, 1) * qnorm(.975) * sd(x) / sqrt(length(x))) / 12
```

```
[1] 5.710 5.738
```

Sample proportions

- In the event that each X_i is 0 or 1 with common success probability p then $\sigma^2 = p(1-p)$
- The interval takes the form

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

- Replacing p by $\hat{p}$ in the standard error results in what is called a Wald confidence interval for p
- Also note that $p(1-p) \leq 1/4$ for $0 \leq p \leq 1$
- Let $\alpha = .05$ so that $z_{1-\alpha/2} = 1.96 \approx 2$ then

$$2\sqrt{\frac{p(1-p)}{n}} \leq 2\sqrt{\frac{1}{4n}} = \frac{1}{\sqrt{n}}$$

- Therefore $\hat{p} \pm \frac{1}{\sqrt{n}}$ is a quick CI estimate for p

Example

- Your campaign advisor told you that in a random sample of 100 likely voters, 56 intent to vote for you.
  - Can you relax? Do you have this race in the bag?
  - Without access to a computer or calculator, how precise is this estimate?
- `1/sqrt(100)=.1` so a back of the envelope calculation gives an approximate 95% interval of `(0.46, 0.66)`
  - Not enough for you to relax, better go do more campaigning!
- Rough guidelines, 100 for 1 decimal place, 10,000 for 2, 1,000,000 for 3.

```
round(1 / sqrt(10 ^ (1 : 6)), 3)
```

```
[1] 0.316 0.100 0.032 0.010 0.003 0.001
```

Poissson interval

- A nuclear pump failed 5 times out of 94.32 days, give a 95% confidence interval for the failure rate per day?
- $X \sim Poisson(\lambda t)$.
- Estimate $\hat{\lambda} = X/t$
- $Var(\hat{\lambda}) = \lambda/t$

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\hat{\lambda}/t}} = \frac{X - t\lambda}{\sqrt{X}} \to N(0,1)$$

- This isn't the best interval.
  - There are better asymptotic intervals.
  - You can get an exact CI in this case.

R code

```
x <- 5; t <- 94.32; lambda <- x / t
round(lambda + c(-1, 1) * qnorm(.975) * sqrt(lambda / t), 3)
```

```
[1] 0.007 0.099
```

```
poisson.test(x, T = 94.32)$conf
```

```
[1] 0.01721 0.12371
attr(,"conf.level")
[1] 0.95
```

In the regression class

```
exp(confint(glm(x ~ 1 + offset(log(t)), family = poisson(link = log))))
```

```
  2.5 %  97.5 %
0.01901 0.11393
```