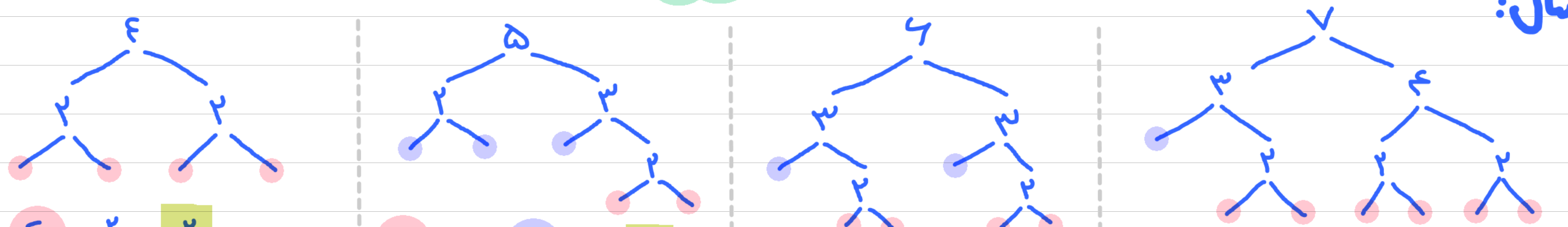
$n = \lfloor \log_{۲} x \rfloor$

$E[x] = \dfrac{n \cdot ۲^{n} + (x - ۲^{n})}{۲^{n}} = n + \dfrac{x - ۲^{n}}{۲^{n}}$

مثال:



$۴ \times \dfrac{۲}{۲^{۲}} = ۲$

$E[۴] = ۲ + \dfrac{(۴-۴)}{۴} = ۲$

$۲ \times \dfrac{۳}{۲^{۳}} + ۳ \times \dfrac{۲}{۲^{۲}} = \dfrac{۹}{۴}$

$E[۵] = ۲ + \dfrac{۱}{۴} = \dfrac{۹}{۴}$

$۴ \times \dfrac{۳}{۲^{۳}} + ۲ \times \dfrac{۲}{۲^{۲}} = \dfrac{۵}{۲}$

$E[۶] = ۲ + \dfrac{۲}{۴} = \dfrac{۵}{۲}$

$۶ \times \dfrac{۳}{۲^{۳}} + ۱ \times \dfrac{۲}{۲^{۲}} = \dfrac{۹}{۴} + \dfrac{۲}{۴} = \dfrac{۱۱}{۴}$

$E[۷] = ۲ + \dfrac{۳}{۴} = \dfrac{۱۱}{۴}$

اثبات: (استقرا)

Claim

For any integer

$$2^n \le x \le 2^{n+1} - 1, \quad x = 2^n + r,\ 0 \le r < 2^n,$$

the expected number of steps is

$r = x - ۲^{n}$

$$\boxed{E(x) = n + \frac{r}{2^n}}.$$

Recurrence

$$E(x) = 1 + \frac{E(\lfloor x/2 \rfloor) + E(\lceil x/2 \rceil)}{2}.$$

Base:

$$E(1) = 0.$$

Induction Hypothesis

For all

$$y = 2^{n-1} + s, \quad 0 \le s < 2^{n-1},$$

we assume:

$$E(y) = (n-1) + \frac{s}{2^{n-1}}.$$

Inductive Step

Let

$$x = 2^n + r, \quad 0 \le r < 2^n.$$

Compute half values:

$$\lfloor x/2 \rfloor = 2^{n-1} + \lfloor r/2 \rfloor,$$
$$\lceil x/2 \rceil = 2^{n-1} + \lceil r/2 \rceil.$$

Apply induction hypothesis:

$$E(\lfloor x/2 \rfloor) = (n-1) + \frac{\lfloor r/2 \rfloor}{2^{n-1}},$$

$$E(\lceil x/2 \rceil) = (n-1) + \frac{\lceil r/2 \rceil}{2^{n-1}}.$$

Plug into recurrence:

$$\begin{aligned} E(x) &= 1 + \frac{1}{2}\left((n-1) + \frac{\lfloor r/2 \rfloor}{2^{n-1}}\right) + \frac{1}{2}\left((n-1) + \frac{\lceil r/2 \rceil}{2^{n-1}}\right) \\ &= 1 + (n-1) + \frac{\lfloor r/2 \rfloor + \lceil r/2 \rceil}{2^n}. \end{aligned}$$

Use identity:

$$\lfloor r/2 \rfloor + \lceil r/2 \rceil = r.$$

So:

$$E(x) = n + \frac{r}{2^n}.$$

Induction complete.
