DETERMINANTS



2*2 matrix is invertible if and only if its determinant is nonzero. extend this useful fact to larger matrices definition for the 3*3 case

$$A = [a_{ij}] \text{ with } a_{11} \neq 0 \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

$$DETERMINA$$

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$A = [a_{11}]$$
—we define det $A = a_{11}$

 $\det A = a_{11}a_{22} - a_{12}a_{21}$

To generalize the definition of the determinant to larger matrices, we'll use 2*2 determinants to rewrite the 3*3 determinant described above.

$$(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$$

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

let Aij denote the submatrix formed by deleting the i th row and j th column of A.

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \qquad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\Delta = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13}$$

In general, an n*n determinant is defined by determinants of (n-1)*(n-1) submatrices.

For $n \ge 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}$ det A_{1j} , with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \dots = -2$$

Given
$$A = [a_{ij}]$$
, the (i, j) -cofactor of A is the number C_{ij} given $C_{ij} = (-1)^{i+j} \det A_{ij}$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

cofactor expansion across the first row of A

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

The cofactor expansion down the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$$

EXAMPLE 3 Compute det A, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} + 0 \cdot C_{51}$$

$$\det A = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \cdot (-2) = -12.$$

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

NUMERICAL NOTE

By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires more than n! multiplications, and 25! is approximately 1.5×10^{25} .

If a computer performs one trillion multiplications per second, it would have to run for more than 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

PROPERTIES OF DETERMINANTS

THEOREM 3

Row Operations

Let A be a square matrix.

- a. If a multiple of one row of A is added to another row to produce a matrix B, then det $B = \det A$.
- b. If two rows of A are interchanged to produce B, then det $B = -\det A$.
- c. If one row of A is multiplied by k to produce B, then det $B = k \cdot \det A$.

EXAMPLE 2 Compute det A, where
$$A = \begin{bmatrix} 2 & -8 & 6 & 6 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot (1)(3)(-6)(1) = -36$$

Suppose a square matrix A has been reduced to an echelon form U If there are r interchanges, then Theorem shows that

$$\det A = (-1)^r \det U$$

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 4

A square matrix A is invertible if and only if det $A \neq 0$.

Theorem 4 adds the statement "det $A \neq 0$ " to the Invertible Matrix Theorem

 $\det A = 0$ when the columns of A are linearly dependent

 $\det A = 0$ when the *rows* of A are linearly dependent.

NUMERICAL NOTES

- Most computer programs that compute det A for a general matrix A use the method of formula (1) above.
- 2. It can be shown that evaluation of an n × n determinant using row operations requires about 2n³/3 arithmetic operations. Any modern microcomputer can calculate a 25 × 25 determinant in a fraction of a second, since only about 10,000 operations are required.

EXAMPLE 4 Compute det A, where
$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$
.

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

Column Operations

- ✓ operations on the columns of a matrix similar to the row operations
- ✓ column operations have the same effects on determinants as row operations.

THEOREM 5

If A is an $n \times n$ matrix, then det $A^T = \det A$.

PROOF n = 1 true for $k \times k$ n = k + 1

cofactor of a_{1i} in A equals the cofactor of a_{i1} in A^T

each statement in Theorem 3 is true when the word *row* is replaced everywhere by column.

A row operation on A^T amounts to a column operation on A.

Determinants and Matrix Products

THEOREM 6

Multiplicative Property

If A and B are $n \times n$ matrices, then det $AB = (\det A)(\det B)$.

EXAMPLE 5 Verify Theorem 6 for
$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$$
 and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB$$

example

verify that $\det EA = (\det E)(\det A)$, where

E is the elementary matrix shown and
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
.

33.
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$
 34.
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

34.
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

35.
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
 36.
$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$36. \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Proof of theorem 3

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

The proof is by induction on the size of A n = k + 1

The action of E on A involves either two rows or only one row.

expand det(EA) across an unchanged row by the action of E, say, row i (B=EA)

$$\det \mathbf{B}_{ij} = \alpha \cdot \det A_{ij}$$
 $\alpha = 1, -1, \text{ or } r$

Proof of theorem 3

$$\det EA = a_{i1}(-1)^{i+1} \det B_{i1} + \dots + a_{in}(-1)^{i+n} \det B_{in}$$
$$= \alpha a_{i1}(-1)^{i+1} \det A_{i1} + \dots + \alpha a_{in}(-1)^{i+n} \det A_{in}$$

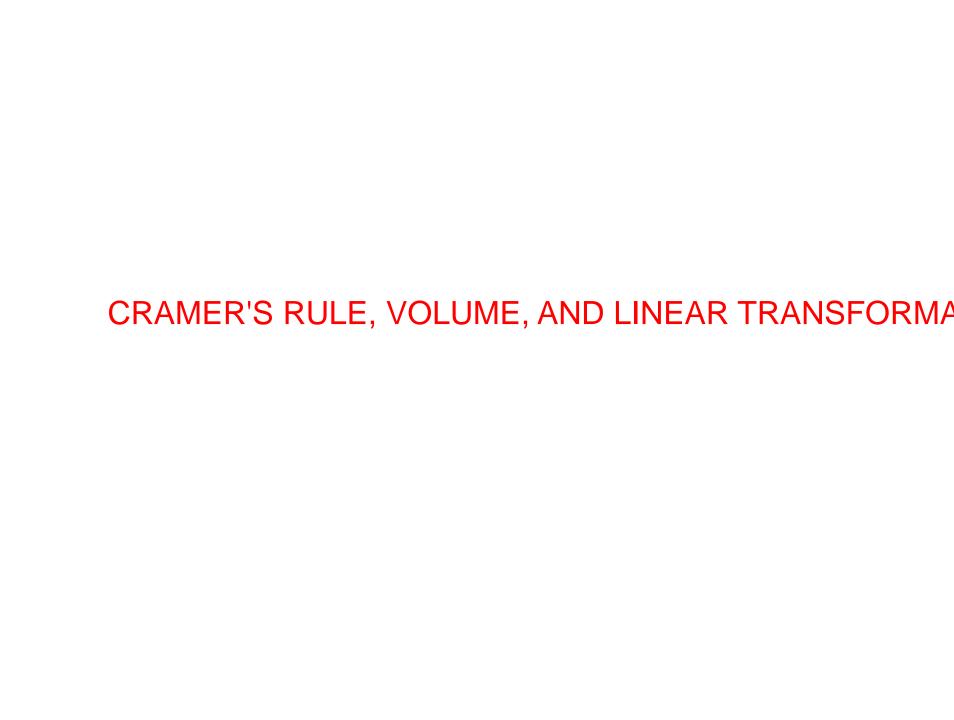
PROOF OF THEOREM 6

A is not invertible, then neither is AB,

$$\det AB = (\det A)(\det B)$$

If A is invertible, then A and the identity matrix I_n are row equivalent

$$A = E_p E_{p-1} \cdots E_1 \cdot I_n = E_p E_{p-1} \cdots E_1$$
 $|AB| = |E_p \cdots E_1 B| = |E_p| |E_{p-1} \cdots E_1 B| = \cdots$
 $= |E_p| \cdots |E_1| |B| = \cdots = |E_p \cdots E_1| |B|$
 $= |A| |B|$



Cramer's Rule

For any $n \times n$ matrix A and any \mathbf{b} in \mathbb{R}^n , let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector \mathbf{b} .

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \mathbf{b} \quad \cdots \quad \mathbf{a}_n]$$

THEOREM 7

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any **b** in \mathbb{R}^n , the unique solution **x** of A**x** = **b** has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \dots, n \tag{1}$$

Cramer's Rule

PROOF

$$A \cdot I_i(\mathbf{x}) = A \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{x} & \cdots & \mathbf{e}_n \end{bmatrix}$$

 $= \begin{bmatrix} A\mathbf{e}_1 & \cdots & A\mathbf{x} & \cdots & A\mathbf{e}_n \end{bmatrix}$
 $= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix} = A_i(\mathbf{b})$
 $(\det A)(\det I_i(\mathbf{x})) = \det A_i(\mathbf{b})$
 $(\det A) \cdot x_i = \det A_i(\mathbf{b})$

Example

EXAMPLE 1 Use Cramer's rule to solve the system

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

Example

EXAMPLE 2 Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$3sx_1 - 2x_2 = 4$$
$$-6x_1 + sx_2 = 1$$

A Formula for A^{-1}

j th column of A^{-1} is a vector **x** that satisfies

$$A\mathbf{x} = \mathbf{e}_j$$

$$\{(i, j)\text{-entry of } A^{-1}\} = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \frac{1}{\det A} \operatorname{adj} A$$

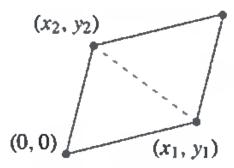
Determinants as Area or Volume

THEOREM 9

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Area of a parallelogram in \mathbb{R}^2 spanned by points $(0,0),\,(a,b),\,(c,d),\,(a+c,b+d)$ is:

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|$$



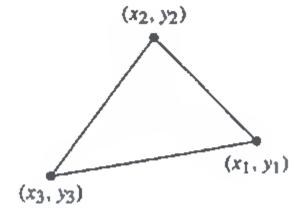
Determinants as Area or Volume

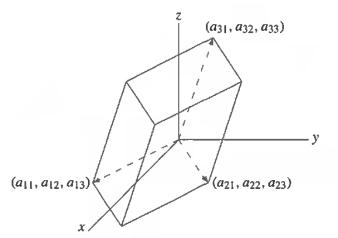
Volume of a parallelogram in \mathbb{R}^3 spanned by points $(x_1,y_1,z_1),$ $(x_2,y_2,z_2),$ (x_3,y_3,z_3) is $\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$

$$egin{array}{c} \mathsf{det} & egin{pmatrix} x_1 & y_1 & z_1 \ x_2 & y_2 & z_2 \ x_3 & y_3 & z_3 \ \end{pmatrix}$$

Area of triangle in \mathbb{R}^2 spanned by the points $(x_1,y_1),\,(x_2,y_2),\,(x_3,y_3)$ is: $egin{array}{c|c} 1&x_1&y_1\ 1&x_2&y_2\ 1&x_3&y_3 \end{array} \end{array}$

$$\left|rac{1}{2}\mathsf{det}\, egin{pmatrix} 1 & x_1 & y_1 \ 1 & x_2 & y_2 \ 1 & x_3 & y_3 \end{pmatrix}
ight|$$





Linear Transformation

THEOREM 10

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in \mathbb{R}^2 , then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$
 (5)

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in \mathbb{R}^3 , then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$
 (6)

$$A = [\mathbf{a}_1 \ \mathbf{a}_2]$$

$$A = [\mathbf{a}_1 \ \mathbf{a}_2]$$
 $S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \le s_1 \le 1, \ 0 \le s_2 \le 1\}$

$$T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) = s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2)$$

= $s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2$

parallelogram determined by the columns of the matrix. Ab1 Ab2

$$\{ \text{area of } T(S) \} = |\det AB| = |\det A| \cdot |\det B|$$

= $|\det A| \cdot \{ \text{area of } S \}$

Linear Transformation

$$\mathbf{p} + S$$

$$T$$
 transforms $\mathbf{p} + S$ into $T(\mathbf{p}) + T(S)$

The conclusions of Theorem 10 hold whenever S is a region in \mathbb{R}^2 with finite area or a region in \mathbb{R}^3 with finite volume.

example

EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$