LINEAR EQUATIONS

References:

- **David C. Lay, Steven R. Lay, and Judi J. McDonald, *Linear Algebra and its applications*, 5th Edition, Pearson, 2015.**
- **Gilbert Strang, *Introduction to linear algeb*ra, welleslay-Cambridge press.**
- **Philip N. Klein, Coding the Matrix: *Linear Algebra through Applications to Computer Science*, 1st Edition, Newtonian Press, 2013.**

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Contents

- Linear Equations in Linear Algebra (linear systems and their solutions, matrices, the matrix equation, linear independence, linear transformations)
- Matrix Algebra (matrix operations, inverse of matrix, matrix factorization, determinants)
- Vector Spaces (vector spaces and subspaces, null space, column space, bases, dimension of a vector space, rank, change of basis)
- Eigenvalues and Eigenvectors (eigenvalues and eigenvectors, characteristic equation, diagonalization, applications)
- Orthogonality and Least Squares (inner products, orthogonal sets, The Gram-Schmidt process, least squares problems, applications)
- Singular Value Decomposition, Principal Component Analysis
- Optimization (vector functions, first and second order derivative, introduction to different types of optimization problems, linear programming, the simplex algorithm)

Grading:

- Homework + Quiz(20%)
- Midterm+ Final(30%+50%)

Linear Equations

A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

b and the **coefficients** $a_1, \ldots, a_n$ are real or complex numbers

$$4x_1 - 5x_2 + 2 = x_1$$

$$4x_1 - 5x_2 = x_1x_2$$

$$x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

$$2x_1 + x_2 - x_3 = 2\sqrt{6}$$

$$x_2 = 2\sqrt{x_1} - 6$$

System of linear equations (or a linear system)

system of linear equations (or a **linear system**)

a collection of one or more linear equations involving the same variables—say, $x_1, \ldots, x_n$

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - 4x_3 &= -7 \end{aligned}$$

A **solution** of the system is a list .s1; s2,…, sn of numbers that makes each equation a true statement when the values s1,…,sn are substituted for x1,…, xn, respectively

$$(5, 6.5, 3)$$

set of all possible solutions is called the **solution set** of the linear system

Two linear systems are called **equivalent** if they have the same solution set.

System of linear equations

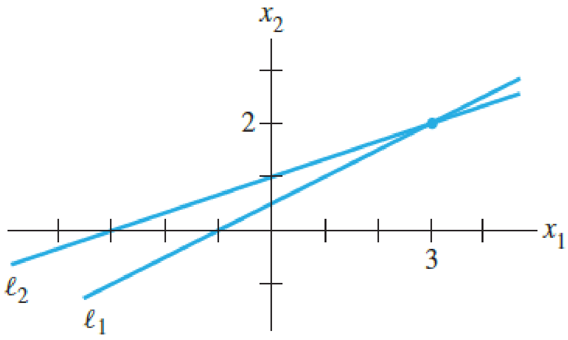
two linear equations in two variables

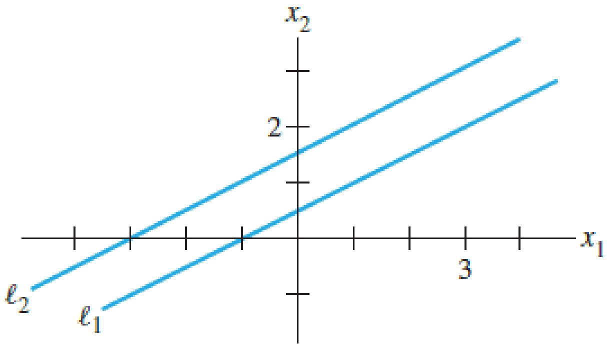
finding the intersection of two lines

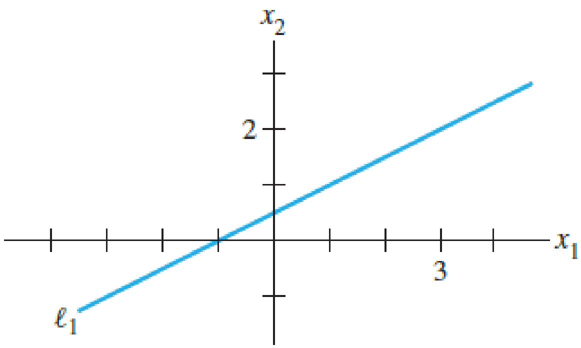
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$







System of linear equations

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

inconsistent

consistent

Matrix Notation

essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

coefficient matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

augmented matrix

size of a matrix

an m*n **matrix** is a rectangular array of numbers with m rows and n columns

Solving a Linear System

systematic procedure, for solving linear systems

replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

- ✓ Row operations can be applied to any matrix
- ✓ Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

Solving a Linear System

row operations on the augmented matrix

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

how to use row operations to determine the size of a solution set, without completely solving the linear system.

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Echelon matrix

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Echelon matrix

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

echelon form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Reduced echelon form

Row Reduction

Row reduction : Any nonzero matrix may transformed by elementary row operations into matrix in echelon form (more than one matrix in echelon form):

reduced echelon form one obtains from a matrix is unique

THEOREM 1

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot

DEFINITION

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix

Row Reduction Algorithm

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange
rows to move this entry into the pivot position

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Row Reduction Algorithm

3. Use row replacement operations to create zeros in all positions below the pivot.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

4. Cover (or ignore) the row containing the pivot position and cover all rows, if any,
above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until

vs to modify.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (-6) \cdot \text{row 3} \\ \leftarrow \text{Row 2} + (-2) \cdot \text{row 3} \\ \end{array}$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Solutions of Linear Systems

$$\begin{aligned} x_1 \quad\quad - 5x_3 &= 1 \\ x_2 + \quad x_3 &= 4 \\ 0 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables.**$_2$ The other variable, x_3, is called a **free variable**.

reduced echelon form places each basic variable in one and only one equation

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \Rightarrow$$

Parametric Descriptions of Solution Sets

example

Find the general solution of the linear system with augmented matrix

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 + 6x_2 \quad + 3x_4 \quad &= 0 \\ x_3 - 4x_4 \quad &= 5 \\ x_5 &= 7 \end{aligned}$$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Existence and Uniqueness

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

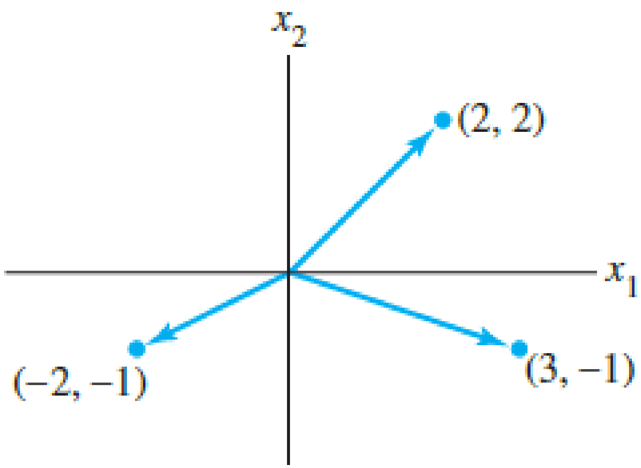
USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

VECTORS

vectors

A matrix with only one column is called a **column vector**, or simply a **vector**



$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Vectors in $\mathbb{R}^n$

zero vector

Equality of vectors in R^n and the operations of scalar multiplication and vector addition in R^n are defined entry by entry just as in R^2.

Vectors

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

Linear Combinations

Given vectors $\mathbf{v}_1; \mathbf{v}_2,\ldots,\mathbf{v}_p$ in R_n and given scalars $c_1; c_2,\ldots,c_p$, the vector **y** defined by

$$\boxed{\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p}$$

is called a **linear combination** of $\mathbf{v}_1,\ldots,\mathbf{v}_p$ with **weights** $c_1,\ldots,c_p$

Example

EXAMPLE 5 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$$

$$x_1 \underset{\mathbf{a}_1}{\underset{\uparrow}{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}}} + x_2 \underset{\mathbf{a}_2}{\underset{\uparrow}{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}}} = \underset{\mathbf{b}}{\underset{\uparrow}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}}$$

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

$$\boxed{[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}\,]}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Vector Equation

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{5}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

One of the key ideas: set of all vectors that can be generated as a linear combination of a fixed set $\{\mathbf{v}_1; : : : ; \mathbf{v}_p\}$ of vectors.

Span

DEFINITION

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated) by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Asking whether a vector **b** is in Span $\{\mathbf{v}_1; : : : ; \mathbf{v}_p\}$

$$\Downarrow$$

$$\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p & \mathbf{b} \end{bmatrix}$$

Geometric Description

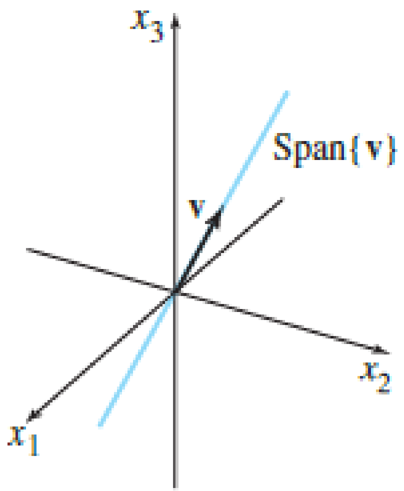


FIGURE 10 Span {**v**} as a line through the origin.

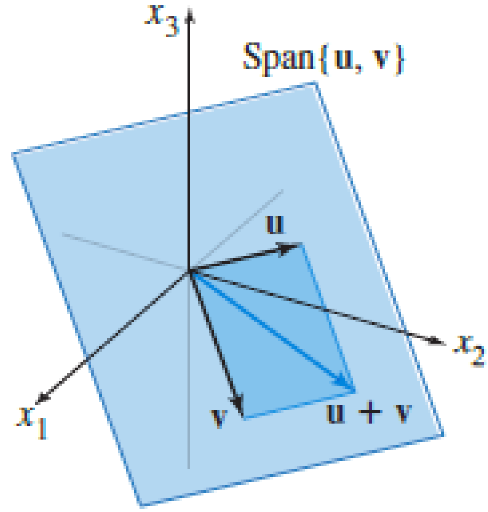


FIGURE 11 Span {**u**, **v**} as a plane through the origin.

LINEAR INDEPENDENCE

DEFINITION

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

Example

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

linearly dependent

LINEAR INDEPENDENCE

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1; : : : ; \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

not say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors