MATRIX ALGEBRA

MATRIX OPERATION

Matrix operation

- If A is an $m \times n$ matrix (m rows and n columns) –then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A.

$$A = [a_1, a_2, \cdots, a_n]$$

$$\begin{array}{cccccc} & & & \text{Column } j & & \\ & & & \downarrow & & \\ \text{Row } i \rightarrow & \left[\begin{array}{c} a_{11} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{array}\right. & \begin{array}{c} \cdots \\ \\ \cdots \\ \\ \cdots \end{array} & \begin{array}{c} a_{1j} \\ \vdots \\ \boxed{a_{ij}} \\ \vdots \\ a_{mj} \end{array} & \left.\begin{array}{c} \cdots a_{1n} \\ \vdots \\ \cdots a_{in} \\ \vdots \\ \cdots a_{mn} \end{array}\right] & = A \\ & \uparrow & & \uparrow & \uparrow & \\ & a_1 & & a_j & a_n & \end{array}$$

diagonal entries identity matrix, I_n

diagonal matrix zero matrix O

- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$ called a column vector

Matrix operation

Equality of two matrices: Two matrices A and B are equal if they have the same size (they are both $m \times n$) and if their entries are all the same.

$$a_{ij} = b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Sum of two matrices: If A and B are $m \times n$ matrices, then their sum $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries in A and B.

- If we call C this sum we can write:

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Matrix operation

scalar multiple of a matrix If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

$$(\alpha A)_{ij} = \alpha a_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

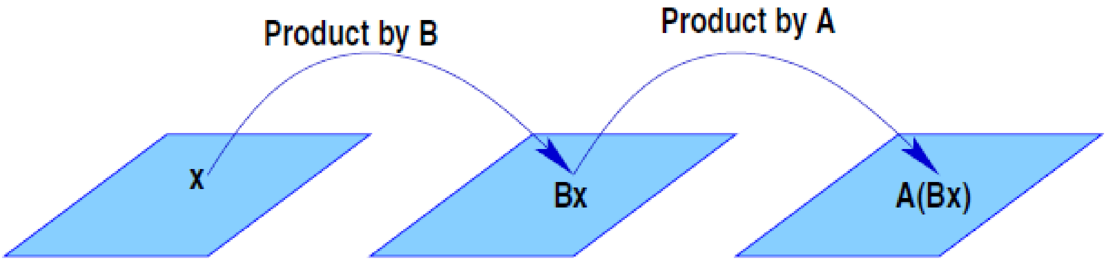
Theorem Let A, B, and C be matrices of the same size, and let α and β be scalars. Then

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $\alpha(A + B) = \alpha A + \alpha B$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = (\alpha\beta)A$

Matrix Multiplication

- When a matrix B multiplies a vector x, it transforms x into the vector Bx.

- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(Bx)$.

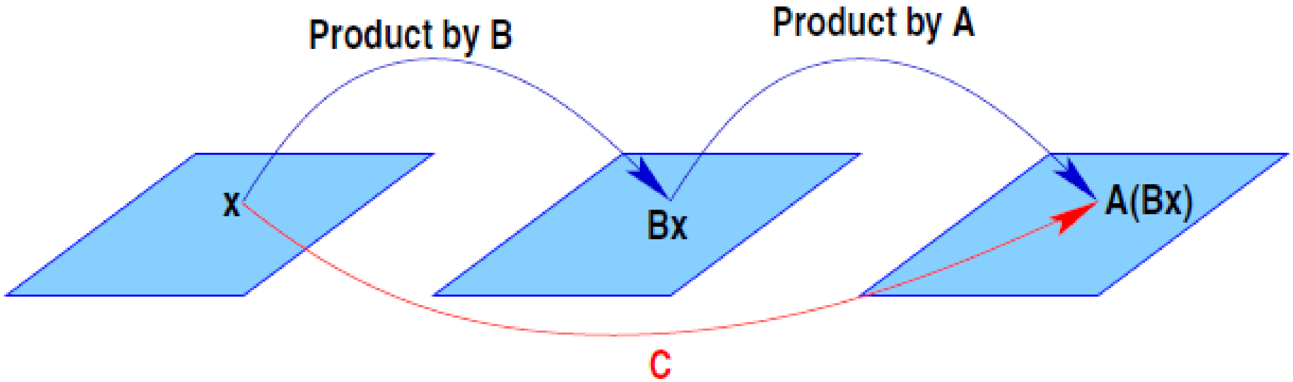


- Thus $A(Bx)$ is produced from x by a composition of mappings– the linear transformations induced by B and A.

Matrix Multiplication

Goal: to represent this composite mapping as a multiplication by a single matrix, call it C for now, so that

$$A(Bx) = Cx$$



Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the matrix whose p columns are $Ab_1, \cdots, Ab_p$. That is:

$$AB = A[b_1, b_2, \cdots, b_p] = [Ab_1, Ab_2, \cdots, Ab_p]$$

Example

EXAMPLE 3 Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

$$AB = A[\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,] = \begin{array}{c} \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{array} \end{array}$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

ROW–COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\text{row}_i(AB) = \text{row}_i(A) \cdot B$$

Theorem Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B + C) = AB + AC$ (left distributive law)
- $(B + C)A = BA + CA$ (right distributive law)
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$ for any scalar α
- $I_m A = A I_n = A$ (product with identity)

WARNINGS:

1. In general, $AB \neq BA$.
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)
3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

AB and BA are usually not the same
If AB = BA, we say that A and B **commute with one another.**

Square matrices, Matrix powers

- Important particular case when $n = m$ - so matrix is $n \times n$
- AA is also a square $n \times n$ matrix and will be denoted by A^2
- More generally, the matrix A^k is the matrix which is the product of k copies of A:

$$A^1 = A; \quad A^2 = AA; \quad \cdots \quad A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

- For consistency define A^0 to be the identity: $A^0 = I_n$,

$$A^l \times A^k = A^{l+k}$$

Transpose of a matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

Transpose of a matrix

Theorem : Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(\alpha A)^T = \alpha A^T$ for any scalar α
- $(AB)^T = B^T A^T$

The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

INVERSE OF A MATRIX

Inverse of a matrix

- An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix B such that $BA = I$ and $AB = I$ where $I = I_n$, the $n \times n$ identity matrix.

- In this case, B is an inverse of A. In fact, B is uniquely determined by A

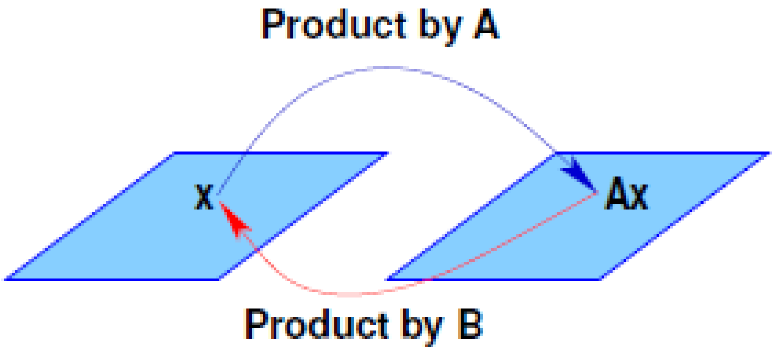
- This unique inverse is denoted by A^{-1} -so that

$$AA^{-1} = A^{-1}A = I$$

Introduction

- We have a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ represented by a matrix A.

- Can we invert this mapping? i.e. can we find a matrix (call it B for now) such that when B is applied to Ax the result is x?



- Example: blurring operation. We want to 'revert' blurring, i.e., to deblur. So: Blurring: A; Deblurring: B.

- B is the inverse of A and is denoted by A^{-1}.

Inverse of a matrix

- Recall that $I_n x = x$ for all x.

- Since we want $A^{-1}(Ax) = x$ for all x this means, we need to have

$$\boxed{A^{-1}A = I_n}$$

- Naturally the inverse of A^{-1} should be A so we also want

$$\boxed{AA^{-1} = I_n}$$

- Finding an inverse to A is not always possible. When it is we say that the matrix A is invertible

Matrix inverse - the 2*2 case

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- If $ad - bc = 0$ then A is not invertible (does not have an inverse)

- The quantity $ad - bc$ is called the determinant of A (det(A))

- The above says that a 2×2 matrix is invertible if and only if det$(A) \neq 0$.

example

EXAMPLE 2 Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

$$\det A = 3(6) - 4(5) = -2 \neq 0$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Matrix inverse - Properties

Theorem If A is invertible, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

proof

Show: If A is invertible then it is one to one, i.e., its columns are linearly independent.

EXAMPLE 4 Use the inverse of the matrix A in Example 2 to solve the system

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Theorem

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and we have

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1} :

$$(A^T)^{-1} = (A^{-1})^T$$

Common notation $(A^T)^{-1} \equiv A^{-T}$

proof

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

There is an important connection between invertible matrices and row operations that leads to a method for computing inverses

an invertible matrix A is row equivalent to an identity matrix

Elementary Matrices

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Elementary Matrices

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

$$E_p \cdots E_1 A$$

- Since row operations are reversible, elementary matrices are invertible
- if E is produced by a row operation on I, then there is another row operation of the same type that changes E back into I

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_1^{-1}E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} A = A$$

Theorem

THEOREM 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

proof

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

Example

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

Another View of Matrix Inversion

row reduction of $[\ A \quad I\ ]$

$$A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{x} = \mathbf{e}_2, \quad \ldots, \quad A\mathbf{x} = \mathbf{e}_n$$

This observation is useful because some applied problems may require finding only one or two columns of A^{-1} In this case, only the corresponding systems need be solved.

CHARACTERIZATIONS
OF
INVERTIBLE MATRICES

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

Each statement in the theorem describes a property of every n*n invertible matrix.

Invertible Matrix Theorem *applies only to square matrices*

EXAMPLE 1 Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Invertible Linear Transformations

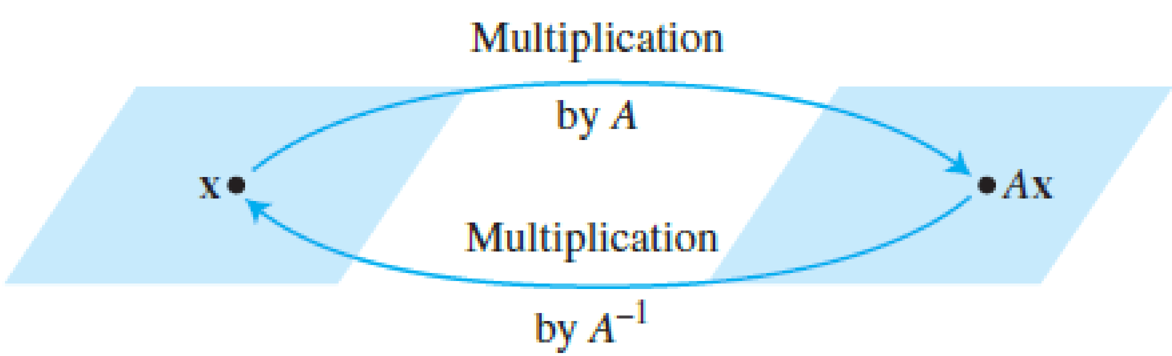
A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

THEOREM 9

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equations (1) and (2).



example

25. Show that if $ad - bc = 0$, then the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. Why does this imply that A is not invertible?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

example

18. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

19. If A, B, and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X? If so, find it.

example

20. Suppose A, B, and X are $n \times n$ matrices with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \tag{3}$$

a. Explain why B is invertible.

b. Solve (3) for X. If you need to invert a matrix, explain why that matrix is invertible.

example

Let $A = \begin{bmatrix} -1 & -7 & -3 \\ 2 & 15 & 6 \\ 1 & 3 & 2 \end{bmatrix}$. Find the third column of A^{-1} without computing the other columns.

example

Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

example

When is a square upper triangular matrix invertible?

When is a square lower triangular matrix invertible?

PARTITIONED MATRICES

Partitioned Matrices

$$A = \left[\begin{array}{rrr|rr|r} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right] \qquad A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Partitioned or block Matrix

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right]$$

Operations on block matrices

Addition

If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum A + B

Multiplication

Partitioned matrices can be multiplied by the usual row–column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B.

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

Partitioned Matrices

Example:

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Example

EXAMPLE 4 Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Verify that

$$AB = \mathrm{col}_1(A)\,\mathrm{row}_1(B) + \mathrm{col}_2(A)\,\mathrm{row}_2(B) + \mathrm{col}_3(A)\,\mathrm{row}_3(B)$$

Block Matrices

THEOREM 10

Column–Row Expansion of *AB*

If A is $m \times n$ and B is $n \times p$, then

$$AB = [\,\mathrm{col}_1(A) \quad \mathrm{col}_2(A) \quad \cdots \quad \mathrm{col}_n(A)\,] \begin{bmatrix} \mathrm{row}_1(B) \\ \mathrm{row}_2(B) \\ \vdots \\ \mathrm{row}_n(B) \end{bmatrix} \tag{1}$$

$$= \mathrm{col}_1(A)\,\mathrm{row}_1(B) + \cdots + \mathrm{col}_n(A)\,\mathrm{row}_n(B)$$

proof

Altinbus

A **block diagonal matrix** is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.

block upper triangular

EXAMPLE 5 A matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible. Find a formula for A^{-1}.

example

Show that $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible and find its inverse.

Compute $X^T X$, where X is partitioned as $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$.

Matrix Factorization

Matrix Factorization

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.

- Suppose we have to solve many linear systems

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}^{(1)}, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}^{(2)}, \quad \cdots, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}^{(p)}$$

where matrix $\boldsymbol{A}$ is the same - but the right-hand sides are different

- Can solve each of them by Gaussian Elimination separately $\longrightarrow$ inefficient

- Can get the inverse $\boldsymbol{A}^{-1}$ then each solution is of the form $\boldsymbol{x}^{(k)} = \boldsymbol{A}^{-1}\boldsymbol{b}^{(k)}$

Matrix Factorization

it is more efficient to solve the first equation in sequence by row reduction and obtain an LU factorization of A at the same time

assume that A is an m*n matrix that can be row reduced to echelon form, *without row interchanges*.

$$A = LU$$

L is an $m \times m$ lower triangular matrix

U is an $m \times n$ echelon form of A

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

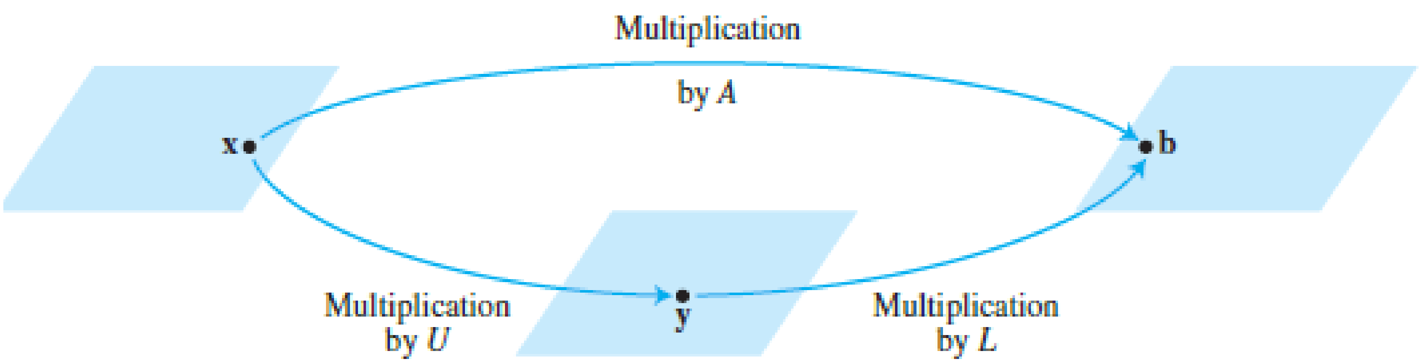
Matrix Factorization

why they are so useful?

$$A\mathbf{x} = \mathbf{b} \Rightarrow L(U\mathbf{x}) = \mathbf{b} \Rightarrow U\mathbf{x} = \mathbf{y}$$

$$\boxed{\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y} \end{aligned}}$$

solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$



example

EXAMPLE 1 It can be verified that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

Use this LU factorization of A to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

$$[\,L \quad \mathbf{b}\,] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [\,I \quad \mathbf{y}\,]$$

$$[\,U \quad \mathbf{y}\,] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

LU decomposition

row reduction of A to an echelon form U amounts to an LU factorization because it produces L with essentially no extra work

LU Factorization Algorithm

Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another row *below it*

there exist unit lower triangular elementary matrices $E_1; \ldots ;E_p$ such that

$$E_p \cdots E_1 A = U$$

$$A = (E_p \cdots E_1)^{-1} U = LU$$

$$L = (E_p \cdots E_1)^{-1}$$

LU Decomposition

row operations which reduce A to U, also reduce L in equation to I

$$E_p \cdots E_1 L = (E_p \cdots E_1)(E_p \cdots E_1)^{-1} = I.$$

ALGORITHM FOR AN LU FACTORIZATION

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I.

Example

EXAMPLE 2 Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Example

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 9 \end{pmatrix}$$