

* حل المسألة 3 :

$$I) p(x, y) = \frac{x+y}{30} \quad x=0,1,2,3, \quad y=0,1,2$$

(1)

$$P(X \leq 2, Y=1) = P(X=0, Y=1) + P(X=1, Y=1) + P(X=2, Y=1)$$

$$= \frac{1}{30} + \frac{2}{30} + \frac{3}{30} = \frac{6}{30} = \frac{1}{5}$$

$$P(X > 2, Y \leq 1) = P(X=3, Y \leq 1)$$

$$= P(X=3, Y=0) + P(X=3, Y=1)$$

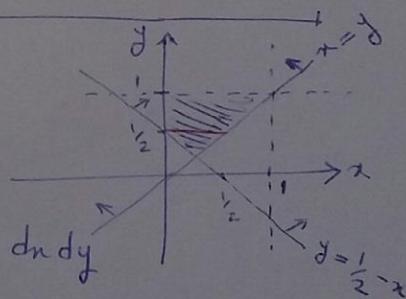
$$= \frac{3}{30} + \frac{4}{30} = \frac{7}{30}$$

$$P(X+Y=4) = P(X=2, Y=2) + P(X=3, Y=1)$$

$$= \frac{4}{30} + \frac{4}{30} = \frac{8}{30}$$

$$II) p(x, y) = \begin{cases} \frac{1}{y} & 0 < x < y < 1 \\ 0 & \text{else} \end{cases}$$

$$P(X+Y > \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{2}-y}^y \frac{1}{y} dx dy + \int_{\frac{1}{2}}^1 \int_0^y \frac{1}{y} dx dy$$



$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(2 - \frac{1}{2y} \right) dy + \int_{\frac{1}{2}}^1 dy = 2y - \frac{1}{2} \ln y \Big|_{\frac{1}{4}}^{\frac{1}{2}} + y \Big|_{\frac{1}{2}}^1$$

$$= \left(1 + \frac{1}{2} \ln 2 \right) - \left(\frac{1}{2} + \frac{1}{2} \ln 4 \right) + \frac{1}{2}$$

$$= 1 - \frac{1}{2} \ln 2$$

$$\text{III) } f_{xy} = \begin{cases} 2 & 0 < x \leq y < 1 \\ 0 & \text{else} \end{cases}$$

$$a) \quad f_{xy} \stackrel{?}{=} f_{yx}$$

$$f_{x|y} = \int f_{xy} dy = \int_x^1 2 dy = 2y \Big|_x^1 = 2(1-x)$$

$$\Rightarrow f_{x|y} = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

$$f_{y|x} = \int f_{xy} dx = \int_0^y 2 dx = 2x \Big|_0^y = 2y$$

$$\Rightarrow f_{y|x} = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

$$f_{x|y} \cdot f_{y|x} = 2(1-x) \cdot 2y = 4y(1-x) \neq 2 = f_{xy}$$

independent \rightarrow no

$$b) \quad p\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{3}{4}\right)$$

$$f_{x|y} = \frac{f_{xy}}{f_y} = \frac{2}{2y} = \frac{1}{y}$$

independent \rightarrow no

$$\Rightarrow f_{x|y} = \begin{cases} \frac{1}{y} & 0 < x \leq y < 1 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} \Rightarrow p\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{3}{4}\right) &= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\frac{3}{4}} dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{4}{3} dx \\ &= \frac{4}{3} x \Big|_{\frac{1}{4}}^{\frac{1}{2}} = \frac{4}{3} \left(\frac{1}{2} - \frac{1}{4}\right) \\ &= \frac{1}{3} \end{aligned}$$

$$IV) f_{X,Y} = \begin{cases} 6x & 0 < x < 1, 0 < y < 1-x \\ 0 & \text{else} \end{cases}$$

$$f_{X,Y} \stackrel{?}{=} f_{X,Y}$$

$$f_X = \int_0^{1-x} 6x dy = 6x y \Big|_0^{1-x} = 6x(1-x) \Rightarrow f_X = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y = \int_0^{1-y} 6x dx = 6 \frac{x^2}{2} = 3x^2 \Big|_0^{1-y} = 3(1-y)^2 \Rightarrow f_Y = \begin{cases} 3(1-y)^2 & 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

$$f_X \cdot f_Y = 6x(1-x) \cdot 3(1-y)^2 = 18x(1-x)(1-y)^2 \neq 6x = f_{X,Y}$$

$$\underbrace{\quad \quad \quad}_{\text{not independent}}$$

$$P(X > 0.3 | Y = 0.5) = ?$$

$$: f_{X|Y}, f_{X,Y} \text{ needed}$$

$$f_{X|Y} = \frac{f_{X,Y}}{f_Y} = \frac{6x}{3(1-y)^2} = \frac{2x}{(1-y)^2} \Rightarrow f_{X|Y} = \begin{cases} \frac{2x}{(1-y)^2} & 0 < x < 1 \\ 0 & 0 < y < 1-x \\ 0 & \text{else} \end{cases}$$

$$P(X > 0.3 | Y = 0.5) = \int_{0.3}^{0.5} \frac{2x}{(0.5)^2} dx = 8 \frac{x^2}{2} = 4x^2 \Big|_{0.3}^{0.5} = 0.64$$

$$\downarrow$$

$$0 < x < 1$$

$$0 < x < 1-y; y=0.5 \Rightarrow 0 < x < 0.5$$

$$\Rightarrow 0.3 < x < 0.5$$

$$\left[\begin{array}{ccc} \text{روز} & \text{سب} & \text{برقال} \\ 3 & 2 & 3 \end{array} \right] \rightarrow 4$$

(2)

X : تعداد برقال ها در این جلسه است

Y : تعداد سب ها

الف) مقادیر ممکن X عبارتند از: 0, 1, 2, 3

$$p_{x,y} = p(X=x, Y=y) = \frac{\binom{3}{x} \binom{2}{y} \binom{3}{4-x-y}}{\binom{8}{4}}; \quad \begin{array}{l} x=0, 1, 2, 3 \\ y=0, 1, 2 \end{array}; \quad 1 \leq x+y \leq 4$$

$$p(X+Y \leq 2) = p(X=0, Y=0) + p(X=0, Y=1) + p(X=0, Y=2) + p(X=1, Y=0) + p(X=1, Y=1) + p(X=2, Y=0)$$

$$\Rightarrow p(X+Y \leq 2) = \frac{1}{\binom{8}{4}} \left[\binom{3}{0} \binom{2}{0} \binom{3}{4} + \binom{3}{0} \binom{2}{1} \binom{3}{3} + \binom{3}{0} \binom{2}{2} \binom{3}{2} + \binom{3}{1} \binom{2}{0} \binom{3}{3} + \binom{3}{1} \binom{2}{1} \binom{3}{2} + \binom{3}{2} \binom{2}{0} \binom{3}{2} \right]$$

$$= \frac{2+3+3+18+9}{70} = \frac{35}{70} = \frac{1}{2}$$

(3) X و Y متغيران عشوائيان

$$f_{X,Y} = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1, x+y \leq 1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow P(X+Y \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} 24xy \, dy \, dx$$

$$= 24 \int_0^{\frac{1}{2}} x \int_0^{\frac{1}{2}-x} y \, dy \, dx$$

$$\left[\frac{y^2}{2} \right]_0^{\frac{1}{2}-x} = \frac{\frac{1}{4} - x^2}{2} = \frac{1-4x^2}{8}$$

$$= 3 \int_0^{\frac{1}{2}} x(1-4x^2) \, dx = 3 \left(\frac{x^2}{2} - x^4 \right) \Big|_0^{\frac{1}{2}}$$

$$= 3 \left(\frac{1}{8} - \frac{1}{16} \right) = \frac{3}{16}$$

$$\Rightarrow f_{X|Y} = \int_0^{1-x} 24xy \, dy = 24x \left(\frac{y^2}{2} \right) = 12xy^2 \Big|_0^{1-x}$$

$$\begin{matrix} \downarrow \\ 0 \leq y \leq 1 \\ y \leq 1-x \end{matrix}$$

$$= 12x(1-x)^2$$

$$\Rightarrow f_{X|Y} = \begin{cases} 12x(1-x)^2 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$c) P(Y < \frac{1}{8} | X = \frac{3}{4}) = ?$$

المarginal $f_{Y|X}$

$$f_{Y|X} = \frac{f_{X,Y}}{f_X} = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2} \Rightarrow f_{Y|X} = \begin{cases} \frac{2y}{(1-x)^2} & 0 \leq x \leq 1, 0 \leq y \leq 1, x+y \leq 1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow P(Y < \frac{1}{8} | X = \frac{3}{4}) = \int_0^{\frac{1}{8}} \frac{2y}{(\frac{1}{4})^2} \, dy = 32 \frac{y^2}{2} \Big|_0^{\frac{1}{8}} = 16y^2 \Big|_0^{\frac{1}{8}} = 16 \left(\frac{1}{64} \right) = \frac{1}{4}$$

(5)