* پاسخ سری 5 :

$$f(x,y)=\begin{cases}\frac{1}{y}e^{-(y+\frac{x}{y})} & x>0\,,\, y>0 \\ 0 & \text{else}\end{cases} \qquad (1)$$

$$Cov(X,Y)=E(XY)-E(X)E(Y)$$

به دست آوردن $f(x)$ راحت نیست پس $E(X)$ را به صورت زیر می‌یابیم:

$$E(X)=\int\int x\, f(x,y)\,dx\,dy$$

$$=\int_0^{+\infty}\int_0^{+\infty} x\left[\frac{1}{y}e^{-(y+\frac{x}{y})}\right]dx\,dy$$

$$=\int_0^{+\infty}e^{-y}\left[(-x-y)e^{-x/y}\right]_0^{+\infty}dy=\int_0^{+\infty}ye^{-y}\,dy=(-y-1)e^{-y}\Big|_0^{+\infty}$$

$$\Rightarrow E(X)=1$$

$$f(y)=\int_0^{+\infty}\frac{1}{y}e^{-(y+\frac{x}{y})}dx=e^{-y}\left[-e^{-\frac{x}{y}}\right]_0^{+\infty}=e^{-y}\quad ; y>0$$

$$\Rightarrow E(Y)=\int_0^{+\infty}ye^{-y}\,dy=-ye^{-y}-e^{-y}\Big|_0^{+\infty}=0+1=1$$

$$\begin{array}{lc} \oplus\, y & e^{-y} \\ \ominus\, 1 & -e^{-y} \\ 0 & e^{-y}\end{array}$$

$$E(XY)=\int_0^{+\infty}\int_0^{+\infty}xy\left(\frac{1}{y}e^{-(y+\frac{x}{y})}\right)dx\,dy=\int_0^{+\infty}ye^{-y}\left[\int_0^{+\infty}\frac{x}{y}e^{-\frac{x}{y}}dx\right]dy$$

$$=\int_0^{+\infty}ye^{-y}(y)\,dy$$

$$=(-y^2-2y-2)e^{-y}\Big|_0^{+\infty}=2$$

$$\Rightarrow Cov(X,Y)=2-(1\times1)=1$$

$$f(x|y)=\frac{f(x,y)}{f(y)}=\frac{\frac{1}{y}e^{-(y+x/y)}}{e^{-y}}=\frac{1}{y}e^{-\frac{x}{y}}\qquad x>0\,,\,y>0$$

$$① \Rightarrow E(X^2|Y=y)=\int_0^{+\infty}\frac{x^2}{y}e^{-x/y}dx=(-x^2-2xy-2y^2)e^{-x/y}\Big|_0^{+\infty}=2y^2$$

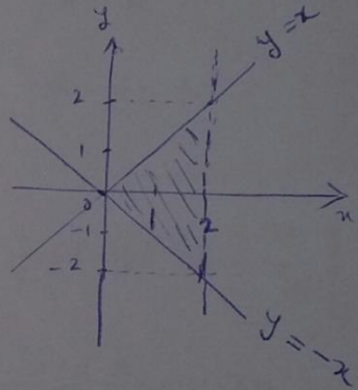
(2)

$$f_{X,Y}(x,y) = \begin{cases} cx(x-y) & -\infty < y < +\infty,\ 0 < x < 2 \\ 0 & \text{else} \end{cases}$$

الف) $$c\int_0^2 \int_{-x}^{x} x(x-y)\,dy\,dx = c\int_0^2 \left[x^2 y - \frac{x}{2} y^2\right]_{-x}^{x} dx$$

$$= c\int_0^2 2x^3\,dx = \frac{c}{2} x^4 \Big|_0^2 = 8c = 1$$

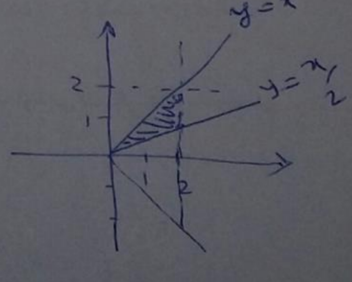
$$\Rightarrow c = 1/8$$



ب) $$f_Y(y) = \begin{cases} \int_y^2 \frac{1}{8} x(x-y)\,dx & 0<y<2 \\ \int_{-y}^2 \frac{1}{8} x(x-y)\,dx & -2<y<0 \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{8}\left[\frac{1}{3}x^3 - \frac{y}{2}x^2\right]_y^2 & 0<y<2 \\ \frac{1}{8}\left[\frac{1}{3}x^3 - \frac{y}{2}x^2\right]_{-y}^2 & -2<y<0 \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{8}\left[\frac{5}{6}y^3 - 2y + \frac{8}{3}\right] & -2<y<0 \\ \frac{1}{8}\left[\frac{1}{6}y^3 - 2y + \frac{8}{3}\right] & 0<y<2 \\ 0 & \text{else} \end{cases}$$



$$P(2Y>X) = \int_0^2 \int_{x/2}^{x} \frac{1}{8} x(x-y)\,dy\,dx$$

$$= \int_0^2 \frac{1}{8}\left(x^2 y - \frac{x}{2} y^2\right)\Big|_{x/2}^{x} dx$$

$$= \frac{1}{64}\int_0^2 x^3\,dx = \frac{1}{16}$$

$$P(Y<\tfrac{1}{2} \mid X=1) = ?$$

ابتدا $f(y|x)$ را می‌یابیم:

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$f(x) = \int_{-x}^{x} \frac{1}{8} x(x-y)\,dy = \frac{1}{8}\left(x^2 y - \frac{x}{2} y^2\right)\Big|_{-x}^{x} = \frac{1}{4} x^3 \quad ; \; 0<x<2$$

$$f(y|x) = \frac{\frac{1}{8} x(x-y)}{\frac{1}{4} x^3} = \frac{1}{2} \cdot \frac{x-y}{x^2} \quad ; \; 0<x<2,\ -x<y<x$$

$$\Rightarrow P(Y<\tfrac{1}{2} \mid X=1) = \int_{-1}^{\frac{1}{2}} \frac{1}{2} \cdot \frac{1-y}{1}\,dy = \int_{-1}^{\frac{1}{2}} \frac{1}{2}(1-y)\,dy = \frac{15}{16}$$

$$f(x,y,z) = \begin{cases} c(x+y)e^{-z} & 0<x<1,\ 0<y<2,\ z>0 \\ 0 & \text{else} \end{cases} \qquad (3$$

الف) $\iiint c(x+y)e^{-z}\,dz\,dy\,dx = 1$

$$\Rightarrow c\int_0^1 \int_0^2 \int_0^{+\infty} (x+y)e^{-z}\,dz\,dy\,dx = 1$$

$$\Rightarrow c\int_0^1 \int_0^2 (x+y)(-e^{-z})\Big|_0^{+\infty}\,dy\,dx = 1$$

$$\Rightarrow c\int_0^1 \int_0^2 (x+y)\,dy\,dx = c\int_0^1 xy + \frac{y^2}{2}\Big|_0^2\,dx$$

$$= c\int_0^1 2x+2\,dx = c(x^2+2x)\Big|_0^1 = 3c = 1$$

$$\Rightarrow c = \frac{1}{3}$$

ادامه ی الف)

$$P(X<Y, Z>1)=\int_0^1\int_x^2\int_1^{+\infty}\frac{1}{3}(x+y)e^{-z}\,dz\,dy\,dx$$

$$=\int_0^1\int_x^2\frac{1}{3}(x+y)\left(-e^{-z}\right)\Big|_1^{+\infty}\,dy\,dx$$

$$=\frac{1}{3}e^{-1}\int_0^1\int_x^2 (x+y)\,dy\,dx$$

$$=\frac{1}{3}e^{-1}\int_0^1 \left(xy+\frac{1}{2}y^2\right)\Big|_x^2\,dx$$

$$=\frac{1}{3}e^{-1}\int_0^1 2x+2-\frac{3}{2}x^2\,dx=\frac{5}{6}e^{-1}$$

ب) $f(x,z)=\int_0^2\frac{1}{3}(x+y)e^{-z}\,dy=\frac{1}{3}e^{-z}\left(xy+\frac{1}{2}y^2\right)\Big|_0^2=\frac{2x+2}{3}e^{-z}$

$$\rightarrow f(x,z)=\begin{cases}\dfrac{2x+2}{3}e^{-z} & 0<x<1,\ z>0\\ 0 & \text{else}\end{cases}$$

$$f_X(x)=\int_0^{+\infty}\frac{2x+2}{3}e^{-z}\,dz=\frac{2x+2}{3}\quad;\quad 0<x<1$$

$$f_Z(z)=\int_0^1\frac{2x+2}{3}e^{-z}\,dx=e^{-z}\left[\frac{x^2+2x}{3}\right]_0^1=e^{-z}\quad;\quad z>0$$

به روش مشابه:

$$f(y,z)=\frac{2y+1}{6}e^{-z}\quad;\quad 0<y<2,\ z>0$$

$$f_Y(y)=\frac{2y+1}{6}\quad;\quad 0<y<2$$

$$f_{XY}(x,y)=\frac{x+y}{3}\quad;\quad 0<x<1,\ 0<y<2$$

با توجه به اینکه: $f(x,y,z)\neq f(x)f(y)f(z)$ پس X، Y، Z از هم مستقل نیستند.
از طرفی $f(x,z)=f(x)f(z)$ پس X و Z مستقل هستند.
" $f(y,z)=f(y)f(z)$ " Y و Z " " .

$f(x,y)\neq f(x)f(y)$ پس X، Y مستقل نیستند.

$$f(x,y)=\begin{cases} \frac{k}{2}y^2e^{-y}x^{y-1} & 0<x<1,\ y>0 \\ 0 & \text{else} \end{cases}$$ (4)

الف) $$1=\int_0^{+\infty}\int_0^1 \frac{k}{2}y^2e^{-y}x^{y-1}\,dx\,dy$$

$$=\frac{k}{2}\int_0^{+\infty} y^2e^{-y}\left(\frac{x^y}{y}\right)\Big|_0^1\,dy$$

$$=\frac{k}{2}\int_0^{+\infty} ye^{-y}\,dy=\frac{k}{2}\left[(-y-1)e^{-y}\right]_0^{+\infty}=\frac{k}{2}(0-(-1))$$

$$\Rightarrow 1=\frac{k}{2}\Rightarrow k=2$$

برای احتمال خواسته شده ابتدا $f_{X|Y}(x|y)$ را می‌یابیم:

$$f(x|y)=\frac{f(x,y)}{f(y)}\ ;\ f(y)=\int_0^1 y^2e^{-y}x^{y-1}\,dx=y^2e^{-y}\left(\frac{x^y}{y}\right)\Big|_0^1=ye^{-y}$$

$$f_Y(y)=\begin{cases} ye^{-y} & y>0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow f(x|y)=\frac{y^2e^{-y}x^{y-1}}{ye^{-y}}=yx^{y-1}\ ;\ 0<x<1,\ y>0$$

$$\Rightarrow P(X<\tfrac{1}{2}\mid Y=2)=\int_0^{1/2}2x\,dx=\frac{1}{4}$$

$$\operatorname{var}(X|Y=y)=E(X^2|Y=y)-\left(E(X|Y=y)\right)^2$$

$$E(X|Y=y)=\int_0^1 yx^y\,dx=\frac{y}{y+1}x^{y+1}\Big|_0^1=\frac{y}{y+1}$$: میانگین

$$E(X^2|Y=y)=\int_0^1 yx^{y+1}\,dx=\frac{y}{y+2}\left(x^{y+2}\right)\Big|_0^1=\frac{y}{y+2}$$

5 $$\Rightarrow \operatorname{var}(X|Y=y)=\frac{y}{y+2}-\left(\frac{y}{y+1}\right)^2=\frac{y}{(y+2)(y+1)^2}$$

(5) الف)

$$var(-4X+12) = 16\, var(X)$$

که داریم:

$$var(X) = E(X^2) - E^2(X)$$

از طرفی

$$E(X(X-4)) = E(X^2) - 4E(X) = 5$$

$$\Rightarrow E(X^2) = 5 + 8 = 13$$

در نتیجه:

$$var(X) = 13 - 4 = \underline{9}$$

$$\sigma_X = \sqrt{var(X)} = \underline{3}$$

ب)

$$E(U) = E(2X - Y) = 2E(X) - E(Y) = 2(1) - 2 = \underline{0}$$

$$E(V) = E(X + 2Y) = E(X) + 2E(Y) = 1 + 2(2) = \underline{5}$$

$$var(U) = var(2X - Y) = 4\,var(X) + var(Y) - 4\,Cov(X,Y)$$

$$= 4(3) + 4 - 4(-1) = 12 + 4 + 4 = \underline{20}$$

$$var(V) = var(X + 2Y) = Var(X) + 4\,var(Y) + 4\,Cov(X,Y)$$

$$= 3 + 4(4) + 4(-1) = 3 + 16 - 4 = \underline{15}$$

$$E(X^2 + 3X - 5) = E(X^2) - 3E(X) - 5 = 13 - 3(1) - 5 = \underline{5}$$

$$\rho(U,V) = \frac{Cov(U,V)}{\sigma_U \cdot \sigma_V} \; ; \; Cov(U,V) = Cov(2X - Y, X + 2Y)$$

$$= 2\,var(X) + 4\,Cov(X,Y) - Cov(X,Y) - 2\,var(Y)$$

$$= 6 + 4(-1) - (-1) - 2(4) = 6 - 4 + 1 - 8 = -5$$

$$\Rightarrow \rho(U,V) = \frac{-5}{\sqrt{20}\sqrt{15}} = \frac{-5}{\sqrt{300}} \simeq -0.29$$

این ضریب همبستگی بیانگر این است که U و V دارای رابطه‌ی خطی ضعیف با شیب منفی هستند.