1 The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (**??**) can be rewritten

$$\begin{aligned} \mathrm{e}_t(m_t)^{-\rho} &= \beta \mathsf{R}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(m_{t+1}) \left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \\ &= \ (1-\wp)\beta \mathsf{R} m_t^{\rho}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(m_{t+1}) m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0 \right] \\ &+ \wp \beta \mathsf{R}^{1-\rho}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t)) \frac{m_t - \mathrm{c}_t(m_t)}{m_t} \right)^{-\rho} \mid \xi_{t+1} = 0 \right]. \end{aligned}$$

Consider the first conditional expectation in (**??**), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-\wp)$. Since $\lim_{m \downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1}) m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-\wp))\Gamma\underline{\psi}\underline{\theta}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-\wp))\Gamma\bar{\psi}\bar{\theta}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1 + \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}. \end{aligned}$$

As noted in the main text, we need the WRIC (**??**) for this to be a convergent sequence:

$$0 \le \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{1}$$

Since $\bar{\kappa}_T = 1$, iterating (1) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{2}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the

FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (1) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} \quad = \beta \mathsf{R}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\, (\mathsf{R}\mathrm{a}_t(m_t)))^{-\rho} \right]$$

and we know from L'Hôpital's rule that $\lim_{m_t \to \infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t \to \infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t \underline{\kappa}_t)^{-\rho} &= \beta \mathsf{R} \left(\underline{\kappa}_{t+1} \mathsf{R}(1 - \underline{\kappa}_t) m_t \right)^{-\rho} \\ \underbrace{\mathsf{R}^{-1} \mathbf{Þ}}_{\equiv \mathbf{Þ}_{\mathsf{R}} = (1 - \underline{\kappa})} \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \mathbf{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \mathbf{Þ}_{\mathsf{R}} \tag{3}$$

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \kappa_{T-n}^{-1} \tag{4}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n \to \infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$\underbrace{c_t \left(1 + \mathbf{Þ}_{\mathsf{R}} + \mathbf{Þ}_{\mathsf{R}}^2 + \cdots \right)}_{= 1 + \mathbf{Þ}_{\mathsf{R}}(1 + \mathbf{Þ}_{\mathsf{R}} \underline{\kappa}_{t+2}^{-1}) \ldots} \quad = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t) \underline{\kappa}_t \tag{5}$$