Beginner's Practical: Hypergraph Conductance-based Clustering

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Hypergraph clustering is a problem of partitioning the nodes of a hypergraph into "true clusters" which are densely connected internally and sparsely connected externally. Conductance is a metric measuring the quality of a cut which is defined as the ratio of the weight of the cut to the minimal volume of one side of the cut.

In this project, we implement a hypergraph clustering algorithm that heuristically tries to minimize the maximum value of the conductance over all cuts between clusters to find a high-quality hypergraph clustering. The algorithm is based on the existing shared-memory parallel algorithm for hypergraph partitioning `Mt-KaHyPar`.

Experimental results show that our implementation is able to find high-quality conductance-wise hypergraph clustering in a short time, but the results are heavily dependent on the parameter k of the estimated number of clusters.

1 INTRODUCTION

Hypergraph clustering is a problem of partitioning the nodes of a hypergraph into "true clusters" which are densely connected internally and sparsely connected externally. It is useful in many applications, such as semi-supervised learning [12], gene expression analysis [11] and image segmentation [6]. Conductance metric, which is defined for a cut as the ratio of the cut weight to the minimal volume of one side of the cut, is widely used in graph clustering [3, 8] but to the best of our knowledge, there are no partitioners directly optimizing conductance-based metric of a hypergraph. However, conductnace has been applied as a metric for evaluating the quality of a hypergraph clustering [1]. Therefore in this project, we implement a heuristic hypergraph clustering algorithm with the objective of minimizing the maximum value of the conductance over all cuts between the clusters while partitioning the hypargraph in at most k clusters.
Our algorithm is based on the existing shared-memory parallel hypergraph k-way partitioner `Mt-KaHyPar` and uses the same multilevel approach as the original algorithm. First, the hypergraph is coarsened in stages by detecting and merging local node clusters, then the coarsest hypergraph is partitioned into k clusters, and finally the initial partitioning is refined by uncoarsening the hypergraph nodes and applying a local search algorithm to improve the given objective. We implement two objective functions for the local search algorithm - *conductance_local* and *conductance_global* - and compare their performance on the circuit-based benchmark set *ISPD98* [2] against a state of the art hypergraph clustering algorithm `HyperModularity`. We also conduct a scaling experiment to evaluate the performance of our algorithm one multiple cores.
In following, we define the used concepts in Section 2, then we shortly describe related work on hypergraph and conductance-based clustering in Section 3. In Section 4, we describe the implementation of the new objectives for optimizing the conductance and their integration in `Mt-KaHyPar`. Afterwards, we present the experimental results of our implementation in Section 5. Finally, we conclude the report with a shoret summary of the results in Section 6.

2 PRELIMINARIES

In this section, we give an overview of the used concepts and notations and at the end define conductance of a hypergraph.

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Hypergraph is a generalization of a graph, where a hyperedge connects one, two or more nodes. Formally, an edge-weighted hypergraph $H = (V, E, \omega)$ consists of a non-empty set of nodes V, a set of hyperedges (also called *nets* or simply *edges*) $E \subseteq 2^V$ and a weight function for hyperedges $\omega : E \to \mathbb{N}$. In case of unweighted hyperedges, we set $\omega(e) = 1$ for all $e \in E$.

k-way clustering of a hypergraph $H = (V, E, \omega)$ is a partitioning of the set of nodes V into k disjoint subsets $V_1, \ldots, V_k$ called *clusters*. Empty clusters are allowed, so k is only an upper bound on the final number of clusters. Our algorithm is allowed to eventially move all the nodes of a cluster to other clusters if it would be beneficial for the objective function.

Cut ∂S of a hypergraph $H = (V, E, \omega)$ is partitioning of the set of nodes V into two disjoint non-empty subsets S and $V \backslash S$. A hyperedge $e \in E$ is called a **cutting edge** of the cut ∂S if it has at least one pin in S and at least one pin in $V \backslash S$. The weight of the cut is defined as the sum of the weights of the cutting edges of the cut ∂S:

$$\omega(\partial S) = \sum_{e \in \partial S} \omega(e)$$

Therefore in case of unweighted hyperedges, the weight of the cut is equal to the number of cutting edges of the cut.

Pin of a hyperedge $e \in E$ is a node $v \in V$ such that $v \in e$. A hyperedge with only one pin is called a **sinle-pin net**. In the implementation, we use number of pins $\text{numPins}_i(e)$ of a hyperedge $e \in E$ in a cluster V_i to decide whether to move a node v from one cluster V_i to another cluster V_j.

Weighted degree of a node $v \in V$ is defined the same way in hypergraphs as in graphs. It is the sum of the weights of all hyperedges $e \in E$ that contain the node v:

$$\deg_\omega(v) = \sum_{e \in E : v \in E} \omega(e)$$

As during the coarsening phase of the algorithm we merge nodes, we also use **original weighted degree** of a node $v' \in V'$ of the coarsened hypergraph $\text{origDeg}_\omega(v')$ which is defined as the sum of the weighted degrees of all nodes of the initial hypergraph $v \in V$ that were merged into v' during the coarsening phase of the algorithm.

Volume of a hypergraph $H = (V, E, c, \omega)$ is defined as the sum of the weighted degrees of all nodes $v \in V$. Analogously, we define the **volume of a cluster** V_i as the sum of the weighted degrees of all nodes $v \in V_i$:

$$\text{vol}(H) = \sum_{v \in V} \deg_\omega(v) \qquad \text{vol}(V_i) = \sum_{v \in V_i} \deg_\omega(v)$$

And for the same reason as for the original weighted degree, we also define the **original volume** of a hypargraph and a cluster:

$$\text{origVol}(H) = \sum_{v \in V} \text{origDeg}_\omega(v) \qquad \text{origVol}(V_i) = \sum_{v \in V_i} \text{origDeg}_\omega(v)$$

This way, the original volume of a cluster or of the whole hypergraph in the coarsened hypergraph is equal to the corresponding volume in the initial - *original* - hypergraph.

Conductance of a cut ∂S of a hypergraph $H = (V, E, \omega)$ is defined as the ratio of the weight of the cut to the minimum volume of one side of the cut. The maximal conductance of a cut In a hypergraph is referred to as the **conductance of the hypergraph**:

$$\varphi(S) = \frac{\omega(\partial S)}{\min\{vol(S), vol(V \backslash S)\}} \qquad \varphi(V) = \max_{\emptyset \subsetneq S \subsetneq V} \varphi(S)$$

To use conductance as a quality metric for a hypergraph clustering with possibly more than two clusters, we define the **conductance of a hypergraph clustering** $V_1, \ldots, V_k$ as the maximum

conductance of all cuts between the clusters. Conveniently, this definition can be simplified to the maximal conductance of a cut ∂V_i over all clusters V_i:

$$\varphi(V_1, \ldots, V_k) := \max_{I \subsetneq \{1,\ldots,k\}: \cup_{i \in I} V_i \neq \emptyset, V} \varphi(\cup_{i \in I} V_i) = \max_{i=1\ldots k: V_i \neq \emptyset, V} \varphi(V_i)$$

A proof of this nice statement can be found in Appendix A.

3 RELATED WORK

For graph clustering, there are several state of the art algorithms that aim to minimize the resulting conductance. Some of them are heuristic like the diffusion based `NIBBLE` [10] and `hk-relax` [7]. Other, such as degree ratio-based `PC_de` [8] have a theoretical guaranatee of the resulting conductance.

As for the hypergraph-clustering, to the best of our knowledge, there are no existing algorithms that directly optimize a conductance-based objective. Instead, state of the art hypergraph clustering algorithms, e. g., `HyperModularity` which we use for comparison with our algorithm, deliver high quality hypergraph clustering by optimizing the modularity of the hypergraph [4, 9].

The algorithm `Mt-KaHyPar` that we use as a base for our implementation also optimizes the modularity of the hypergraph in its coarsening stage. But as whole, it could be seen as a framework for hypergraph partitioning with a user-implemented objective, which is optimized during the initial partitioning of the coarsest hypergraph and later in the uncoarsening stage consisting of label propagation and FR-refinement x[5].

4 IMPLEMENTATION

To implement our algorithm, we introduced a new objective function for the `Mt-KaHyPar` algorithm, which provides a framework for hypergraph partitioning. Also, we introduced a clustering mode (calles `cluster`) for the `Mt-KaHyPar` algorithm, which relaxes weight constarints for the clusters. The idea and most of the implementation of this mode was taken with a premission from a fork of `Mt-KaHyPar` by Adil Chhabara: https://github.com/adilchhabra/mt-kahypar

For a new objective function for `Mt-KaHyPar`, we needed to implement efficient calculation of the conductance of the clustering, formulate an initial partitioning algorithm to apply to the coarsest hypergraph and a define a gain function for the local search algorithm. In this section, we describe our approaches to these tasks.

4.1 Efficient Conductance Calculation

Conductance calculation is needed to evaluate the quality of the clustering and to stop the local search algorithm when no more moves are worsening the conductance of the clustering.

Firstly, to be able to efficiently calculate the conductance of a clustering, we keep all original weighted degrees, original volumes and cut weights up to date. This is done by adjusting their values during the changes of the clustering or (un)coarsening of the hypergraph. Here are both these changes:

- **(Un)contraction** of a group of nodes $v_1 \ldots v_c$ into (from) one node v':
  - $\text{origDeg}_\omega(v') = \text{origDeg}_\omega(v_1) + \cdots + \text{origDeg}_\omega(v_c)$. Hence, atomical substraction (addition) is enough to update the original weighted degrees; It is important to note that this operation doesn't increase the run-time complexity of the algorithm, as to contract the nodes algorithm needs to go through them.
  - original volumes and cut weights should't be updated, as uncontracted nodes $v_1 \ldots v_c$ are placed in the cluster of their representative node v'.
- **Node move** of v from one cluster V_i to another cluster V_j:

– for original volumes we simpty need one atomical addition and one atomical substraction:

$$\text{origVol}(V_i) -= \text{origDeg}_\omega(v), \qquad \text{origVol}(V_i) += \text{origDeg}_\omega(v)$$

– updating cut weight is a bit trickier, but conveniently, `Mt-KaHyPar` maintains the number of pins $\text{numPins}_t(e)$ of each hyperedge e in each cluster V_t, so the following adjustments for each incident to v hyperedge e are enough to update the cut weights:
  * if $1 = \text{numPins}_i(e) < size(e)$:
    e is a cutting net for the cut $\partial(V_i)$, v is its last pin in V_i, therefore e won't be a cutting edge after removal of v: cut weight of $\partial(V_i)$ shuld be decreased by the weight of the hyperedge e,
  * else if $1 < \text{numPins}_i(e) < size(e)$:
    e was and still will be a cutting net for the cut $\partial(V_i)$, therefore no changes from this edge are needed for the cut weight of $\partial(V_i)$.
  * else if $1 < \text{numPins}_i(e) = size(e)$:
    e was not a cutting net for the cut $\partial(V_i)$, but it will be after the move of v, therefore the weight of the hyperedge e should be added to the cut weight of $\partial(V_i)$.

  Again, this operation doesn't increase the run-time complexity of the algorithm, as to move a node the algorithm needs to go through all its incident hyperedges and update the number of pins in each cluster.

Maintaining this information makes calculating of the maximal conductance of a cut $\partial(V_i)$ possible. But to do it more efficient than going through all the partitions each time we need to calculate the conductance of the clustering, we implement an additional data structure: **ConductancePriorityQueue** - a priority queue with an entery for each cluster V_i. This entery is a fraction with $\omega(\partial(V_i))$ in the numerator and $\min(\text{origVol}(V_i), \text{origVol}(V) - \text{origVol}(V_i))$ in the denominator. These enteries are updated by changes of the cut weights and volumes of the clusters. To make sure, that our priority queue (max 2 heap) is always sorted, we use a lock by every change of its entery. We also use lazy updates of the priority queue to avoid unnecessary usage of the lock. This is done by storing the deltas of the cut weights and volumes of the clusters, when it is clear, that the update is not yet finished (e. g., when the new cut weight would be higher that the now volume of a cluster).

To store and compare the fractions, we also use another new class **NonnegativeFraction**, that supports comparison of two nonnegative fractions without a risk of an overflow. This is achieved by tricks with integral diviion and comparing recipocal rest fractions in the case of the same integral part of the fraction. Furthermore, for the case of an empty cluster, we make sure, that a fraction with 0 in the denominator is always not than any other fraction without a zero in the denominator.

With all this information, we can efficiently calculate the conductance of the clustering by sinmpy looking at the top element of the ConductancePriorityQueue. Note that we use the original volumes, thus the computed conductance of the clustering is the value of the conductance for the clustering of the original hypergraph with nodes assigned to clusters of their representatives in the current coarsened hypergraph. This way we always optimize the final value of the conductance of the clustering. If we would use the volumes of the coarsened hypergraph, the conductance of the clustering would be different, as during the coarsening phase the algorithm merges nodes into clusters, and therefore reduces the volumes of the clusters.

4.2 Initial Partitioning

Initial partitioning algorithm is applied after the coarsening phase of the algorithm (which works exactly the same way as in the original `Mt-KaHyPar` algorithm) to the coarsest hypergraph. Our first but less successful approach was to use a *singleton* partitioning, where each node is assigned to a separate cluster. The idea is, that during the uncoarsening phase, the algorithm will merge some of the clusters together, to minimize the conductance of the resulting partition. However, this approach turned out to be inefficient, as most of the time the resulting clustering had conductance 1.

The second approach was to use a *random* partitioning, where each node of the coarsest hypergraph is randomly assigned to one of the k clusters. For a better qulity of the initial partitioning, we run the random partitioning algorithm 10 times and choose the one with the lowest conductance. This approach was more successful, so it is default in our *cluster*-mode.

4.3 Gain function for Label Propagation

Gain function for the local search algorithm is needed to decide whether to move a node v from one cluster V_i to another cluster V_j of the current partition during the uncoarsening phase. We again tried two approaches. They are implemented as separate objectives, but differ only in the way of calculating the gain of a move.

The first one is called *conductance_local*. Here, the gain of moving a node v from one cluster V_i to another cluster V_j is defined as the difference between the maximal conductance of the cuts ∂V_i and ∂V_j after and before the move.

The second one is called *conductance_global*. The gain of a move here is defined as the difference between the conductance of the partition after and before the move, i. e., the difference between the maximal conductance of all the cuts ∂V_t after and before the move. This is done by looking at the top tree elements of the ConductancePriorityQueue: if the maximal conductance of the clustering will be changed, the new maximum will be one of the clusters V_i, V_j or the cluster with the highest cut conductance apart from these two. As this third cluster is guaranteed to be one of the top three elements of the ConductancePriorityQueue, we simply compute the conductance of the clustering after the move in a constant time and therefore the gain of one move is also computed in a constant time.

In both cases, non-negative gain means that executing the move will make the maximal conductance of the clustering higher. However, as the label propagation algorithm first computes gains for moves of all nodes to all their neighbouring partitions, and then moves active nodes in parallel in the cluster with the best gain, it can so happen that some or all of the precomputed gains are incorrect. Because of this, the algorithm uses **attributed gains** to track the real change in the objective function, i. e., conductance of the clustering. If this change gets bad after a round of node moves, the label propagation will stop and the algorithm will start a new round of uncoarsening.

The original `Mt-KaHyPar` algorithm was implemented with a cut-based objective function, so the attributed gain of a move was atomically updated for each incident hyperedge of the moved node. In our case, **the attributed gain of a move** is updated for each move only once: after the update of the cut weight and the original volumes. To account for concurrent changes of cut weights, during each move a difference between the new and the old cut weights and volumes made by the move is used to get the attributed gain of the concrete move. The **attributed gains** of a move is then calculated as the difference in the conductance of the clustering exactly the same way as in *conductance_global* approach and also runs in constant time for each move.

	k	HyperModularity		Conductance-based clustering			
				conductance_local, random		conductance_global, random	
		φ	Run- and I/O time	φ	Run- and I/O time	φ	Run- and I/O time
ibm02	11	0,3211	18,0384 s + 0,0363 s	**0,2662**	0,2471 s + 0,0175 s	0,9231	0,3123 s + 0,0176 s
ibm03	27	**0,1976**	10,4540 s + 0,0485 s	0,2640	0,2920 s + 0,0190 s	0,9444	0,3105 s + 0,0209 s
ibm04	28	**0,2132**	9,4372 s + 0,0571 s	0,2902	0,3500 s + 0,0215 s	0,9231	0,4271 s + 0,0193 s
ibm05	14	0,3704	13,1119 s + 0,0573 s	**0,2595**	0,4120 s + 0,0222 s	0,8636	0,4514 s + 0,0210 s
ibm06	26	**0,2399**	11,9333 s + 0,2838 s	0,2581	0,4565 s + 0,0213 s	0,9259	0,4611 s + 0,0217 s
ibm07	30	0,2361	19,4188 s + 0,0859 s	**0,2200**	0,5388 s + 0,0291 s	1,0000	0,5722 s + 0,0297 s
ibm08	26	0,3210	108,7750 s + 0,0896 s	**0,2393**	0,6975 s + 0,0330 s	1,0000	0,6873 s + 0,0297 s
ibm09	40	**0,1595**	18,1716 s + 0,1136 s	0,2193	0,6194 s + 0,0342 s	0,9063	0,6618 s + 0,0297 s
ibm10	29	**0,2105**	91,1637 s + 0,1474 s	0,2114	0,8727 s + 0,0404 s	0,8636	0,8972 s + 0,0297 s
ibm11	36	0,2033	29,0756 s + 0,1424 s	**0,1853**	0,8045 s + 0,0401 s	0,8495	0,9233 s + 0,0406 s
ibm12	29	**0,1977**	136,1306 s + 0,3668 s	0,2497	0,9654 s + 0,0429 s	1,0000	0,9946 s + 0,0427 s
ibm13	29	**0,1044**	67,8915 s + 0,4204 s	0,1928	1,0230 s + 0,0499 s	0,8211	1,1253 s + 0,0481 s
ibm14	34	0,1960	180,9870 s + 0,5436 s	**0,1927**	1,7571 s + 0,0774 s	0,9412	1,9409 s + 0,0766 s
ibm15	38	**0,1414**	321,1060 s + 0,4440 s	0,1708	2,2087 s + 0,0973 s	0,9111	2,3470 s + 0,0948 s
ibm16	37	0,2339	746,5931 s + 0,4849 s	**0,1753**	2,4574 s + 0,1036 s	0,8333	2,6329 s + 0,1015 s
ibm17	47	**0,2063**	789,4657 s + 0,4700 s	0,2071	2,9459 s + 0,1047 s	0,9412	2,9660 s + 0,1068 s
ibm18	48	**0,0944**	977,1867 s + 0,5143 s	0,1464	2,8057 s + 0,1083 s	0,8889	3,1900 s + 0,1114 s

Table 1. Comparison of our conductance-based clustering with `HyperModularity`. k for each instance is the number of clusters found by the `HyperModularity` algorithm. The best results are marked in bold.

This concludes the overview of our adjustments to the `Mt-KaHyPar` algorithm. A more detailed (but even more technical) description of the implementation can be found in the `Changes Guide.md` on the GitHub repository of this project.

5 EXPERIMENTAL RESULTS

We conducted tree sets of experiments to evaluate the performance of our implementation. All the experiments were conducted on a machine consisting of a sixteen-core Intel Xeon Silver 4216 processor running at 2.1 GHz, 100 GB of main memory, 16 MB of L2-Cache, and 22 MB of L3-Cache running Ubuntu 20.04.1.

5.1 Comparison against a state of the art hypergraph clustering algorithm

As no other conductance-based hypergraph clustering algorithm is known to us, we compare our implementation with the state of the art modularity based hypergraph clustering algorithm `HyperModularity`. Conductance of the found clusterings is calculated separately for both our algorithm and `HyperModularity`. Our implementation was run on one cpu, as `HyperModularity` is not parallelized. Experiment was conducted on the circuit-based benchmark set `ISPD98`, which was previously used to evaluate the conductance-quality of a hypergraph clustering algorithm [1]. The benchmark set consists of 18 hypergraphs with increasing sizes, ranging from 14111 nets with 50566 pins in the smallest hypergraph `ibm01` to 201920 nets with 819697 pins in the largest one `ibm18` [2].

To begin with, as `HyperModularity` decides the number of clusters k automatically, we ran it first and then ran our algorithm with the same k in different configurations. In the Table I 1 we show the results of the comparison of `HyperModularity` against *conductance_local* and *conductance_global* with the random initial partitioner. We can see, that *conductance_local* gives roughlt the same

Instance	*conductance_local*	
	Conductance	**Run- and I/O-time**
ibm01	0,3590	0,188 s + 0,0147 s
ibm02	0,4436	0,459 s + 0,016 s
ibm03	0,3634	0,3712 s + 0,0177 s
ibm04	0,4012	0,4396 s + 0,0197 s
ibm05	0,4044	0,6242 s + 0,0234 s
ibm06	0,3540	0,6637 s + 0,0244 s
ibm07	0,3290	0,8266 s + 0,0275 s
ibm08	0,3675	0,8152 s + 0,0321 s
ibm09	0,2687	0,6863 s + 0,0316 s
ibm10	0,3073	0,9924 s + 0,0400 s
ibm11	0,2926	0,9065 s + 0,0403 s
ibm12	0,3347	1,0801 s + 0,0405 s
ibm13	0,2683	1,1451 s + 0,0468 s
ibm14	0,2557	2,1130 s + 0,0768 s
ibm15	0,2453	2,4405 s + 0,0958 s
ibm16	0,2151	2,5286 s + 0,0992 s
ibm17	0,2514	3,0733 s + 0,1061 s
ibm18	0,2302	3,0714 s + 0,1084 s

Table 2. *conductance_local* mode with the random IP and $k = 100$. In all cases, exactly 100 clusters were found.

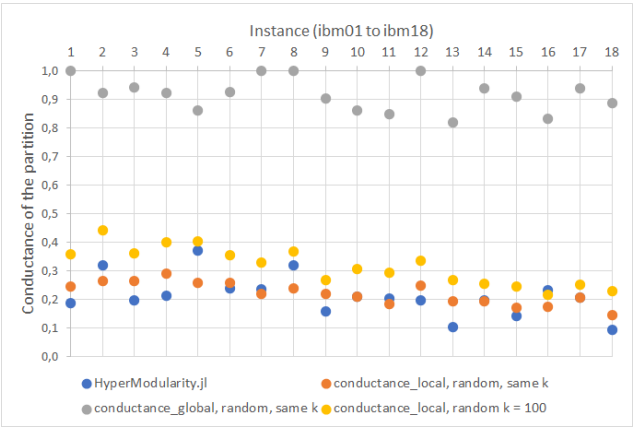

Fig. 1. Comparison of conductance of the partition.

conductance as the `HyperModularity`, while *conductance_global* gives a much worse result. Time-wise, both our implementations are much faster than `HyperModularity`.

To make the comparison more fair, we also ran *conductance_local* with the preset number of clusters $k = 100$ and the random initial partitioner. The results are shown in the Table 2. We can see, that this way *conductance_local* gives a worse result than in the previous run, but it is still close to the `HyperModularity`.

Figure 1 and Figure 2 summarize the conductance and the run time of this comparison.

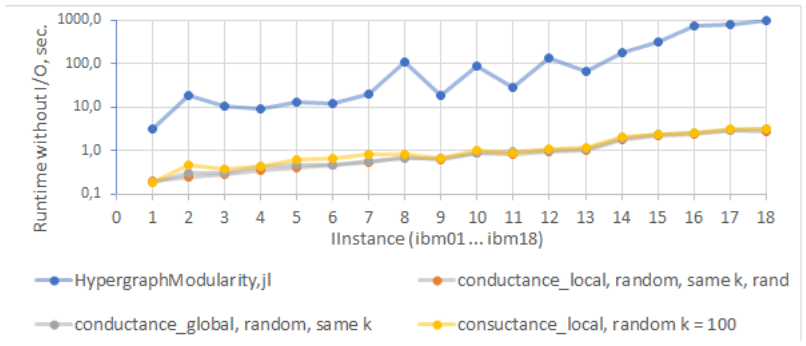

Fig. 2. Comparison of the run time against `HyperModularity`. The y-acsis is logarithmic

Threads	Conductance	Run- and I/O-time
1	0.5967	131.5407 s + 3.4846 s
2	0.7283	74.2843 s + 2.1009 s
4	0.9369	46.9272 s + 1.6002 s
8	0.6478	25.4402 s + 1.4921 s
16	0.7473	18.9490 s + 1.5178 s

Table 3. Performance of *conductance_local* with $k = 100$ by increasing number of threads.

Analogous experiments were run with the singleton initial partitioner, but the resulting conductance of the clustering was always 1 except for rare 0.625 and 0.750. Therefore we don't include the results of these experiments in the report.

5.2 Scalability of the algorithm

The scalability of our implementation was tested only on the best configuration (*conductance_local* with random initial partitioner) in two ways. First, we ran our algorithm on a large circuit-based hypergraph `circuit5M.mtx.hgr` with 5558326 hyperedges and 59524291 pins using 1, 2, 4, 8 and 16 threads. The results are shown in the Table 3. We can see, that the running time of the algorithm decreases by about factor 2 with each next thread. Unfortunately, the conductance of the clustering is not getting better with the increase of the number of threads.

The second test was run on the same hypergraph, but with a different number of clusters k. We ran the algorithm with $k = 100$, $k = 200$, $k = 400$, $k = 800$ and $k = 1600$. The results are shown in the Table 4. We can see, that the running time of the algorithm grows with the exponential increase of k, but not exponentially. Unfortunately, the conductance of the clustering is getting worse and the number of clusters is not seemingly converging to some optimal number of "true" clusters. This is probably due to the fact, that with k greater than the number of nodes in the coarsest hypergraph, random initial partitioner works similarily to singleton.

6 CONCLUSION

The experimantal results show, that our implementation can find clustering with conductance comparable to a state of the art hypergraph clustering and needs significantly less time to do so, but the result heavily depends on the parameter k and the initial partitioner.

k	Conductance	Run- and I/O-time	# clusters
100	0.6483	42.3435 s + 5.1161 s	100
200	0.8326	50.0950 s + 1.6660 s	198
400	0.8597	46.2647 s + 1.6059 s	376
800	0.8272	61.7254 s + 5.12642 s	623
1600	0.9367	80.9240 s + 3.4388 s	770

Table 4. The performance of *conductance_local* with random on 4 threads with increasing k

A APPENDIX

Theorem A.1. *Concuctance of a k-way partition $V_1, \ldots, V_k$ is the maximal conductance of a cut V_i for $i = 1, \ldots k$ with $V_i \neq \emptyset, V$.*

Proof. Without loss of generality, we can assume that the clusters V_i are non-empty (otherwise, we could remove the empty clusters from the partition whothout changing on eitther side of the equation discussed). Per definition of conductance of a k-way partition, there exists a subset $\emptyset \subsetneq I \subsetneq \{1, \ldots, k\}$ such that $\varphi(V_1, \ldots V_k) = \varphi(\cup_{i \in I} V_i)$. Without loss of generality, we can further assume that the volume of the union of the clusters V_i for $i \in I$ is less than or equal to the volume of its complement, i. e., the volume of the union of all the clusters V_i for $i \notin I$. Then we can write:

$$\begin{aligned}
\varphi(V_1, \ldots, V_k) &= \frac{\omega(\partial(\cup_{i \in I} V_i))}{\min(\mathrm{vol}(\cup_{i \in I} V_i), \mathrm{vol}(\cup_{i \notin I} V_i))} = \frac{\sum\{\omega(e) : e - \text{a cutting edge of } \partial(\cup_{i \in I} V_i)\}}{\mathrm{vol}(\cup_{i \in I} V_i)} \\
&\leq \frac{\sum_{i \in I} \omega(\partial V_i)}{\sum_{i \in I} \mathrm{vol}(V_i)} = \frac{\sum_{i \in I} \mathrm{vol}(V_i) \cdot \frac{\omega(\partial V_i)}{\mathrm{vol}(V_i)}}{\sum_{i \in I} \mathrm{vol}(V_i)} \leq \max_{i \in I} \frac{\omega(\partial V_i)}{\mathrm{vol}(V_i)} \\
&\leq \max_{i \in I} \frac{\omega(\partial V_i)}{\min(\mathrm{vol}(V_i), \mathrm{vol}(V \setminus V_i))} = \max_{i \in I} \varphi(V_i) \\
&\leq \varphi(V_1, \ldots, V_k)
\end{aligned}$$

Thus, all abothe written inequalities are equalities, which means that the conductance of the k-way partition is equal to the maximal conductance of a cut V_i for $i = 1, \ldots k$.

It is interesting to note that with this we have also proven that the conductance of a hypergraph partition is equal to the maximum ratio of the weight of a cut V_i to the volume of the cluster V_i over all the clusters $V_1, \ldots V_k$. This property, however, was not used in the implementation due to late discovery. □

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