Statistical Machine Learning – Exercise 2 – Michael Meidlinger

1 Bayes Classifier

a) We have $p(x,y) = p(y|x)p(x) = p(x|y)P(y)$ (1)

- c_1 is NOT equivalent. In fact, $p(x)$ is independent of y so that $\arg\max_y p(x)$ can be defined to output any arbitrary number
- c_2 is NOT equivalent. The output is independent of x always the least likely label
- c_3 is equivalent because of (1) and the fact that $p(x)$ is positive and independent of y
- c_4 is NOT equivalent, since $p(x|y) = p(y|x)\frac{p(x)}{p(y)}$

b.) Using a similar argumentation as above we have

c_5, c_7, c_8, c_9, c_{10}, c_{12} equivalent, the rest not

2 Gaussian Discriminant Analysis

a) $$c(x) = \arg\max_{y\in\mathcal{Y}} p(y|x) = \arg\max_{y\in\mathcal{Y}} P(x|y)\,p(y) \stackrel{|y\in\{-1,1\}|}{=} \operatorname{sign}\left(\log\frac{p(x|1)p(y=1)}{p(x|-1)p(y=-1)}\right)$$

$$= \operatorname{sign}\left(\log\left(\exp\left(-\tfrac{1}{2}(\underline{x}-\mu_1)^T\underline{\underline{\Sigma}}^{-1}(\underline{x}-\mu_1) + \tfrac{1}{2}(\underline{x}-\mu_{-1})^T\Sigma^{-1}(\underline{x}-\mu_{-1}\right) + \log\frac{p(y=1)}{p(y=-1)}\right)\right)$$

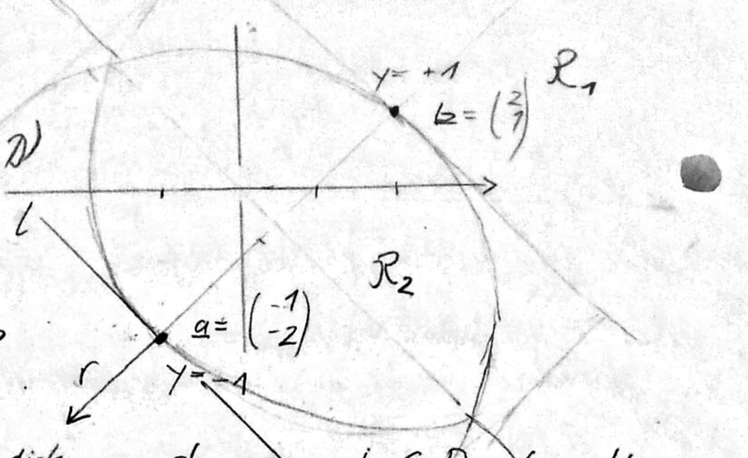
quadratic $\underline{x}$ terms cancel, quadratic μ terms can be dropped (independent of y)

$$= \operatorname{sign}\Big(\underbrace{\underline{x}^T\Sigma^{-1}\mu_1 - \underline{x}^T\Sigma^{-1}\mu_{-1}}_{\underline{x}^T\underbrace{(\Sigma^{-1}\mu_1 - \Sigma^{-1}\mu_{-1})}_{:=\underline{w}}} + \underbrace{\log\frac{p(y=1)}{p(y=-1)}}_{\theta}\Big)$$

$$= \operatorname{sign}(\langle\underline{x},\underline{w}\rangle + \theta)$$ q.e.d.

b.) With a few examples, the exact distributions cannot be estimated precisely. Thus one has to resort to parametric models of the distribution.

3 Robustness of the Perceptron



The robustnes ρ of a classifier g (with respect to $\mathcal{D}$) is the largest ammount by which we can perturb the training samples without changing the predictions of g:

$$g(x^i + \varepsilon) = g(x^i) \quad \forall i \quad \forall \varepsilon : \|\varepsilon\| < \rho$$

For a linear classifier ρ is the smallest distance of any $x^i \in D_x$ from the decision boundary. For the specifed training data with points $x^i \in \mathbb{R}^2$ we can distinguish the following cases:

a.) $\begin{pmatrix} a \\ b \end{pmatrix}$ is an element of the ray $r \Rightarrow D$ is not linearly separable and the perceptron algorithm won't converge

b.) $\begin{pmatrix} a \\ b \end{pmatrix}$ is an element of $R_2 := \left\{ \underline{x} \in \mathbb{R}^2 \,\middle|\, \|\underline{x} - \underline{a}\| < \frac{\|\underline{b}-\underline{a}\|}{2} \wedge \|\underline{x} - \underline{b}\| < \frac{\|\underline{b}-\underline{a}\|}{2} \right\}$

For that case, ρ is given by

$$\rho = \frac{\|\underline{x} - \underline{a}\|}{2} = \frac{1}{2} \left\| \begin{pmatrix} a+1 \\ b+2 \end{pmatrix} \right\| = \frac{1}{2} \sqrt{(a+1)^2 + (b+2)^2}$$

c.) $\begin{pmatrix} a \\ b \end{pmatrix}$ is in $\Rightarrow$ $\rho = \frac{1}{2} \frac{\|\underline{b} - \underline{a}\|}{2} = \frac{1}{2} \frac{\sqrt{3^2 + 3^2}}{2 \cdot 9} = \frac{3}{\sqrt{2}}$

d.) $\begin{pmatrix} a \\ b \end{pmatrix} \in R_2 \setminus \{R_1 \cup R_2 \cup r\}$

The decision boundary is the line $d: \frac{x_0 + x}{2} + \lambda (\underline{b} - \underline{x})$ where

$$\underline{x}_0 = \underset{\underline{v} \in \{\underline{a}, \underline{b}\}}{\arg\min} \|\underline{v} - \underline{x}\| \quad \Rightarrow \quad \rho = \frac{\left(\frac{\underline{x} - \underline{x}_0}{2}\right) \times (\underline{b} - \underline{x})}{\|\underline{b} - \underline{x}\|} = \Bigg\{$$

- ρ_{max} is thus given by $3/\sqrt{2}$

4 Perceptron Training as Convex Optimization

Comparing Algorithm 1 with "Stochastic Gradient Descent" (ml 2014_09.pdf) we identify:

$$w_{t+1} \leftarrow w_t - \eta_t \overbrace{v}^{-n\nabla f_i(w_t) \text{ with random } i} \iff w_{t+1} \leftarrow w_t + yx$$

$$\hookrightarrow \quad -\eta_t n \nabla_w f_i(w_t) = \begin{cases} y^i x^i, & y\langle w_t, x\rangle \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \eta_t = const = 1 \quad , \quad [x]^+ = \begin{cases} x, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_i(w) = \frac{1}{n}[-y^i\langle w, x^i\rangle]^+ \Rightarrow f(w) = \sum_{i=1}^n f_i(w) = \overbrace{\frac{1}{n}\sum_{i=1}^n [-y^i\langle w, x^i\rangle]^+}^{\text{Average of inner products is the cost function to be minimized}}$$

Advantages:

Shortcomings:

5 Hard-Margin SVM Dual

$$\min_{w\in\mathbb{R}^d, b\in\mathbb{R}} \frac{1}{2}\|w\|^2 \quad \text{subject to} \quad y^i(\langle w, x^i\rangle + b) \geq 1$$

We compute the Lagrangian:

$$\mathcal{L}(w, b, \alpha) = \frac{1}{2}\|w\|^2 + \overbrace{\sum_i \alpha_i(1 - y^i(\langle w, x^i\rangle + b))}^{\sum_i \alpha_i - \sum_i \alpha_i y^i\langle w, x^i\rangle - b\sum_i \alpha_i y^i} \quad , \quad h(\alpha) = \min_{(w,b)} \mathcal{L}(w, b, \alpha)$$

$$\Rightarrow \quad 0 = \partial_w \mathcal{L} = w - \sum_i \alpha_i y_i x^i \Rightarrow w = \sum_i \alpha y^i x^i$$

$$0 = \partial_b \mathcal{L} = \sum_i \alpha_i y^i$$

$\hookrightarrow$ Insert back:

$$h(\alpha) = \frac{1}{2}\left\|\sum_i \alpha_i y^i x^i\right\|^2 + \sum_i \alpha_i - \overbrace{\sum_i \alpha_i y^i \left\langle \sum_j \alpha_j y^j x^j, x^i\right\rangle}^{\|\sum_i \alpha_i y^i x^i\|^2}$$

$$= -\frac{1}{2}\left\|\sum_i \alpha_i y^i x^i\right\|^2 + \sum_i \alpha_i$$

$$= -\frac{1}{2}\left\langle \sum_i \alpha_i y^i x^i, \sum_j \alpha_j y^j x^j\right\rangle + \sum_i \alpha_i = -\frac{1}{2}\sum_i\sum_j \alpha_i\alpha_j y^i y^j\langle x^i, x^j\rangle + \sum_i \alpha_i$$

Dual Problem: $\min_{\alpha > 0} h(\alpha)$ subject to

$$\sum_i \alpha_i y^i = 0 \quad ,$$

6 Missing Proofs

$= \max\{f_1(w), \ldots, f_K(w)\}$

a) We need to show $f_k(w) \geq f_k(w_0) + \langle v, w - w_0 \rangle \Longrightarrow f(w) \geq f_k(w_0) + \langle v, w - w_0 \rangle \quad \forall k$

↳ Use assumption

$$f(w) = \max_{k'=1,\ldots,K} f_{k'}(w) \geq f_k(w) \geq \underbrace{f_k(w_0)}_{= f(w_0) \text{ (assumption)}} + \langle v, w - w_0 \rangle = f(w_0) + \langle v, w - w_0 \rangle \quad \square$$

b)
$$\|\underbrace{w_{t+1}}_{w_t - \eta v} - w^*\|^2 = \|w_{t+1}\|^2 + \|w^*\|^2 - 2\langle w_{t+1}, w^* \rangle = \|w_t - w^*\|^2$$

$$= \|w_t\|^2 + \|w^*\|^2 - 2\langle w_t, w^* \rangle + 2\eta \langle v, w^* \rangle + \eta^2 \|v\|^2 \quad (1)$$

$$\|w_t - w^*\|^2 = \|w_t\|^2 + \|w^*\|^2 - 2\langle w_t, w^* \rangle \quad (2)$$

v is a subgradient at $w_t \Longleftrightarrow f(w) \geq f(w_t) + \langle v, w - w_t \rangle$, in particular, for $w = w_{t+1}$

$$f(w_{t+1}) - f(w_t) \geq \langle v, -\eta v \rangle \Longleftrightarrow f(w_t) - f(w_{t+1}) \leq \eta \|v\|^2 \quad (3)$$

$$f(\theta w_{t+1} + (1-\theta) w_t) = f(-\eta \theta v + w_t)$$

$$\leq \theta f(w_{t+1}) + (1-\theta) f(w_t) = f(w_t) + \theta [f(w_{t+1}) - f(w_t)]$$

We now show $\|w_{t+1} - w^*\| < \|w_t - w^*\|$ by showing $\|w_t - w^*\|^2 - \|w_{t+1} - w^*\|^2 > 0$, which is equivalent to ((1), (2)) showing

$$\|v\|^2 \eta < 2\langle v, w^* \rangle$$