Assignment 2, question 2. Adm no 89431

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1 Question 2: Generalized Black–Scholes PDE under Stochastic Volatility

We are given the dynamics of the asset price and variance processes: $\mathrm{dS}_t = \mu S_t\,dt + \sigma_t S_t\,dZ_t$,
$dv_t = \kappa(\theta - v_t)\,dt + \gamma\sqrt{v_t}\,dW_t$, where Z_t and W_t are Wiener processes with correlation

$$\mathrm{Cov}(dZ_t, dW_t) = \rho\,dt,$$

and $v_t = \sigma_t^2$.

Derivation

Let the option price be $U(S, v, t)$. Applying Itô's Lemma for a two-dimensional diffusion gives: $\mathrm{dU} = \partial U \frac{}{\partial t dt + \frac{\partial U}{\partial S}\,dS + \frac{\partial U}{\partial v}\,dv \quad + 12\,\frac{\partial^2 U}{\partial S^2}\,(dS)^2 + 12\,\frac{\partial^2 U}{\partial v^2}\,(dv)^2 + \frac{\partial^2 U}{\partial S \partial v}\,dS\,dv.}$

Using the variance of increments:

$$(dS)^2 = \sigma_t^2 S^2 dt = vS^2 dt, \quad (dv)^2 = \gamma^2 v\,dt, \quad dS\,dv = \rho\sigma_t S \cdot \gamma\sqrt{v}\,dt = \rho\gamma v S\,dt,$$

we obtain $\mathrm{dU} = \partial U \frac{}{\partial t dt + \frac{\partial U}{\partial S}\left(\mu S\,dt + \sigma_t S\,dZ\right) + \frac{\partial U}{\partial v}\left(\kappa(\theta - v)\,dt + \gamma\sqrt{v}\,dW\right) \quad + 12 v S^2\,\frac{\partial^2 U}{\partial S^2}\,dt + 12\gamma^2 v\,\frac{\partial^2 U}{\partial v^2}\,dt + \rho\gamma v S\,\frac{\partial^2 U}{\partial v \partial S}\,dt.}$

Risk-neutral valuation

Under the risk-neutral measure, the drift of S is rS instead of μS, and the option value discounted at the risk-free rate r must be a martingale. Incorporating the market price of volatility risk λ, the drift of v becomes:

$$\Big[\kappa(\theta - v) - \lambda v\Big]\,dt.$$

Eliminating the stochastic terms and requiring the drift of the discounted option price to vanish leads to the generalized Black–Scholes PDE: $1 \frac{}{2vS^2\,\frac{\partial^2 U}{\partial S^2} + \rho\gamma v S\,\frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\gamma^2 v\,\frac{\partial^2 U}{\partial v^2} + rS\,\frac{\partial U}{\partial S} + \Big[\kappa(\theta - v) - \lambda v\Big]}$