Branch and Price with PySCIPOpt

School on Column Generation 2025

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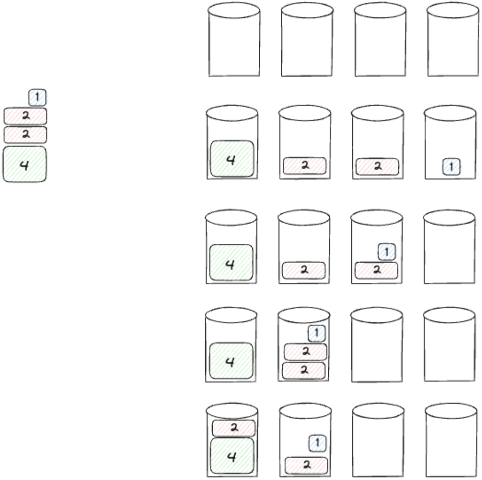
Zuse Institute Berlin, University of Porto, RWTH Aachen University

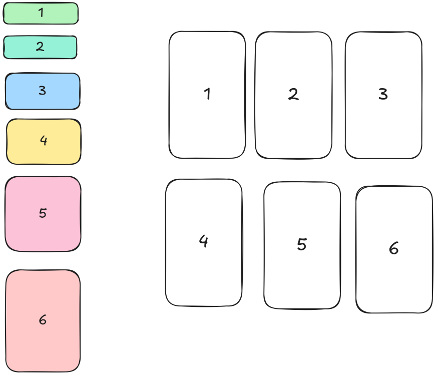
Bin Packing

- Need to store items in bins
- Items have weight and bins have capacity
- Use minimum bins with items not exceeding their capacity



Compact Formulation

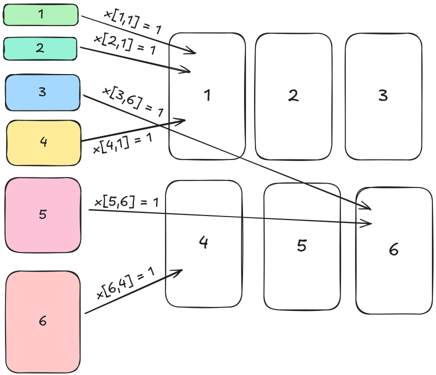
How can we formulate this?



Compact Formulation

How can we formulate this?

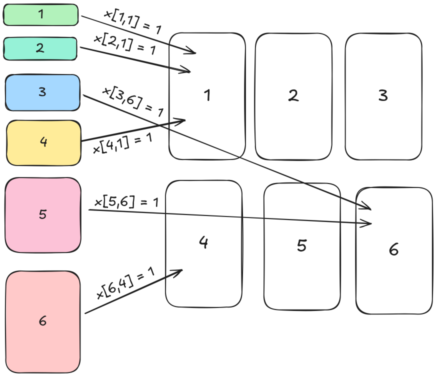
- Variable saying where each item is packed



Compact Formulation

How can we formulate this?

- Variable saying where each item is packed
- Enforce all items are packed



Compact Formulation

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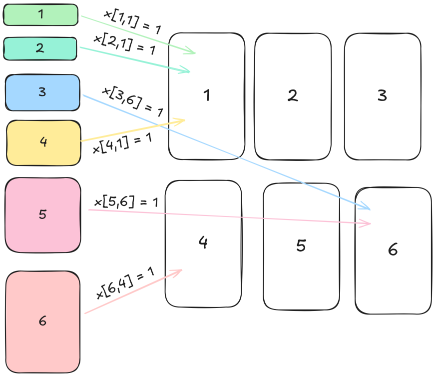
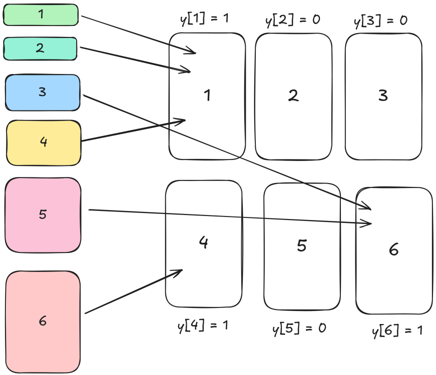
- Variable saying where each item is packed
- Enforce all items are packed
- Capacity constraints



Compact Formulation

How can we formulate this?

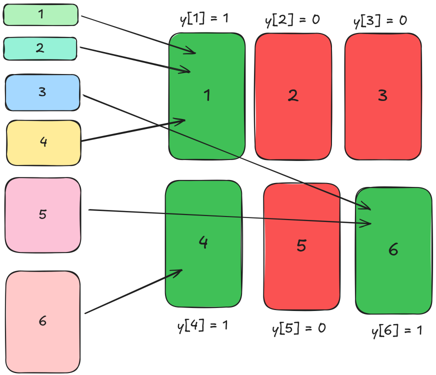
- Variable saying where each item is packed
- Enforce all items are packed
- Capacity constraints
- Variable saying whether a bin is being used



Compact Formulation

How can we formulate this?

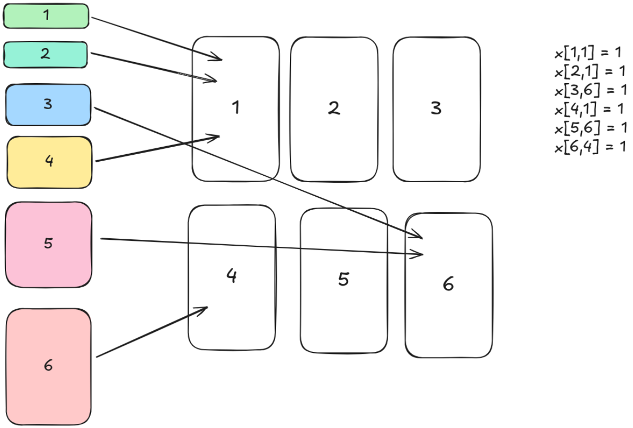
- Variable saying where each item is packed
- Enforce all items are packed
- Capacity constraints
- Variable saying whether a bin is being used
- Minimize the number of used bins



Compact formulation and its poor scaling

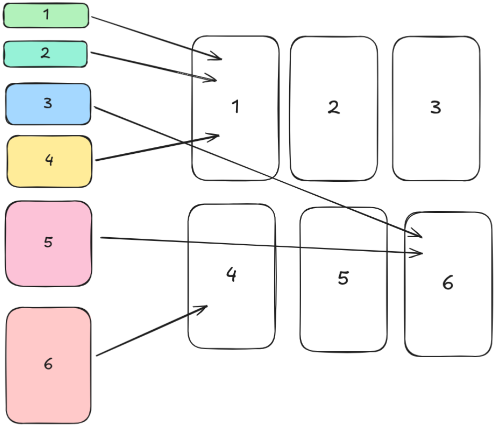
DEMO

Why doesn’t this work?



It doesn’t seem complicated…

Why doesn't this work?



x[1,1] = 1
x[2,1] = 1
x[3,6] = 1
x[4,1] = 1
x[5,6] = 1
x[6,4] = 1

x[1,2] = 0
x[1,3] = 0
x[5,3] = 0
x[6,6] = 0
...

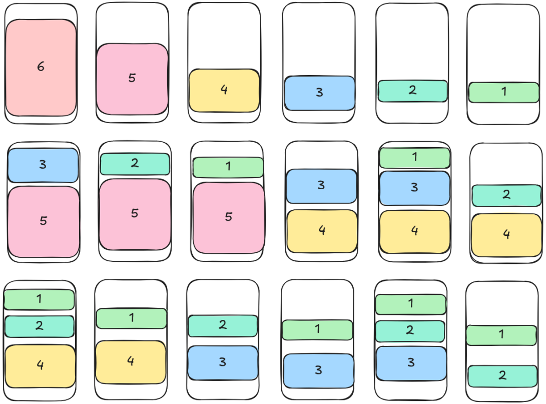
Extended Formulation: Modeling with Packings

We need a new formulation. Let's change our perspective.

Extended Formulation: Modeling with Packings

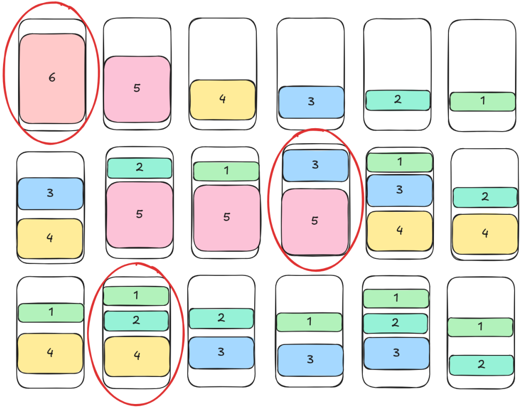
We need a new formulation. Let's change our perspective.

Let's look at all the ways of doing this (packings) and choose the best combination.



What does a solution look like?

p[1] = 1
p[2] = 0
...
p[10] = 1
...
p[14] = 1
...
p[20] = 0



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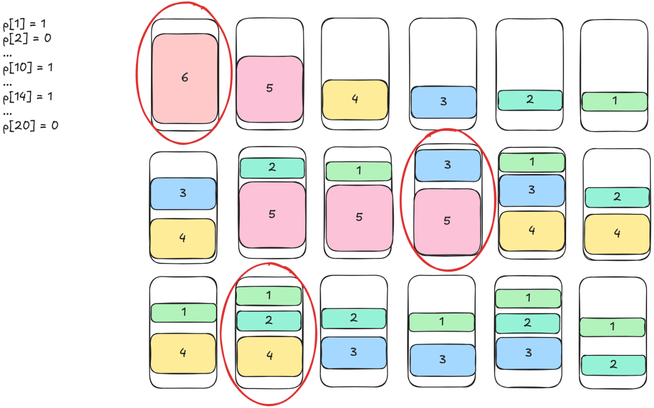

Problem: There is an exponential number of packings.

Integer Master Problem

For a list of all feasible packings $\mathcal{P}$, $a_i^p = 1$ if item $i \in \mathcal{I}$ is in packing p.

$$
\begin{aligned}
\min \quad & \sum_{p \in \mathcal{P}} z_p \\
\text{s.t.} \quad & \sum_{p \in \mathcal{P}} a_i^p z_p = 1, \forall i \in \mathcal{I} \quad (\pi_i) \\
& z_p \in \{0, 1\}, \forall p \in \mathcal{P}
\end{aligned}
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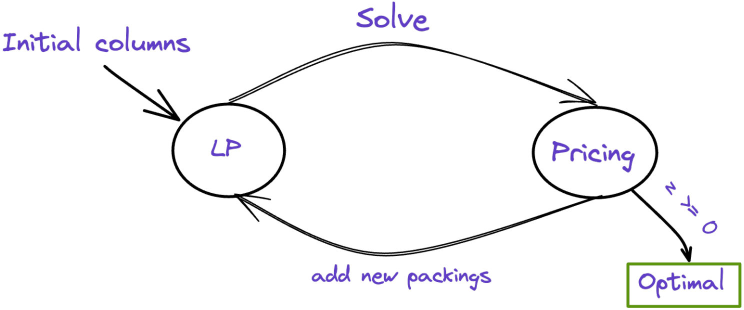
- Problem: There is an exponential number of packings.
- Solution: **Branch and Price!**

Game Plan

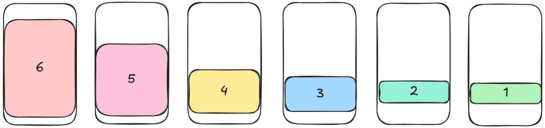
1. Solve the LP relaxation with Column Generation.
2. Embed in a branch-and-bound scheme to get optimal integer solution.
3. Improve our solver!

First Step: Solving the LP relaxation



Initial Columns

We need to initialize the RMP with some packings.



Let's go with the simplest way of assigning one item per packing.

Generating new columns

$$
\begin{aligned}
\min \quad & \sum_{p \in \mathcal{P}'} z_p \\
\text{s.t.} \quad & \sum_{p \in \mathcal{P}'} a_i^p z_p = 1, \forall i \in \mathcal{I} \quad (\pi_i) \\
& 0 \leq z_p \leq 1, \forall p \in \mathcal{P}'
\end{aligned}
$$

How can we know which columns to add?

Generating new columns

$$\begin{aligned} \min \quad & \sum_{p \in \mathcal{P}'} z_p \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}'} a_i^p z_p = 1, \forall i \in \mathcal{I} \quad (\pi_i) \\ & 0 \le z_p \le 1, \forall p \in \mathcal{P}' \end{aligned}$$

How can we know which columns to add? **Reduced Cost < 0**

$$\text{minimize} \quad \underbrace{1}_{\text{obj. fn. coefficient}} \quad - \quad \overbrace{\sum_{i \in \mathcal{I}} a_i^p \pi_i}^{\text{column coefs. * dual vector}}$$

Pricing for Bin Packing

$$
\begin{aligned}
\text{minimize} \quad & 1 - \sum_{i \in \mathcal{I}} a_i \pi_i \\
\text{subject to} \quad & \sum_{i \in \mathcal{I}} s_i a_i \leq C \\
& a_i \in \{0, 1\}, \quad \forall i \in \mathcal{I}
\end{aligned}
$$

Pricing for Bin Packing

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Let's rework the objective function.

Pricing for Bin Packing

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$$\min 1 - \sum_{i \in \mathcal{I}} a_i \pi_i = 1 + \min - \sum_{i \in \mathcal{I}} a_i \pi_i$$

Pricing for Bin Packing

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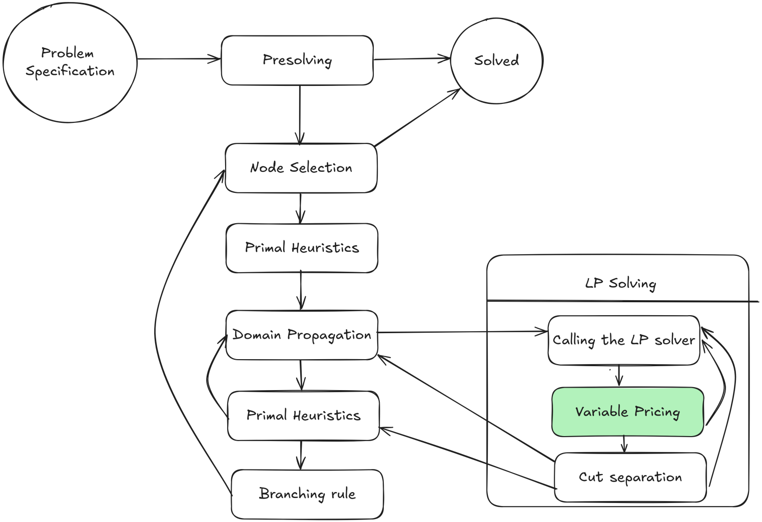
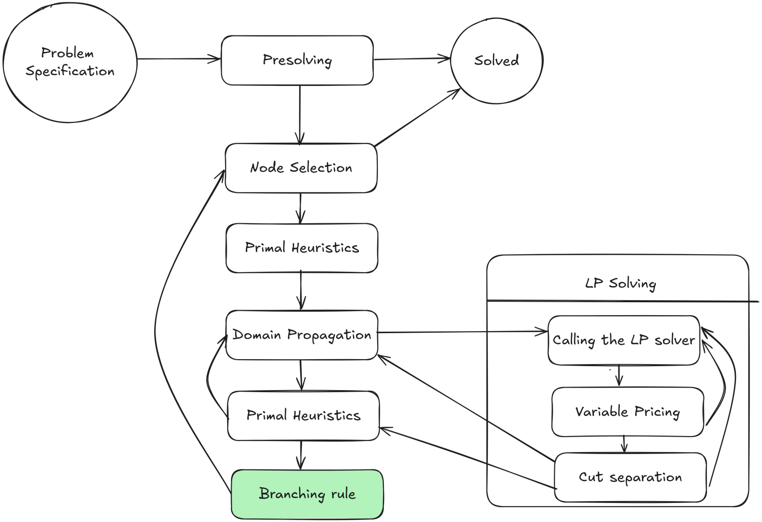
Let's rework the objective function.

$\min 1 - \sum_{i \in \mathcal{I}} a_i \pi_i = 1 + \min - \sum_{i \in \mathcal{I}} a_i \pi_i = 1 - \max \sum_{i \in \mathcal{I}} a_i \pi_i \rightarrow 1$ - **Knapsack!**

How to implement this in SCIP?



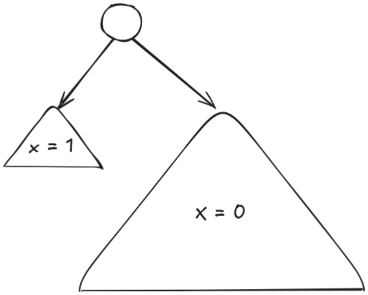
Getting Integer Solutions



Branching on master variables

Branching on master variable x has 2 options:

1. $x = 1$: we force the packing. Very restrictive
2. $x = 0$: we forbid the packing. Not restrictive at all.

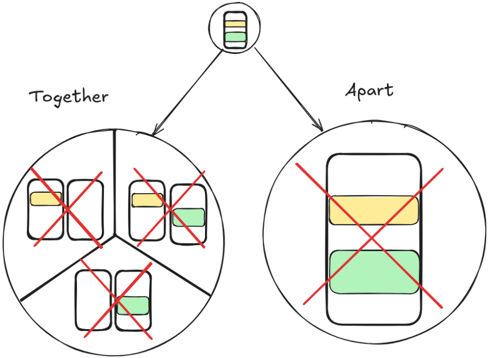


Leads to very unbalanced trees...

Ryan Foster Branching

Here there are two options:

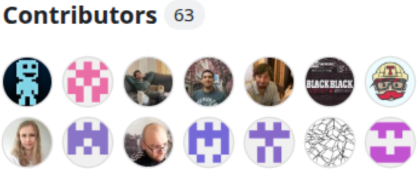
1. Forbid two items from appearing in a new packing
2. Ensure that they appear in the same packing



PySCIPOpt Intermission

As an open-source solver, SCIP (and PySCIPOpt!) appreciates the help of its users!

It's your chance to be our 900th star :)



Many contributors started as users!