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rans(eXtreme) Analysis Framework for Compressible Hydrodynamic Simulations

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ABSTRACT
ransX ..

Key words: turbulence – analysis – RANS

1 INTRODUCTION

Analysis of hydrodynamic simulations was in past focused either on calculation of simple averaged quantities (like rms vel, Fourier spectra etc.) (citations), mathematical exploration of RANS equations only (citations), or few individual flux calculations (citations). Complete RANS analysis has only been done by Meakin, Viallet, Mocak, Campbell, Hirchi, Arnett. The presented RANS framework called ransX[1] is supposed to promote and strealime such complete RANS analysis of 3D hydrodynamic fully compressible multi-elements simulations of turbulence in stars (no rotation, no magnetic fields).

We obtain our 1D RANS equations by introducing two types of averaging: statistical averaging and horizontal averaging (Besnard et al. 1992; Viallet et al. 2013). In practice, statistical averages are computed by performing a time average (the ergodic hypothesis). Therefore, the combined average of a quantity q is defined as

$$\overline{q}(r,t) = \frac{1}{T\Delta\Omega}\int_{t-T/2}^{t+T/2} q(r,\theta,\phi,t')\ d\Omega\ dt' \tag{1}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle in spherical coordinates, T is the averaging time period, and $\Delta\Omega$ is total solid angle being averaged over.

The flow variables are then decomposed into mean and fluctuation $q = \overline{q} + q'$, noting that $\overline{q'} = 0$ by construction. Similarly, we introduce Favre (or density weighted) averaged quantities by

$$\widetilde{q} = \frac{\overline{\rho q}}{\overline{\rho}} \tag{2}$$

which defines a complimentary decomposition of the flow into mean and fluctuations according to $q = \widetilde{q} + q''$. Here, q'' is the Favrian fluctuation and its mean is zero when Favre averaged $\widetilde{q''} = 0$.

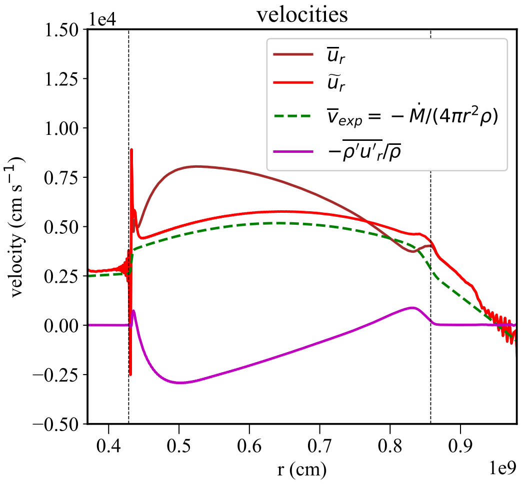

Figure 1. Reynolds and Favre velocities derived from 3D oxygen burning shell model.

Such decomposition gives us the advantage not only to understand compressible hydrodynamics better but to look at some traditional Reynolds mean fields from a new perspective. For example, it allows us to decompose mean Reynolds velocity $\overline{u}_r$ into its background expansion ($\widetilde{u}_r = v_{exp}$ how do we prove this mathematically?) and density scaled turbulent mass flux $-\overline{\rho' u'_r}/\overline{\rho} = \overline{u''}_r$ (Fig.1).

$$\overline{u}_r = \widetilde{u}_r - \overline{u''}_r \tag{3}$$

For a more complete elaboration on the algebra of these averaging procedures we refer the reader to Chassaing et al. (2010).

2 GENERAL FORM OF RANS EQUATIONS

The RANS equations terms can be split in general into:

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[1] https://github.com/mmicromegas/ransX

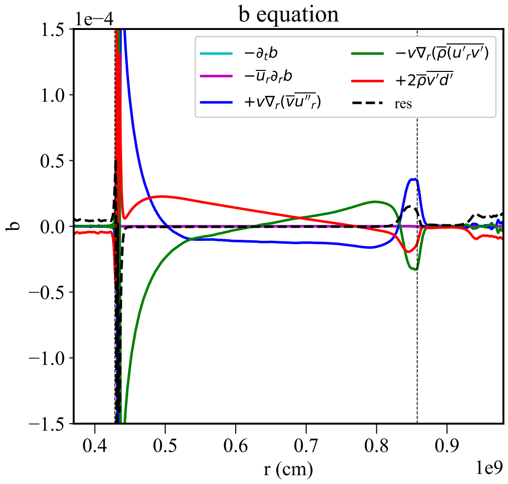

Figure 2. Example RANS equation for density-specific volume covariance.

- time-dependence $\mathcal{T}$
- redistribution (the div terms) $\mathcal{R}$
- production/destruction $\mathcal{P}, \mathcal{D}$
- geometry terms $\mathcal{G}$
- numerical residuals $\mathcal{N}$

General form of RANS equations is:

$$\mathcal{T} = \mathcal{R} + \mathcal{P} + \mathcal{D} + \mathcal{G} + \mathcal{N} \tag{4}$$

3 TIME-DEPEDENCE TERMS

4 REDISTRIBUTION TERMS

5 PRODUCTION/DESTRUCTION TERMS

6 GEOMETRY TERMS

7 NUMERICAL RESIDUALS

8 SOFTWARE IMPLEMENTATION

All presented equations (including their Cartesian geometry equivalents) were implemented into a software framework done in Python (https://github.com/mmicromegas/ransX), which relies on a specific space-time averaged field output currently programmed into hydrodynamic PROMPI code only, but which can easily be implemented in any other hydrodynamic code too. Using the ransX framework is then just a matter of working with the specific hydrodynamic output.

REFERENCES

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Chassaing P., Antonia R., Anselmet F., Joly L., Sarkar R., 2010, Variable Density Fluid Turbulence. Kluwer Academic Publishers

Viallet M., Meakin C., Arnett D., Mocák M., 2013, ApJ, 769, 1

This paper has been typeset from a TEX/LATEX file prepared by the author.

[t]

Table 1. 1D RANS equations in Lagrangian form.

$$\widetilde{D}_t\overline{\rho} = -\overline{\rho}\widetilde{d} + \mathcal{N}_\rho \quad (1)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r + \mathcal{N}_{ur} \quad (2)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\theta = -\nabla_r\widetilde{R}_{\theta r} - \overline{G_\theta^M} - (1/r)\overline{\partial_\theta P} + \mathcal{N}_{u\theta} \quad (3)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{u}_\phi = -\nabla_r\widetilde{R}_{\phi r} - \overline{G_\phi^M} + \mathcal{N}_{u\phi} \quad (4)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_I = -\nabla_r(f_I + f_T) - \overline{P}\ \overline{d} - W_P + \mathcal{S} + \mathcal{N}_{\epsilon I} \quad (5)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_K = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon k} \quad (6)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_I + f_T + f_k + f_P) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + \mathcal{S} + \overline{\rho}\widetilde{D}_t(\widetilde{u}_i\widetilde{u}_i/2) + \mathcal{N}_{\epsilon t} \quad (7)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha = -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}}_\alpha^{\rm nuc} + \mathcal{N}_\alpha \quad (8)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{s} = -\nabla_r f_s + \overline{(\nabla\cdot F_T)/T} + \overline{\mathcal{S}/T} + \mathcal{N}_s \quad (9)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{A} = -\nabla_r f_A - \overline{\rho A^2\Sigma_\alpha(\dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{N}_A \quad (10)$$
$$\overline{\rho}\widetilde{D}_t\widetilde{Z} = -\nabla_r f_Z - \overline{\rho Z A\Sigma_\alpha(\dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \overline{\rho A\Sigma_\alpha(Z_\alpha\dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{N}_Z \quad (11)$$
$$\overline{D}_t\overline{P} = -\nabla_r f_P - \Gamma_1\overline{P}\ \overline{d} + (1-\Gamma_1)W_P + (\Gamma_3 - 1)\mathcal{S} + (\Gamma_3 - 1)\nabla_r f_T + \mathcal{N}_P \quad (12)$$
$$\overline{D}_t\overline{T} = -\nabla_r f_T + (1-\Gamma_3)\overline{T}\ \overline{d} + (2-\Gamma_3)\overline{T'd'} + \overline{(\nabla\cdot F_T)/\rho c_v} + \overline{(\tau_{ij}\partial_i u_j)/\rho c_v} + \overline{\epsilon_{\rm nuc}/c_v} + \mathcal{N}_T \quad (13)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{k} = -\nabla_r(f_k + f_P) - \widetilde{R}_{ir}\partial_r\widetilde{u}_i + W_b + W_P + \mathcal{N}_k \quad (14)$$

$$\overline{\rho}\widetilde{D}_t\overline{u''_r} = -(\overline{\rho' u'_r u'_r}/\overline{\rho})\partial_r\overline{\rho} + (\widetilde{R}_{rr}/\overline{\rho})/\partial_r\overline{\rho} - \overline{\rho}\nabla_r(\overline{u''_r}\ \overline{u''_r}) + \nabla_r\overline{\rho' u'_r u'_r} - \overline{\rho}\overline{u''_r}\nabla_r\overline{u}_r + \overline{\rho}\overline{u'_r d''} - b\partial_r\overline{P} + \overline{\rho' v\partial_r P'} + \mathcal{G}_a + \mathcal{N}_a \quad (15)$$
$$\overline{D}_t b = +\overline{v}\nabla_r\overline{\rho}\overline{u''_r} - \overline{\rho}\nabla_r(\overline{u'_r v'}) + 2\overline{\rho}\overline{v'd'} + \mathcal{N}_b \quad (16)$$

$$\overline{\rho}\widetilde{D}_t(f_I/\overline{\rho}) = -\nabla_r f_I^r - f_I\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{\epsilon}_I - \overline{\epsilon''_I}\partial_r\overline{P} - \overline{\epsilon''_I\partial_r P'} - \overline{u''_r(Pd)} + \overline{u''_r(\mathcal{S} + \nabla\cdot F_T)} + \mathcal{G}_I + \mathcal{N}_{fI} \quad (17)$$
$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{s} - \overline{s''}\partial_r\overline{P} - \overline{s''\partial_r P'} + \overline{u''_r(\mathcal{S} + \nabla\cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \quad (18)$$
$$\overline{\rho}\widetilde{D}_t(f_\alpha/\overline{\rho}) = -\nabla_r f_\alpha^r - f_\alpha\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_\alpha - \overline{X''_\alpha}\partial_r\overline{P} - \overline{X''_\alpha\partial_r P'} + \overline{u''_r\rho\dot{X}_\alpha^{\rm nuc}} + \mathcal{G}_\alpha + \mathcal{N}_{f\alpha} \quad (19)$$
$$\overline{\rho}\widetilde{D}_t(f_A/\overline{\rho}) = -\nabla_r f_A^r - f_A\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{A} - \overline{A''}\partial_r\overline{P} - \overline{A''\partial_r P'} - \overline{u''_r\rho A^2\Sigma_\alpha\dot{X}_\alpha^{\rm nuc}/A_\alpha} + \mathcal{G}_A + \mathcal{N}_{fA} \quad (20)$$
$$\overline{\rho}\widetilde{D}_t(f_Z/\overline{\rho}) = -\nabla_r f_Z^r - f_Z\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{Z} - \overline{Z''}\partial_r\overline{P} - \overline{Z''\partial_r P'} - \overline{u''_r\rho Z A\Sigma_\alpha(\dot{X}_\alpha^{\rm nuc}/A_\alpha)} - \overline{u''_r\rho A\Sigma_\alpha(Z_\alpha\dot{X}_\alpha^{\rm nuc}/A_\alpha)} + \mathcal{G}_Z + \mathcal{N}_{fZ} \quad (21)$$
$$\widetilde{D}_t f_{pr} = -\nabla_r f_p^r - f_{pr}\partial_r\overline{u}_r + \overline{u'_r u''_r}\partial_r\overline{P} + \Gamma_1\overline{u'_r Pd} + (\Gamma_3 - 1)\overline{u'_r\rho\varepsilon_{nuc}} + \overline{P'u''_r d''} - \overline{P'G_r^M/\rho} - \overline{P'\partial_r P/\rho} + \mathcal{N}_{fpr} \quad (22)$$
$$\overline{\rho}\widetilde{D}_t(f_h/\overline{\rho}) = -\nabla_r f_h^r - f_h\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{h} - \overline{h''}\partial_r\overline{P} - \overline{h''\partial_r P'} - \Gamma_1\overline{u''_r(Pd)} + \Gamma_3\overline{u''_r(\mathcal{S} + \nabla\cdot F_T)} + \mathcal{G}_h + \mathcal{N}_h \quad (23)$$
$$\overline{\rho}\widetilde{D}_t(f_s/\overline{\rho}) = -\nabla_r f_s^r - f_s\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{s} - \overline{s''}\partial_r\overline{P} - \overline{s''\partial_r P'} + \overline{u''_r(\mathcal{S} + \nabla\cdot F_T)/T} + \mathcal{G}_s + \mathcal{N}_{fs} \quad (24)$$

$$\overline{\rho}\widetilde{D}_t\sigma_\alpha = -\nabla_r(\overline{\rho X''_\alpha X''_\alpha u''_r}) - 2f_\alpha\partial_r\widetilde{X}_\alpha + 2\overline{X''_\alpha\rho\dot{X}_\alpha^{\rm nuc}} + \mathcal{N}_{\sigma_\alpha} \quad (25)$$

Definitions:

ρ density

T temperature

P pressure

u_r, u_θ, u_ϕ velocity components

$\mathbf{u}$ velocity

$d = \nabla \cdot \mathbf{u}$ dilatation

ϵ_I specific internal energy

$\epsilon_K = (1/2)u_i u_i$ specific kinetic energy

ϵ_t specific total energy

s specific entropy

$v = 1/\rho$ specific volume

X_α mass fraction of isotope α

$\dot{X}_\alpha^{nuc}$ rate of change of X_α

A_α number of nucleons in isotope α

Z_α charge of isotope α

A mean number of nucleons per isotope

Z mean charge per isotope

g_r gravitational acceleration

$\mathcal{S} = \overline{\rho}\epsilon_{\rm nuc}$ (q) nuclear energy production (cooling function)

$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)

$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate

$\widetilde{R}_{ij} = \overline{\rho}\widetilde{u''_i u''_j}$ Reynolds stress tensor

$\widetilde{k} = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy

$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy

$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy

$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy

$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizonatal turbulent kinetic energy

$f_k = (1/2)\overline{\rho}\widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux

$f_k^r = (1/2)\overline{\rho}\widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux

$f_k^\theta = (1/2)\overline{\rho}\widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux

$f_k^\phi = (1/2)\overline{\rho}\widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux

$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux

$W_p = \overline{P'd''}$ turbulent pressure dilatation

$W_b = \overline{\rho}\overline{u''_r}\widetilde{g}_r$ buoyancy energy

Table 2. Definitions (continued):

$f_P = \overline{P'u'_r}$ acoustic flux	$F_T = \chi \partial_r T$ full heat flux (χ thermal conductivity)
$f_T = \overline{\chi \partial_r T}$ turbulent heat flux	
$f_I = \overline{\rho}\widetilde{\epsilon''_I u''_r}$ internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X''_\alpha u''_r}$ X_α flux
$f_s = \overline{\rho}\widetilde{s''u''_r}$ specific entropy flux	$b = \overline{v'\rho'}$ density-specific volume covariance
$f_A = \overline{\rho}\widetilde{A''u''_r}$ A flux	$\mathcal{N}_\rho$ numerical effect
$f_Z = \overline{\rho}\widetilde{Z''u''_r}$ Z flux	$\mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}$ numerical effect
$f_\tau = f^r_\tau + f^\theta_\tau + f^\phi_\tau$ viscous flux	$\mathcal{N}_{\epsilon I} = +\varepsilon_k$ numerical effect
$f^r_\tau = -\overline{\tau'_{rr}u'_r}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f^\theta_\tau = -\overline{\tau'_{\theta r}u'_\theta}$ viscous flux	$\mathcal{N}_{\epsilon t}$ numerical effect
$f^\phi_\tau = -\overline{\tau'_{\phi r}u'_\phi}$ viscous flux	$\mathcal{N}_\alpha$ numerical effect
$f^h_\tau = f^\theta_\tau + f^\phi_\tau$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k/T}$ numerical effect
$f^r_I = \overline{\rho}\epsilon''_I\widetilde{u''_r u''_r}$ radial flux of f_I	$\mathcal{N}_A$ numerical effect
$f^r_s = \overline{\rho}s''\widetilde{u''_r u''_r}$ radial flux of f_s	$\mathcal{N}_Z$ numerical effect
$f^r_\alpha = \overline{\rho}\alpha''\widetilde{u''_r u''_r}$ radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f^r_A = \overline{\rho}A''\widetilde{u''_r u''_r}$ radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f^r_\tau - \varepsilon^r_k$ numerical effect
$f^r_Z = \overline{\rho}Z''\widetilde{u''_r u''_r}$ radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f^h_\tau - \varepsilon^h_k$ numerical effect
$\mathcal{G}^r_k = -(1/2)\overline{G^R_{rr}} - \overline{u''_r G^M_r}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect
$\mathcal{G}^\theta_k = -(1/2)\overline{G^R_{\theta\theta}} - \overline{u''_\theta G^M_\theta}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}^\phi_k = -(1/2)\overline{G^R_{\phi\phi}} - \overline{u''_\phi G^M_\phi}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I \tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}^h_k = +\mathcal{G}^\theta_k + \mathcal{G}^\phi_k$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j/T} - \varepsilon_s$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G^M_r}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{X''_\alpha \tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_I = -\overline{G^I_r} - \overline{\epsilon''_I G^M_r}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G^\alpha_r} - \overline{\epsilon''_\alpha G^M_r}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G^A_r} - \overline{A''G^M_r}$	$\mathcal{G}_Z = -\overline{G^Z_r} - \overline{Z''G^M_r}$
$\varepsilon^r_k = \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r}$	$\overline{G^M_r} = -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r}$
$\varepsilon^\theta_k = \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta}$	$\overline{G^M_\theta} = +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)}$
$\varepsilon^\phi_k = \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi}$	$\overline{G^M_\phi} = +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)}$
$\varepsilon_k = (1/2)(\varepsilon^r_k + \varepsilon^\theta_k + \varepsilon^\phi_k)$	$\overline{G^R_{rr}} = -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r}$
$\varepsilon^h_k = (1/2)(\varepsilon^\theta_k + \varepsilon^\phi_k)$	$\overline{G^R_{\theta\theta}} = +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)}$
$\varepsilon_a = \overline{\rho' v \nabla_r \tau'_{rr}}$	$\overline{G^R_{\phi\phi}} = +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)}$
$\varepsilon_I = \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I}$	$\overline{G^I_r} = -\overline{\rho\epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho\epsilon''_I u''_\phi u''_\phi/r}$
$\varepsilon_s = \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''}$	$\overline{G^s_r} = -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r}$
$\varepsilon_\alpha = \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha}$	$\overline{G^\alpha_r} = -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r}$
$\varepsilon_A = \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''}$	$\overline{G^A_r} = -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r}$
$\varepsilon_Z = \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''}$	$\overline{G^Z_r} = -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r}$

Differential operators

$$\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2.) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta.) + \frac{1}{r\sin\theta}\partial_\phi(.)$$